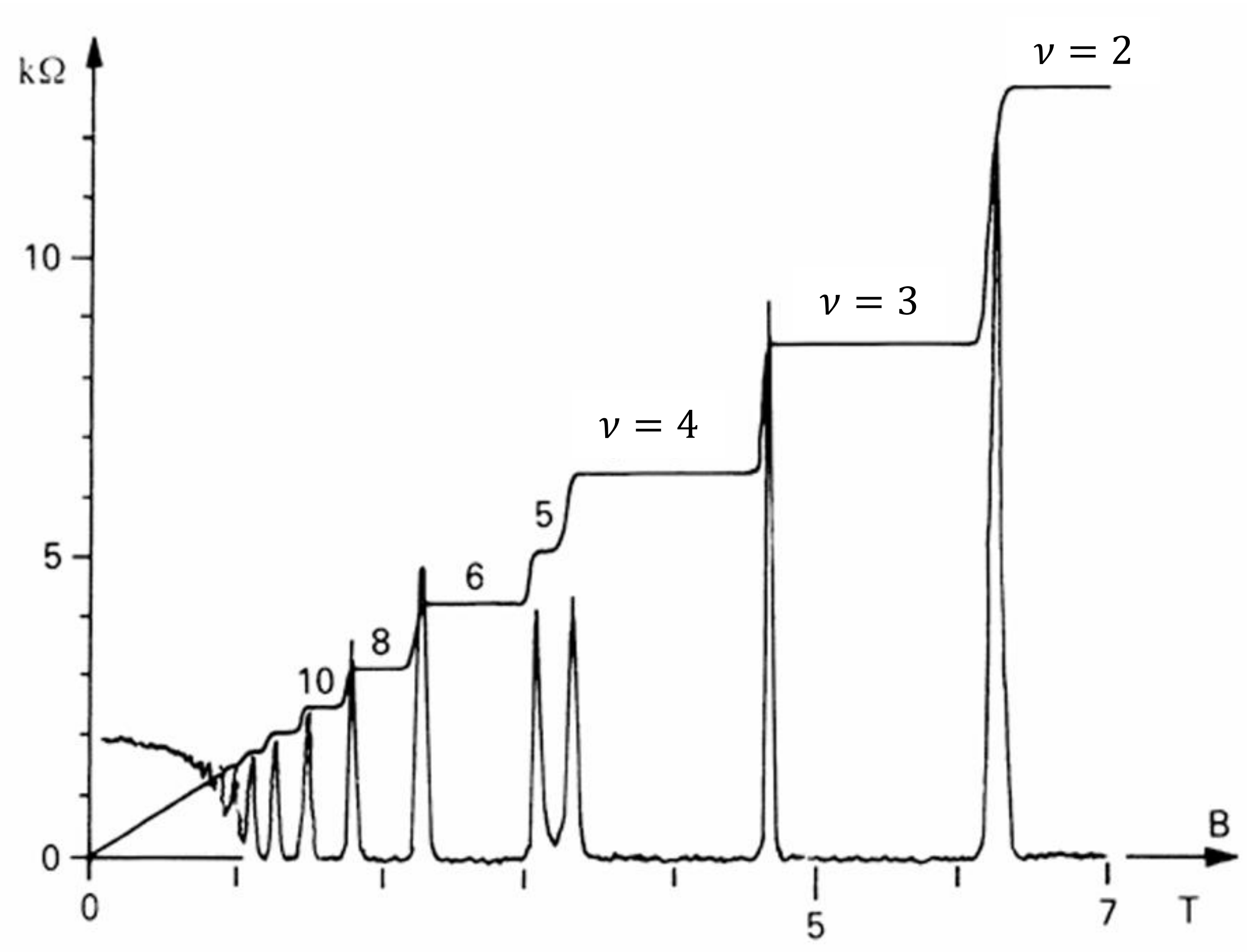
Quantum Hall Effect

Talker: ***

Feb.6 2025

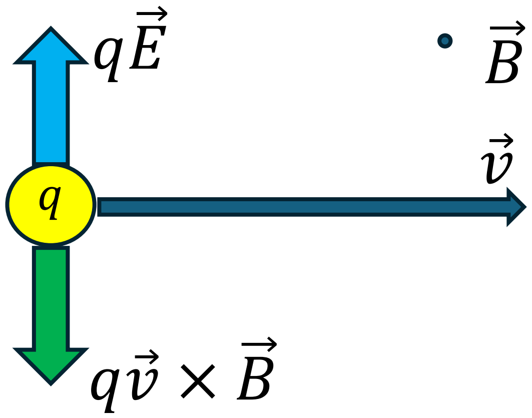


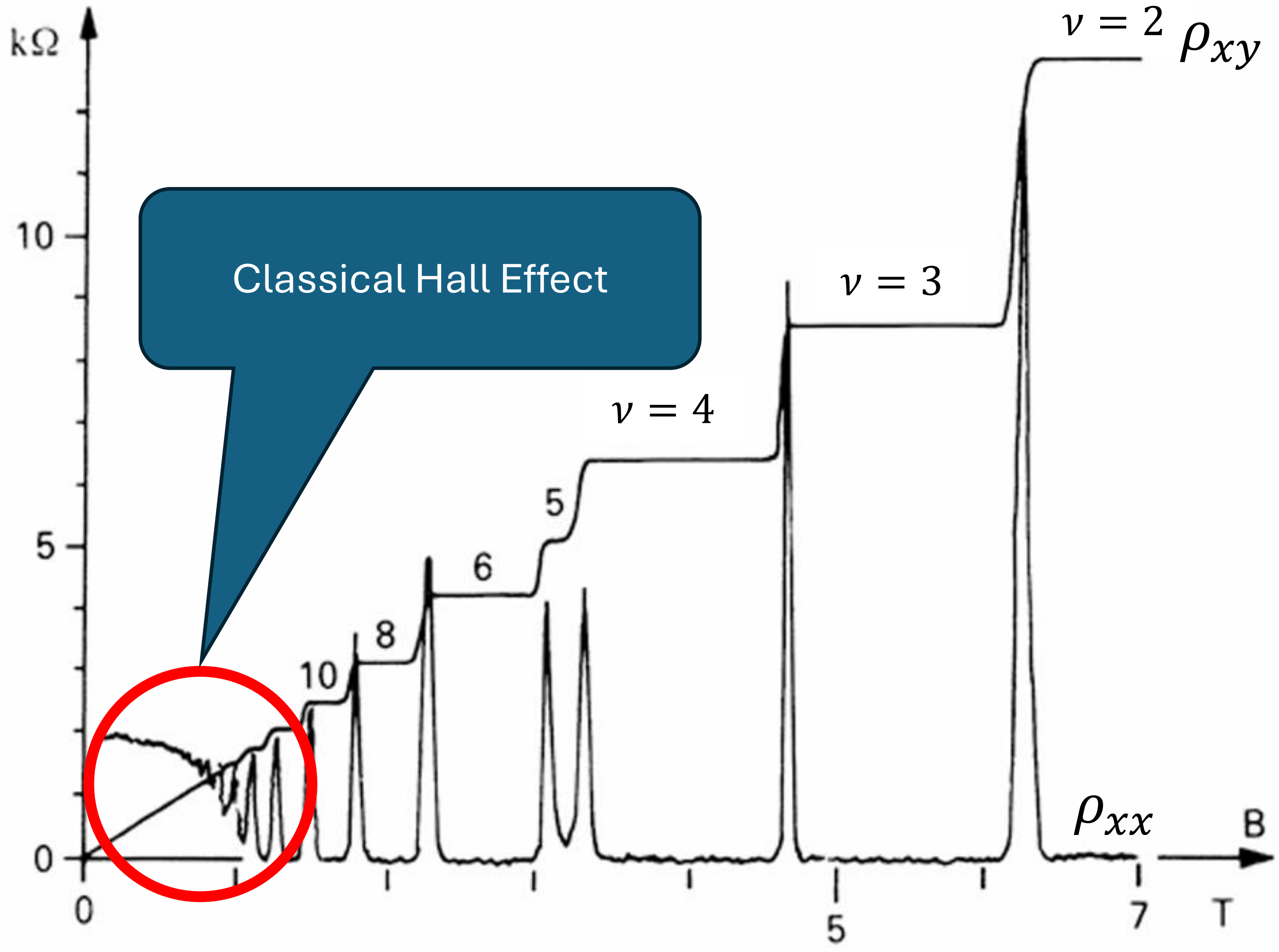
References

- (Lecture Notes) The Quantum Hall Effect by David Tong https://www.damtp.cam.ac.uk/user/tong/qhe/qhe.pdf

Recap: What we learned at High School

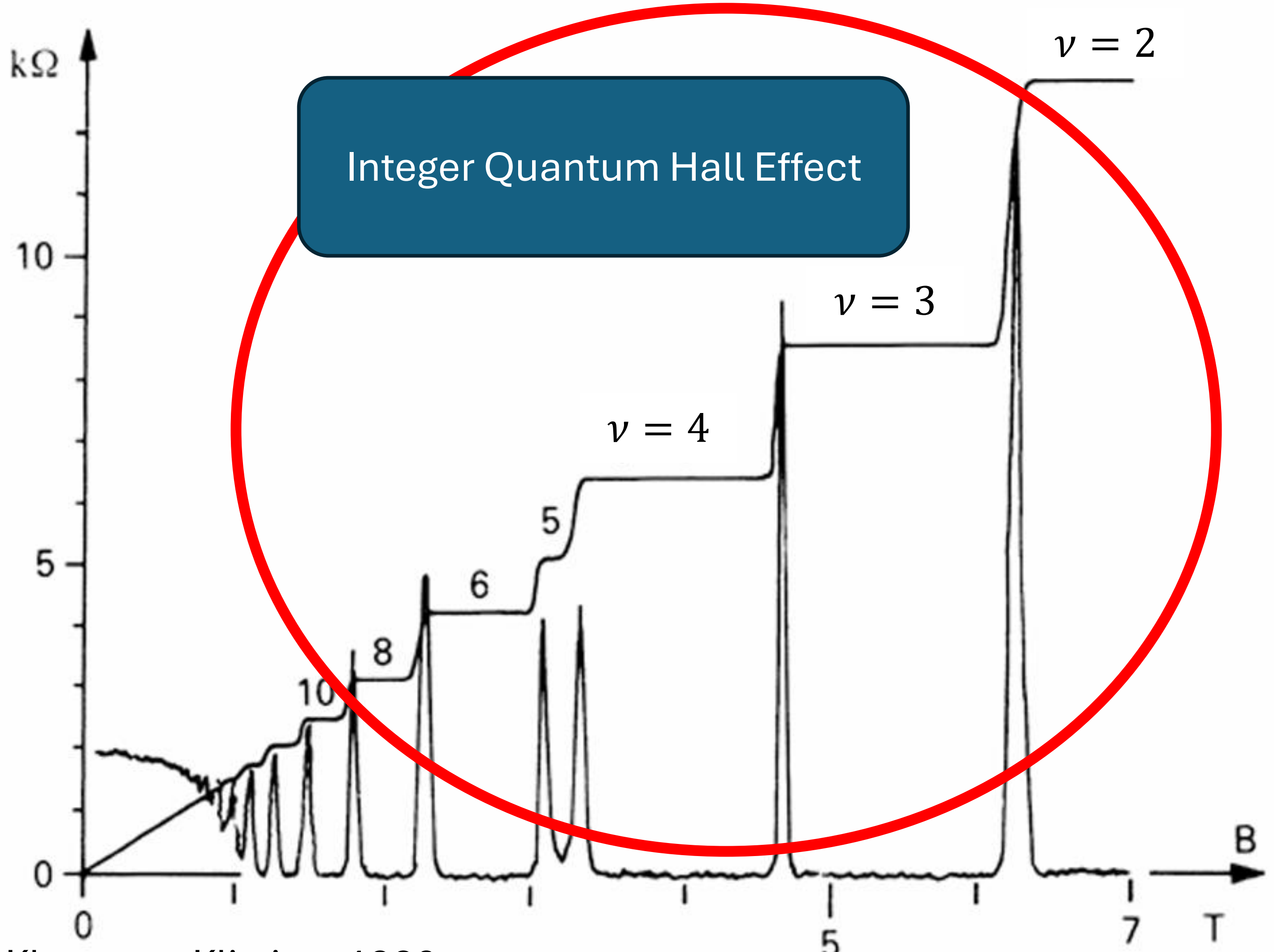
- $F_B = qvB$
- $F_E = qE$
- $F_B = F_E \rightarrow v = \frac{E}{B}$
- $\rho_{xy} = \frac{E}{j} = \frac{E}{nqv} = \frac{B}{nq} \propto B$





Deviation from classical HE at large B at 2D

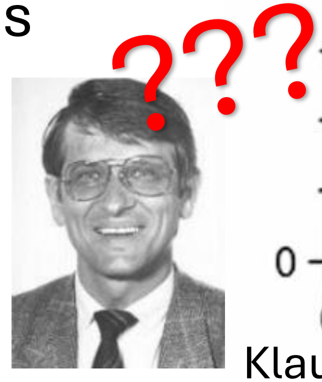
- Requirement
  - 2D material
  - More “dirty” the material is, more prominent plateau effect shows.
- Experiment setup
  - Si MOSFET
  - Electron trapped in the inversion band of width ~30A at the interface between insulator and semi conductor
  - $n \sim 10^{11} - 10^{12} cm^{-2}$

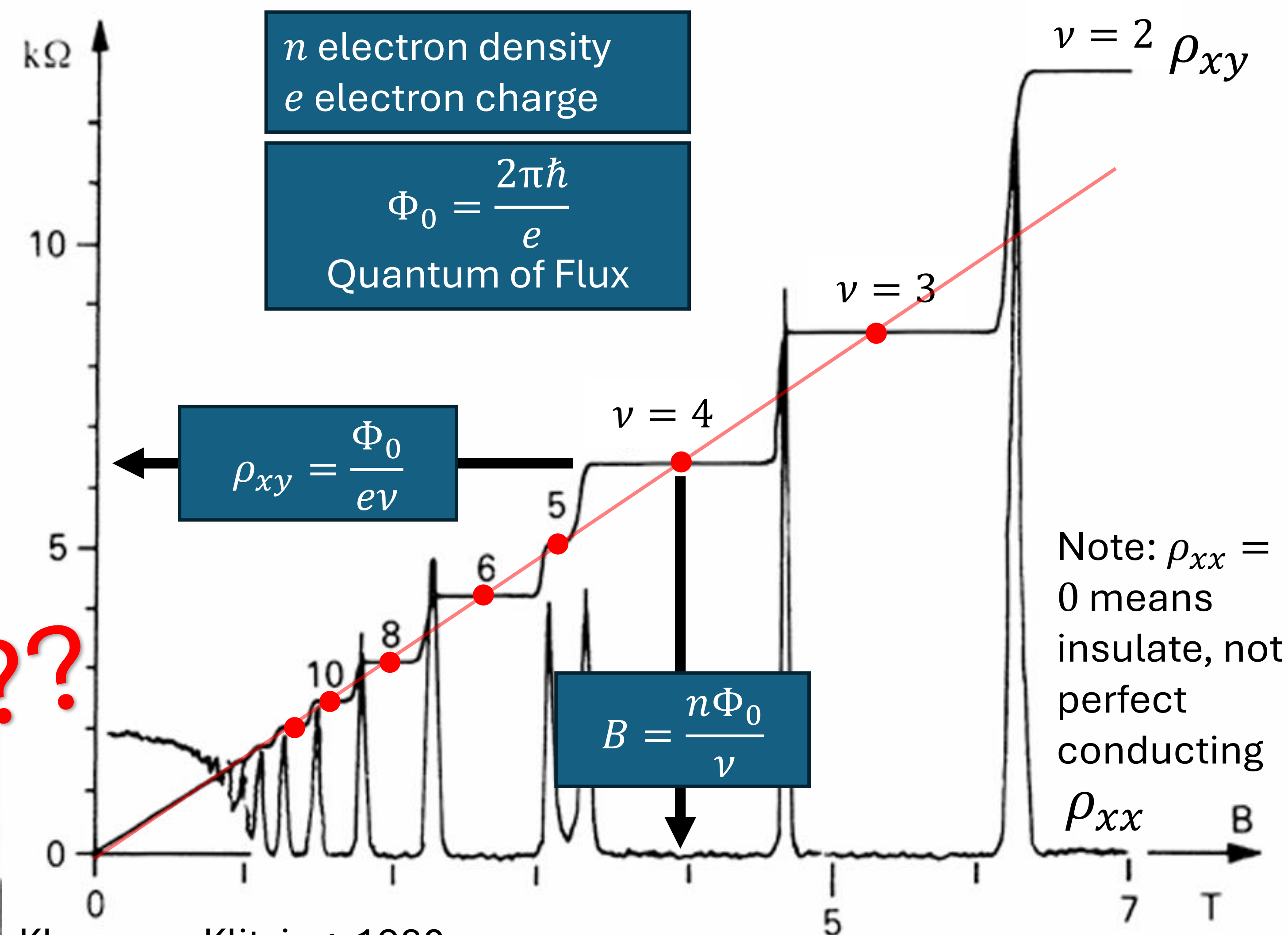


Klaus von Klitzing, 1980

Deviation from classical HE at large B at 2D

- ρ_{xy} no longer increases linearly as we increase B. Instead, it stuck at plateaus, followed by sudden jumps to catch up B
- centers and heights at inverse integer intervals $1/\nu$
- widths are less predictable, vary w.r.t. material
- only conductive at hops between plateaus
- How to understand it?

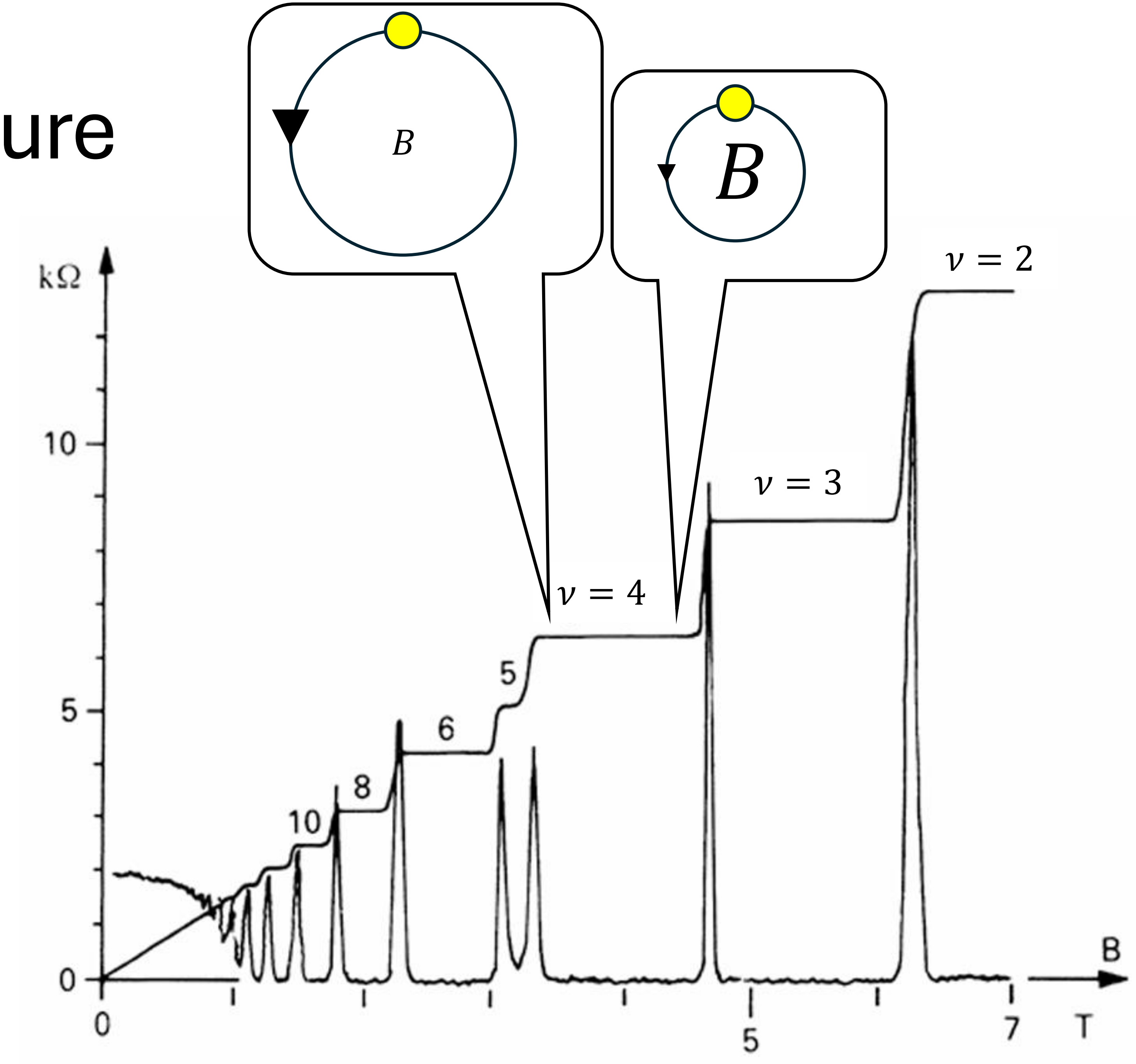




Klaus von Klitzing, 1980

Semiclassical Picture

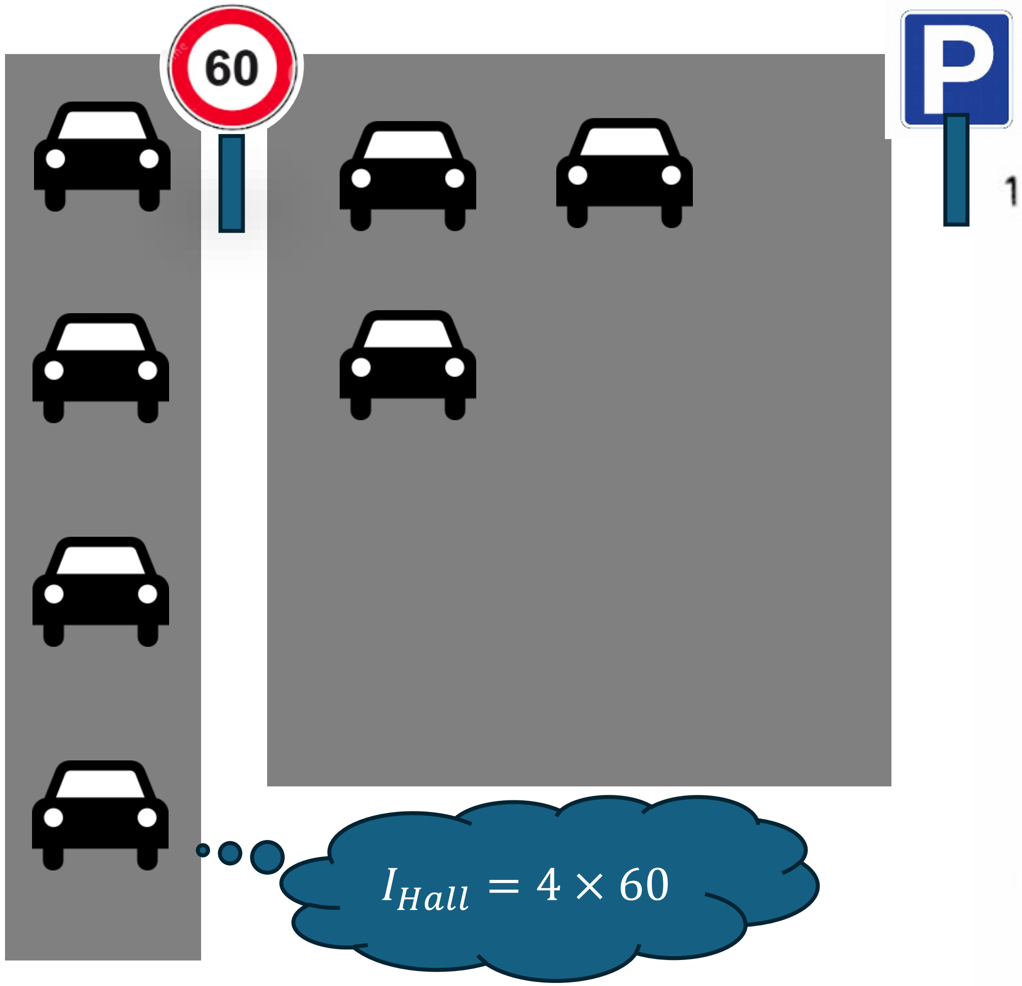
- Hint1: How much space does an electron take?
  - $r = \frac{mv}{eB} \rightarrow n_n \sim \frac{1}{\pi r^2} = \frac{eB}{h} = \frac{B}{\Phi_0}$
- Hint2: How fast does electron drift?
  - $v = \frac{E}{B}$
- Hint3: In solid state material, we saw:
  - Conduction bands
  - Valence bands

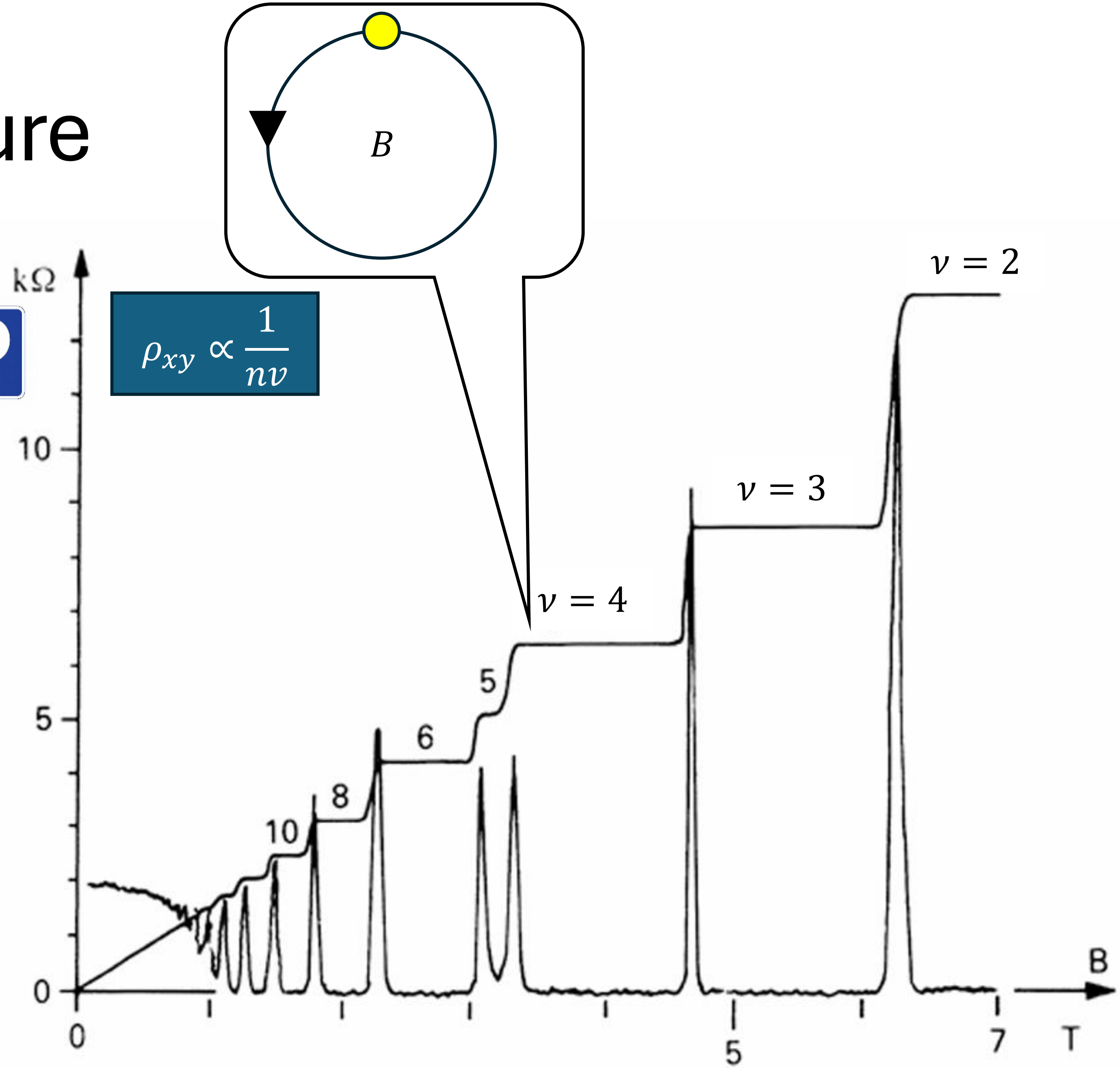


Semiclassical Picture

$n \propto B$

$\nu \propto B^{-1}$





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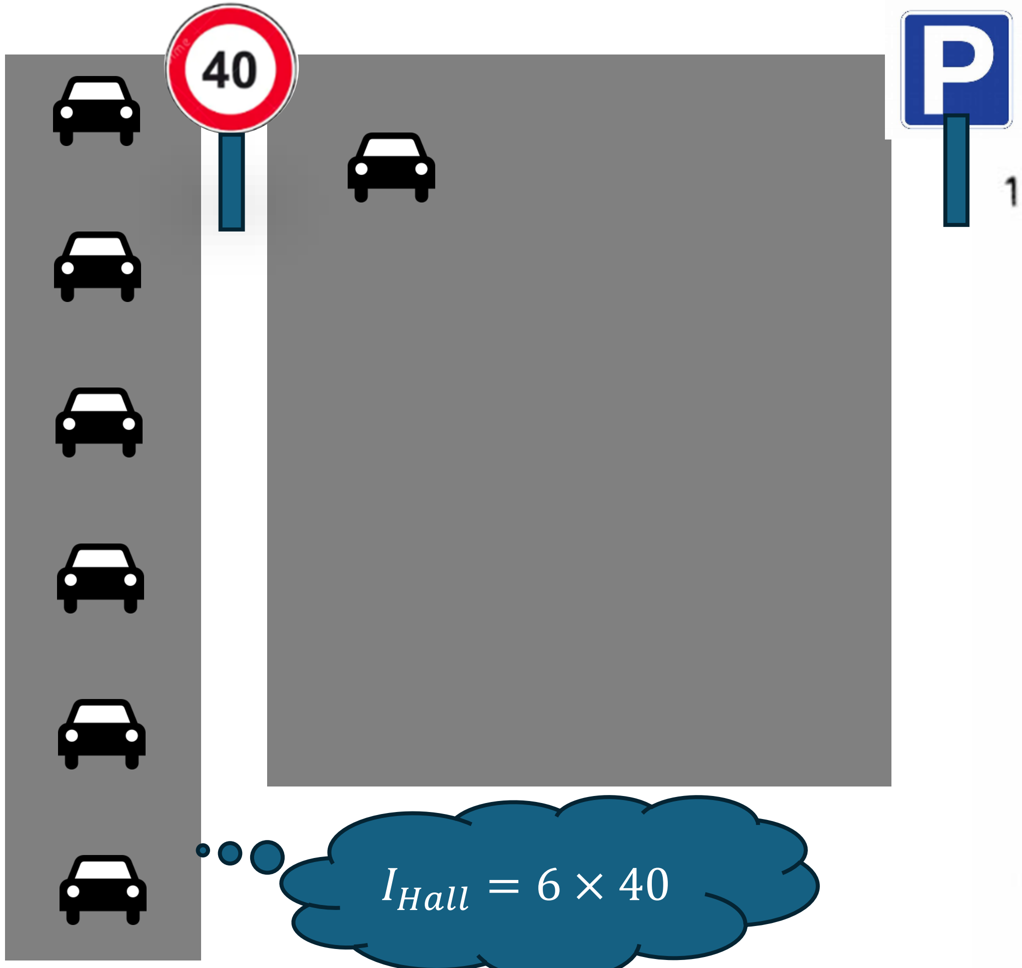
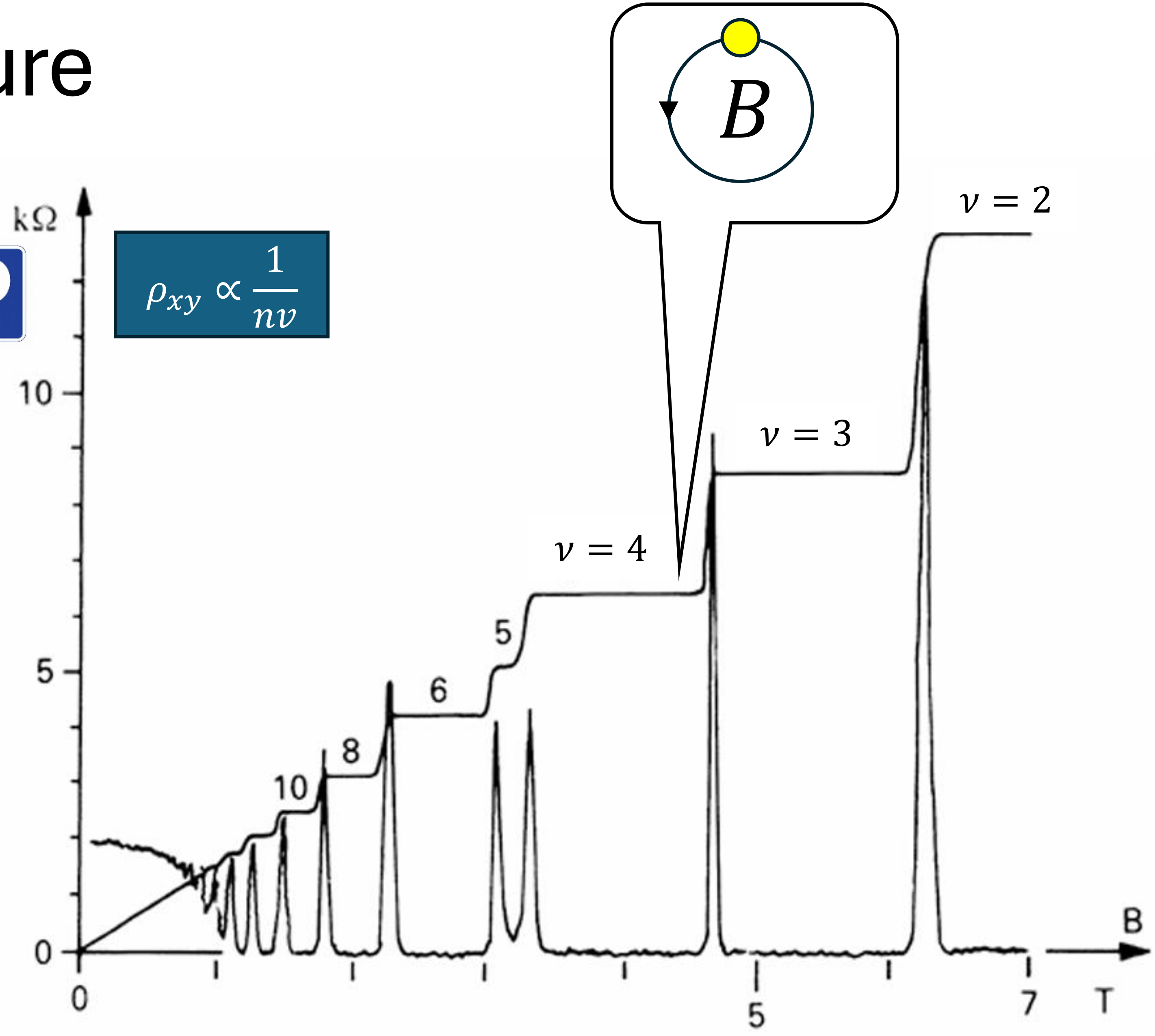


$I_{Hall} = 6 \times 40$

$\rho_{xy} \propto \frac{1}{n\nu}$



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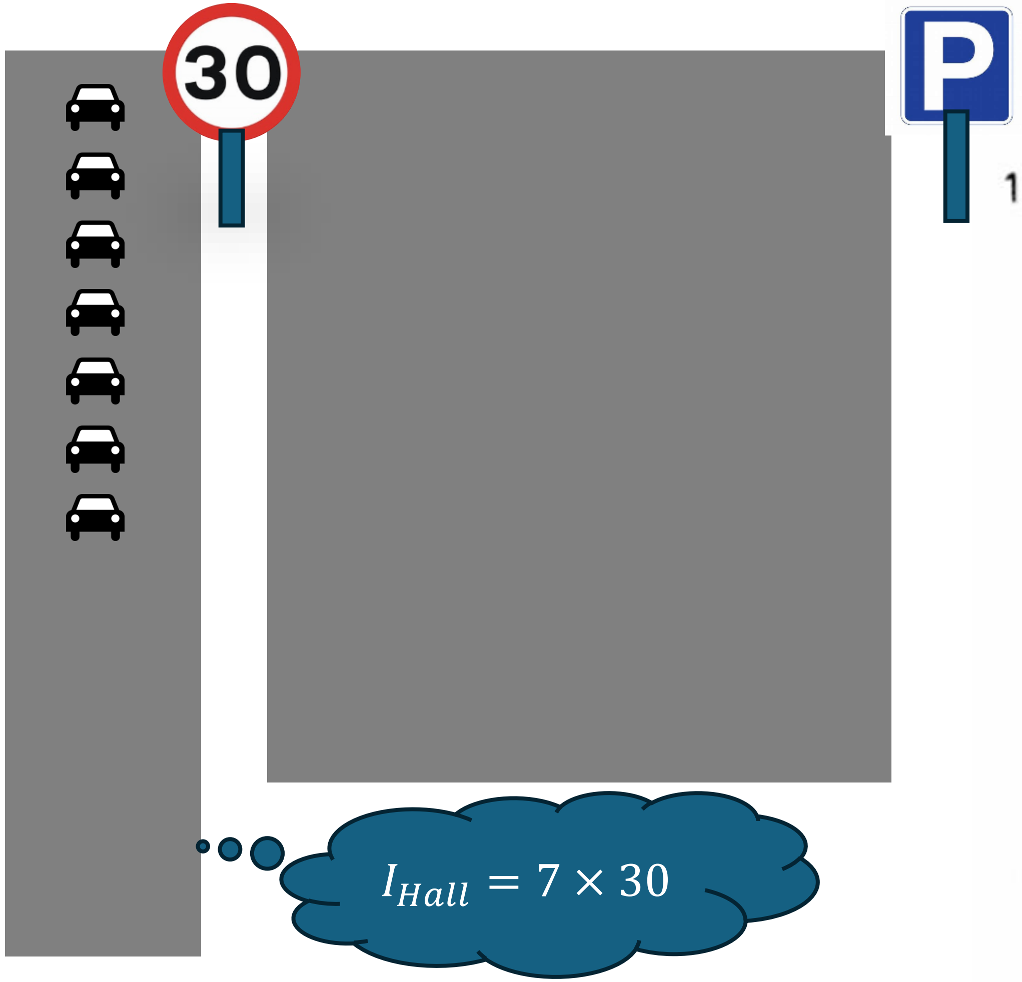

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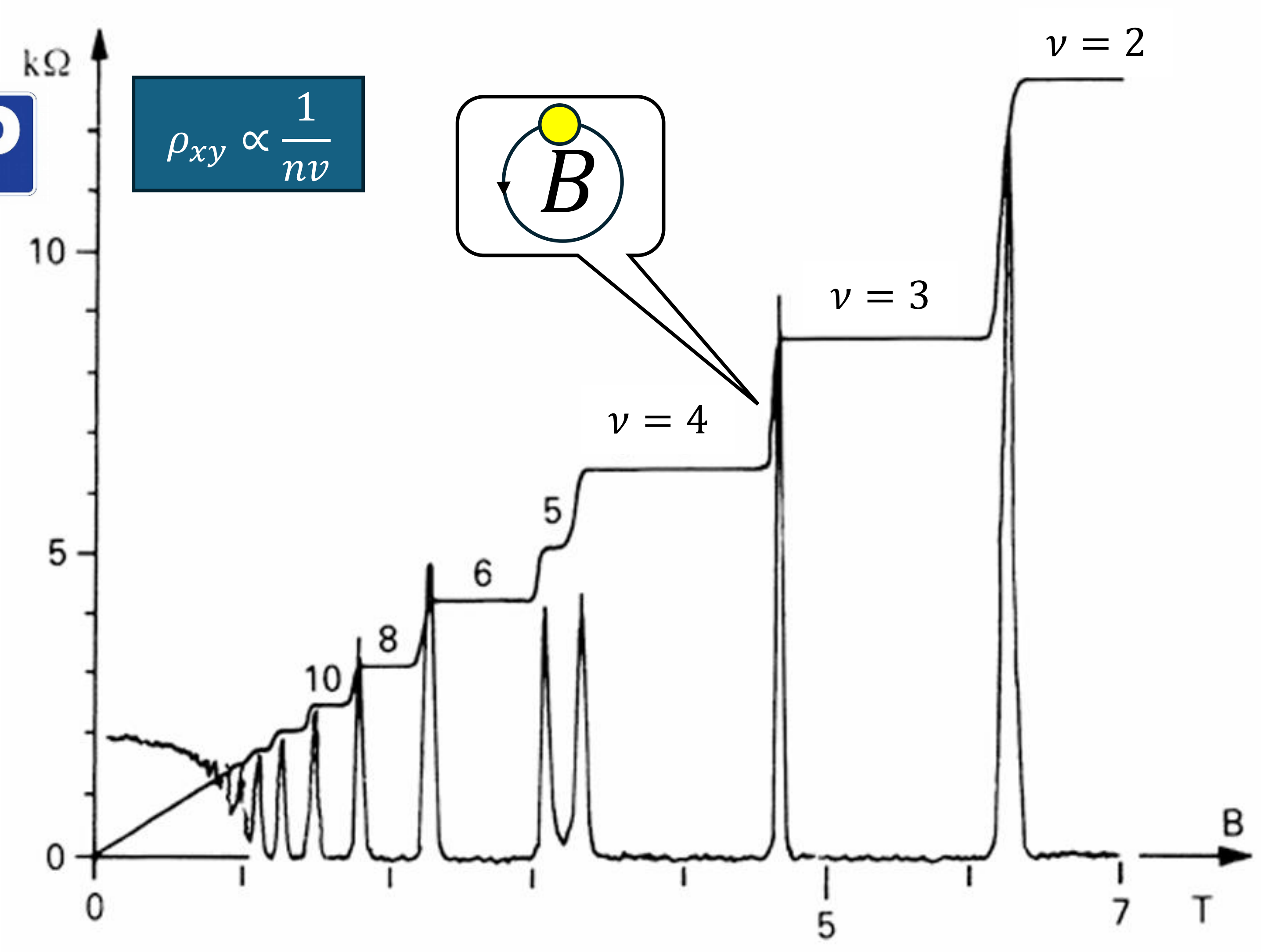

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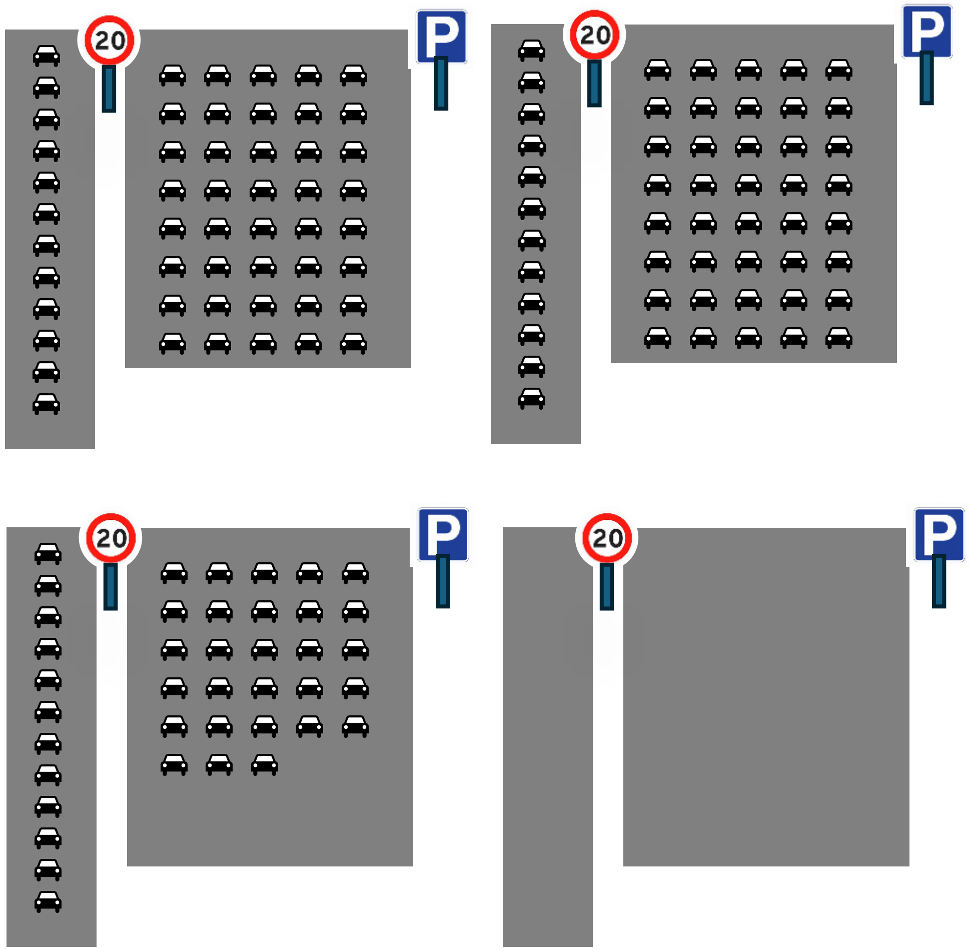
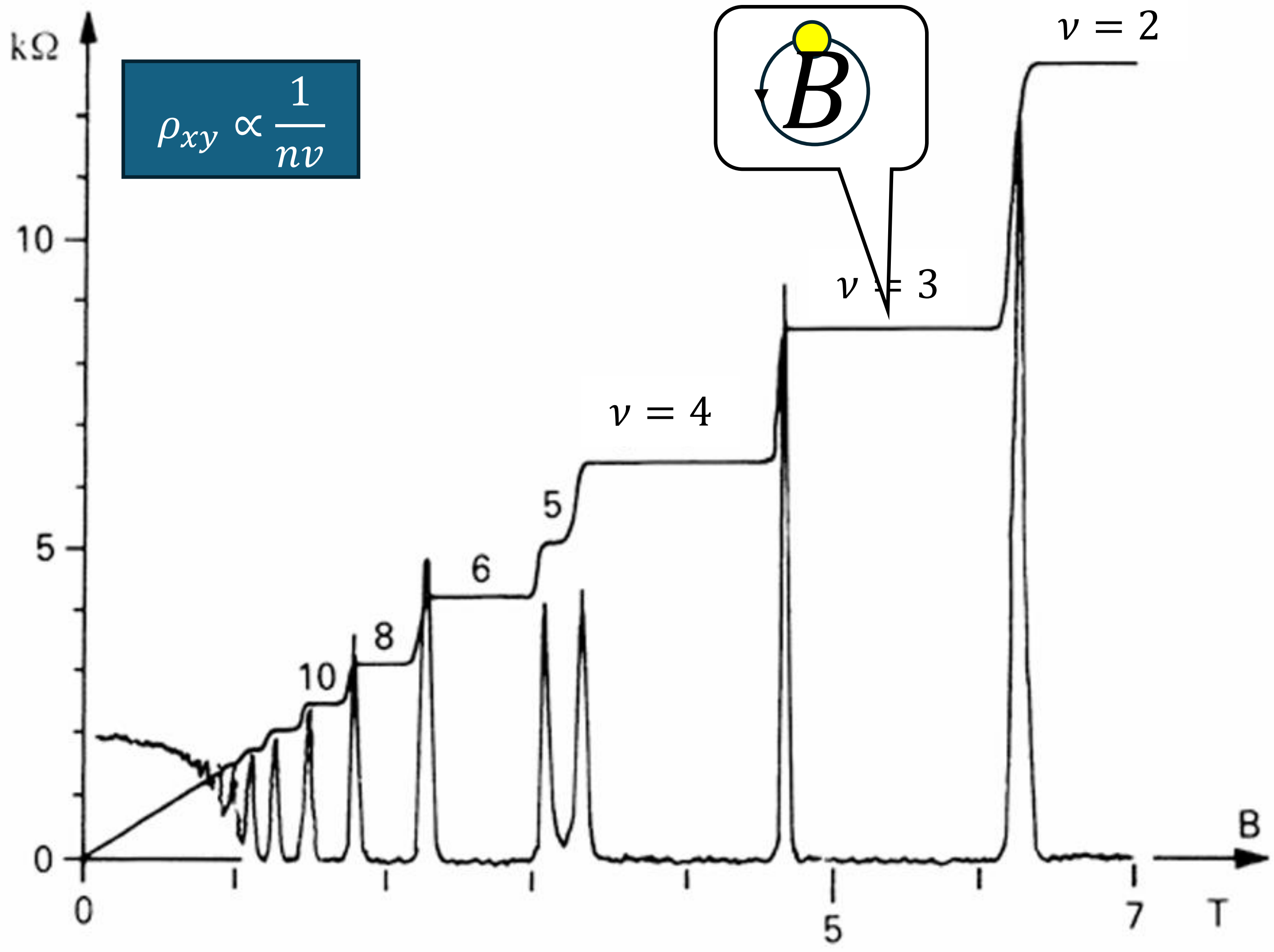
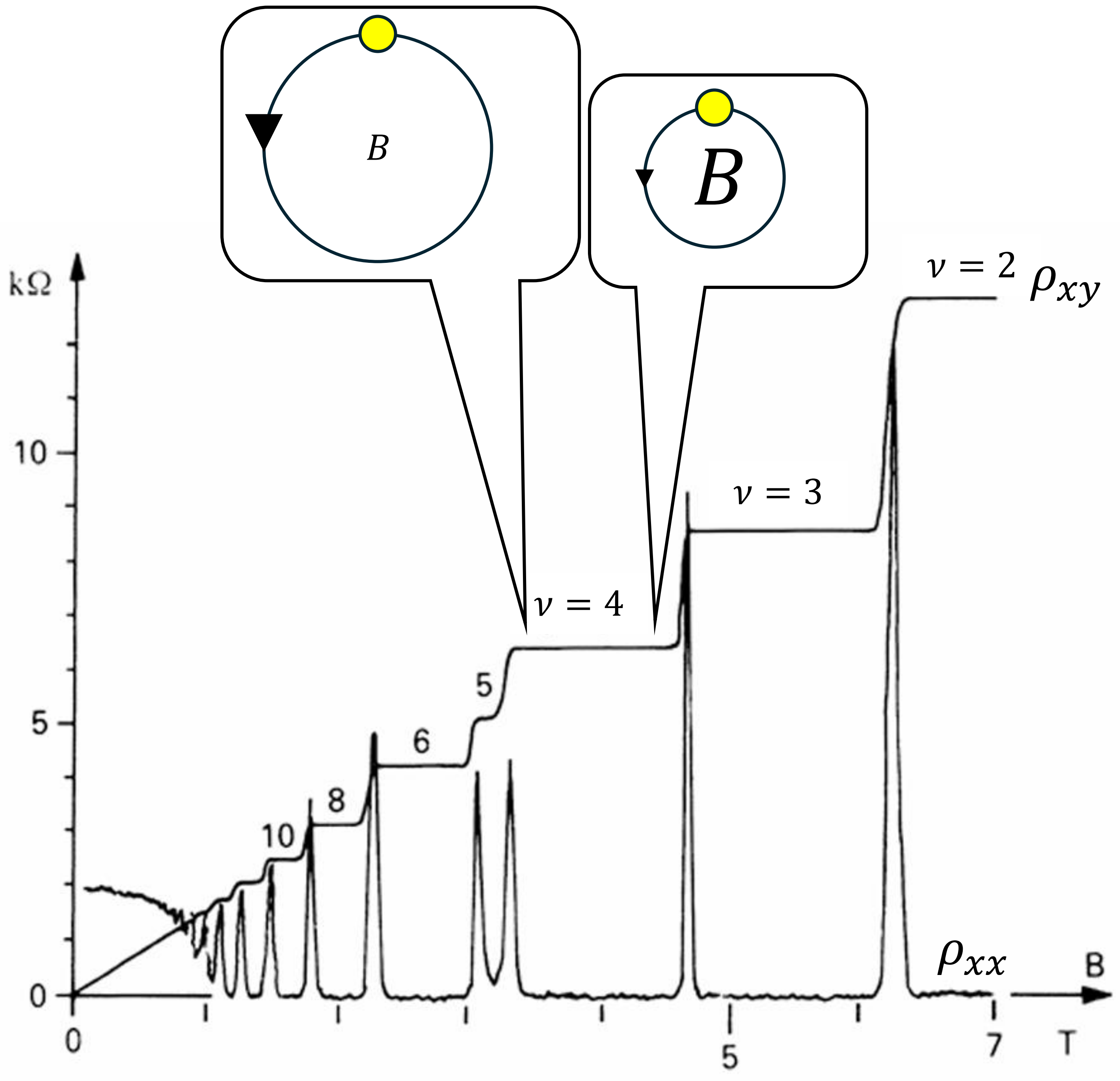




Conclusion

- When we tune B
- $v_{Hall} \propto 1/B$
- $n_{conduct} \propto B$
- $I_{Hall} = \text{constant} \times \nu$
- ν: fully filled conduct bands
- Plateau: fermi level @ valence band, ν unchanged
- Hops: fermi level @ conduct band, ν changes drastically
- Spikes: sudden emerge of longitudinal conductivity σ_{xx} supports our hypothesis that certain conduct band is half-filled

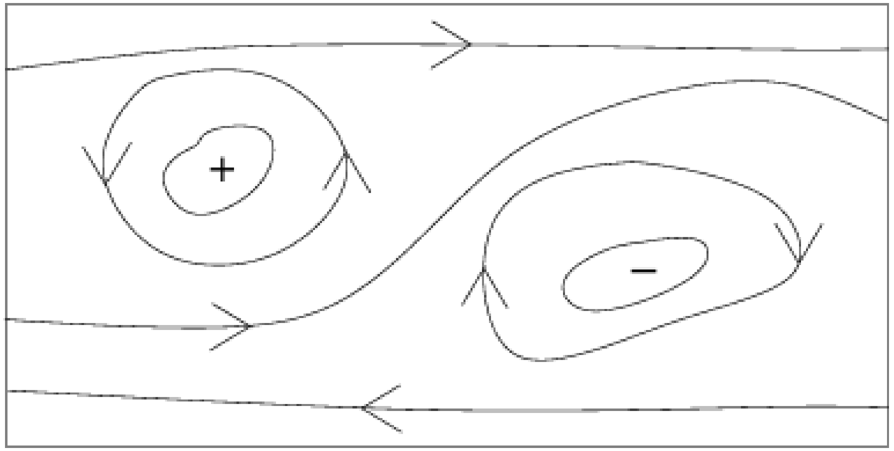
- But, wait…



Two assumptions we need

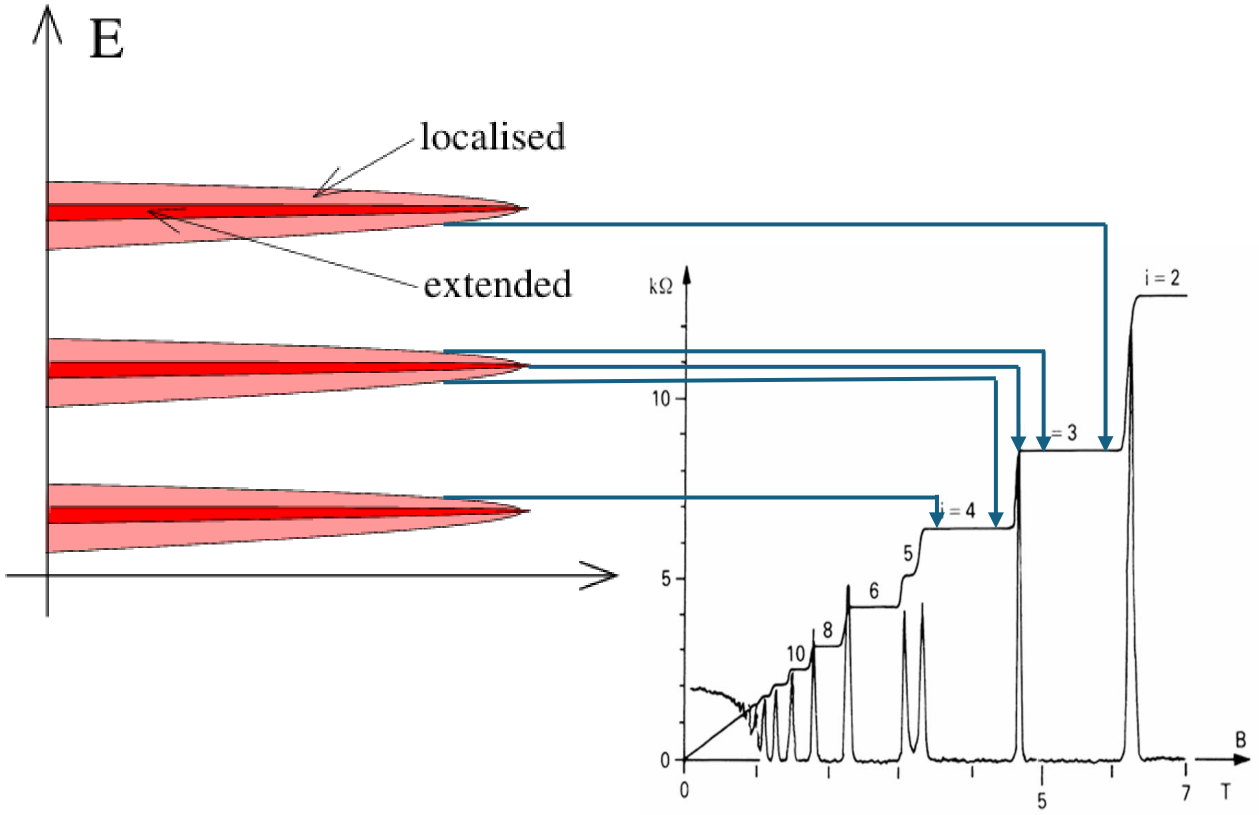
- 1. Does the band structure of a material in strong magnetic field resembles valence bands (localized orbits) and conductive bands (extensive orbits)?
  - **Physics: Impurity localizes the Landau Level**
- 2. $\frac{\partial}{\partial B} j_{\mathrm{Hall}} = \partial_B\left(\frac{1}{2\pi}\int_{\mathrm{band}} v_{\mathrm{Hall}}(k)dk\right) = 0$ was demonstrated for free electron. Does it hold true after the spectrum was drastically deformed into sets of valence and conductive bands?
  - **Physics: the integral is a topological invariant, thus robust against local deformation**

Impurity Localizes Landau Levels



- At strong B, $l_B = \sqrt{\frac{h}{eB}} \ll$ potential gradient
- Instant center operator
  - $X = x - \frac{\pi_y}{m\omega_B}, Y = \cdots$
- Equation of motion
  - $i\hbar\dot{X} = [X, H + V] = il_B^2 \partial_y V, \ldots$
- Cyclone drifts along the left-hand wall of equipotential
  - $(\dot{X}, \dot{Y}) \propto E \times B$
- At each landau level, we expected:
  - $E > E_{avg}$: orbits localized around local maximum
  - $E < E_{avg}$: orbits localized around local minimum
  - $E \approx E_{avg}$: extensive orbits navigating saddle points

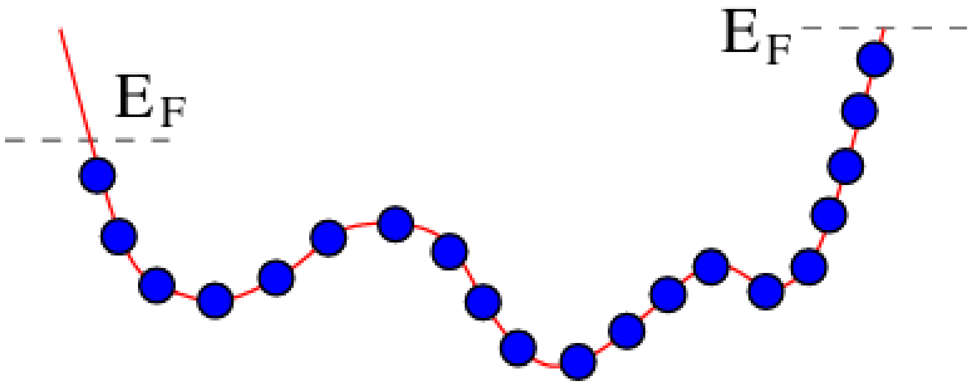
$* \pi = p - qA = m\dot{x}$ mechanical momentum



1D Example of Integral Invariant

- Landau gauge $A = (0, xB), V = V(x)$
  - preserves ∂_y symmetry
- State with k_y localized at $x = -k_y l_B^2$
  - $H_k = \frac{1}{2m} p_x^2 + \frac{1}{2} m\omega_B^2 \left(x + p_y l_B^2\right)^2$
  - $\psi_{n,k_y} \sim e^{ik_y y} e^{-\frac{x^2}{2l_B^2}} H_n(x)$
  - $E_n \approx \hbar\omega_B \left(n + \frac{1}{2}\right) + V(-k_y l_B^2)$
- Drift velocity = Slope of local potential
  - $v_y = \frac{E}{B} = \frac{1}{B} \partial_x V(x)$
- The total current is a **total derivative**
  - $J_y = q \int \frac{dk}{2\pi} v_y(k) = \frac{q^2}{2\pi\hbar} \int dx \frac{\partial V}{\partial x} = \frac{q^2}{2\pi\hbar} V_x$

$l_B = \sqrt{h/qB}, \omega_B = qB/m$



- The total derivative indicates Hall Current is a topological invariant, robust against perturbation of basin shape
- Each fully filled landau level contributes $\frac{e^2}{2\pi\hbar}$
- No matter how it was deformed

If we can write an observable as an integral over a total derivative (curvature term), then that observable is being topologically protected and have consistent values among different materials

Kubo Formula and Chern Number

- Overview
- **Berry Curvature**: for each state, its contribution to Hall conductivity is equivalent to the Berry Curvature inside the momentum space
- **Brillouin Zone**: atomic lattice breaks the momentum space into Brillouin Zones / Energy Bands
- **Chern Number**: Integrating the Berry Curvature in a certain Brillouin Zone gives an integer C

$$\sigma_{xy} = -\frac{e^2}{2\pi\hbar}C$$

Kubo Formula and Chern Number

- Hall Conductivity is the **linear response** of Current Operators

$$\sigma_{xy}(\omega) = \frac{1}{\hbar\omega}\int_{-\infty}^{0} dt\, e^{-i\omega t}\langle 0|\left[J_x(t), J_y(0)\right]|0\rangle$$

- Kubo Formular

$$\sigma_{xy}(\omega \to 0) = i\hbar \sum_{n\neq 0} \frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle - J_x \leftrightarrow J_y}{(E_n - E_0)^2}$$

- TKNN Invariant

$$\sigma_{xy} = i\hbar \sum_{a,k;b,k'} \frac{\langle u_{a,k}|\tilde{J}_y|u_{b,k'}\rangle\langle u_{b,k'}|\tilde{J}_x|u_{a,k}\rangle - J_x \leftrightarrow J_y}{(E_n - E_0)^2}$$

a, k filled states
b, k' empty states

$$= \frac{iq^2}{\hbar}\int_{T^2} \frac{dk^2}{(2\pi)^2}\left\langle \partial_{k_y} u_{a,k} \middle| \partial_{k_x} u_{a,k} \right\rangle - x \leftrightarrow y$$

$$= -\frac{iq^2}{2\pi\hbar}\sum_a \frac{1}{2\pi}\iint F_{ij}^{(a)} dS^{ij}$$

Berry Curvature $F_{ij} = \langle \partial_{k_i}\psi(k)|\partial_{k_j}\psi(k)\rangle$

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Example

- $\tilde{H}(k) = \sin k_x \sigma_1 + \sin k_y \sigma_2 + (m + \cos k_x + \cos k_y)\sigma_3$
  - 2-component Dirac fermion in 2+1d
  - "Dirac-Chern insulator"
  - $C = \frac{1}{2\pi} \iint d^2k \langle \partial_{k_x} u_{\alpha,y} | \partial_{k_y} u_{\alpha,x} \rangle - x \leftrightarrow y$
    - $= \begin{cases} -1 & -2 < m < 0 \\ 1 & 0 < m < 2 \\ 0 & |m| > 2 \end{cases}$

Lattice Model with Magnetic Field

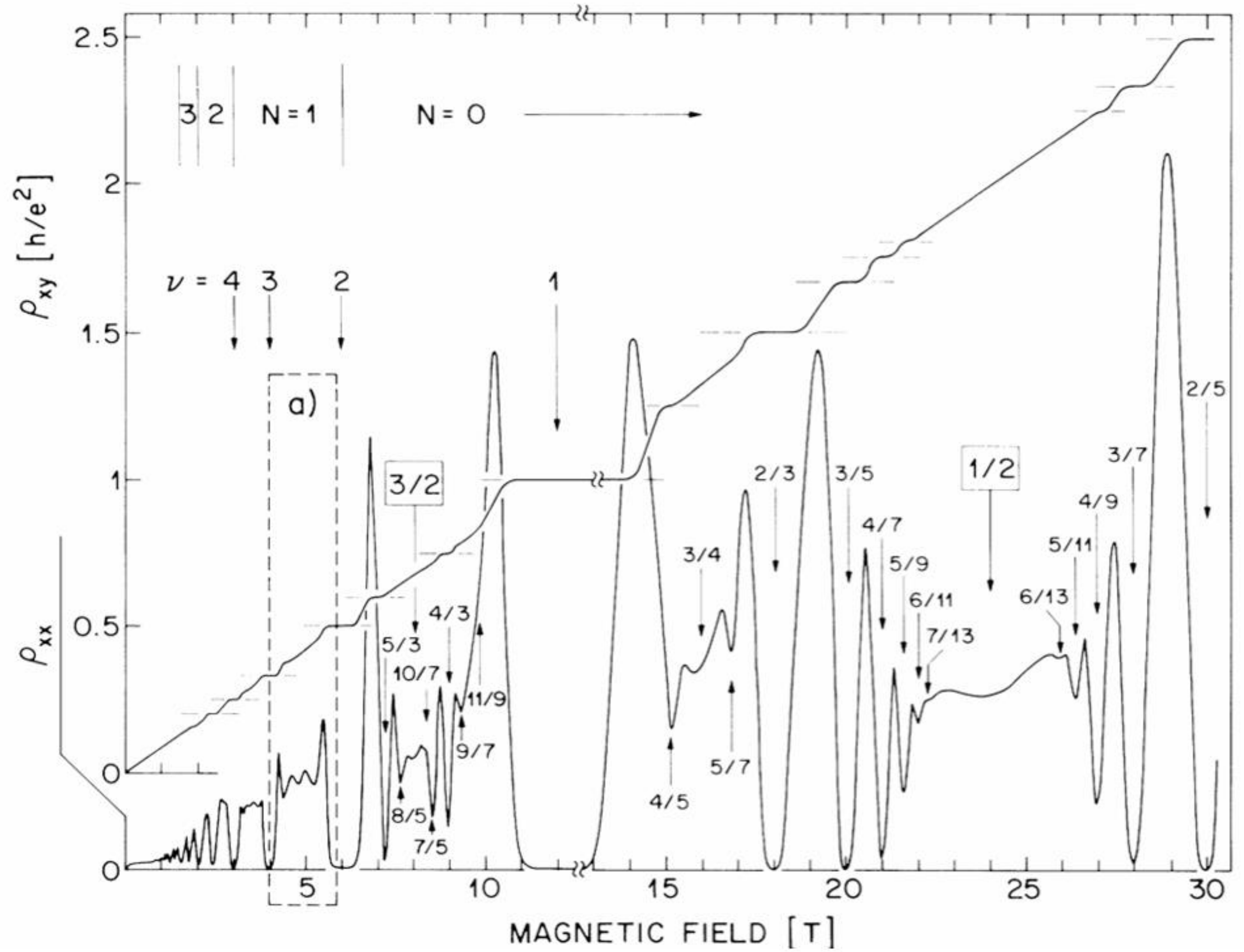
- $H = -t\left(T_x + T_x^\dagger + T_y + T_y^\dagger\right), T_i = \exp\left(\partial_i - i\frac{e}{\hbar}A_i\right) = \sum|x\rangle e^{-i\frac{e}{\hbar}A_i}\langle x + e_i|$ $\Phi_0 = h/e$
- Dirac Quantization Condition $T_y^{-1}T_x^{-1}T_yT_x|\psi\rangle \sim |\psi\rangle$ requires $BA \in 2\pi\mathbb{Z}\,\Phi_0$
- When $BL_xL_y = \frac{p}{q}\Phi_0$, lattice translation symmetry breaks: $\mathbb{Z} \to q\mathbb{Z}$ into supercells
- After Fourier transform (Harper Equation)

$$2\cos(k_1a + \tfrac{p}{q}2\pi r)\tilde{\psi}_r(k) + e^{ik_2a}\tilde{\psi}_{r+1}(k) + e^{-ik_2a}\tilde{\psi}_{r-1}(k) = -\frac{E(k)}{t}\tilde{\psi}_r(k)$$

- Finding integer solution of linear Diophatine equation: $r = qs_r + pt_r, \quad |t_r| \le q/2$ $C_r = t_r - t_{r-1}$
  - $t_0 = 0$
- Examples:
  - $\Phi = p\Phi_0$: single band, σ_{xy} vanishes
  - $p/q = 11/7$:
    - $(s_r, t_r) =$(-3,2), (5,-3), (2,-1), (-1,1), (-4,3), (4,-2), (1,0)
  - the Hall Conductivity varies between negative and positive by the sequence
    - 2, -3, -1, 1, 3, -2, 0

Fractional Quantum Hall Effect

- Stronger B
- In contrast to IQHE, sample cleaner, FQHE effect stronger



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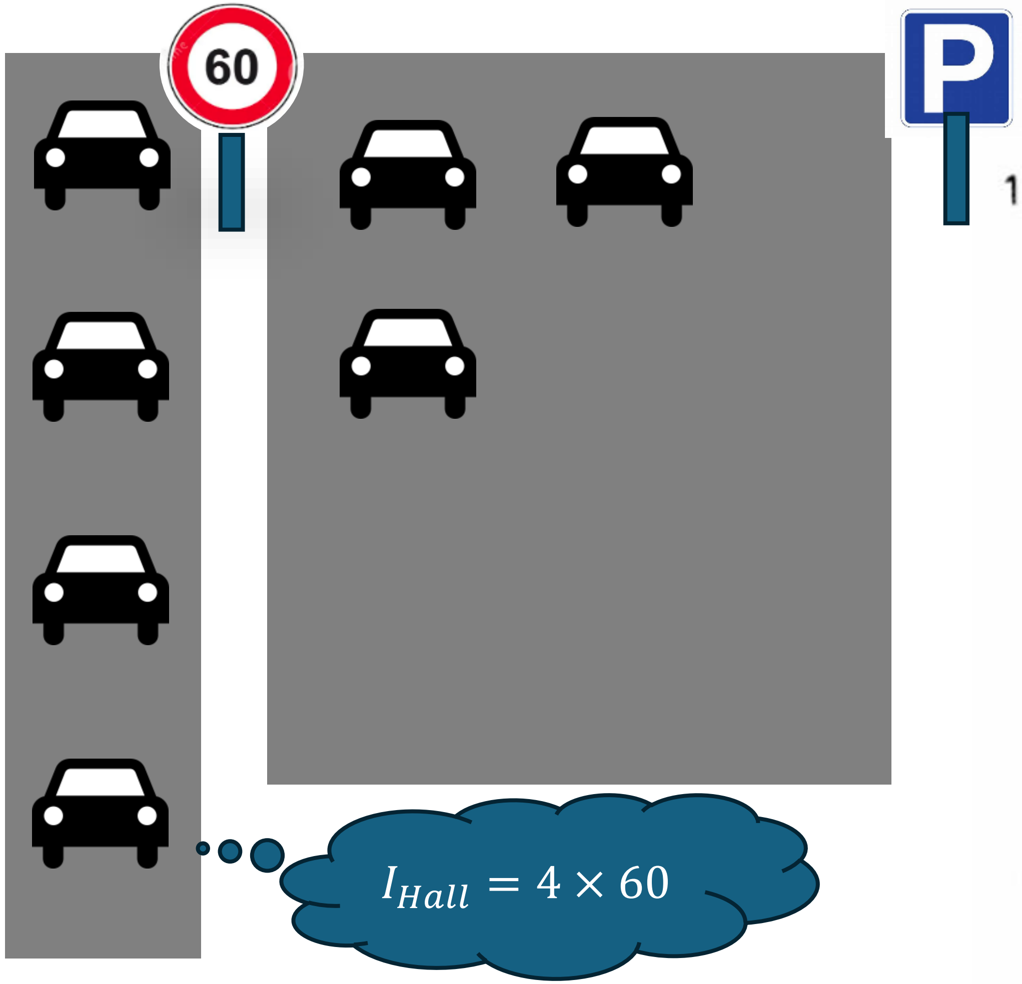
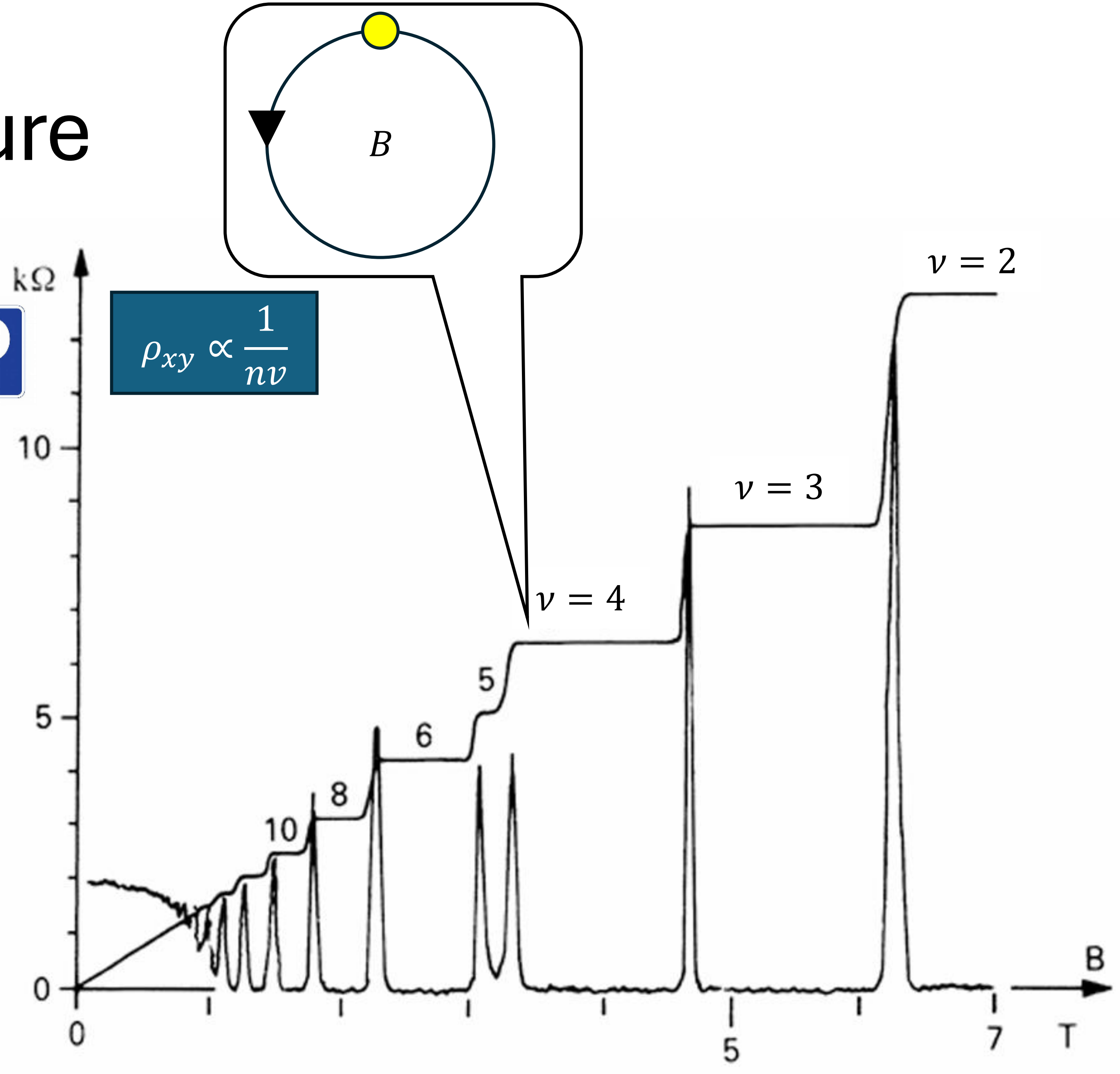
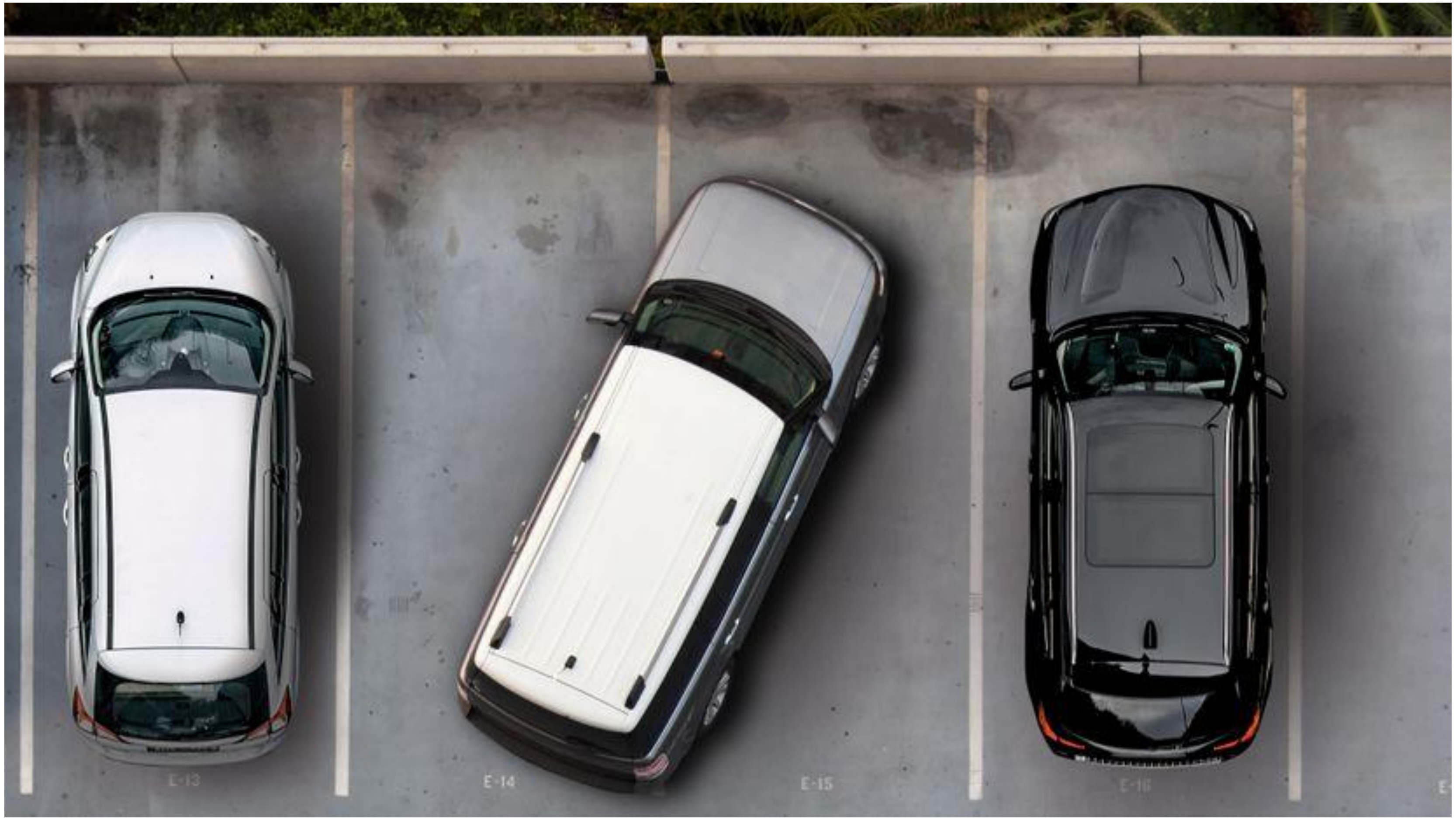




$$\nu = 4/3$$



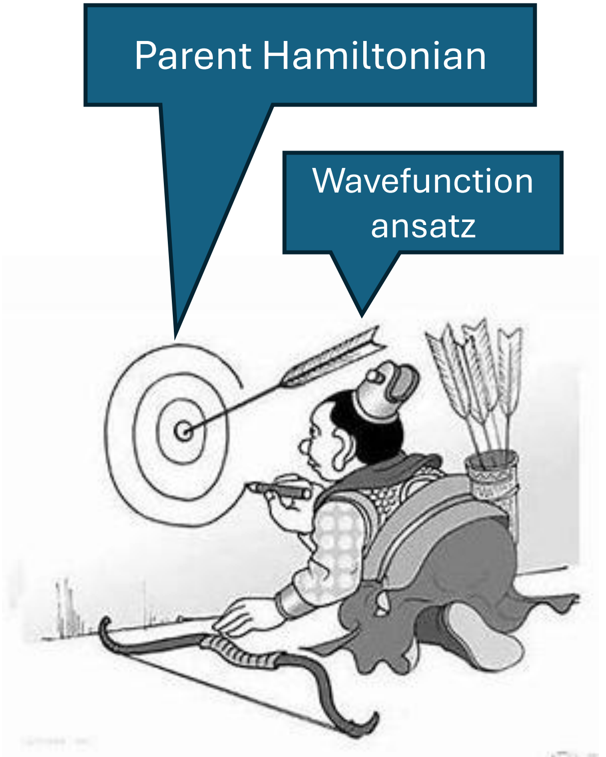
What we want

- Write down a collective motion of N fermions, which
- When from a distance, looks like N/m free fermions

"choreographing the dance of 100 billion infinitesimal particles"

Methodology

- instead of writing down a model Hamiltonian and solve it
- we first write down a wavefunction ansatz,
- then construct a parent Hamiltonian that annihilates the wavefunction ansatz
- which is usually a sum of non-commuting local projectors
- finally, we argue if the physical system falls into the "universality class" of such model Hamiltoians



Laughlin Wavefunction

$$\psi^{(m)}(z_i) = \prod_{i<j} \left(z_i - z_j\right)^m e^{-\frac{\sum_i |z_i|^2}{4l_B^2}}$$

$\nu = \frac{1}{m}$: Filling rate

m is also the relative angular momentum between any pair of particles

- It is an intuitive generalization of Landau Ground State
  - Single particle wf of Lowest Landau Level:

$$\psi \sim z^{m-1} e^{-\frac{|z|^2}{4l_B^2}}$$

  - Many free particles: Vandermonde determinant

$$\det\left(z_j^{i-1}\right) = \prod_{i<j} (z_i - z_j)$$

  - Is the $m = 1$ case of $\psi^{(m)}$

Parent Hamiltonian

- Laughlin state have property that relative angular momentum between any pair of particles is m
- To construct the parent Hamiltonian, we simply gapped out the subspace where any pair of particles have relative angular momentum $m_{ij} < m$

$$H = \sum_{\langle i,j \rangle} \sum_{m'=0}^{m-1} P_{i,j}^{(m')} + \omega J$$

- $m_{ij} > m \Rightarrow$ particles cannot be too close
- $\omega J \Rightarrow$ find the most compact possible state

Excitations

$$\tilde{\psi}(z_i;\eta_k,\eta_l') = \left(\left(\prod_{i,k}(z_i-\eta_k)\right)\left(\prod_{i,l}(2\partial_i-\bar{\eta}_l')\right)\left(\prod_{i<j}\left(z_i-z_j\right)^m\right)\right)e^{-\frac{\sum_i |z_i|^2}{4l_B^2}}$$

- η_j, η_k' are the locations of quasi-holes and quasi-particles
  - Quasi-hole at $\eta \to$ a $\prod_i (z_i-\eta)$ prefactor
  - Quasi-particle at $\eta' \ \to$ a $\prod_i (2\partial_i-\bar{\eta}')$ prefactor
  - ∂_i only acts on polynomial part of ψ, not the exponential factor
- Why η, η' resembles particle-like DoF?
  1. m overlapped quasi-holes simulates a virtual particle, pushing away other particles
  2. Energy of those states replicates repulsive/attractive potential between particles/holes of charge q/m (see next slide)

Plasma Analogy

- Inner product of Laughlin WF = Partition function of classical plasma

$$\langle\psi|[\cdot]|\psi\rangle = \int d^{2N}z\,[\cdot]|\psi(z)|^2 = \int d^{2N}z\,[\cdot]e^{-\beta U(z)}$$

$$U = -\sum_{i<j}\log|z_i - z_j| + \frac{1}{4m}\sum_i |z_i|^2$$

  - First term = coulomb interaction between particles
  - Second term = background potential ($\rho_0 = \frac{1}{2\pi l_B^2 m}$)
- At charge neutral, the plasma analogy depicts the particle distribution of ground state Laughlin function
- Quasi hole: repulsive potential from am impurity of charge -1/m
  - $\Delta U = -\frac{1}{m}\sum_i \log|z_i - \eta|$
- In plasma model with added charge -1/m particles, the screening effect

$$\partial_\eta U_{plasma} \approx 0,\, U_{plasma} = U(z,z) + U(z,\eta) + U(\eta,\eta) + U(z) + U(\eta)$$

- Compared to the "correct" plasma model,

$$U_{Laughlin} = U(z,z) + U(z) + U(z,\eta)$$

- The missing terms simulates the interaction of **free -1/m charged particles**

$$\partial_\eta U_{Laughlin} = -\partial_\eta(U(\eta,\eta) + U(\eta))$$

Thanks