Finite Element Methods for Option Pricing

Pricing American Basket Options

Filippo Felice Boggetti   Andrea Cesari   Isacco Lotti

Abstract

This work applies the finite element method (F.E.M.) to pricing American basket options. After outlining the flexibility of F.E.M. in solving this class of problems and arguing the choice of the instrument to price, we present the variational form of the linear complementarity problem associated with the multi-dimensional Black Scholes. Barely tackling the mathematical foundations of the F.E.M., we depict the weak form, the discretized problem, and the Primal-Dual active set algorithm employed to solve it. Finally, the pricing results of the implementation in FreeFem++ are commented.

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1 American Put on the Best of Two Assets

The instrument we decided to price is an American Put option on a basket of two assets S_1 and S_2, more precisely we study the American put on the maximum of the two:

$$P_0(S_1, S_2) = (K - max(S_1, S_2))_+ \ . \tag{1}$$

The choice of such payoff structure, is due to the significant gain in efficiency obtained by pricing this instrument trough a deterministic method relatively to a simulation-based one. Pricing exotic options like this with Montecarlo would not only be highly computationally intensive, but would also require shaping the binomial tree to price different instruments with similar payoffs. Conversely, finite element is robust for different payoff functions provided that they are square integrable. Results presented in this work can therefore been extended from the specific case in (1) to a broad class of instruments.

1.1 Early Exercise Premium

To have a benchmark for the price of the American put we calculate the price for the European option with the same payoff. The European put on the maximum of two assets can be replicated trough put-call parity as

$$p_{max}(S_1, S_2, K, T) = Ke^{-rT} - c_{max}(S_1, S_2, 0, T) + c_max(S_1, S_2, K, T) \ .$$

Where c_{max} represents the price of a European call on the best of the same two assets:

$$c_{max}(S_1, S_2, 0, T) = S_2 e^{(q_2-r)T} + S_1 e^{(q_1-r)T} N(d_1) - S_2 e^{(q_2-r)T} N(d_1 - \sigma\sqrt{T}) \ .$$

Exploiting the closed form solution for the price of the European basket put, we then compare it to the one of the American option, deriving the early exercise premium.

2 Variational Inequality

In this section, we aim to find a variational formulation for the Parabolic problem whose solution is the price for the American Put on the best of two assets. Using the standard notation, and indicating t as the time to maturity of the option, we generalize the set of inequalities for a plain vanilla American Put [1] to the d-dimensional case:

$$\begin{aligned} \left(\tfrac{\partial P}{\partial t} - LP + rP\right)(\boldsymbol{S}, t) \geq 0, \quad 0 < t \leq T,\ \boldsymbol{S} \in \mathbb{R}_+^d, \\ P(\boldsymbol{S}, t) \geq P_\circ(\boldsymbol{S}), \quad \boldsymbol{S} \in \mathbb{R}_+^d, \\ \left(\tfrac{\partial P}{\partial t} - LP + rP\right)(P - P_\circ)(\boldsymbol{S}, t) = 0, \quad \boldsymbol{S} \in \mathbb{R}_+^d, \\ P(\boldsymbol{S}, 0) = P_\circ(\boldsymbol{S}), \quad \boldsymbol{S} \in \mathbb{R}_+^d. \end{aligned} \tag{2}$$

Where L is the partial differential operator:

$$LP = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \boldsymbol{\Xi}_{i,j} S_i S_j \frac{\delta^2 P}{\delta S_i \delta S_j} + r \sum_{i=1}^{d} S_i \frac{\delta P}{\delta S_i}$$

and with $\boldsymbol{\Xi}_{i,j}(\boldsymbol{S}, t) = \rho_{i,j} \sigma_i(\boldsymbol{S}, t) \sigma_j(\boldsymbol{S}, t)$.

Here, we assume that the volatilities are functions of the variables S, t and that r is a function of time. Assuming that the coefficients are regular enough, we write the operator in divergence form:

$$Lu = \frac{1}{2} \sum_{i=1}^{d} \frac{\partial}{\partial S_i} \left(\sum_{j=1}^{d} \Xi_{i,j} S_i S_j \frac{\partial u}{\partial S_j} \right) + \sum_{j=1}^{d} \left(r(t) S_j - \frac{1}{2} \sum_{i=1}^{d} \frac{\partial}{\partial S_i} \left(\Xi_{i,j} S_i S_j \right) \right) \frac{\partial u}{\partial S_j}$$

.

Multiplying $-Lu + ru$ by a test function v, integrating on $\mathbb{R}_+^d$ and performing suitable integrations by part, we obtain the bilinear form

$$\begin{aligned} a_t(u, v) =& \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_{\mathbb{R}_+^d} \Xi_{i,j} S_i S_j \frac{\partial u}{\partial S_j} \frac{\partial v}{\partial S_i} \\ & - \sum_{j=1}^{d} \int_{\mathbb{R}_+^d} \left(r(t) S_j - \frac{1}{2} \sum_{i=1}^{d} \frac{\partial}{\partial S_i} \left(\Xi_{i,j} S_i S_j \right) \right) \frac{\partial u}{\partial S_j} v + r(t) \int_{\mathbb{R}_+^d} uv. \end{aligned} \tag{3}$$

3 Finite Element Method

For a weak formulation in an infinite-dimensional function space V, such as:

$$V = H^1(\Omega) = \left\{ w \in L^2(\Omega) : \nabla w \in L^2(\Omega)^d \right\}$$

the *finite element method* consists of choosing a finite-dimensional subspace V_h of V, the space of continuous piecewise affine functions on a triangulation of Ω, and solving the problem with test and trial functions in V_h instead of V.

Hence, we fundamentally need to transform our set of variational inequalities in (2) in a problem stated in weak form. Having done so we need to define a space of functions which is suitable for our problem, that will turn out to be a *weighted Sobolev* space, and find a triangulation of Ω over which we will define the nodal basis. This will define our finite dimensional space V_h that will approximate, as well as one desire, the infinite dimensional space V.

3.1 Function Spaces

Suppose that Ω is an open set in $\mathbb{R}^n$.

Definition 3.1. *A function $f \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable with respect to x_i if there exists a function $g_i \in L^1_{\mathrm{loc}}(\Omega)$ such that*

$$\int_\Omega f \partial_i \phi dx = -\int_\Omega g_i \phi dx \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

The function g_i is called the weak $i-th$ partial derivative of f, and is denoted by $\partial_i f$. Thus, for weak derivatives, the integration by parts formula

$$\int_\Omega f \partial_i \phi dx = -\int_\Omega \partial_i f \phi dx$$

holds by definition for all $\phi \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $L^1_{loc}(\Omega)$, the weak derivative of a function, if it exists, is unique up to pointwise almost everywhere equivalence.

Sobolev spaces consist, intuitively, of functions whose weak derivatives belong to L^p. These spaces provide one of the most useful settings for the analysis of PDEs.

Definition 3.2. *Suppose that Ω is an open set in $\mathbb{R}^n, k \in \mathbb{N}$, and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $f : \Omega \to \mathbb{R}$ such that*

$$\partial^\alpha f \in L^p(\Omega) \quad \text{for } 0 \leq |\alpha| \leq k.$$

We write $W^{k,2}(\Omega) = H^k(\Omega)$.

We work with specific Sobolev spaces, namely with *weighted* Sobolev spaces and indeed we define V as:

$$V = \left\{ v : v \in L^2\left(\mathbb{R}_+^2\right), S_i \frac{\partial v}{\partial S_i} \in L^2\left(\mathbb{R}_+^2\right), i = 1, 2 \right\} \tag{4}$$

and we call $\mathcal{K}$ the subset of V :

$$\mathcal{K} = \{v \in V, v \geq P_\circ \text{ in } \mathbb{R}_+\}$$

3.2 Weak Form

A weak formulation for the set if inequalities in (2) can be achieved after making the following assumptions

- The functions $\Xi_{i,j}, 1 \leq i, j \leq d$, are bounded by a positive constant $\bar{\sigma}^2$ independent of $\boldsymbol{S}$ and t :

$$\|\Xi_{i,j}\|_{L^\infty\left(\mathbb{R}_+^d\right))} \leq \bar{\sigma}^2$$

- There exists a positive constant $\underline{\sigma}$ such that for all $q \in \mathbb{R}^d$,

$$\sum_{i=1}^{d} \sum_{j=1}^{d} \Xi_{i,j} q_i q_j \geq \underline{\sigma}^2 |q|^2$$

- There exists a positive constant M such that, for $i = 1, \ldots, d$,

$$\left\| \sum_{i=1}^{d} \frac{\partial}{\partial S_i} \left(\Xi_{i,j} S_i \right) \right\|_{L^\infty\left(\mathbb{R}_+^d\right)} \leq M$$

- The function r is nonnegative and bounded by a constant.

With the assumptions above, the bilinear form a_t defined in (3) is continuous on $V \times V$ and there exists a constant $\bar{c}$ independent of t such that, for any $v, w \in V$,

$$a_t(v, w) \leq \bar{c}|v|_V |w|_V$$

Moreover, there is a uniform Gärding's inequality for a_t : there exists a positive constant $\underline{c}$ and a non-negative constant λ such that, for any $v \in V$,

$$a_t(v, v) \geq \underline{c}|v|_V^2 - \lambda \|v\|_{L^2\left(\mathbb{R}_+^d\right)}^2$$

From these considerations, we see that a variational formulation to (2) is as follows: Find $P \in \mathcal{C}^0([0, T]; L^2(\mathbb{R}_+)) \cap L^2(0, T; \mathcal{K})$, such that $\frac{\delta P}{\delta t} \in L^2(0, T; V')$, satisfying

$$\begin{aligned} P(\boldsymbol{S}, 0) = P_\circ(\boldsymbol{S}), \quad \boldsymbol{S} \in \mathbb{R}_+^d \\ \forall v \in \mathcal{K}, \quad \left(\tfrac{\delta P}{\delta t}(t), v - P(t)\right) + a(P(t), v - P(t)) \geq 0 \ . \end{aligned} \tag{5}$$

The solution to this variational inequality evolution problem exists and is unique (see [6]).

3.3 Discretization

We deal with a simple implementation of the finite element method for approximating the pricing function of an option on a basket containing two assets. Hence we need to discretize time and a space of dimension $d = 2$.

3.3.1 The Time Semi-Discrete Problem

We introduce a partition of the interval $[0, T]$ into subintervals $[t_{m-1}, t_m]$, $1 \leq m \leq M$, such that $0 = t_0 < t_1 < \cdots < t_m = T$. We denote by δt_m the length $t_m - t_{m-1}$, and by δt the maximum of the $\delta t_m, 1 \le m \le M$.

For simplicity, we assume that $P_\circ \in V$, where V is given by (4). We discretize (5) by means of an implicit Euler scheme, *i.e.* we look for $P^m \in V, m = 0, \dots, M$, such that $P^0 = P_\circ$, and for $m = 1, \dots, M$, $\forall v \in V$,

$$\frac{1}{\delta t_m}\left(P^m - P^{m-1}, v\right)_{L^2(\Omega)} + a_{t_m}\left(P^m, v\right) = 0$$

where a_{t_m} is given by (3). This scheme is first order.

Remark 3.3. *If $P_\circ$ does not belong to V, then we first have to approximate $P_\circ$ by a function in V, at the cost of an additional error.*

3.3.2 The Full Discretization: Lagrange Finite Elements

As we already pointed out, discretization with respect to S_1, S_2 consists of replacing V with a finite dimensional subspace $V_h \subset V$. We choose V_h as a space of continuous piecewise polynomial (we choose affine ones) functions on a triangulation of Ω : for a positive real number h, consider a partition $\mathcal{T}_h$ of Ω into nonoverlapping closed triangles, ($\mathcal{T}_h$ is the set of all the triangles forming the partition) such that:

1. $\bar{\Omega} = \cup_{K \in \mathcal{T}_h} K$
2. For all $K \neq K'$ two triangles of $\mathcal{T}_h, K \cap K'$ is either empty, either a vertex of both K and K', or a whole edge of both K and K'.
3. For all $K \neq K'$ two triangles of $\mathcal{T}_h, K \cap K'$ is either empty, either a vertex of both K and K', or a whole edge of both K and K'.

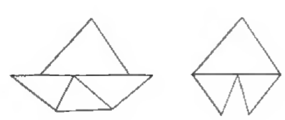


Figure 1: These cases are ruled out

Remark 3.4. *If Ω is not polygonal but has a smooth boundary, it is possible to find a set $\mathcal{T}_h$ of nonoverlapping triangles of diameters less than h such that the distance between $\bar{\Omega}$ and $\cup_{K\in\mathcal{T}_h} K$ scales like h^2.*

For a positive integer k, we introduce the spaces

$$W_h = \left\{ w_h \in \mathcal{C}^0(\bar{\Omega}) : w_h|_K \in \mathcal{P}^k, \forall K \in \mathcal{T}_h \right\}, \quad V_h = \left\{ v_h \in W_h, \, v_h|_{\Gamma_0} = 0 \right\}$$

As we already pointed out, we focus on the case when $k = 1$, *i.e.* the functions in W_h are locally affine. It is clear that V_h is a finite dimensional subspace of V.

Assuming that $P_\circ \in V_h$, the full discretization of the variational formulation consists of finding $P_h^m \in V_h, m = 0, \ldots, M$, such that $P_h^0 = P_\circ$, and

$$\forall v_h \in V_h, \quad \frac{1}{\delta t_m} \left(P_h^m - P_h^{m-1}, v_h \right)_{L^2(\Omega)} + a_{t_m} \left(P_h^m, v_h \right) = 0.$$

Note that not always is possible to compute $a_{t_m}(u_h, v_h)$ for $u_h, v_h \in V_h$ analytically, as in the case of non-constant coefficients. This can be solved by using quadrature formulas which induces an additional but controlled source of error.

4 Semi–Smooth Newton Method

For any positive constant c, the discretized problem is equivalent to finding $P^m \in V$ and a Lagrange multiplier $\lambda \in V'$ such that

$$\begin{aligned} \forall v \in V, \quad \tfrac{1}{\delta t_m}(P^m - P^{m-1}, v) + a_{t_m}(P^m, v) - (\lambda, v) = 0 \\ \lambda = max(0, \lambda - c(P^m - P^0)) \ . \end{aligned}$$

To be consistent with the standard formulation (see [5]) we define

$$\begin{aligned} \tilde{a}_{t_m}(P^m, v) = -\tfrac{1}{\delta t_m}(P^{m-1}, v) + a_{t_m}(P^m, v) \\ f = -\tfrac{P^m}{\delta t_m} \ , \end{aligned}$$

and we reformulate the problem as finding $P^m \in V$ and a Lagrange multiplier $\lambda \in V'$ such that

$$\begin{aligned} &\forall v \in V, \quad \tilde{a}(P, v) - (\lambda, v) = (f, v) \\ &\lambda = max(0, \lambda - c(P^m - P^0)) \ . \end{aligned} \tag{6}$$

Denoting by $A_{ij} := \tilde{a}(w^i, w^j)$, Newton's algorithm applied to (6) gives

1. Choose $c > 0, P^{m,0}, \lambda^0 \geq 0$, set $k = 0$.
2. Find
$$A_k := S \ : \lambda^k(S) + c(P^{m,k}(S) - P^0(S)) > 0$$
3. Set
$$P^{m,k+1} = arg \min_{P \in V} \left\{ \frac{1}{2}\tilde{a}(P^{m,k+1}, v) - (f, v) : P^{m,k} = P^{m,0} \text{ on } A_k \right\}$$
4. Set
$$\lambda^{k+1} = f - AP^{m,k+1}$$

   Set $k = k + 1$ and go to 2.

The beauty of the method is that its implementation is almost painless from an implicit finite element or finite difference solver with the following "TGV" (très grande valeur) trick:
To compute the solution v_h of

$$\overline{a}(v_h, w^j) = (\overline{f}, w^j) \ \forall j \notin K, \ v_h(q^j) = \phi(q^j), \ j \in K, \ j = 1...I$$

- Compute $A_{ij} = \overline{a}(w^i, w^j)$ and $f_j = (f, w^j) \ \forall i, j$.
- Reset $A_{ii} = 10^{30}, \ f_i = 10^{30}\phi(q^j) \ \forall i \in K$
- Reset $Av = f$.

Although it look as if the modification of A would ruin its condition number in reality it does not because it is essentially equivalent to the elimination of the rows and columns of indices in K. It would be equivalent to rows and columns elimination if instead of 10^{30} the largest integer number that the computer can store is used so that the addition of $\sum_{j \neq i} A_{ij}v_i$ does not affect the significant digits.

The algorithm converges locally super linearly provided A is an M-matrix (see [4]). A is a M-matrix if mass-lumping (a reduced integration formula with Gauss points at the vertices) is applied to the part of A which comes from $(w^i, w^j)/\delta t$ and provided the triangulation has no angle greater than $\pi/2$. If it is not the case the method still converges provided $P^{m,0}, \lambda_0$ is not too far from the solution. The semi-smooth Newton method converges if applied to the continuous problem, before discretization, but the precision is proportional to c^{-1}, so c must be large.

5 Results

In our work, the aforementioned procedures were applied to the pricing of the American put on the best of two assets; in this section, we will outline the implementation of the algorithm and present the results.

5.1 FreeFem++

The pricing algorithm was implemented using FreeFem++, a partial differential equation solver for non-linear multi-physics systems in 1D, 2D, 3D and 3D border domains through the finite element method, using the FreeFem++ language, a C++ idiom; the resulting code shows near optimal execution speed compared with compiled C++ implementations programmed directly.

FreeFem++ allows the user to write problem description by their variational formulations, with access to the internal vectors and matrices if needed, and comes with an automatic mesh generator, based on the Delaunay–Voronoi algorithm, and metric-based anisotropic mesh adaptation.

The latter capability played a crucial role in our code, greatly enhancing its computational efficiency.

Our code, which we attach to this report, is schematized by the flowchart in Figure 2.

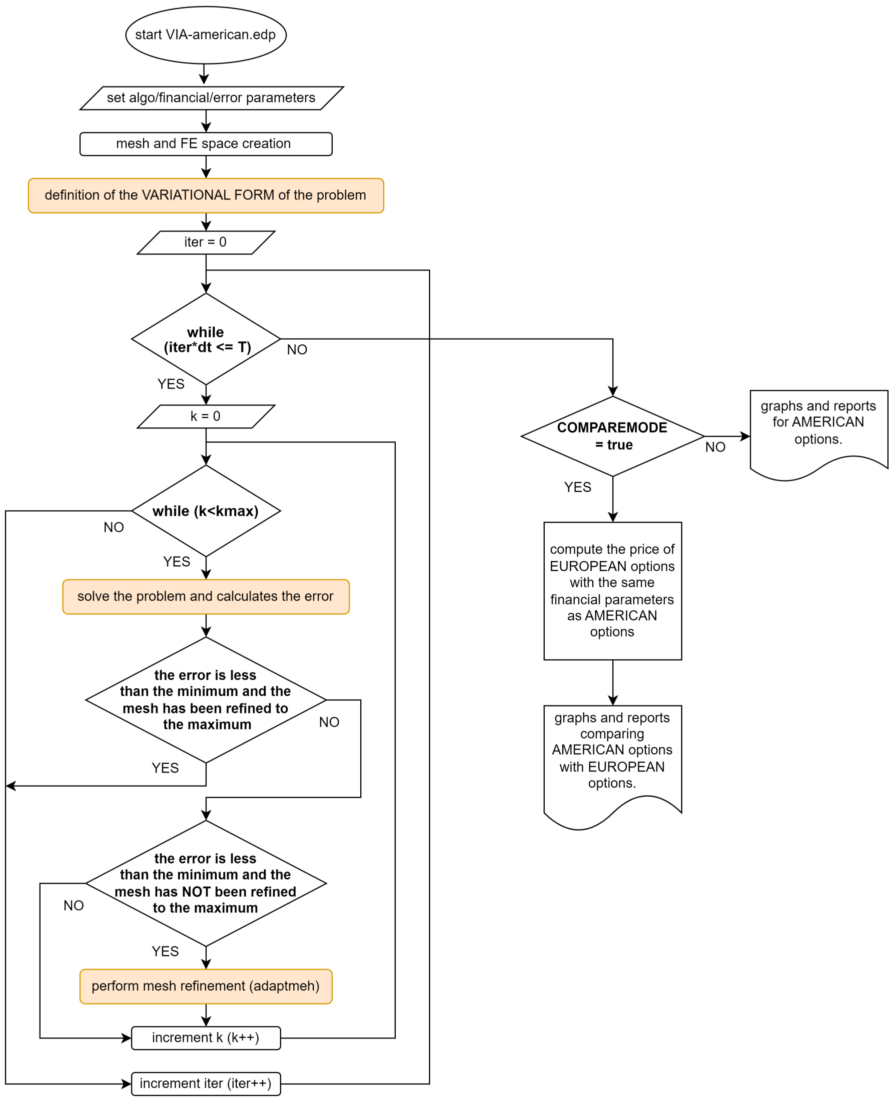
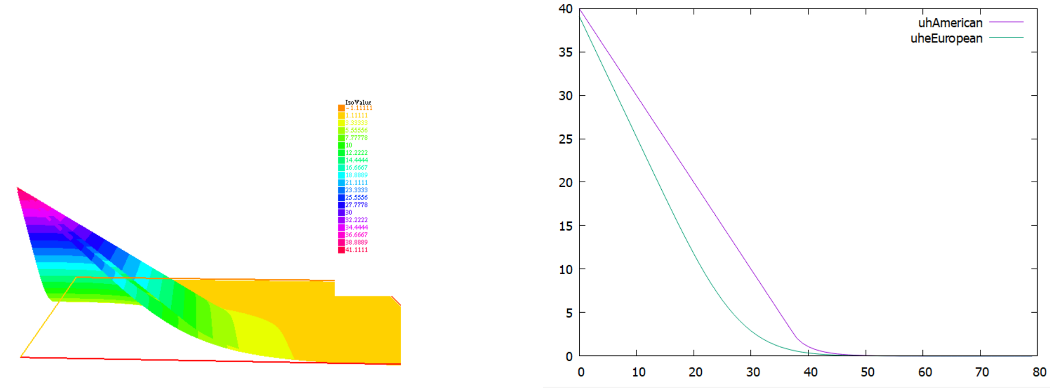


Figure 2: Flowchart

5.2 Pricing Results

In this section we present the pricing of a specific American put on the best of two assets, characterised by: $K = 40$, $\sigma_{S_1} = 0.35$, $\sigma_{S_2} = 0.3$, $\rho = -0.3$ and $r = 2\%$. Consistently with the payoff structure, in Figure 3a we observe that the smaller the maximum between the two underlying S_1

and S_2, the more in the money the put option and the higher its price.



(a) Price American vs European

(b) American vs European diagonal section

Figure 3

The shape of the early exercise premium in Figure 4 has a peak along the diagonal of the exercise region: taking a vertical section for S_2, the value of the American is fixed at $K - S_2$ until $S_1 \leq S_2$, while the value of the European is higher when $S_1 \ll S_2$ and decreases when S_1 gets close to S_2, since the probability of the European expiring out of the money increases. A specular dynamics explains the shape for $S_1 > S_2$.

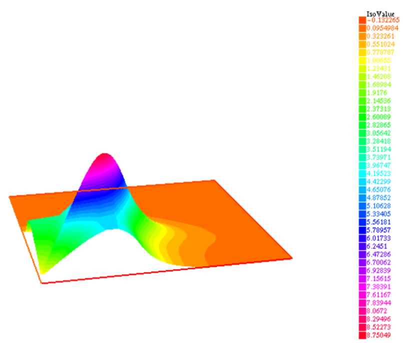


Figure 4: Early exercise premium

The time value of the put (Figure 5) is maximised when one of the two underlying assets is at K and the other is considerably lower, in this case the exotic payoff approximates a plain vanilla put at the money. The exercise region extends from where the put is deep in the money to where, along the diagonal, the price of the two assets $S_1 = S_2$ approaches K from below. Although a plain vanilla put attains its maximum value as S approaches K, the structure of the basket payoff results in the continuation value being more negatively influenced by volatility. As a consequence, exercise is optimal even within this proximity of $S_1 = S_2$ to the strike price.

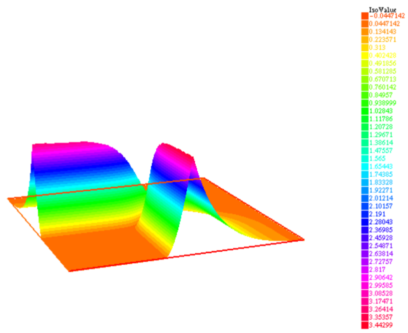
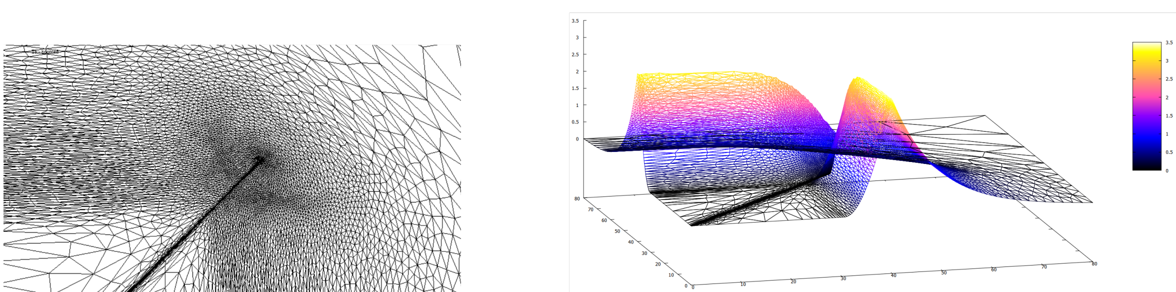


Figure 5: Time value

Figure 6a shows the adapted mesh for the problem and Figure 6b shapes it on the time value. The initial partition is designed in order to have only vertexes and edges of triangles along the diagonal, where the price function is angular due to the switch in the maximum of the two underlying assets. The adapted mesh is particularly refined in correspondence of K for both S_1 and S_2, where time value peaks.



(a) Mesh 2D

(b) Mesh on time value

Figure 6

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