Three-Way Multiattribute Decision-Making Based on Outranking Relations

Jianming Zhan, Haibo Jiang, and Yiyu Yao

***Abstract*—In contrast to two-way decisions (2WD), three-way decisions (3WD) can effectively reduce decision risks by utilizing a new delayed decision option. This article incorporates 3WD into multiattribute decision-making (MADM) based on an outranking relation. We construct the outranked set for each alternative and introduce a hybrid information table that combines an MADM matrix with a loss function table. We propose three strategies to design a new 3WD model for MADM. The rationality and effectiveness of the proposed 3WD method are demonstrated by solving a problem of enterprise project investment target selections. Finally, we provide the comparative analysis and two experimental evaluations. The results show that the proposed 3WD method is effective and practically useful.**

***Index Terms*—Decision-theoretic rough fuzzy set (DTRFS), multiattribute decision-making, outranking relation, three-way decision.**

I. Introduction

MULTIATTRIBUTE decision-making (MADM) is one of the most important topics of decision-making theory. The main task of MADM is to evaluate the performance of alternatives in multiattribute environments [1]. A decision maker (DM) evaluates each alternative based on a set of attributes to produce an evaluation matrix. Many decision-making models have been proposed to assist a DM to obtain a reasonable decision result. These models are typically based on traditional two-way decision (2WD). Recently, Yao [2]–[4] proposed new theory named three-way decision (3WD) via thinking in three. Three-way decision can effectively reduce the decision risks by introducing a third delayed decision option. The emergence of 3WD opens up a new avenue and offers new opportunities for studying MADM. The main objective of the present article is to explore this new direction of research. We use the well-known method ELECTRE-I to establish outranking relations on a universe and discuss the corresponding 3WD model in an ordered information table (OIT). We apply the proposed 3WD method for MADM to the problem of EPI target selections.

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Jianming Zhan and Haibo Jiang are with the Department of Mathematics, Hubei Minzu University, Enshi 445000, China (e-mail: zhanjianming@hotmail.com; 1872482530@qq.com).

Yiyu Yao is with the Department of Computer Science, University of Regina, Regina, SK S4S 0A2, Canada (e-mail: yyao@cs.uregina.ca).

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A. Brief Review of Three-Way Decisions

The theory of 3WD may be described and explained in a trisecting-acting-outcome conception [4], that is, we divide a whole into three parts and act on the three parts for achieving a desired outcome. In the context of decision-making, the three parts correspond to the three options of acceptance, rejection, and delayed decisions. The theory was proposed by Yao [2], [3] originally to provide a sound semantical interpretation of decision-theoretic rough sets (DTRS). We have seen a fast growing and extensive studies on a more general theory and applications of 3WD. The theory has been applied in many areas, including malware analysis [5], social networks [6], multiclass statistical recognition [7], medical decisions [8], cognitive concept learning [9], [10], and so on.

Earlier studies on 3WD mainly concentrate on DTRS [2], involving investigations on conditional probabilities, loss functions, and pairs of thresholds on the conditional probabilities. Yao and Zhou [11] proposed naive Bayesian rough set model and gave a simple approach to estimate conditional probability. Herbert and Yao [12] studied approaches to determining a loss function and decision thresholds based on game-theoretic rough sets. Jia *et al.* [13] proposed an optimal representation of DTRS model and an adaptive method to calculate a pair of thresholds. Deng and Yao [14] combined information entropy to characterize the uncertainty of three regions and provided a method to automatically calculate a pair of thresholds. Liu *et al.* [15] proposed a new classification method based on the logistic regression and DTRS, and calculated a pair of thresholds and conditional probability in DTRS. Liang *et al.* [16]–[18] determined loss functions and discussed the calculation methods of a pair of thresholds based on hesitant fuzzy numbers and linguistic assessments, respectively.

Subsequent studies focus on a wider sense of 3WD that moves beyond rough set theory, as well as new applications of 3WD. Yao [19] introduced a general evaluation-based model of 3WD. Hu [20] proposed the concept of 3WD spaces in an attempt to establish a unified framework of 3WD. Qian *et al.* [21] and Mandal and Ranadive [22] investigated multigranulation 3WD. Ciucci *et al.* [23] explored the connections between three-valued logics and extended classic logic. Liu *et al.* [24] proposed a new 3WD model for incomplete information tables. Li *et al.* [10] studied three-way concept learning by multigranularity from the viewpoint of cognition. Campagner *et al.* [25] proposed a set of learning algorithms to treat and exploit ambiguity in data based on 3WD. Sun *et al.* [26] introduced a 3WD method to conflict analysis and resolution by means of probabilistic rough sets over two universes. Fang and Min [27] discussed cost-sensitive approximate attribute reductions by utilizing 3WD. Li *et al.* [28] put forward a sequential 3WD method for cost-sensitive face recognition. Yao [3], [4], [29] systematically investigated the connections among 3WD, cognitive computing, granular computing, and formal concept analysis.

B. Brief Review of MADM

One of the purposes of MADM is to find an optimal alternative from all feasible alternatives in a limited time according to certain criteria or attributes [1], [30]. An attribute is associated with a weight to indicate its relative importance. MADM has been widely used in supplier selections [31], [32], medical diagnosis [33], talent introductions [34], [35], and many other fields. A number of methods have been proposed to deal with MADM problems, including utility theory-based methods and outranking methods [36]–[39]. These methods search for a ranking of alternatives toward a high quality decision-making.

However, the traditional MADM methods only determine the optimal alternative according to the attribute values, but they ignore the costs/losses of alternatives in the procedure of decision-making. In addition, these methods belong to 2WD (acceptance or rejection). In fact, with the increasing uncertainty and complexity of decision-making problems, we hope to gather more information to further determine the decision results. This idea is consistent with the central idea of 3WD. Therefore, utilizing 3WD to solve MADM problems has become a promising research issue in recent years. Hu *et al.* [40] introduced a novel TODIM method-based 3WD model for medical treatment selections. Jia and Liu [41] investigated a novel 3WD model under multiple criteria environments and used it to solve the problem of project investment selections. Pang *et al.* [42] proposed a novel data-driven GDM method under interval-valued intuitionistic uncertain linguistic environment based on the idea of multigranulation and 3WD. Liu *et al.* [43] established a novel MADM three-way model for intuitionistic fuzzy numbers and utilized it to supplier selection problems. Although these studies have been applied to various uncertain decision-making problems, the 3WD model based on outranking relations is rarely mentioned since the decision information tables are usually expressed as an OIT. In view of this reason, we plan to investigate this topic in this article.

C. Motivations and the Structure of This Article

This article is motivated by these existing researches. There are three main motivations and contributions of this article.

First, existing studies on 3WD considered preference information among alternatives based on the notion of dominance relations [44]. However, a dominance relation in an OIT may be too restrictive. As some alternatives cannot be dominated by others, decision results may not converge when solving some MADM problems. In contrast, an outranking relation can make up for the deficiency of the dominance relation [39]. For this reason, by virtue of the ELECTRE-I method, we establish an outranking relation on a universe and the corresponding 3WD method for solving MADM problems.

Second, traditional MADM methods only consider the information table and ignore the loss functions of alternatives. Some existing 3WD models use the same cost or loss functions of alternatives when taking three actions with respect to (w.r.t.) two states. On the other hand, in actual decision-making problems, each alternative may have different loss functions. Based on initial studies by Liu *et al.* [24], [44], we use a hybrid information table that considers both the MADM matrix and different loss functions for individual alternatives.

Third, many classic MADM methods are developed based on 2WD, that is, the decision result is an either-or problem. 3WD can better explain the outcome of decisions in some situations. Accordingly, we introduce a novel 3WD method in MADM problems. The method not only divides alternatives into three regions but also gives a complete ranking according to the corresponding losses of alternatives.

The rest of this article is organized as follows. Section II introduces the basic concepts of outranking relations, the ELECTRE-I method, and 3WD in rough sets. Section III discusses the outranked set based on an outranking relation. A hybrid information table is proposed by integrating a MADM matrix with a loss function table, which gives rise to a new 3WD model. Section IV applies the proposed 3WD method to the problem of EPI target selections. Sections V and VI give a comparison analysis and two experimental evaluations to illustrate the effectiveness and superiority of the proposed 3WD method. Finally, Section VII concludes this article.

II. Outranking Relations and 3WD

In this section, we introduce some basic notions of outranking relations, the ELECTRE-I method and 3WD.

A. Outranking Relations and the ELECTRE-I Method

Let $U = \{u_1, u_2, \ldots, u_n\}$ be a set of n alternatives, $\mathbf{C} = \{C_1, C_2, \ldots, C_m\}$ be a set of m attributes, and $W = (w_1, w_2, \ldots, w_m)$ be the weight vector of attributes, where $0 \leq w_j \leq 1$ and $\sum_{j=1}^{m} w_j = 1$. The value of the alternative u_i w.r.t. the attribute C_k is denoted by $C_k(u_i) (i = 1, 2, \ldots, n, k = 1, 2, \ldots, m)$ and all values can be represented as an $n \times m$ MADM matrix [1]. We always want to choose an optimal alternative from U by evaluating and ranking all alternatives under the m attributes. Kaliszewski *et al.* [1] utilized the dominance relation to solve this problem. They defined the dominance relation as: $u_i \succeq u_j \Leftrightarrow C_k(u_i) \geq C_k(u_j)$ for all $C_k \in \mathbf{C}$, where $u_i \succeq u_j$ means that u_i is dominating u_j. However, there are some alternatives that cannot be dominated by others. In contrast, an outranking relation can make up for the deficiency of the dominance relation [39].

Definition II.1: Based on an MADM matrix, for any two alternatives u_i and u_j, if $C_k(u_i) \geq C_k(u_j)$ for some C_k such that a DM believes that u_i is superior to or indifferent from u_j, then there exists an outranking relation between u_i and u_j, denoted as $u_i \mathcal{S} u_j$, meaning u_i is at least as good as u_j.

There are many methods to design an outranking relation. Two common methods to construct an outranking relation are the ELECTRE method and the PROMETHEE method. In the following, we will introduce an outranking relation which is built by the ELECTRE method.

The ELECTRE method or the ELECTRE-I method was proposed by Roy [39] to process some choice problems in MADM. The outranking relations among alternatives are used to provide a choice for DMs.

After normalizing an MADM matrix, the ELECTRE-I method constructs an outranking relation based on two concepts, i.e., the concordance and the nondiscordance. A concordance test is carried out by calculating two concordance indices that are computed by three index sets. For any two alternatives $u_i, u_j \in U$, the three index sets are given by

$$\begin{aligned} C_{ij}^{+} &= C^{+}(u_i, u_j) = \{k \mid k = 1, 2, \ldots, m, C_k(u_i) > C_k(u_j)\} \\ C_{ij}^{=} &= C^{=}(u_i, u_j) = \{k \mid k = 1, 2, \ldots, m, C_k(u_i) = C_k(u_j)\} \\ C_{ij}^{-} &= C^{-}(u_i, u_j) = \{k \mid k = 1, 2, \ldots, m, C_k(u_i) < C_k(u_j)\} \end{aligned} \tag{1}$$

and the two concordance indices are defined by

$$C_{ij} = \left(\sum_{k \in C_{ij}^{+}} w_k + \sum_{k \in C_{ij}^{=}} w_k \right) \Bigg/ \sum_{k=1}^{n} w_k = \sum_{k \in C_{ij}^{+}} w_k + \sum_{k \in C_{ij}^{=}} w_k, \; D_{ij} = \sum_{k \in C_{ij}^{+}} w_k \Bigg/ \sum_{k \in C_{ij}^{-}} w_k. \tag{2}$$

Set $\delta \in (0.5, 1]$, if $C_{ij} \geq \delta$ and $D_{ij} \geq 1$, we say an outranking relation $u_i \mathcal{S} u_j$ is validated by the concordance test.

A nondiscordance test is carried out by a discordance index defined as follows:

$$G_{ij} = \max_{k=1,2,\ldots,m} \left\{ \frac{C_k(u_j) - C_k(u_i)}{p} \right\} \tag{3}$$

where p denotes a given veto threshold. If $G_{ij} \geq 1$, an outranking relation $u_i \mathcal{S} u_j$ is invalid, otherwise it passes the nondiscordance test.

If the outranking relation $u_i \mathcal{S} u_j$ passes both the concordance test and the nondiscordance test, i.e., $C_{ij} \geq \delta$, $D_{ij} \geq 1$, and $G_{ij} \geq 1$, then the alternative u_i outranks the alternative u_j and the outranking relation $u_i \mathcal{S} u_j$ holds.

B. *Three-Way Decision Based on DTRS*

In the formulation of DTRS, there are two states and three actions [2]. Let A denote a subset of alternatives $A \subseteq U$, we have two states $S = \{A, \neg A\}$ for each alternative, namely, in A and not in A, respectively. Let $[u]$ denote the equivalence class containing $u \in U$ under an equivalence relation. The set of three actions is given by $\mathcal{A} = \{a_P, a_B, a_N\}$, that is, classifying an alternative u by deciding $u \in POS(A)$, deciding $u \in BND(A)$ and deciding $u \in NEG(A)$, respectively. Finally, let $\lambda_{PP}, \lambda_{BP}$, and λ_{NP} represent the losses incurred for choosing actions a_P, a_B, and a_N, respectively, when $u \in A$. Similarly, let λ_{PN}, λ_{BN}, and λ_{NN} denote the losses caused for adopting the same actions when $u \notin A$. For an element u, the expected losses $\mathcal{R}(a_\diamond|[u])(\diamond = P, B, N)$ of choosing three different actions are calculated as follows:

$$\begin{aligned} \mathcal{R}(a_P|[u]) &= \lambda_{PP}\mathcal{P}(A|[u]) + \lambda_{PN}\mathcal{P}(\neg A|[u]) \\ \mathcal{R}(a_B|[u]) &= \lambda_{BP}\mathcal{P}(A|[u]) + \lambda_{BN}\mathcal{P}(\neg A|[u]) \\ \mathcal{R}(a_N|[u]) &= \lambda_{NP}\mathcal{P}(A|[u]) + \lambda_{NN}\mathcal{P}(\neg A|[u]) \end{aligned} \tag{4}$$

where $\mathcal{P}(A|[u])$ denotes that the conditional probability of an alternative is in A given that it is in $[u]$.

In view of the Bayesian decision procedure, the minimum-risk decision rules are obtained as follows:

(P) If $\mathcal{R}(a_P|[u]) \leq \mathcal{R}(a_B|[u]), \mathcal{R}(a_P|[u]) \leq \mathcal{R}(a_N|[u])$, then $u \in POS(A)$

(B) If $\mathcal{R}(a_B|[u]) \leq \mathcal{R}(a_P|[u]), \mathcal{R}(a_B|[u]) \leq \mathcal{R}(a_N|[u])$, then $u \in BND(A)$

(N) If $\mathcal{R}(a_N|[u]) \leq \mathcal{R}(a_P|[u]), \mathcal{R}(a_N|[u]) \leq \mathcal{R}(a_B|[u])$, then $u \in NEG(A)$. (5)

By the property of the probability function, we have $\mathcal{P}(A|[u]) + \mathcal{P}(\neg A|[u]) = 1$. According to the real-life semantic explanation of the loss functions, we may assume that $\lambda_{PP} \leq \lambda_{BP} < a_{NP}$ and $\lambda_{NN} \leq \lambda_{BN} < a_{PN}$. In addition, under the condition $(\lambda_{PN} - \lambda_{BN})(\lambda_{NP} - \lambda_{BP}) > (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$, we have $\alpha > \beta$. The decision rules (P)–(N) can be simplified as follows:

(P) If $\mathcal{P}(A|[u]) \geq \alpha$, then we have $u \in POS(A)$

(B) If $\beta < \mathcal{P}(A|[u]) < \alpha$, then we have $u \in BND(A)$

(N) If $\mathcal{P}(A|[u]) \leq \beta$, then we have $u \in NEG(A)$ (6)

where

$$\begin{aligned} \alpha &= \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})} \\ \beta &= \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}. \end{aligned} \tag{7}$$

III. Novel 3WD Model Based on Outranking Relations

In this section, a novel 3WD model based on outranking relations is introduced. We propose a decision-theoretic rough fuzzy set (DTRFS) model based on an outranking relation, which serves as a basis of a novel 3WD model to MADM.

A. *DTRFS Based on Outranking Relations*

In this section, based on the outranking relation established by the ELECTRE-I method, we define the outranked set of any alternative and put forward a DTRFS.

Definition III.1: According to an MADM matrix and an outranking relation $\mathcal{S}$ determined by the ELECTRE-I method, we define an outranked set of any alternative $u \in U$

$$[u]_{\mathcal{S}} = \{v \mid u\mathcal{S}v \wedge v \in U\}. \tag{8}$$

If $v \in [u]_{\mathcal{S}}$, we have $u\mathcal{S}v$. Let $U/\mathcal{S} = \{[u]_{\mathcal{S}} | u \in U\}$ denote a possibly overlapping classification of all alternatives w.r.t. **C**

TABLE I
MADM MATRIX OF STUDENT EVALUATIONS

$X/\mathbf{C}$	C_1	C_2	C_3	C_4	C_5	C_6
u_1	96	93	90	71	85	72
u_2	92	82	78	72	94	78
u_3	85	97	84	89	92	80
u_4	90	83	89	71	78	85
u_5	98	85	82	85	69	75
u_6	85	89	78	68	84	88
u_7	84	92	81	80	72	77
u_8	82	85	70	75	93	67

in an MADM matrix. In general, $U/\mathcal{S} = \{[u]_{\mathcal{S}} | u \in U\}$ is a covering of U.

Example III.2: Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ denote the set of 8 students and $\mathbf{C} = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ denote the set of 6 courses. The weights of six courses are identical. The evaluation results are presented in Table I, where the scores of each student w.r.t. 6 courses range from 0 to 100.

First of all, we normalize the MADM matrix of student evaluations through $C_j(u_i)/100$ $(i = 1, 2, \ldots, 8; j = 1, 2, \ldots, 6)$. Then, by means of the ELECTRE-I method with $\delta = 0.6$ and $p = 0.2$, we have $u_1\mathcal{S}u_1$, $u_1\mathcal{S}u_4$, $u_1\mathcal{S}u_6$, $u_1\mathcal{S}u_7$, $u_1\mathcal{S}u_8$, $u_2\mathcal{S}u_2$, $u_2\mathcal{S}u_6$, $u_2\mathcal{S}u_8$, $u_3\mathcal{S}u_1$, $u_3\mathcal{S}u_3$, $u_3\mathcal{S}u_5$, $u_3\mathcal{S}u_6$, $u_3\mathcal{S}u_7$, $u_3\mathcal{S}u_8$, $u_4\mathcal{S}u_4$, $u_4\mathcal{S}u_7$, $u_5\mathcal{S}u_5$, $u_6\mathcal{S}u_6$, $u_6\mathcal{S}u_8$, $u_7\mathcal{S}u_7$, $u_8\mathcal{S}u_8$.

Finally, based on Definition III.1, we calculate the outranked set for every $u_i \in U (i = 1, 2, \ldots, 8)$ as follows: $[u_1]_{\mathcal{S}} = \{u_1, u_4, u_6, u_7, u_8\}$, $[u_2]_{\mathcal{S}} = \{u_2, u_6, u_8\}$, $[u_3]_{\mathcal{S}} = \{u_1, u_3, u_5, u_6, u_7, u_8\}$, $[u_4]_{\mathcal{S}} = \{u_4, u_7\}$, $[u_5]_{\mathcal{S}} = \{u_5\}$, $[u_6]_{\mathcal{S}} = \{u_6, u_8\}$, $[u_7]_{\mathcal{S}} = \{u_7\}$, $[u_8]_{\mathcal{S}} = \{u_8\}$.

From the main concepts of the ELECTRE-I method and the definition of the outranked set, we state a useful result.

Proposition III.3: In an MADM matrix, suppose that $[u]_{\mathcal{S}}$ is the outranked set of $u \in U$. The following statements hold.

1) For any $u \in U$, there is $u \in [u]_{\mathcal{S}}$, namely, $[u]_{\mathcal{S}}$ satisfies the reflexivity;
2) For any $u, v, w \in U$, if $v \in [u]_{\mathcal{S}}$ and $w \in [v]_{\mathcal{S}}$, then we have $w \in [u]_{\mathcal{S}}$.

Proof: It can be directly obtained by Definition III.1. ■

We now introduce two approximations and three decision regions of a fuzzy concept via the outranked set.

Definition III.4: Given an MADM matrix and a pair of thresholds $0 \leq \beta < \alpha \leq 1$, for any fuzzy set $X \in \mathcal{F}(U)$, its the upper and lower approximations w.r.t. an outranking relation $\mathcal{S}$ are defined as follows:

$$\overline{\mathcal{S}}_{(\alpha,\beta)}(X) = \{u \mid \mathcal{P}(X|[u]_{\mathcal{S}}) > \beta\}$$
$$\underline{\mathcal{S}}_{(\alpha,\beta)}(X) = \{u \mid \mathcal{P}(X|[u]_{\mathcal{S}}) \geq \alpha\} \tag{9}$$

where $\mathcal{P}(X|[u]_{\mathcal{S}}) = (\sum_{v\in[u]_{\mathcal{S}}} X(v))/|[u]_{\mathcal{S}}|$ and $|\diamond|$ represents the cardinality of a set. Moreover, we have $\mathcal{P}(X|[u]_{\mathcal{S}}) + \mathcal{P}(\neg X|[u]_{\mathcal{S}}) = 1$. Based on the two approximations $\overline{\mathcal{S}}_{(\alpha,\beta)}(X)$ and $\underline{\mathcal{S}}_{(\alpha,\beta)}(X)$, a new DTRFS is obtained, which is denoted by $(U, \overline{\mathcal{S}}_{(\alpha,\beta)}(X), \underline{\mathcal{S}}_{(\alpha,\beta)}(X))$.

The upper and lower approximations of the fuzzy set X generate three decision regions as follows:

$$POS(X) = \underline{\mathcal{S}}_{(\alpha,\beta)}(X) = \{u \mid \mathcal{P}(X|[u]_{\mathcal{S}}) \geq \alpha\}$$
$$BND(X) = \overline{\mathcal{S}}_{(\alpha,\beta)}(X) - \underline{\mathcal{S}}_{(\alpha,\beta)}(X) = \{u \mid \alpha < \mathcal{P}(X|[u]_{\mathcal{S}}) < \beta\}$$
$$NEG(X) = U - \overline{\mathcal{S}}_{(\alpha,\beta)}(X) = \{u \mid \mathcal{P}(X|[u]_{\mathcal{S}}) \leq \beta\}. \tag{10}$$

They are three pair-wise disjoint subsets of U.

Example III.5: (Continued From Example III.2) Suppose that a fuzzy set X stands for "good students" and the membership function of the fuzzy set is given as follows:

$$X = \frac{0.82}{u_1} + \frac{0.77}{u_2} + \frac{0.94}{u_3} + \frac{0.67}{u_4} + \frac{0.55}{u_5} + \frac{0.65}{u_6} + \frac{0.54}{u_7} + \frac{0.49}{u_8}.$$

Based on Example III.2, we have $\mathcal{P}(X \mid [u_1]_{\mathcal{S}}) = 0.634$, $\mathcal{P}(X \mid [u_2]_{\mathcal{S}}) = 0.6367$, $\mathcal{P}(X \mid [u_3]_{\mathcal{S}}) = 0.665$, $\mathcal{P}(X \mid [u_4]_{\mathcal{S}}) = 0.605$, $\mathcal{P}(X \mid [u_5]_{\mathcal{S}}) = 0.55$, $\mathcal{P}(X \mid [u_6]_{\mathcal{S}}) = 0.57$, $\mathcal{P}(X \mid [u_7]_{\mathcal{S}}) = 0.54$, $\mathcal{P}(X \mid [u_8]_{\mathcal{S}}) = 0.49$.

Let $\alpha = 0.6$ and $\beta = 0.5$. Based on Definition III.4, the upper and lower approximations of X can be constructed as follows: $\overline{\mathcal{S}}_{(0.6,0.5)}(X) = \{u \mid \mathcal{P}(X \mid [u]_{\mathcal{S}}) \geq 0.5\} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$, $\underline{\mathcal{S}}_{(0.6,0.5)}(X) = \{u \mid \mathcal{P}(X \mid [u]_{\mathcal{S}}) \geq 0.6\} = \{u_1, u_2, u_3, u_4\}$.

The three regions of X are given by

$$POS(X) = \{u_1, u_2, u_3, u_4\}, BND(X) = \{u_5, u_6, u_7\}, NEG(X) = \{u_8\}.$$

B. Three-Way Decision Models Based on MADM

Various 3WD models have been investigated in the contexts of different information tables, e.g., OIT [44], incomplete information tables [24], multisource information tables [45], etc. There are a few studies that use an information table and loss function simultaneously. Liu and Liang [24], [44] introduced a hybrid information table by integrating an information table with a loss function table. Based on their idea, we introduce a hybrid information table by considering both an MADM matrix and a loss function table.

Definition III.6: According to 3WD models and the main concept of MADM, a three-way based MADM table is defined as 3WMADMT=(MADMM, LFT), in which MADMM $= (U, \mathbf{C}, V, W)$ represents an MADM matrix and LFT $= (U, \lambda_{\diamond\diamond})$ represents a loss function table of alternatives in a universe. For any $u \in U$, the value of u in 3WMADMT can be written as $m + 6$ vectors: $f(u) = (C_1(u), C_2(u), \ldots, C_m(u), \lambda_{PP}(u), \lambda_{BP}(u), \lambda_{NP}(u), \lambda_{NN}(u), \lambda_{BN}(u), \lambda_{PN}(u))$.

A 3WMADMT can be represented as in a tabular form as shown in Table II. Based on real-life semantic explanations of decision-making, the loss functions for $u_i (i = 1, 2, \ldots, n)$ satisfy the following two conditions: (I) $\lambda_{PP}(u_i) \leq \lambda_{BP}(u_i) < \lambda_{NP}(u_i)$; (II) $\lambda_{NN}(u_i) \leq \lambda_{BN}(u_i) < \lambda_{PN}(u_i)$.

TABLE II
THREE-WAY-BASED MADM TABLE

$\mathbb{U}$	C_1	C_2	$\cdots$	C_m	λ_{PP}	λ_{BP}	λ_{NP}	λ_{NN}	λ_{BN}	λ_{PN}
u_1	a_{11}	a_{12}	$\cdots$	a_{1m}	$\lambda_{PP}(u_1)$	$\lambda_{BP}(u_1)$	$\lambda_{NP}(u_1)$	$\lambda_{NN}(u_1)$	$\lambda_{BN}(u_1)$	$\lambda_{PN}(u_1)$
u_2	a_{21}	a_{22}	$\cdots$	a_{2m}	$\lambda_{PP}(u_2)$	$\lambda_{BP}(u_2)$	$\lambda_{NP}(u_2)$	$\lambda_{NN}(u_2)$	$\lambda_{BN}(u_2)$	$\lambda_{PN}(u_2)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
u_n	a_{n1}	a_{n1}	$\cdots$	a_{nm}	$\lambda_{PP}(u_n)$	$\lambda_{BP}(u_n)$	$\lambda_{NP}(u_n)$	$\lambda_{NN}(u_n)$	$\lambda_{BN}(u_n)$	$\lambda_{PN}(u_n)$
W	w_1	w_2	$\cdots$	w_m	*	*	*	*	*	*

Remark III.7: Since each attribute plays a different role in a decision-making process, we need to incorporate attribute weights. Comparing with the hybrid information tables in [24], [44], the proposed 3WMADMT not only takes into account both information table and loss function table, but also considers the weight of each attribute.

Since $U/\mathcal{S} = \{[u]_{\mathcal{S}} | u \in U\}$ is a covering on U, different alternatives in an outranked set may have different loss functions. On the basis of existing methods and reasonable semantic interpretations in MADM, there are three possible strategies for aggregating loss functions in $[u]_{\mathcal{S}}$.

Definition III.8: Let $(U, \mathbf{C}, V, W, \lambda_{\diamond\diamond})$ be a 3WMADMT and $\lambda_{\diamond\diamond}(u)(\diamond = P, B, N)$ be the loss function of any $u \in U$. The loss function of $[u]_{\mathcal{S}}$ for any $u \in U$ can be aggregated by the following three strategies.

$$(1) \text{ Optimistic aggregated lo ss function: } \lambda^{Opt}_{\diamond\diamond}([u]_{\mathcal{S}}) = \min_{v \in [u]_{\mathcal{S}}} (\lambda_{\diamond\diamond}(v)).$$

$$(2) \text{ Pessimistic aggregated loss function: } \lambda^{Pes}_{\diamond\diamond}([u]_{\mathcal{S}}) = \max_{v \in [u]_{\mathcal{S}}} (\lambda_{\diamond\diamond}(v)).$$

$$(3) \text{ Average aggregated loss function: } \lambda^{Ave}_{\diamond\diamond}([u]_{\mathcal{S}}) = \frac{\sum_{v \in [u]_{\mathcal{S}}} \lambda_{\diamond\diamond}(v)}{|[u]_{\mathcal{S}}|}.$$

According to Definitions III.6 and III.8, we have the next proposition.

Proposition III.9: If the loss function of every alternative u in Table II satisfies the conditions (I) and (II), we have the following properties.

1) $\lambda^{Opt}_{PP}([u]_{\mathcal{S}}) \le \lambda^{Opt}_{BP}([u]_{\mathcal{S}}) < \lambda^{Opt}_{NP}([u]_{\mathcal{S}})$, $\lambda^{Opt}_{NN}([u]_{\mathcal{S}}) \le \lambda^{Opt}_{BN}([u]_{\mathcal{S}}) < \lambda^{Opt}_{PN}([u]_{\mathcal{S}})$.
2) $\lambda^{Pes}_{PP}([u]_{\mathcal{S}}) \le \lambda^{Pes}_{BP}([u]_{\mathcal{S}}) < \lambda^{Pes}_{NP}([u]_{\mathcal{S}})$, $\lambda^{Pes}_{NN}([u]_{\mathcal{S}}) \le \lambda^{Pes}_{BN}([u]_{\mathcal{S}}) < \lambda^{Pes}_{PN}([u]_{\mathcal{S}})$.
3) $\lambda^{Ave}_{PP}([u]_{\mathcal{S}}) \le \lambda^{Ave}_{BP}([u]_{\mathcal{S}}) < \lambda^{Ave}_{NP}([u]_{\mathcal{S}})$, $\lambda^{Ave}_{NN}([u]_{\mathcal{S}}) \le \lambda^{Ave}_{BN}([u]_{\mathcal{S}}) < \lambda^{Ave}_{PN}([u]_{\mathcal{S}})$.

Proof: It can be directly obtained by the conditions (I), (II) and Definition III.8. ■

Based on Definitions III.6 and III.8, the expected losses $\mathcal{R}^{\bullet}(a_\diamond|[u]_{\mathcal{S}})(\bullet = \{Opt, Pes, Ave\}; \diamond = \{P, B, N\})$ associated with taking different actions (a_P, a_B and a_N) in 3WMADMT can be expressed as follows.

For the optimistic strategy, the expected losses $\mathcal{R}(a_\diamond|[u]_{\mathcal{S}})$ can be expressed as

$$\begin{aligned}
&\mathcal{R}^{Opt}(a_P|[u]_{\mathcal{S}}) \\
&\quad = \lambda^{Opt}_{PP}([u]_{\mathcal{S}})\mathcal{P}(X|[u]_{\mathcal{S}}) + \lambda^{Opt}_{PN}([u]_{\mathcal{S}})\mathcal{P}(\neg X|[u]_{\mathcal{S}}) \\
&\mathcal{R}^{Opt}(a_B|[u]_{\mathcal{S}}) \\
&\quad = \lambda^{Opt}_{BP}([u]_{\mathcal{S}})\mathcal{P}(X|[u]_{\mathcal{S}}) + \lambda^{Opt}_{BN}([u]_{\mathcal{S}})\mathcal{P}(\neg X|[u]_{\mathcal{S}}) \\
&\mathcal{R}^{Opt}(a_N|[u]_{\mathcal{S}}) \\
&\quad = \lambda^{Opt}_{NP}([u]_{\mathcal{S}})\mathcal{P}(X|[u]_{\mathcal{S}}) + \lambda^{Opt}_{NN}([u]_{\mathcal{S}})\mathcal{P}(\neg X|[u]_{\mathcal{S}}).
\end{aligned} \tag{11}$$

Since $\mathcal{P}(X \mid [u]_{\mathcal{S}}) + \mathcal{P}(\neg X \mid [u]_{\mathcal{S}}) = 1$, we have

$$\begin{aligned}
&\mathcal{R}^{Opt}(a_P|[u]_{\mathcal{S}}) \\
&\quad = \lambda^{Opt}_{PP}([u]_{\mathcal{S}})\mathcal{P}(X|[u]_{\mathcal{S}}) + \lambda^{Opt}_{PN}([u]_{\mathcal{S}})(1 - \mathcal{P}(X|[u]_{\mathcal{S}})) \\
&\mathcal{R}^{Opt}(a_B|[u]_{\mathcal{S}}) \\
&\quad = \lambda^{Opt}_{BP}([u]_{\mathcal{S}})\mathcal{P}(X|[u]_{\mathcal{S}}) + \lambda^{Opt}_{BN}([u]_{\mathcal{S}})(1 - \mathcal{P}(X|[u]_{\mathcal{S}})) \\
&\mathcal{R}^{Opt}(a_N|[u]_{\mathcal{S}}) \\
&\quad = \lambda^{Opt}_{NP}([u]_{\mathcal{S}})\mathcal{P}(X|[u]_{\mathcal{S}}) + \lambda^{Opt}_{NN}([u]_{\mathcal{S}})(1 - \mathcal{P}(X|[u]_{\mathcal{S}})).
\end{aligned} \tag{12}$$

In view of the Bayesian decision procedure, the decision rules with minimum-risk criterion are given as follows:

(P1) Decision $u \in POS(X)$ if $\mathcal{R}^{Opt}(a_P|[u]_{\mathcal{S}}) \le \mathcal{R}^{Opt}(a_B|[u]_{\mathcal{S}})$ and $\mathcal{R}^{Opt}(a_P|[u]_{\mathcal{S}}) \le \mathcal{R}^{Opt}(a_N|[u]_{\mathcal{S}})$.

(B1) Decision $u \in BND(X)$ if $\mathcal{R}^{Opt}(a_B|[u]_{\mathcal{S}}) \le \mathcal{R}^{Opt}(a_P|[u]_{\mathcal{S}})$ and $\mathcal{R}^{Opt}(a_B|[u]_{\mathcal{S}}) \le \mathcal{R}^{Opt}(a_N|[u]_{\mathcal{S}})$.

(N1) Decision $u \in NEG(X)$ if $\mathcal{R}^{Opt}(a_N|[u]_{\mathcal{S}}) \le \mathcal{R}^{Opt}(a_P|[u]_{\mathcal{S}})$ and $\mathcal{R}^{Opt}(a_N|[u]_{\mathcal{S}}) \le \mathcal{R}^{Opt}(a_B|[u]_{\mathcal{S}})$.

Furthermore, according to 3WD, decision rules (P1)–(N1) can be simplified into

$$\begin{aligned}
&(\mathrm{P1}') \text{ Decide } u \in POS(X), \text{ if } \mathcal{P}(X|[u]_{\mathcal{S}}) \ge \alpha' \\
&\qquad \text{and } \mathcal{P}(X|[u]_{\mathcal{S}}) \ge \gamma,' \\
&(\mathrm{B1}') \text{ Decide } u \in BND(X), \text{ if } \mathcal{P}(X|[u]_{\mathcal{S}}) \le \alpha' \\
&\qquad \text{and } \mathcal{P}(X|[u]_{\mathcal{S}}) \ge \beta,' \\
&(\mathrm{N1}') \text{ Decide } u \in NEG(X), \text{ if } \mathcal{P}(X|[u]_{\mathcal{S}}) \le \beta' \\
&\qquad \text{and } \mathcal{P}(X|[u]_{\mathcal{S}}) \le \gamma,'
\end{aligned}$$

where the values of α', β', and γ' are computed by

$$\begin{aligned}
\alpha' &= \frac{\lambda^{Opt}_{PN}([u]_{\mathcal{S}}) - \lambda^{Opt}_{BN}([u]_{\mathcal{S}})}{(\lambda^{Opt}_{PN}([u]_{\mathcal{S}}) - \lambda^{Opt}_{BN}([u]_{\mathcal{S}})) + (\lambda^{Opt}_{BP}([u]_{\mathcal{S}}) - \lambda^{Opt}_{PP}([u]_{\mathcal{S}}))} \\
\beta' &= \frac{\lambda^{Opt}_{BN}([u]_{\mathcal{S}}) - \lambda^{Opt}_{NN}([u]_{\mathcal{S}})}{(\lambda^{Opt}_{BN}([u]_{\mathcal{S}}) - \lambda^{Opt}_{NN}([u]_{\mathcal{S}})) + (\lambda^{Opt}_{NP}([u]_{\mathcal{S}}) - \lambda^{Opt}_{BP}([u]_{\mathcal{S}}))} \\
\gamma' &= \frac{\lambda^{Opt}_{PN}([u]_{\mathcal{S}}) - \lambda^{Opt}_{NN}([u]_{\mathcal{S}})}{(\lambda^{Opt}_{PN}([u]_{\mathcal{S}}) - \lambda^{Opt}_{NN}([u]_{\mathcal{S}})) + (\lambda^{Opt}_{NP}([u]_{\mathcal{S}}) - \lambda^{Opt}_{PP}([u]_{\mathcal{S}}))}.
\end{aligned} \tag{13}$$

By considering a well-defined boundary region, we assume that

$$\frac{(\lambda^{Opt}_{BP}([u]_{\mathcal{S}}) - \lambda^{Opt}_{PP}([u]_{\mathcal{S}}))}{(\lambda^{Opt}_{PN}([u]_{\mathcal{S}}) - \lambda^{Opt}_{BN}([u]_{\mathcal{S}}))} < \frac{(\lambda^{Opt}_{NP}([u]_{\mathcal{S}}) - \lambda^{Opt}_{BP}([u]_{\mathcal{S}}))}{(\lambda^{Opt}_{BN}([u]_{\mathcal{S}}) - \lambda^{Opt}_{NN}([u]_{\mathcal{S}}))}.$$

It implies that $0 \leq \beta' < \gamma' < \alpha' \leq 1$. In this situation, after tie-breaking, we have the following simplified decision rules:

(P1$''$) Decide $u \in POS(X)$, if $\mathcal{P}(X|[u]_{\mathcal{S}}) \geq \alpha,'$

(B1$''$) Decide $u \in BND(X)$, if $\beta' < \mathcal{P}(X|[u]_{\mathcal{S}}) < \alpha,'$

(N1$''$) Decide $u \in NEG(X)$, if $\mathcal{P}(X|[u]_{\mathcal{S}}) \leq \beta'$.

Similarly, for the pessimistic strategy, we have decision rules (P2)–(N2). For the average strategy, we have decision rules (P3)–(N3). Therefore, for different decision-making problems, one may use one of the three strategies according to the needs of DMs.

IV. Applications of the New 3WD Model to MADM

In this section, we investigate a comprehensive MADM method based on the proposed DTFRS and corresponding 3WD model.

A. Description of the Problem

Enterprises inevitably face investment expansion behaviors in the development process. A sound investment target selection helps companies to reduce investment risks and obtain higher economic benefits.

Suppose that $U = \{u_1, u_2, \ldots, u_n\}$ is a finite set of n kinds of investment projects and $\mathbf{C} = \{C_1, C_2, \ldots, C_m\}$ is a discrete set of m different assessment attributes. Moreover, $V = \{C_j(u_i) \in R \mid u_i \in U, C_j \in \mathbf{C}; i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}$ denotes the evaluation set of investment project u_i in U w.r.t. the attribute C_j in the attribute set $\mathbf{C}$. Assume that $w_j \in (0, 1)$ represents the weight of the attribute C_j and satisfies $\sum_{j=1}^{m} = 1$. We use $W = (w_1, w_2, \ldots, w_m)$ to denote a weight vector of all evaluation attributes. In this article, we name $(U, \mathbf{C}, V, W)$ an MADM matrix for EPI target selections. The problem for enterprises is to decide an optimal investment project from all possible investment projects. The board of directors of an enterprise evaluates all possible investment projects based on previous investment experiences and determines a comprehensive evaluation set $X \in \mathcal{F}(U)$. In general, the fuzzy set X represents a concept of "good project." We express the final conclusion as the optimal investment projects, excluding investment projects and the investment projects that need additional evaluation. Therefore, the MADM problem in the case of EPI target selections can be described as deciding the positive region $POS(X)$, negative region $NEG(X)$, and boundary region $BND(X)$ according to $(U, \mathbf{C}, V, W)$. Suppose that $\lambda_{PP}(u)$, $\lambda_{BP}(u)$, and $\lambda_{NP}(u)$ are the loss functions for taking actions a_P, a_B, and a_N when a project u is considered to be a good project, while $\lambda_{PN}(u)$, $\lambda_{BN}(u)$, and $\lambda_{NN}(u)$ are the loss functions for taking actions a_P, a_B and a_N when u is considered to be a bad project.

B. Application of the 3WD Model

First, we construct a 3WMADMT by using the MADM matrix for EPI target selections $(U, \mathbf{C}, V, W)$ and the loss function of each alternative. Then, we determine the concept of the "good project" X described by a fuzzy membership function.

Second, we normalize the MADM matrix. Suppose that I_1 is the set of benefit attributes and I_2 is the set of cost attributes.

If $j \in I_1$, then we have

$$A_{ij} = \frac{C_j(u_i)}{\max\limits_{1 \leq i \leq n} C_j(u_i)}. \tag{14}$$

If $j \in I_2$, then we have

$$A_{ij} = \frac{\min\limits_{1 \leq i \leq n} C_j(u_i)}{C_j(u_i)} \tag{15}$$

where $A_{ij} \in [0, 1]$ represents the evaluation value of the project u_i with respect to the attribute C_j.

Third, according to the normalized 3WMADMT and by means of Definition III.1, we calculate the outranked set $[u]_{\mathcal{S}}$ of every project u in U. We determine the conditional probability $\mathcal{P}(X|[u]_{\mathcal{S}})$ of the outranked set of the project u in U to the fuzzy concept of the "good project" X. By Definition III.8, we aggregate the loss functions of the outranked set $[u]_{\mathcal{S}}$ of the project u in U by three strategies.

Furthermore, based on the above steps, for each outranked set $[u]_{\mathcal{S}}$ of the project u in U, the expected loss $\mathcal{R}^{\bullet}(a_{\diamond}|[u]_{\mathcal{S}})(\diamond = \{P, B, N\}; \bullet = \{Opt, Pes, Ave\})$ can be obtained as follows:

$$\begin{aligned}
&\mathcal{R}^{\bullet}(a_P|[u]_{\mathcal{S}}) \\
&\quad = \lambda^{\bullet}_{PP}([u]_{\mathcal{S}})\mathcal{P}(X|[u]_{\mathcal{S}}) + \lambda^{\bullet}_{PN}([u]_{\mathcal{S}})\mathcal{P}(\neg X|[u]_{\mathcal{S}}) \\
&\mathcal{R}^{\bullet}(a_B|[u]_{\mathcal{S}}) \\
&\quad = \lambda^{\bullet}_{BP}([u]_{\mathcal{S}})\mathcal{P}(X|[u]_{\mathcal{S}}) + \lambda^{\bullet}_{BN}([u]_{\mathcal{S}})\mathcal{P}(\neg X|[u]_{\mathcal{S}}) \\
&\mathcal{R}^{\bullet}(a_N|[u]_{\mathcal{S}}) \\
&\quad = \lambda^{\bullet}_{NP}([u]_{\mathcal{S}})\mathcal{P}(X|[u]_{\mathcal{S}}) + \lambda^{\bullet}_{NN}([u]_{\mathcal{S}})\mathcal{P}(\neg X|[u]_{\mathcal{S}}).
\end{aligned} \tag{16}$$

According to the Bayesian decision procedure, the decision rules with minimum-risk are given as follows, $(\bullet = \{Opt, Pes, Ave\}; k = \{1, 2, 3\})$:

(Pk) Decision $u \in POS(X)$ if $\mathcal{R}^{\bullet}(a_P|[u]_{\mathcal{S}}) \leq \mathcal{R}^{\bullet}(a_B|[u]_{\mathcal{S}})$ and $\mathcal{R}^{\bullet}(a_P|[u]_{\mathcal{S}}) \leq \mathcal{R}^{\bullet}(a_N|[u]_{\mathcal{S}})$.

(Bk) Decision $u \in BND(X)$ if $\mathcal{R}^{\bullet}(a_B|[u]_{\mathcal{S}}) \leq \mathcal{R}^{\bullet}(a_P|[u]_{\mathcal{S}})$ and $\mathcal{R}^{\bullet}(a_B|[u]_{\mathcal{S}}) \leq \mathcal{R}^{\bullet}(a_N|[u]_{\mathcal{S}})$.

(Nk) Decision $u \in NEG(X)$ if $\mathcal{R}^{\bullet}(a_N|[u]_{\mathcal{S}}) \leq \mathcal{R}^{\bullet}(a_P|[u]_{\mathcal{S}})$ and $\mathcal{R}^{\bullet}(a_N|[u]_{\mathcal{S}}) \leq \mathcal{R}^{\bullet}(a_B|[u]_{\mathcal{S}})$.

According to earlier studies [17], [33], the semantics of these three rules can easily be explained in terms of their related costs. Therefore, for any project $u (u \in U)$, the related cost of the decision rules (Pk)–(Nk) is calculated as

$$\text{cost}(u) = \begin{cases} \mathcal{R}^{\bullet}(a_P|[u]_{\mathcal{S}}) & \text{if } u \in POS(X) \\ \mathcal{R}^{\bullet}(a_B|[u]_{\mathcal{S}}) & \text{if } u \in BND(X) \\ \mathcal{R}^{\bullet}(a_N|[u]_{\mathcal{S}}) & \text{if } u \in NEG(X). \end{cases} \tag{17}$$

The equation indicates that the final decision (action) of every project corresponds to an appropriate cost. If the decision is $u \in POS(X)$, then the project u takes action a_P with related cost $\mathcal{R}^{\bullet}(a_P|[u]_{\mathcal{S}})$. Analogously, if the decision is $u \in BND(X)$ or $u \in NEG(X)$, then the project u takes action a_B or a_N with related cost $\mathcal{R}^{\bullet}(a_B|[u]_{\mathcal{S}})$ or $\mathcal{R}^{\bullet}(a_N|[u]_{\mathcal{S}})$.

Finally, we give a ranking of projects $u_1, u_2, \ldots, u_n$ according to the following three rules: (a) given the preference of the DMs for (Pk)–(Nk) and the idea of 3WD in [17], [33], we consider a priority order (Pk) $\succ$ (Bk) $\succ$ (Nk), which means that the projects classified as $POS(X)$ dominate the projects classified as $BND(X)$, and the projects classified as these two regions dominate the projects classified as $NEG(X)$; (b) for these three regions, we rank the corresponding projects within their respective regions using the related costs of the projects, where the related cost of every project is generated based on (17); (c) we obtain a complete ranking of all projects $u_1, u_2, \ldots, u_n$ based on the previous two rules.

C. Algorithm of the Decision-Making Method

To illustrate the 3WD method for EPI target selections based on MADM, we summarize the steps by presenting an algorithm.

Input: A 3WMADMT for EPI target selections $(U, \mathbf{C}, V, W, \lambda_{\diamond\diamond})$ and the fuzzy concept of X.
Output: The ranking of all projects and an optimal project.
Step 1: Normalize the 3WMADMT for EPI target selections by (14) and (15).
Step 2: Calculate the outranked set $[u]_{\mathcal{S}}$ of each project u by Definition III.1.
Step 3: Determine the conditional probability $\mathcal{P}(X|[u]_{\mathcal{S}})$ of the outranked set of the project u to X.
Step 4: Aggregate the loss functions of the outranked set $[u]_{\mathcal{S}}$ for the project u based on Definition III.8.
Step 5: Calculate the expected losses $\mathcal{R}^{\bullet}(a_{\diamond}|[u]_{\mathcal{S}})$ of projects for three strategies.
Step 6: Determine the decision rules (Pk)–(Nk) (k = 1, 2, 3) of each project and calculate the related cost of each project.
Step 7: Rank all projects and determine the optimal project according to three rules (a)–(c).

Remark IV.1: In the algorithm of the above decision-making method, the complexity of the algorithm is to calculate the algorithm complexity of each step and combine the complexity of all steps into a whole to obtain the global complexity. Step 1 calculates the normalized 3WMADMT by (14) and (15), with the complexity of $O(nm)$. Step 2 calculates the outranked set $[u]_{\mathcal{S}}$ of each project, with the complexity of $O(n^2m)$. Step 3 calculates the conditional probability $\mathcal{P}(X|[u]_{\mathcal{S}})$ of the outranked set of the project u to X, with the complexity of $O(n^2)$. Step 4 calculates the loss functions of the outranked set $[u]_{\mathcal{S}}$ for each project u by three strategies, with the complexity of $O(n^2)$. Step 5 calculates expected losses $\mathcal{R}^{\bullet}(a_{\diamond}|[u]_{\mathcal{S}})$ of projects, with the complexity of $O(n)$. Step 6 determines the decision rules of each project and calculate the related cost of each project, with the complexity of $O(n)$. Step 7 ranks all projects with the complexity of $O(n^2)$. Therefore, the complexity of the algorithm equals to $O(n^2m)$.

In order to have a better understanding of the algorithm, the flow chart of the proposed 3WD method for EPI target selections is presented in Fig. 1.

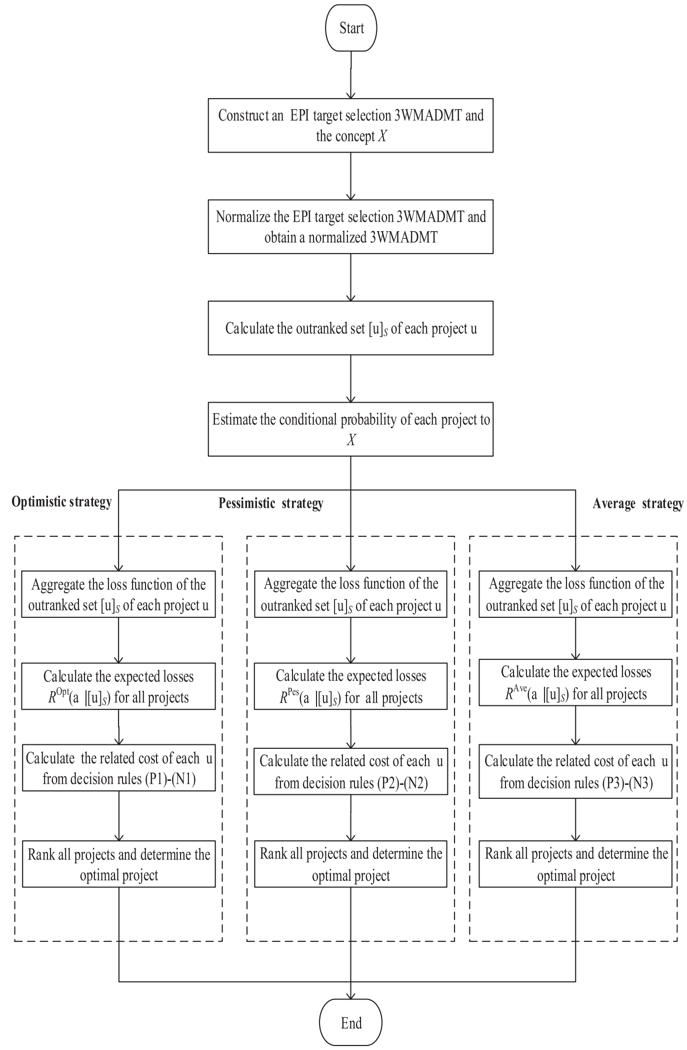

Fig. 1. Flowchart of the proposed 3WD method.

D. Illustrative Example

In order to illustrate the validity of the new established decision-making method and the algorithm, we study an actual MADM example of EPI target selections.

1) Case Description: Assume that there is an enterprise who has a sum of idle capital. The enterprise intends to choose and invest some investment projects from some possible projects. The director of the enterprise department has identified nine possible investment projects $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\}$, in which u_1: real estate project; u_2: education project; u_3: culture project; u_4: food project; u_5: computer project; u_6: high-tech project; u_7: stock project; u_8: arms project; u_9: car project. Five attributes are selected as evaluation attributes $\mathbf{C} = \{C_1, C_2, C_3, C_4, C_5\}$, where C_1 is market development prospect; C_2 is expected benefits; C_3 is social-political influence; C_4 is environmental influence; C_5 is energy conservation. Among the five attributes, market development prospect, expected benefits, social-political influence and energy conservation are benefit attributes. Environmental influence is the cost attribute. The weight vector of five evaluation attributes

TABLE III
3WMADMT OF EPI TARGET SELECTIONS

U	C_1	C_2	C_3	C_4	C_5	λ_{PP}	λ_{BP}	λ_{NP}	λ_{NN}	λ_{BN}	λ_{PN}
u_1	86	75	77	48	66	2	3.25	3.75	1.55	3.35	5.5
u_2	91	84	78	48	65	3	4.15	4.75	1.75	2.25	4.5
u_3	87	74	68	47	71	2.5	3.75	5	2.75	4.25	6.15
u_4	81	87	74	34	68	1.55	2.25	2.75	1	2.45	3
u_5	86	77	82	51	59	3	3.25	5.75	2.25	3.5	5.5
u_6	87	83	70	43	74	2	2.75	4.5	3	3.75	4.5
u_7	95	86	75	42	70	1.75	2.25	3.15	2	2.75	3.5
u_8	93	90	87	49	76	0.75	1.5	1.75	1.25	1.5	2.25
u_9	90	80	85	40	59	2.5	3.15	3.45	2.75	3	3.5
W	0.0408	0.4183	0.2653	0.1083	0.1673	*	*	*	*	*	*

TABLE IV
3WMADMT OF EPI TARGET SELECTIONS

U	C_1	C_2	C_3	C_4	C_5	λ_{PP}	λ_{BP}	λ_{NP}	λ_{NN}	λ_{BN}	λ_{PN}
u_1	0.90	0.83	0.89	0.71	0.87	2	3.25	3.75	1.55	3.35	5.5
u_2	0.96	0.93	0.90	0.71	0.85	3	4.15	4.75	1.75	2.25	4.5
u_3	0.92	0.82	0.78	0.72	0.94	2.5	3.75	5	2.75	4.25	6.15
u_4	0.85	0.97	0.85	1.00	0.89	1.55	2.25	2.75	1	2.45	3
u_5	0.90	0.85	0.94	0.67	0.78	3	3.25	5.75	2.25	3.5	5.5
u_6	0.91	0.92	0.80	0.79	0.98	2	2.75	4.5	3	3.75	4.5
u_7	1.00	0.96	0.86	0.81	0.92	1.75	2.25	3.15	2	2.75	3.5
u_8	0.98	1.00	1.00	0.69	1.00	0.75	1.5	1.75	1.25	1.5	2.25
u_9	0.95	0.89	0.98	0.84	0.78	2.5	3.15	3.45	2.75	3	3.5
W	0.0408	0.4183	0.2653	0.1083	0.1673	*	*	*	*	*	*

$W = (0.0408, 0.4183, 0.2653, 0.1083, 0.1673)$. A project has two possible states: a good project or a bad project. In general, the enterprise should decide which investment projects should be invested, which projects should not be invested, and which projects should be further researched and made decisions via gathering more information. Therefore, the director of the enterprise department needs to divide nine possible investment projects into three areas, namely, acceptance a_P, rejection a_B, and noncommitment a_N, which correspond to the projects that should be invested, the projects that should not be invested, and the projects that need further investigation. The evaluation results of nine projects over five attributes and the loss functions of all projects are shown in Table III, which is a 3WMADMT of EPI target selections.

2) Process of EPI Target Selections: In what follows, we use the algorithm for EPI target selections in Section IV-C to process the case study that is depicted in Section IV-D1.

We normalize the 3WMADMT of EPI target selections by (14) and (15) to obtain a normalized 3WMADMT of EPI target selections as Table IV.

By means of the ELECTRE-I method (set $\delta = 0.6$) and Definition III.1, we calculate the outranked set $[u_i]_{\mathcal{S}}(i = 1, 2, \ldots, 9)$ of every project u_i. The results are listed as follows: $[u_1]_{\mathcal{S}} = \{u_1, u_3\}$, $[u_2]_{\mathcal{S}} = \{u_1, u_2, u_3, u_5, u_6, u_9\}$, $[u_3]_{\mathcal{S}} = \{u_3\}$, $[u_4]_{\mathcal{S}} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_9\}$, $[u_5]_{\mathcal{S}} = \{u_1, u_3, u_5\}$, $[u_6]_{\mathcal{S}} = \{u_1, u_3, u_5, u_6\}$, $[u_7]_{\mathcal{S}} = \{u_1, u_2, u_3, u_5, u_6, u_7, u_9\}$, $[u_8]_{\mathcal{S}} = \{u_1, u_2, u_3, u_5, u_6, u_7, u_8, u_9\}$, $[u_9]_{\mathcal{S}} = \{u_1, u_3, u_5, u_9\}$.

Suppose that a fuzzy set X represents the concept of "good projects" and

$$X = \frac{0.54}{u_1} + \frac{0.65}{u_2} + \frac{0.53}{u_3} + \frac{0.78}{u_4} + \frac{0.48}{u_5} + \frac{0.68}{u_6} + \frac{0.74}{u_7} + \frac{0.84}{u_8} + \frac{0.72}{u_9}.$$

We calculate the conditional probability of each project's outranked set to the fuzzy concept X. The results are given as follows: $\mathcal{P}(X|[u_1]_{\mathcal{S}}) = 0.535$, $\mathcal{P}(X|[u_2]_{\mathcal{S}}) = 0.6$, $\mathcal{P}(X|[u_3]_{\mathcal{S}}) = 0.53$, $\mathcal{P}(X|[u_4]_{\mathcal{S}}) = 0.6257$, $\mathcal{P}(X|[u_5]_{\mathcal{S}}) = 0.5167$, $\mathcal{P}(X|[u_6]_{\mathcal{S}}) = 0.5575$, $\mathcal{P}(X|[u_7]_{\mathcal{S}}) = 0.62$, $\mathcal{P}(X|[u_8]_{\mathcal{S}}) = 0.6475$, and $\mathcal{P}(X|[u_9]_{\mathcal{S}}) = 0.5675$.

TABLE V
OPTIMISTIC AGGREGATED LOSS FUNCTIONS FOR THE OUTRANKED SET OF PROJECTS

$U/\mathcal{S}$	$\lambda_{PP}^{Opt}([u_{\mathcal{S}}])$	$\lambda_{BP}^{Opt}([u_{\mathcal{S}}])$	$\lambda_{NP}^{Opt}([u_{\mathcal{S}}])$	$\lambda_{NN}^{Opt}([u_{\mathcal{S}}])$	$\lambda_{BN}^{Opt}([u_{\mathcal{S}}])$	$\lambda_{PN}^{Opt}([u_{\mathcal{S}}])$
$[u_1]_{\mathcal{S}}$	2	3.25	3.75	1.55	3.35	5.5
$[u_2]_{\mathcal{S}}$	3	2.75	3.45	1.55	2.25	3.5
$[u_3]_{\mathcal{S}}$	2.5	3.75	5	2.75	4.25	6.15
$[u_4]_{\mathcal{S}}$	1.55	2.25	2.75	1	2.25	3
$[u_5]_{\mathcal{S}}$	2	3.25	3.75	1.55	3.35	5.5
$[u_6]_{\mathcal{S}}$	2	2.75	3.75	1.55	3.35	4.5
$[u_7]_{\mathcal{S}}$	1.75	2.25	3.15	1.55	2.25	3.5
$[u_8]_{\mathcal{S}}$	0.75	1.5	1.75	1.25	1.5	2.25
$[u_9]_{\mathcal{S}}$	2.5	3.15	3.45	1.55	3	3.5

TABLE VI
PESSIMISTIC AGGREGATED LOSS FUNCTIONS FOR THE OUTRANKED SET OF PROJECTS

$U/\mathcal{S}$	$\lambda_{PP}^{Pes}([u_{\mathcal{S}}])$	$\lambda_{BP}^{Pes}([u_{\mathcal{S}}])$	$\lambda_{NP}^{Pes}([u_{\mathcal{S}}])$	$\lambda_{NN}^{Pes}([u_{\mathcal{S}}])$	$\lambda_{BN}^{Pes}([u_{\mathcal{S}}])$	$\lambda_{PN}^{Pes}([u_{\mathcal{S}}])$
$[u_1]_{\mathcal{S}}$	2.5	3.75	5	2.75	4.25	6.15
$[u_2]_{\mathcal{S}}$	3	4.15	5.75	3	4.25	6.15
$[u_3]_{\mathcal{S}}$	2.5	3.75	5	2.75	4.25	6.15
$[u_4]_{\mathcal{S}}$	3	4.15	5.75	3	4.25	6.15
$[u_5]_{\mathcal{S}}$	3	3.75	5.75	2.75	4.25	6.15
$[u_6]_{\mathcal{S}}$	3	3.75	5.75	3	4.25	6.15
$[u_7]_{\mathcal{S}}$	3	4.15	5.75	3	4.25	6.15
$[u_8]_{\mathcal{S}}$	3	4.15	5.75	3	4.25	6.15
$[u_9]_{\mathcal{S}}$	3	3.75	5.75	2.75	4.25	6.15

TABLE VII
AVERAGE AGGREGATED LOSS FUNCTIONS FOR THE OUTRANKED SET OF PROJECTS

$U/\mathcal{S}$	$\lambda_{PP}^{Ave}([u_{\mathcal{S}}])$	$\lambda_{BP}^{Ave}([u_{\mathcal{S}}])$	$\lambda_{NP}^{Ave}([u_{\mathcal{S}}])$	$\lambda_{NN}^{Ave}([u_{\mathcal{S}}])$	$\lambda_{BN}^{Ave}([u_{\mathcal{S}}])$	$\lambda_{PN}^{Ave}([u_{\mathcal{S}}])$
$[u_1]_{\mathcal{S}}$	2.25	3.5	4.375	2.15	3.8	5.825
$[u_2]_{\mathcal{S}}$	2.5	3.3833	4.5333	2.3417	3.35	4.9417
$[u_3]_{\mathcal{S}}$	2.5	3.75	5	2.75	4.25	6.15
$[u_4]_{\mathcal{S}}$	2.3643	3.2214	4.2786	2.15	3.2214	4.6643
$[u_5]_{\mathcal{S}}$	2.5	3.4167	4.8333	2.1833	3.7	5.716
$[u_6]_{\mathcal{S}}$	2.375	3.25	4.75	2.3875	3.7125	5.4125
$[u_7]_{\mathcal{S}}$	2.3929	3.2214	4.3357	2.2929	3.2643	4.7357
$[u_8]_{\mathcal{S}}$	2.1875	3.0062	4.0125	2.1625	3.0438	4.425
$[u_9]_{\mathcal{S}}$	2.5	3.35	4.4875	3.325	3.525	5.1625

TABLE VIII
EXPECTED LOSSES OF PROJECTS FOR THREE STRATEGIES

$\mathcal{R}(a_\diamond\|[u]_{\mathcal{S}})/U$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
$\mathcal{R}^{Opt}(a_P\|[u]_{\mathcal{S}})$	3.628	2.6	4.216	2.093	3.692	3.106	2.415	1.279	2.649
$\mathcal{R}^{Opt}(a_B\|[u]_{\mathcal{S}})$	3.297	2.55	3.985	2.25	3.298	3.016	2.25	1.5	3.085
$\mathcal{R}^{Opt}(a_N\|[u]_{\mathcal{S}})$	2.727	2.69	3.943	2.095	2.687	2.777	2.542	1.574	2.628
$\mathcal{R}^{Pes}(a_P\|[u]_{\mathcal{S}})$	4.197	4.26	4.216	4.179	4.523	4.394	4.197	4.11	4.362
$\mathcal{R}^{Pes}(a_B\|[u]_{\mathcal{S}})$	3.983	4.19	3.985	4.187	3.992	3.971	4.188	4.185	3.966
$\mathcal{R}^{Pes}(a_N\|[u]_{\mathcal{S}})$	3.954	4.65	3.943	4.721	4.3	4.533	4.705	4.781	4.453
$\mathcal{R}^{Ave}(a_P\|[u]_{\mathcal{S}})$	3.912	3.477	4.216	3.225	4.055	3.719	3.283	2.976	3.652
$\mathcal{R}^{Ave}(a_B\|[u]_{\mathcal{S}})$	3.64	3.37	3.985	3.221	3.553	3.455	3.238	3.02	3.426
$\mathcal{R}^{Ave}(a_N\|[u]_{\mathcal{S}})$	3.34	3.657	3.943	3.482	3.552	3.705	3.559	3.36	3.552

According to Definition III.8, we use three strategies to calculate three aggregated loss functions of each project's outranked set. The results are summarized in Tables V–VII.

For three strategies, we calculate the expected losses of projects by utilizing (16) (see Table VIII).

In order to better show the expected losses of projects for three strategies, Fig. 2 depicts the results of Table VIII.

Based on the decision rules (Pk)–(Nk) (k = 1, 2, 3), we calculate the decision rule and the related cost of each project $u_i \in U(i = 1, 2, \ldots, 9)$, which are given in Table IX.

From Table IX, we have the following conclusions.

1) For the optimistic strategy and by the decision rules (P1)–(N1), we can obtain the following three regions: $POS^{Opt}(X) = \{u_4, u_8\}$, $BND^{Opt}(X) = \{u_2, u_7\}$,

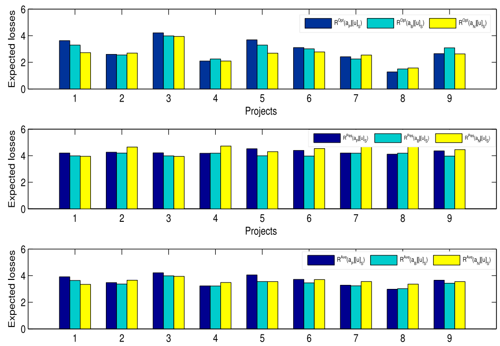

Fig. 2. Expected losses of projects for three strategies.

TABLE IX
DECISION RULE AND COST OF EACH PROJECT FOR THREE STRATEGIES

U	Optimistic strategy		Pessimistic strategy		Average strategy	
	Decision rule	Cost(u)	Decision rule	Cost(u)	Decision rule	Cost(u)
u_1	N1	2.727	N2	3.954	N3	3.34
u_2	B1	2.55	B2	4.19	B3	3.37
u_3	N1	3.943	N2	3.943	N3	3.943
u_4	P1	2.093	P2	4.179	B3	3.221
u_5	N1	2.687	B2	3.992	N3	3.552
u_6	N1	2.777	B2	3.971	B3	3.455
u_7	B1	2.25	B2	4.188	B3	3.238
u_8	P1	1.279	P2	4.11	P3	2.976
u_9	N1	2.628	B2	3.966	B3	3.426

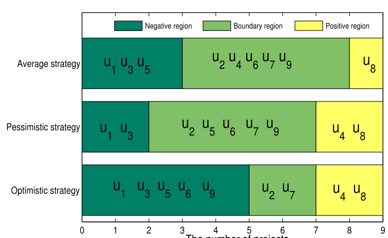

Fig. 3. Decision regions for three strategies.

$NEG^{Opt}(X) = \{u_1, u_3, u_5, u_6, u_9\}$. Thus, the enterprise can decide that $\{u_4, u_8\}$ should be invested, $\{u_2, u_7\}$ should not be invested, and $\{u_1, u_3, u_5, u_6, u_9\}$ should be further studied via collecting more information.

2) For the pessimistic strategy and by the decision rules (P2)–(N2), we can obtain the following three regions: $POS^{Pes}(X) = \{u_4, u_8\}$, $BND^{Pes}(X) = \{u_2, u_5, u_6, u_7, u_9\}$, and $NEG^{Pes}(X) = \{u_1, u_3\}$. Thus, the enterprise can decide that $\{u_4, u_8\}$ should be invested, $\{u_1, u_3\}$ should not be invested, and $\{u_2, u_5, u_6, u_7, u_9\}$ should be further studied via collecting more information.

3) For the average strategy and by the decision rules (P3)–(N3), we can obtain the following three regions: $POS^{Ave}(X) = \{u_8\}$, $BND^{Ave}(X) = \{u_2, u_4, u_6, u_7, u_9\}$, $NEG^{Ave}(X) = \{u_1, u_3, u_5\}$. Thus, the enterprise can decide that $\{u_8\}$ should be invested, $\{u_1, u_3, u_5\}$ should not be invested, and $\{u_2, u_4, u_6, u_7, u_9\}$ should be further studied via collecting more information.

These conclusions are illustrated in Fig. 3.

TABLE X
DECISION RESULTS OF THREE STRATEGIES

Strategies	Ranking	Optimal project
Opt	$u_8 \succ u_4 \succ u_7 \succ u_2 \succ u_9 \succ u_5 \succ u_1 \succ u_6 \succ u_3$	u_8
Pes	$u_8 \succ u_4 \succ u_9 \succ u_6 \succ u_5 \succ u_7 \succ u_2 \succ u_3 \succ u_1$	u_8
Ave	$u_8 \succ u_4 \succ u_7 \succ u_2 \succ u_9 \succ u_6 \succ u_1 \succ u_5 \succ u_3$	u_8

TABLE XI
RANKINGS OF PROJECTS DETERMINED BY DIFFERENT METHODS

The methods	Ranking results
Proposed method (optimistic strategy)	$u_8 \succ u_4 \succ u_7 \succ u_2 \succ u_9 \succ u_5 \succ u_1 \succ u_6 \succ u_3$
Proposed method (average strategy)	$u_8 \succ u_4 \succ u_7 \succ u_2 \succ u_9 \succ u_6 \succ u_1 \succ u_5 \succ u_3$
Proposed method (pessimistic strategy)	$u_8 \succ u_4 \succ u_9 \succ u_6 \succ u_5 \succ u_7 \succ u_2 \succ u_3 \succ u_1$
PROMETHEE II method	$u_8 \succ u_4 \succ u_7 \succ u_9 \succ u_2 \succ u_6 \succ u_5 \succ u_1 \succ u_3$
ELECTRE II method	$u_8 \succ u_7 \approx u_9 \succ u_2 \approx u_4 \succ u_5 \approx u_6 \succ u_1 \approx u_3$
TOPSIS method	$u_8 \succ u_4 \succ u_7 \succ u_9 \succ u_6 \succ u_2 \succ u_5 \succ u_1 \succ u_3$
WAA operator method	$u_8 \succ u_4 \succ u_7 \succ u_9 \succ u_6 \succ u_2 \succ u_5 \succ u_1 \succ u_3$

Finally, based on decision rules Pk $\succ$ Bk $\succ$ Nk and the related cost of each project, we can obtain the ranking of nine projects and the optimal project. The results for three strategies are shown in Table X.

Remark IV.2: From Table X, we can find that the optimal projects of three strategies are identical, i.e., the project u_8. In addition, we observe that there are some differences among the rankings of nine projects for three strategies. The reason for the differences may be the adoption of different strategies. Therefore, DMs can choose different strategies according to their subjective preferences.

V. COMPARATIVE ANALYSIS AND DISCUSSIONS

In this section, the validity and superiority of the proposed 3WD method are illustrated through the comparative analysis and corresponding discussions with several existing methods.

A. Comparison Analysis

In this section, we plan to compare the proposed method with the existing MADM methods based on three numerical examples. Three numerical examples are the illustrative example introduced in Section IV-D1), the practical MADM example cited from [41] and a simple example of energy project selection (EPS) problem, respectively.

1) A Comparison Analysis With MADM Methods Based on the Illustrative Example in Section IV-D1: In this section, based on the problem of EPI target selections introduced in Section IV-D1, we plan to demonstrate the effectiveness and superiority of the proposed method by comparing the results with several existing methods.

Since there is no well-matched 3WD method to address the problem of EPI target selections introduced in Section IV-D1, the four existing MADM methods [36], [38], [46], [47] are used to address the problem of EPI target selections. The ranking results of projects derived from different MADM methods are presented in Table XI and Fig. 4.

From the results in Table XI and Fig. 4, we can see that although the rankings of nine projects obtained by the four

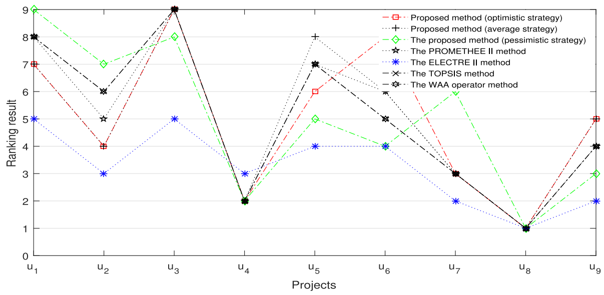

Fig. 4. Rankings of projects by different methods.

TABLE XII
SCCs OF THE RANKINGS OBTAINED BY DIFFERENT METHODS

The Methods	Proposed method (optimistic)	Proposed method (average)	Proposed method (pessimistic)	PROMETHEE II	WAA	TOPSIS	ELECTRE II
Proposed method (optimistic)	1.000	0.933	0.633	0.933	0.933	0.876	0.567
Proposed method (average)	–	1.000	0.667	0.967	0.967	0.933	0.567
Proposed method (pessimistic)	–	–	1.000	0.800	0.800	0.850	0.500
PROMETHEE II	–	–	–	1.000	1.000	0.983	0.600
WAA	–	–	–	–	1.000	0.983	0.600
TOPSIS	–	–	–	–	–	1.000	0.583
ELECTRE II	–	–	–	–	–	–	1.000

different MADM methods are not exactly identical, the optimal project is identical, i.e., the project u_8. This is consistent with the decision result determined by the proposed method based on three strategies, which illustrates that the proposed method is reasonable and effective. Moreover, we find that the ELECTRE II method cannot establish a strictly order relationship among some projects, for example, the indiscernibility between projects u_7 and u_9. In contrast, the proposed method obtains a strictly order relationship among projects for three strategies, which illustrates that the proposed method is more effective to cope with some MADM problems than the ELECTRE II method.

Furthermore, to compare the rankings obtained by different methods, we adopt the Spearman's correlation coefficient (SCC) [48] to show the statistical significance of the ranking results of different methods and demonstrate the effectiveness of the proposed method. Based on the ranking results in Table XI, the SCCs of the rankings obtained by different methods are shown in Table XII.

Remark V.1: It is generally believed that an SCC greater than 0.8 means a strong similarity between two methods [48]. The results in Table XII show that SCCs between the proposed method (based on three strategies) and three classic methods (PROMETHEE II, WAA, and TOPSIS) are significantly greater than 0.8, which illustrates a strong relationship between our proposed method and three classic methods. Therefore, we can conclude that the decision result from the proposed method is credible and effective.

2) A Comparison Analysis With MADM Methods Based on the Practical MADM Example in [41]: In this section, to further demonstrate the effectiveness of the developed 3WD method, we will cite an example from [41]. Next, we will briefly introduce the following example.

Example V.2: (Cited from [41]) An investment company plans to select and invest some investment projects, the potential eight alternatives are $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$. There are five attributes: C_1, C_2, C_3, C_4 and C_5. According to [41], C_1, C_4 and C_5 are benefit attributes; C_2 and C_3 are cost attributes. The weighted vector of these attributes $W = (0.3, 0.1, 0.3, 0.2, 0.1)$. In addition, based on Table XV and Table XVIII in [41], a 3WMADMT is list in the following Table XIII:

TABLE XIII
3WMADMT OF EPI TARGET SELECTIONS IN [41]

U	C_1	C_2	C_3	C_4	C_5	λ_{PP}	λ_{BP}	λ_{NP}	λ_{NN}	λ_{BN}	λ_{PN}
u_1	0.8	0.4	0.3	0.8	0.9	0	0.301	0.76	0	0.094	0.24
u_2	0.9	0.5	0.5	0.7	0.6	0	0.2605	0.67	0	0.1345	0.33
u_3	0.3	0.4	0.6	0.4	0.3	0	0.1505	0.38	0	0.2445	0.62
u_4	0.5	0.2	0.2	0.7	0.6	0	0.2665	0.67	0	0.1285	0.33
u_5	0.7	0.6	0.6	0.5	0.8	0	0.2175	0.55	0	0.1775	0.45
u_6	0.4	0.8	0.7	0.7	0.3	0	0.157	0.4	0	0.238	0.6
u_7	0.9	0.5	0.1	0.8	0.7	0	0.3215	0.82	0	0.0735	0.18
u_8	0.6	0.8	0.8	0.3	0.4	0	0.139	0.36	0	0.256	0.64
W	0.3	0.1	0.3	0.2	0.1	*	*	*	*	*	*

In the process of our proposed method, we need to calculate the conditional probability $\mathcal{P}(X|[u]_{\mathcal{S}})$ of the outranked set of the project u to the fuzzy concept of the "good project" X that is consistent with the concept of "worth investing" C in [41]. However, there is no exact form of the fuzzy set in [41]. In the following, we will give a method to determine the fuzzy concept of the "good project" X. First, according to (14) and (15), a normalized 3WMADMT is obtained, then we define the membership function of each project: $X(u_i) = \sum_{j=1}^{5} w_j A_{ij}$. Based on this method, we give the fuzzy set of the "good project" $X = \frac{0.7167}{u_1} + \frac{0.6417}{u_2} + \frac{0.3333}{u_3} + \frac{0.6584}{u_4} + \frac{0.5306}{u_5} + \frac{0.4095}{u_6} + \frac{0.9178}{u_1} + \frac{0.3820}{u_2}$.

In what follows, the proposed method is used to address the problem in [41]. The decision results are shown in Tables XIV and XV.

TABLE XIV
THREE REGIONS OF PROJECTS FOR THREE STRATEGIES

Strategies	Positive region	Boundary region	Negative region
Optimistic strategy	u_7	u_1, u_2, u_4	u_3, u_5, u_6, u_8
Average strategy	u_7	u_1, u_2, u_4	u_3, u_5, u_6, u_8
Pessimistic strategy	u_7	u_1, u_2, u_4	u_3, u_5, u_6, u_8

TABLE XV
RANKINGS OF PROJECTS FOR THREE STRATEGIES

Strategies	Ranking	Optimal project
Optimistic strategy	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_3 \succ u_6 \succ u_5 \succ u_8$	u_7
Pessimistic strategy	$u_7 \succ u_2 \succ u_4 \succ u_1 \succ u_3 \succ u_5 \succ u_6 \succ u_8$	u_7
Average strategy	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_3 \succ u_6 \succ u_5 \succ u_8$	u_7

In the following, we intend to compare our method with the existing MADM methods from two points. First, this example is cited from [41], so we can directly compare with the method in [41]. Furthermore, we plan to compare the decision results with the four classic MADM methods.

1) In [41], Jia and Liu assumed that the conditional probabilities $\mathcal{P}(X|u_i)$ of all projects are the same and set it from 0.00 to 1.00 in steps of 0.05. Here, we select $\mathcal{P}(X|u_i) = 0.3$ and $\mathcal{P}(X|u_i) = 0.7$ to compare with our proposed method. Based on the results of [41], the decision results of eight projects are listed in Tables XVI and XVII.

TABLE XVI
THREE REGIONS OF EIGHT PROJECTS IN [41]

Conditional probabilities	Positive region	Boundary region	Negative region
$\mathcal{P}(X\|u_i) = 0.3$	u_7	u_1, u_2, u_4	u_3, u_5, u_6, u_8
$\mathcal{P}(X\|u_i) = 0.7$	u_1, u_2, u_4, u_5, u_7	u_3, u_8	$\emptyset$

TABLE XVII
RANKINGS OF EIGHT PROJECTS BASED ON THE METHOD IN [41]

Strategies	Ranking	Optimal project
Based on α	$u_7 \succ u_1 \succ u_2 \succ u_4 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	u_7
Based on β	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	u_7
Based on γ	$u_7 \succ u_1 \succ u_2 \approx u_4 \succ u_5 \succ u_6 \succ u_3 \succ u_8$	u_7

TABLE XVIII
SCCS OF THE RANKINGS OBTAINED BY THE PROPOSED METHOD AND THE METHOD IN [41]

	Proposed method (optimistic)	Proposed method (pessimistic)	Proposed method (average)	Method in [41] (based on α)	Method in [41] (based on β)	Method in [41] (based on γ)
Proposed method (optimistic)	1	0.881	0.905	0.833	0.810	0.833
Proposed method (pessimistic)	–	1	0.976	0.905	0.929	0.905
Proposed method (average)	–	–	1	0.881	0.905	0.881
Method in [41] (based on α)	–	–	–	1	0.976	0.990
Method in [41] (based on β)	–	–	–	–	1	0.990
Method in [41] (based on γ)	–	–	–	–	–	1

Based on the results obtained by the proposed method and the method in [41], we have the following conclusions.

a) According to the results of Tables XV and XVII, we find that the optimal projects are consistent, i.e., the project u_7. Furthermore, based on the ranking results in Tables XV and XVII, the SCCs between the proposed method (for three strategies) and the method in [41] are shown in Table XVIII. The results in Table XVIII indicate that the SCCs between the proposed method (for three strategies) and the method in [41] are significantly greater than 0.8. Therefore, it can demonstrate the efficiency of the proposed method.

b) Compared with the regions of different projects in Tables XIV and XVI, when $\mathcal{P}(X|u_i) = 0.3$ in [41], the decision regions of eight projects are identical for the proposed method and the method in [41]. However, when $\mathcal{P}(X|u_i) = 0.7$ in [41], there are more projects of Table XVI that are divided into the positive region. In general, in the procedure of decision-making, DMs do not want to bear the cost from a wrong rejection or acceptance. In this situation, more ambiguous selections will emerge. At this point, the best option of DMs is to further investigate the candidates. Compared with the decision results of Table XIV, the proposed method divides the eight projects into three reasonable decision regions and more projects into the boundary region. Therefore, compared with the decision regions of the method in [41], our proposed method is effective and credible.

2) The decision results obtained from the four classic MADM methods are listed in Table XIX.

From Tables XV and XIX, we can find that the optimal projects obtained by the proposed method and four classic MADM methods are consistent, i.e., the project u_7. In addition, from Table XIV, we find that the proposed method can divide the eight projects into three decision regions. Compared with the four classic MADM methods, the proposed method not only determines an optimal project but also obtains a reasonable classification of projects. It implies that the proposed method is more effective and reasonable.

TABLE XIX
DECISION RESULTS DETERMINED BY FOUR CLASSIC MADM METHODS

The methods	Ranking results
PROMETHEE II method	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_8 \succ u_3$
ELECTRE II method	$u_7 \approx u_1 \succ u_4 \succ u_2 \approx u_5 \succ u_3 \approx u_6 \approx u_8$
TOPSIS method	$u_7 \succ u_1 \succ u_2 \succ u_4 \succ u_5 \succ u_6 \succ u_8 \succ u_3$
WAA operator method	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$

TABLE XX
3WMADMT OF EPS PROBLEM

U	C_1	C_2	C_3	C_4	λ_{PP}	λ_{BP}	λ_{NP}	λ_{NN}	λ_{BN}	λ_{PN}
u_1	0.80	0.69	0.64	0.74	0	0.3262	0.712	0	0.1338	0.288
u_2	0.65	0.85	0.72	0.67	0	0.3285	0.717	0	0.283	0.1315
u_3	0.73	0.77	0.78	0.61	0	0.3285	0.717	0	0.283	0.1315
u_4	0.82	0.68	0.64	0.75	0	0.3285	0.717	0	0.283	0.1315
u_5	0.54	0.96	0.57	0.82	0	0.3285	0.717	0	0.283	0.1315
u_6	0.88	0.62	0.70	0.69	0	0.3285	0.717	0	0.283	0.1315
W	0.2	0.2	0.3	0.3	$*$	$*$	$*$	$*$	$*$	$*$

TABLE XXI
RANKINGS OF PROJECTS DETERMINED BY THE DIFFERENT METHODS

The methods	Ranking results	Optimal project
Proposed method (optimistic strategy)	$u_4 \succ u_1 \succ u_3 \succ u_6 \succ u_2 \succ u_5$	u_4
Proposed method (pessimistic strategy)	$u_4 \succ u_1 \succ u_3 \succ u_6 \succ u_2 \succ u_5$	u_4
Proposed method (average strategy)	$u_4 \succ u_1 \succ u_3 \succ u_6 \succ u_2 \succ u_5$	u_4
The method in [41] (based on α)	$u_2 \approx u_3 \approx u_4 \approx u_5 \approx u_6 \succ u_1$	None
The method in [41] (based on β)	$u_2 \approx u_3 \approx u_4 \approx u_5 \approx u_6 \succ u_1$	None
The method in [41] (based on γ)	$u_2 \approx u_3 \approx u_4 \approx u_5 \approx u_6 \succ u_1$	None
PROMETHEE II method	$u_1 \approx u_2 \approx u_3 \approx u_4 \approx u_5 \approx u_6$	None
ELECTRE II method	$u_1 \approx u_2 \approx u_3 \approx u_4 \approx u_5 \approx u_6$	None
TOPSIS method	$u_4 \succ u_3 \succ u_5 \approx u_6 \succ u_2 \succ u_1$	u_4
WAA operator method	$u_1 \approx u_2 \approx u_3 \approx u_4 \approx u_5 \approx u_6$	None

3) Comparison analysis with MADM methods based on a simple example of EPS problem.

In order to further verify the advantages of our proposed 3WD method, we give another simple example of the EPS problem.

Example V.3: Suppose that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ denotes the set of 6 alternatives (energy projects) and $\mathbf{C} = \{C_1, C_2, C_3, C_4\}$ denotes the set of 4 attributes. The weight vector of five attributes $W = (0.2, 0.2, 0.3, 0.3)$. The evaluation results of six energy projects w.r.t. four attributes and the loss function of each alternative are presented in Table XX. Moreover, assume that the fuzzy set X represents a concept of the "good project." Obviously, $X \in \mathcal{F}(U)$ and $X = \frac{0.73}{u_1} + \frac{0.58}{u_2} + \frac{0.62}{u_3} + \frac{0.74}{u_4} + \frac{0.51}{u_5} + \frac{0.62}{u_6}$.

According to the proposed method and the five existing MADM methods, we can obtain the rankings of six projects. The results are presented in Table XXI.

From Table XXI, we have the conclusion that the proposed method establishes a strictly partial ordering relation among alternatives for the three strategies. However, the five existing MADM methods cannot establish a strictly partial ordering relation among alternatives, e.g., we have $u_1 \approx u_2 \approx u_3 \approx u_4 \approx u_5 \approx u_6$ by the PROMETHEE II method. Besides, from the perspective of optimal project, the proposed method is able to acquire an optimal project for three strategies. However, the four existing MADM methods (the method in [41], PROMETHEE II, ELECTRE II, and WAA operator) cannot obtain an optimal project from six evaluation energy projects. Thus, we can conclude that the proposed method is superior to the five exiting MADM methods.

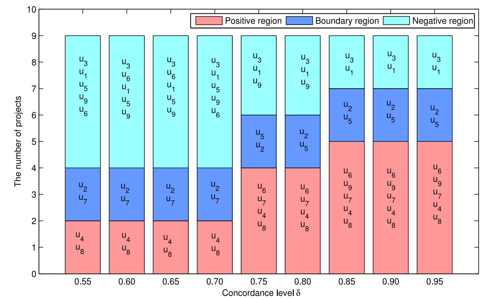

Fig. 5. Results of the sensitivity analysis of δ by the proposed method for the optimistic strategy.

B. Discussions

Based on the comparative analyses, we have several findings.

1) The main differences between the proposed method in this article and the existing methods are presented as follows.
   a) The proposed method not only considers the MADM matrix, but also considers the loss functions of alternatives in the procedure of decision-making. In addition, the proposed method ranks the corresponding projects within their respective regions by using the related costs of the projects in the problem of EPI target selections. Nevertheless, the traditional MADM methods, such as the PROMETHEE II method, the ELECTRER II method, the WAA operator method, and the TOPSIS method, only consider the MADM matrix and do not consider the cost or loss of decisions.
   b) The proposed method is a novel MADM method by virtue of 3WD models, which divides alternatives into three decision regions. For some traditional MADM methods only intend to prioritize alternatives and select an optimal alternative. For example, in the problem of EPI target selections cited from [41], we have $u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_6 \succ u_8 \succ u_3$ by means of the PROMETHEE II method and acquire an optimal project u_7. However, they cannot categorize alternatives.
   c) The proposed method objectively determines the number of alternatives in different regions based on Bayesian minimum-cost decision rules and the related costs of alternatives. In some existing MADM methods, the number of acceptable or rejected alternatives is determined by DMs' subjective judgments.
   d) Although the optimal project between the proposed method and the method in [41] is consistent for the example cited from [41], the approach to determine the conditional probability in the process of decision-making is different. In [41], Jia and Liu subjectively set that the conditional probabilities of all the projects are the same. In the proposed method, the conditional probabilities of all the projects are different and determined by the conditional probability formula.
2) Compared with five existing MAMD methods, the advantages of the established method are summarized as follows.
   a) The proposed method considers both information table and loss functions in 3WD and gives a comprehensive decision-making rule to MADM problems with the aid of Bayesian minimum risk decision rule, which can reduce decision-making costs and obtain an optimal alternative with a minimum cost/loss.
   b) The concordance level δ in the construction of the outranked set is based on DMs' willingness to take the risk of approving one alternative is superior to another, which can be adjusted according to the attitude of DMs. Therefore, depending on different decision environments and DMs, the decision results are very good in flexibility and extension.
   c) According to the different risk preferences of DMs, the proposed method in this article includes three strategies, namely, the optimistic strategy, the pessimistic strategy, and the average strategy. This illustrates that the proposed method can express the subjective preferences of experts.

VI. Experimental Evaluations

In this section, two experiments are carried out to verify the stability and effectiveness of the proposed method.

A. Experimental Evaluation Based on Sensitivity Analysis

For the proposed method, there is an important parameter: the concordance level δ. To verify the effects of different concordance levels to the decision results, we conduct the following experiment. This experiment is based on the problem of EPI target selections presented in Section IV-D1.

For the nine projects in the problem of EPI target selections presented in Section IV-D1, we change the concordance level δ from 0.55 to 0.95 in steps of 0.05 and make decisions according to the proposed 3WD method. For three strategies, the results of the experiments are shown in Figs. 5–7.

From the results of Figs. 5–7, more projects are put into the positive region with the increase of δ. For the optimistic strategy, with the increase of δ, DMs have a higher possibility to divide the project into the positive region. Therefore, DMs will select more projects to invest and reduce the number of excluded investment projects. For the average strategy and the pessimistic strategy, with the increase of δ, DMs have a higher possibility to divide the project into the positive region. From the perspective of the theory of rough sets, the deterministic rules are increasing and the uncertain rules are decreasing in the action

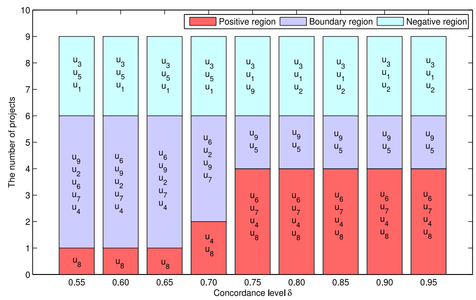

Fig. 6. Results of the sensitivity analysis of δ by the proposed method for the average strategy.

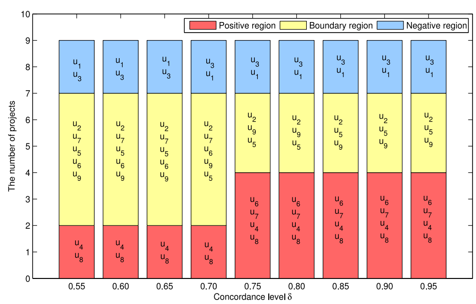

Fig. 7. Results of the sensitivity analysis of δ by the proposed method for the pessimistic strategy.

of decision-making. Generally speaking, the proposed 3WD method can support the needs of different decision strategies and provide us more diverse decision results. Moreover, for the three strategies, the optimal projects in three strategies are identical with the increase of δ and the ranking results are the same in most case with the increase of δ. This implies that the proposed method is stable and effective.

B. Experimental Evaluation Based on Weight Analysis

There are two classes of approaches to obtain the weight vector of attributes: the subjective approach and the objective approach. Each class has its own advantages. To further verify the effectiveness of the proposed method, we conduct an experiment based on the problem of EPI target selections cited from [41].

1) Decision Result Based on the Subjective Weights: For the problem of EPI target selections cited from [41], Jia and Liu determined the weight vector of five evaluation attributes based on subjective approach and $W = (0.3, 0.1, 0.3, 0.2, 0.1)$. The decision results obtained from the proposed method and the method in [41] are shown in Tables XIV-XVII. Compared with the method in [41], we find that the proposed method is effective and credible. Furthermore, we compare the decision results with four classic MADM methods, which are stated in Section V-B2. Through comparative analysis and discussions, we find that the proposed method is more effective and reasonable.

2) Decision Result Based on the Objective Weights: In the following, we utilize entropy measures [49] to acquire the objective weights of attributes.

TABLE XXII
DECISION REGIONS OF PROJECTS OBTAINED BY THE PROPOSED METHOD AND THE METHOD IN [41]

Methods	Positive region	Boundary region	Negative region
Proposed method(optimistic)	u_7	u_1, u_2, u_4	u_3, u_5, u_6, u_8
Proposed method(average)	u_7	u_1, u_2, u_4	u_3, u_5, u_6, u_8
Proposed method(pessimistic)	u_7	u_1, u_2, u_4	u_3, u_5, u_6, u_8
Method in [41]($\mathcal{P}(X\|u_i) = 0.3$)	u_7	u_1, u_4	u_2, u_3, u_5, u_6, u_8
Method in [41]($\mathcal{P}(X\|u_i) = 0.7$)	$u_1, u_2, u_3, u_4, u_5, u_7$	u_6, u_8	$\emptyset$

TABLE XXIII
RANKINGS OF PROJECTS DERIVED FROM DIFFERENT METHODS

The methods	Ranking results
Proposed method (optimistic)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_3 \succ u_8 \succ u_6 \succ u_5$
Proposed method (average)	$u_7 \succ u_2 \succ u_4 \succ u_1 \succ u_3 \succ u_8 \succ u_6 \succ u_5$
Proposed method (pessimistic)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_3 \succ u_8 \succ u_6 \succ u_5$
Method in [41] (based on α)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_6 \succ u_8$
Method in [41] (based on β)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_6 \succ u_8$
Method in [41] (based on γ)	$u_7 \succ u_1 \succ u_4 \succ u_2 \succ u_5 \succ u_3 \succ u_6 \succ u_8$
PROMETHEE II method	$u_7 \succ u_4 \succ u_1 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$
TOPSIS method	$u_7 \succ u_4 \succ u_1 \succ u_5 \succ u_2 \succ u_8 \succ u_3 \succ u_6$
WAA operator method	$u_7 \succ u_4 \succ u_1 \succ u_2 \succ u_5 \succ u_6 \succ u_3 \succ u_8$
ELECTRE II method	$u_7 \approx u_1 \succ u_4 \succ u_2 \approx u_5 \succ u_3 \approx u_6 \approx u_8$

For the problem of EPI target selections cited from [41], we can obtain the entropy weight vector of five attributes based on the entropy weight method [50]. The weight vector $W = (0.1525, 0.1835, 0.3633, 0.1183, 0.1823)$.

Based on the evaluation information of Table XV (in [41]), the decision regions and rankings of eight projects determined by the proposed method and some existing methods are shown in Tables XXII and XXIII.

From the results of Tables XXII and XXIII, we have two findings.

1) Compared with the method in [41], the decision regions of the proposed method (based on the three strategies) have more projects divided into boundary region. From the perspective of risk aversion, it is easier to divide projects into boundary region. It implies that the DMs need to collect more information before making final decisions. Thus, when the DM is a risk-averse individual, the decision regions obtained by the proposed method is more reasonable.
2) From the results in Table XXIII, the optimal projects determined by the proposed method and the five existing methods are identical, i.e., the project u_7. Furthermore, the rankings of eight projects are consistent in most the ranking positions of projects. Therefore, the proposed method is more effective and credible.

VII. CONCLUSION

In this article, we have proposed a novel 3WD model based on outranking relations with the aid of the well-known MADM method ELECTRE-I. The main contributions of this article and further research topics are summarized as follows:

An outranking relation based on the ELECTRE-I method was introduced and the outranked set was discussed. A hybrid information table was proposed by integrating MADM matrix with loss function table and corresponding 3WD model has been investigated. By applying the proposed 3WD method to the problem of EPI target selections, we have demonstrated that the 3WD method is effective. Further support to the proposed method was also given by the comparative analysis and two experimental evaluations.

There are several research topics deserving further exploration. The combination of 3WD with some traditional MADM methods [36], [38] is also a promising research direction. In a hybrid information table, the loss functions based on fuzzy values [51], linguistic terms [34], [52], and other uncertain evaluation values can be studied.

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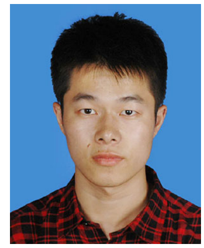
Jianming Zhan received the B.Sc. degree from the Hubei Minzu University, Enshi, China, in 1996, and the M.Sc. degree from the Hubei University, Wuhan, China, in 1999, both in mathematics.

Since 1999, he has been a Professor with the Hubei Minzu University, Enshi, China. He has authored or coauthored more than 200 research publications. His current research interests include fuzzy algebras, fuzzy rough sets, and decision-making methods.



Haibo Jiang received the B.Sc. degree in mathematics from Fuyang Normal University, Fuyang, China, in 2017. He is currently working toward the M.Sc. degree in mathematics with the Hubei Minzu University, Enshi, China.

He has authored or coauthored three papers in the international journals, such as IEEE TFS, INS, and JMLC. His current research interests include decision-making methods and fuzzy rough sets.

Yiyu Yao received the B.E. degree in computer science from Xi'an Jiaotong University, Xi'an, China, in 1983, and the M.Sc. and Ph.D. degrees in computer science from University of Regina, Regina, SK, Canada, in 1988 and 1991, respectively.

He is a professor of computer science with the University of Regina, Canada. His research interests include three-way decision, granular computing, rough sets, formal concept analysis, information retrieval, data mining, and Web intelligence. He proposed a theory of three-way decision, a decision-theoretic rough set model, and a triarchic theory of granular computing. He has published over 400 papers. He was a highly cited researcher from 2015 to 2019.