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Three-way recommendation model based on shadowed set with uncertainty invariance

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ABSTRACT

Recommender systems are an effective tool to resolve information overload by enabling the selection of the subsets of items from a universal set based on user preferences. The operation of most of recommender systems depends on the prediction ratings, which may introduce a degree of uncertainty into the process of recommendation. However, systems equipped with only two strategies lack the flexibility to address such uncertain decision-making problems. Thus, the presence of far-fetched recommendations accompanied by uncertainties often decreases recommendation quality. To resolve this issue, a three-way recommendation model based on a novel shadowed set is proposed in this paper to reduce decision-making risk and improve quality. To this end, a neighborhood rough set model is first introduced into three-way recommendation to determine similar user to active users with respect to the original rating decision system. This helps to avoid the uncertainty generated during the assignment of prediction rating. Subsequently, the optimal neighborhood radius is defined to overcome the subjectivity associated with the construction of the aforementioned neighborhood with a subjective parameter. Following this, a novel shadowed set model, based on neighborhood memberships of boundary users, is proposed to partition all users into positive region, negativity region and boundary region. This facilitates the adoption of different decisions by recommender systems for users in different regions. Finally, the effectiveness and reliability of the proposed model are verified on two Movielens datasets via comparison analyses.

1. Introduction

Recommender systems (RSs) garnered significant academic attention in the nineties owing to the advances in computer and web technologies [3]. Even at present, they are being widely studied to resolve the issue of information overload and have found successful applications in numerous fields, such as e-commerce [1], music, news [54], movies [4], and tag recommendation [34]. The topics associated with RSs are diverse, owing to their capability of using various types of user preferences and user requirement data to carry out different types of recommendation tasks. The most well-known methods in RSs include content-based filtering [5,16,35], hybrid and data mining-based approaches [30,45], model-based

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collaborative filtering [39], memory-based collaborative filtering [41], and knowledge-based methods [11]. Although existing recommendation methods are capable of providing useful and personal recommendations to their users, most suffer from certain disadvantages, which are summarized as follows.

(1) Most RSs focus on the recommendations based on prediction ratings. Thus, the N items corresponding to the highest prediction ratings are recommended to users. However, the recommendation of the same number of items to users in all instances is inconsistent with the underlying principle personalized recommendations.

(2) Using uncertain and sparse datasets to estimate the user ratings for certain items increases the uncertainty of the model, which, in turn, introduces uncertain decisions into recommendation. Most existing RSs employ binary recommendation merely deciding whether or not to recommend each item to the user and lack effective strategies to handle the case of uncertain users. Certain far-fetched recommendations in such cases may reduce the recommendation quality.

The rough set introduced by Pawlak [36] is a competent mathematical tool to process incomplete and vague information. It has been widely used in several domains, such as data mining [2,52], knowledge discovery [42,56], decision analysis [33,55], and uncertain management and uncertain reasoning [50]. Yao et al. [4] developed a game-theoretic rough set for RSs, which utilizes the rough set model to identify users similar to active users. RSs based on the aforementioned rough set first search for an equivalence class (comprising similar users) corresponding to the identical rating pattern of every active user and then implement recommendations according to the preferences of these similar users. However, such recommendation models are incapable of predicting a user's ratings or recommending items to active users, when an equivalence relation between that user and any user from the training set cannot be established successfully.

To improve fault-tolerance, the classical rough set was extended to the neighborhood rough set in [19]. In contrast to equivalence classes in rough sets, basic granules in neighborhood rough sets are constructed based on neighborhood relations [20,49]. Therefore, compared to rough sets, neighborhood rough sets lend themselves to more extensive applications [43] and are more effective in dealing with uncertainty [15]. They have been successfully applied to tasks in multiple fields, including multi-label classification [48], three-way classification [49,51], and feature selection [23]. To address the difficulty of identification of similar users using a rough set model, a neighborhood rough set is introduced in this study to construct an appropriate neighborhood corresponding to each active user. All users in a neighborhood are deemed to be similar to the one at its center (the active user), and the radius of each active user's neighborhood is dependent on their spatial location. RSs identify similar users based on neighborhood relations associated with the original decision system of ratings. In addition to overcoming the issue caused by uncertainty arising from prediction ratings, the proposed model also identifies similar users corresponding to a greater number of active users.

Unlike binary decision, three-way decision (3WD) [46] provides a third decision: deferment decision. This is a favorable methodology to resolve uncertain problems. Recently, motivated by developments in artificial intelligence, a series of three-way approaches has been applied to handle uncertainty problems in machine learning-based tasks [7], such as spam filtering [64], face recognition [27], classification errors [59,61,62], and decision analysis [8,26]. For recommender system, Zhang et al. [53,54] developed a three-way recommender system based on random forests, and regression-based three-way recommendation. Liu et al. [31] developed a matrix factorization-based dynamic granularity recommendation with 3WD. Huang et al. [22] proposed method to improve three-way recommendation quality from a cost-sensitive view. Compared to the traditional binary recommendation models, the three-way recommendation models provide deferment as a recommendation decision, which may be more suitable for uncertain users (boundary users).

The primary task in three-way recommendation is the determination of a pair of recommendation thresholds (α^*, β^*). To this end, all of the aforementioned references [22,31,53,54]focus on cost matrix to calculate (α^*, β^*). However, the accurate measurement of certain cost matrices may be exceedingly difficult or even impossible in certain applications. In such cases, values are assigned to the cost matrices in a subjective manner based on expert experience. In addition, some other recommender tasks do not require consideration of costs.

Thus, a novel method is required to address the aforementioned limitations of the cost matrix for calculating thresholds. The shadowed set, introduced by Pedrycz [37,38], is an important tool to deal with cases involving uncertain information. Besides providing three-way approximations of fuzzy sets, it also provides an objective function to calculate a pair of thresholds, which can be effectively applied to the establishment of three-way decision. Recently, the shadowed set has also garnered significant attention, both in theoretical and practical applications. Yao [47] proposed a generalized three-way approximation framework, Deng and Yao [13] proposed a 0.5 shadowed set, and Zhang and Yao [63] proposed a novel shadowed set based on game theory. Further, Andrea [9] proposed an entropy− based shadowed set, Zhou [65] proposed a constrained shadowed set alongside a fast optimization algorithm, Zhang [57,58] proposed fuzzy entropy-based game theoretic and interval shadowed sets, etc [17,60,67]. Most of the existing shadowed set models focus on the determination and interpretation of thresholds, and few studies have considered the distribution of the actual data. Thus, although the aforementioned models facilitate the clustering of uncertain data [66], they may not be suitable for supervised learning tasks. To address this shortcoming, a novel shadowed set model, the uncertainty invariance-based shadowed set model (UISS), is proposed in this paper to partition all users into certain regions (positive or negative region) and uncertain region (boundary region). This enables RSs to decide on whether to recommend a particular item to users belonging to certain regions, or to adopt a deferment strategy for users belonging to boundary region.

In this paper, a novel three-way recommendation model based on the shadowed set with uncertainty invariance (3WR-UISS) is proposed to resolve the two aforementioned issues with existing RSs. In the 3WR-UISS model, each active user is considered to be a neighborhood center during the construction of neighborhoods, and all users belong to a neighborhood are deemed to be similar to the at the center. Subsequently, the UISS model is constructed based on neighborhood memberships. Following this, based on the UISS, all users are portioned into three regions: positive, negative, and uncertain region. Finally, RSs adopt different decisions corresponding to users in different regions, i.e., recommendations are made corresponding to users that are partitioned into positive regions, recommendations are deferred corresponding to uses that are classified into boundary regions and recommendations are withheld corresponding to users that are classified into negative region. Compared to other kinds of recommendation models, the 3WR-UISS model exhibits the following advantages:

(1) The 3WR-UISS model is based on the distribution of the recommendation datasets and does not involve the prediction of ratings. Thus, it does not require the specification of any parameters in advance.

(2) To achieve better personal recommendations, neighborhood rough set model is introduced into three-way recommendation to find similar users. In this model, different users correspond to different neighborhoods, and each user's preference for items is determined by their neighborhood, rather than that of their nearest neighbor.

(3) The novel shadowed set model based on neighborhood memberships enables more reasonable decision-making for uncertain users and improves the recommendation quality.

The remainder of this paper is organized as follows. In Section 2, the preliminary definitions related to the study are briefly reviewed. In Section 3, the framework of the 3WR-UISS model and implementation algorithms are introduced. Experimental comparisons are presented and extensively analyzed in Section 4. Finally, the conclusion and prospective directions for future research are presented in Section 5.

2. Preliminaries

In this section, some basic concepts related to the study, such as decision system, neighborhood rough set and shadowed set are briefly reviewed.

2.1. Decision system

The concept of decision system is widely used in data mining [24] and machine learning [28].

Definition 1. (Decision system) [32] A decision system is a five-tuple: $S = (U, C, d, V = \{V_a | a \in C \cup \{d\}\}, f)$, where
(1) $U = \{x_1, x_2, ..., x_m\}$ denotes a nonempty set called the universe;
(2) $C = \{a_1, a_2, ..., a_n\}$ denotes a conditional attribute set;
(3) d denotes a decision attribute;
(4) V_a denotes a value for each $a \in C \cup \{d\}$;
(5) $f : U \to V_a$ denotes an information function for each $a \in C \cup \{d\}$.

In particular, in the absence of decision attributes, a decision system is simply an information system, denoted by $IS = (U, C, V, f)$.

2.2. Neighborhood rough set

Definition 2. (Neighborhood information granule) [20] For any $x_i \in U$, the neighborhood $N(x_i, \delta)$ of x_i is defined as follows:

$$N(x_i, \delta) = \{x_j | x_j \in U, \Delta(x_i, x_j) \le \delta\}, \tag{1}$$

where $\Delta(x_i, x_j)$ denotes the distance between x_i and x_j, and $\delta (\delta \ge 0)$ denotes the radius of the neighborhood.

A multitude of distance functions may be used, and a detailed investigation of this topic is available in [20]. $NDS = (U, C \cup \{d\}, N)$ denotes as a neighborhood decision system if when there is exists an attribute in the decision system generating a neighborhood relation, N, on U. More generally, $NIS = (U, C, N)$ is called a neighborhood information system.

Definition 3. (Neighborhood rough set) [20] For any subset $X \subseteq U$ in $NIS = (U, C, N)$, two subsets of objects, called the lower and upper approximations of X in NIS, are defined as follows:

$$\begin{aligned} &\underline{N}X = \{x_i | N(x_i, \delta) \subseteq X, x_i \in U\}, \\ &\overline{N}X = \{x_i | N(x_i, \delta) \cap X \neq \emptyset, x_i \in U\}. \end{aligned} \tag{2}$$

Based on the lower and upper approximations, the positive, negative, and boundary regions of any concept, X, are defined as follows [46]:

$$\begin{aligned} &POX(X) = \underline{N}X, \\ &NEG(X) = (\overline{N}X)^c, \\ &BND(X) = \overline{N}X - \underline{N}X. \end{aligned} \tag{3}$$

Definition 4. (Neighborhood membership function) [44] For any subset $X \subseteq U$ in $NIS = (U, C, N)$, the membership degree of arbitrary $x \in U$ in X is defined as follows:

$$\mu_X^{\delta}(x) = \frac{Card(N(x,\delta) \cap X)}{Card(N(x,\delta))}, \tag{4}$$

where $Card(\cdot)$ denotes the cardinality of a set.

Uncertainty measures provide principled methodologies to analyze uncertain data and reveal the substantive characteristics of datasets. Shannon [40] first introduced the definition of the entropy of a system as a measure of the uncertainty of its actual structure. Later, the Gini impurity index was proposed by Breiman et al. [6] as a measure of the purity of data in classification and regression trees. Furthermore, a large number of entropy measures have been proposed by researchers [10,12,29]. Moveover, for measuring uncertainty and fuzziness in rough set theory, neighborhood entropy based on Shannon entropy and the Gini impurity index was introduced by Liang et al. [29] to evaluate the uncertainty of the neighborhood information system from the perspective of knowledge granularity.

Definition 5. (Neighborhood entropy) [29] For any subset $X \subseteq U$ in a neighborhood information system (U, C, N), the neighborhood entropy $e(N)$ in X is defined as follows:

$$e(N) = \sum_{i=1}^{n} \mu_X^{\delta}(x_i)(1 - \mu_X^{\delta}(x_i)), i = 1, 2, ..., |U|, \tag{5}$$

where $\mu_X^{\delta}(x_i)$ denotes the neighborhood membership degree of x_i in X.

2.3. Shadowed set

Given a membership function μ_X, for any data instance $x \in U$, its membership $\mu_X(x)$ in X is briefly denoted as $\mu(x)$. With the triple elements of the shadowed set, the membership value, $\mu(x)$, of each instance, x, is mapped into a triplet set, $\{0, [0, 1], 1\}$.

Definition 6. (Shadowed set) [37] Let α and β be two real numbers $(0 \le \beta < \alpha \le 1)$. A shadowed set S on universe U is defined as a mapping from U to the set $\{0, [0, 1], 1\}$, that is, $S : U \to \{0, [0, 1], 1\}$. The mapping S is defined as follows,

$$S_{\mu_X}(x) = \begin{cases} 1 & \mu(x) \ge \alpha, \\ 0 & \mu(x) \le \beta, \\ [0,1] & \beta < \mu(x) < \alpha. \end{cases} \tag{6}$$

As depicted in Fig. 1, the classical shadowed set provides a three-way approximation scheme to transform the universe of a fuzzy set into three disjoint areas: elevated area, reduced area, and shadowed area. It is evident that the transformation from fuzzy memberships to a shadowed set redistributes the uncertainty—the uncertainty in the elevated and reduced area is reduced, while that in the shadowed area is increased. Based on this observation, Pedrycz established an objective function to maintain a constant overall level of uncertainty, formulated as follows:

$$V_{(\beta,\alpha)}(\mu_X) = |\sum_{\mu(x)\ge a} (1 - \mu(x)) + \sum_{\mu(x)\le\beta} \mu(x) - \mathrm{Card}(\{x \in \mathrm{U} | \beta < \mu(x) < \alpha\})|. \tag{7}$$

To reduce the complexity of calculations, Pedrycz [37] assumed the relationship between α and β to be $\alpha + \beta = 1$. In practice, it is difficult to identify the optimal thresholds that satisfy the aforementioned objective function. Therefore, the problem becomes an optimization problem. For any membership function, the optimal thresholds can be obtained by minimizing the objective function, as follows:

$$\arg \min_{(1-\alpha,\alpha)} V_{(1-\alpha,\alpha)}(\mu_A). \tag{8}$$

To measure the uncertainty corresponding to the shadowed area, Zhang et al. [57] defined interval fuzzy entropy and proposed an interval shadowed set model on its basis.

Definition 7. (Interval fuzzy entropy) [57] Let $[a, b]$ be a subinterval of $[0, 1]$. The membership degree $\mu(x)$ of an element x is any value in $[a, b]$ with equal probability. The fuzzy entropy of x is called the interval fuzzy entropy and is defined as follows:

$$\overline{e}_{[a,b]}(x) = \frac{1}{b-a} \int_a^b \mu(x)(1 - \mu(x)) d\mu(x). \tag{9}$$

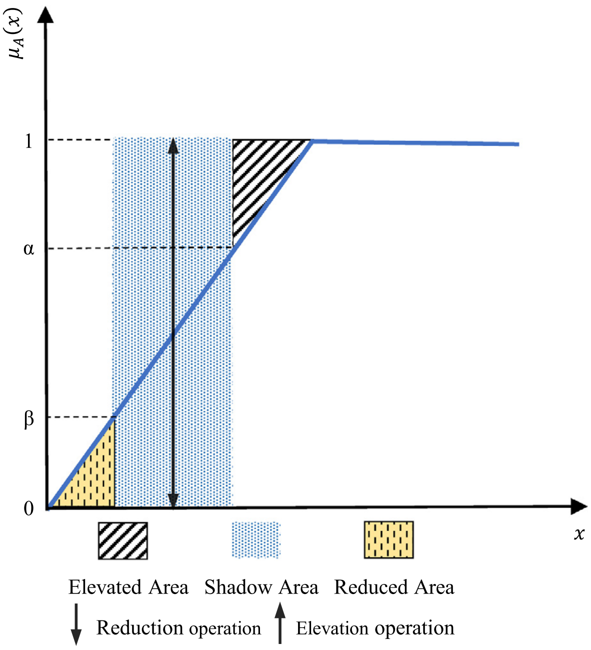


Fig. 1. Pedrycz's classical shadowed set.

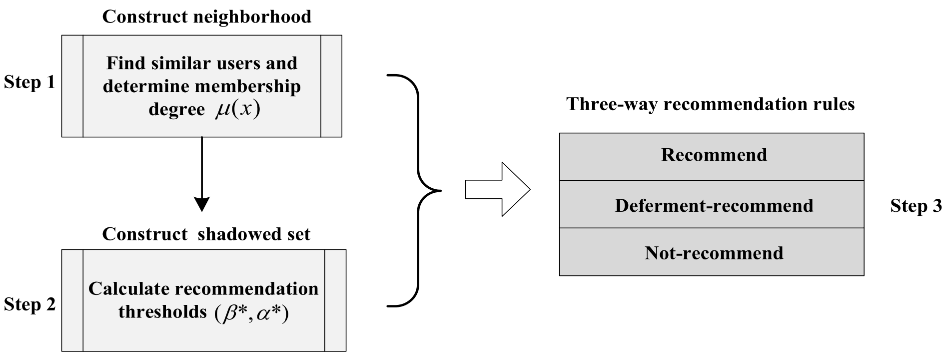


Fig. 2. Framework of 3WR-UISS.

3. Constructing the three-way recommendation model

To resolve issues regarding uncertainty in RSs more efficiently and enable the application of three-way recommendation systems in more diverse fields, a novel three-way recommendation model based on a shadowed set with uncertainty invariance (3WR-UISS) is proposed in this section to handle uncertain data and improve recommendation quality. As depicted in Fig. 2, the construction of the 3WR-UISS model is divided into the following three steps.

(1) Users similar to each active user are identified via the neighborhood method and the neighborhood membership, $\mu(x)$, of each active user (i.e., the user's preference) is determined on the basis of the preferences of similar users for each item.

(2) A novel shadowed set model is constructed based on the neighborhood membership function to obtain a pair of recommendation thresholds, β^* and α^*.

(3) In light of the three-way recommendation rules, an item is recommended if $\mu(x) \geq \alpha^*$, and it is not recommended if $\mu(x) \leq \beta^*$. Otherwise, if $\beta^* < \mu(x) < \alpha^*$, the 3WR-UISS defers the recommendation owing to the necessity to obtain more recommendation information before making a decision.

These three steps are described in greater detail in the following subsections.

3.1. Decision system in recommendation learning

Formally, the structural data used during recommendation learning can be represented as a decision system, which is discussed in Example 1.

Table 1
A decision system in recommendation learning.

U	m_1	m_2	m_3	m_4	m_5	m_6	m_7	$m_8(d)$
x_1	4	5	5	5	5	1	4	N
x_2	4	0	0	0	0	2	0	N
x_3	0	5	5	5	4	0	4	Y
x_4	3	0	0	0	0	0	0	N
x_5	0	1	0	0	0	0	0	N
x_6	0	0	0	0	2	0	0	N
x_7	0	2	5	0	4	5	5	Y
x_8	5	1	0	4	5	5	5	Y
x_9	4	0	2	4	4	5	0	Y
x_{10}	3	5	5	4	4	0	4	Y
x_{11}	0	0	1	5	0	0	0	$*$
x_{12}	5	1	4	3	2	0	4	$*$
x_{13}	4	5	5	0	5	0	4	$*$
x_{14}	4	5	0	0	2	5	0	$*$
x_{15}	0	5	5	5	4	5	0	$*$

Example 1. A decision system for recommendation learning is presented in Table 1, in which each row records the ratings assigned by each user to eight movies. Users in the universe with known decision labels can be considered to belong to the training set, U_{train}, and the users with unknown labels (demarcated by "*") can be considered to constitute a testing set, U_{test}. Obviously, $U = U_{train} \cup U_{test} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\} \cup \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}$. $C = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7\}$ is the conditional attribute set and $d = \{m_8\} = \{Y, N\}$ is the decision attribute, where "Y" denotes that the user likes m_8, and "N" denotes that the user dislikes m_8.

In this paper, a decision system is taken as input in the process of recommendation, and on this basis, a neighborhood rough set model is introduced to construct the neighborhood for active users.

3.2. Constructing neighborhoods for active users

By the definition of neighborhood rough set, all users belonging to a neighborhood are deemed to be similar to that at the center of the neighborhood. The neighborhood of any active user is constructed by grouping their neighboring users into a set. For any user, $x \in U$, the radius of their neighborhood is denoted by δ_x, which determines the size of the neighborhood. If δ_x is too small, no users will be deemed to be similar to the active user. On the contrary, the preferences of users in the neighborhood will no longer faithfully reflect the preferences of the active user if δ_x is too large.

Thus, the fundamental problem during the construction of neighborhoods is the determination of a reasonable neighborhood radius. Typically, the radius, δ_x, of the neighborhood is set by researchers [20,21], which is inevitably subjective. Therefore, the development of an automatic assignment method in accordance with the distribution of actual data is essential. To resolve this issue, the optimal neighborhood radius of each object is determined separately in this study, and the average of the optimal neighborhood radii of all objects is taken to be the neighborhood radius of the training and testing sets. For clarity, the neighborhood granule size is defined first.

Definition 8. (Neighborhood granule size) For any subset $X \subseteq U$ in a neighborhood information system (U, C, N), the neighborhood granule size of object $x \in U$ is defined as follows:

$$\rho(x, \delta) = \frac{Card(N(x, \delta))}{Card(U)}, \tag{10}$$

where $N(x, \delta)$ denotes neighborhood information granule.

Definition 9. (Optimal neighborhood radius) Let $(U, C \bigcup \{d\}, N)$ be a neighborhood decision system and $U/\{d\} = \{X_1, X_2, ...X_i, ...X_m\}$. Then the optimal neighborhood radius for $x \in X_i$ is defined as follows:

$$\delta_x^* = \min\{\delta | \max\{\rho(x, \delta) | N(x, \delta) \subseteq X_i\}\}, \tag{11}$$

where $N(x, \delta)$ denotes neighborhood information granule, $\rho(x, \delta)$ denotes the size of neighborhood granule, and m is the cardinality of domain of decision attribute.

In this model, δ_x^* is not specified by any expert; instead, it is determined by the spatial location of the object. Fig. 3 illustrates the determination of δ_x^* in a 2-D numerical space, where "•" and "∘" denote two distinct class labels. Let us consider the object, x_1, as an example. First, the radius, $\delta_{x_1} \in [0, \delta_3]$, is selected such that objects in the neighborhood of radius, δ_{x_1}, belong to the same decision class. Then, $\delta_{x_1} \in [\delta_2, \delta_3]$ is selected to maximize the size of the neighborhood

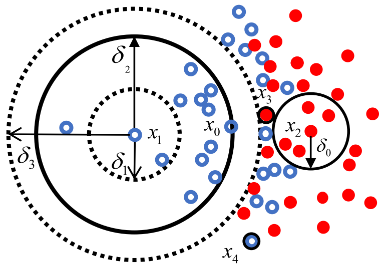


Fig. 3. The determination of the optimal neighborhood radius for x_1.

Table 2
Similarity matrix.

U_{train}	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	0	0.96	0.376	0.964	0.988	0.96	0.732	0.687	0.732	0.177
x_2	0.96	0	1	0.198	0.407	0.435	0.865	0.779	0.595	0.901
x_3	0.376	1	0	0.956	0.878	0.865	0.687	0.86	0.851	0.281
x_4	0.964	0.198	0.956	0	0.281	0.32	0.905	0.869	0.699	0.878
x_5	0.988	0.407	0.878	0.281	0	0.198	0.851	0.956	0.784	0.878
x_6	0.96	0.435	0.865	0.32	0.198	0	0.808	0.892	0.715	0.865
x_7	0.732	0.865	0.687	0.905	0.851	0.808	0	0.732	0.742	0.687
x_8	0.687	0.779	0.86	0.869	0.956	0.892	0.732	0	0.502	0.753
x_9	0.732	0.595	0.851	0.699	0.784	0.715	0.742	0.502	0	0.774
x_{10}	0.177	0.901	0.281	0.878	0.878	0.865	0.687	0.753	0.774	0

granule. Finally, the minimum δ_2 is selected to be the optimal radius within the range, $[\delta_2, \delta_3]$, i.e., $\delta^*_{x_1} = \delta_2$. Similarly, $\delta^*_{x_2} = \delta_0, \delta^*_{x_3} = 0, \delta^*_{x_4} = 0$. Example 2 is presented below to illustrate the determination of the optimal neighborhood radius in RSs.

Example 2. (Continued) Example 1. The optimal neighborhood radius is identified for each user in the training set, U_{train}.

Step 1: As presented in Table 2, the similarity matrix is obtained by calculating the standardized Euclidean distances between each pair of users in U_{train}.

Step 2: The optimal neighborhood radius for each user is obtained by following Definition 9. Subsequently, U_{train} is partitioned into two decision classes: $X_1 = \{x_3, x_7, x_8, x_9, x_{10}\}$ and $X_2 = U_{train} - X_1 = \{x_1, x_2, x_4, x_5, x_6\}$. The nearest neighbor of the user, x_1, is x_{10}, but they belong to different decision classes. Hence, $\delta^*_{x_1} = 0$. In order of distance from x_2, its neighbors are x_4, x_5, x_6, x_9, ..., and x_3. x_4, x_5, x_6, and x_2 belong to the same decision class, which is different from that of x_9. Therefore, the optimal neighborhood radius of x_2 is taken to be the distance between x_2 and x_6, i.e., $\delta^*_{x_2} = 0.435$. Similarly, $\delta^*_{x_3} = 0.281$, $\delta^*_{x_4} = 0.32$, $\delta^*_{x_5} = 0.407$, $\delta^*_{x_6} = 0.435$, $\delta^*_{x_7} = 0.687$, $\delta^*_{x_8} = 0.502$, $\delta^*_{x_9} = 0.502$, and $\delta^*_{x_{10}} = 0$.

The optimal neighborhood radii corresponding to several objects are 0 or close to 0, such as x_3 and x_4 in Fig. 3, and x_1 in Example 2. Objects belonging to each neighborhood are related by an equivalence relation if $\delta^*_x = 0$. In this case, the neighborhood rough set model degenerates to Pawlak's model. Therefore, from the perspective of the entire sample space, it is necessary to choose a more suitable neighborhood radius than the optimal neighborhood radius during the construction of the neighborhood. In this study, the average, $\bar{\delta}$, of the optimal neighborhood radii is taken to be the radius of the neighborhoods for both the training and testing sets. $\bar{\delta}$ is defined as follows:

$$\bar{\delta} = \frac{1}{Card(U_{train})} \sum_{x_i \in U_{train}} \delta^*_{x_i}. \tag{12}$$

Although the radii of the neighborhoods of all objects are identical, the neighborhood sizes are different. Following the definition of the neighborhood rough set, $\bar{\delta}$ serves as the threshold between boundary regions and certain regions. In other words, $\forall x \in U_{train}$, x is classified into the boundary region if $\delta^*_x < \bar{\delta}$ and it is classified into a certain region if $\delta^*_x \geq \bar{\delta}$.

Fig. 4 illustrates the construction of the neighborhood with radius, $\bar{\delta}$, in a 2-D numerical space, where X_1 and X_2 denote two distinct decision classes denoted by"•" and "∘", respectively. Let us suppose that X_1 is a target concept. All $\bar{\delta}$-neighbors of x_1 exhibit class labels identical to those of positive objects, and all $\bar{\delta}$-neighbors of x_2 exhibit class labels identical to those of negative objects. The samples in $N(x_3, \bar{\delta})$ are derived from different decision classes. Obviously, $\mu^{\bar{\delta}}_{X_1}(x_1) = 1$, $\mu^{\bar{\delta}}_{X_1}(x_2) = 0$, and $\mu^{\bar{\delta}}_{X_1}(x_3)$ are fuzzy concepts, $\mu^{\bar{\delta}}_{X_1}(x_3) \in (0, 1)$. The process of calculating neighborhood memberships is illustrated in Example 3.

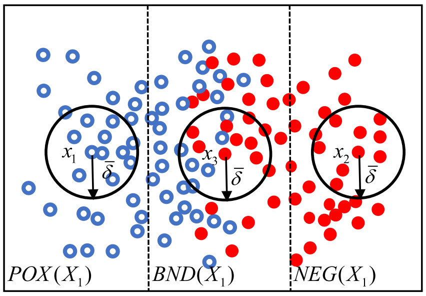


Fig. 4. Construction neighborhood with $\bar{\delta}$.

Example 3. (Continued) Example 2. Based on the aforementioned analysis, this example illustrates the identification of boundary users and the calculation of membership degrees of all users.

Step 1: The neighborhood radius is calculated using Eq. (12), $\bar{\delta} = 0.357$. x_1, x_3, x_4, and x_{10} are classified into the boundary region because their optimal neighborhood radii are less than $\bar{\delta}$.

Step 2: The neighborhoods of boundary users with the radius, $\bar{\delta}$ are constructed. They are $N(x_1, 0.357) = \{x_j | x_j \in U_{train}, \Delta(x_1, x_j) \le 0.357\} = \{x_1, x_{10}\}$, $N(x_3, 0.357) = \{x_3, x_{10}\}$, $N(x_4, 0.357) = \{x_2, x_4, x_5, x_6\}$, and $N(x_{10}, 0.357) = \{x_{10}\}$.

Step 3: The neighborhood membership degree of each user is calculated by following Definition 4. They are determined to be $\mu^{\bar{\delta}}_{X_1}(x_1) = 0.5$, $\mu^{\bar{\delta}}_{X_1}(x_3) = 1$, $\mu^{\bar{\delta}}_{X_1}(x_4) = 0$, and $\mu^{\bar{\delta}}_{X_1}(x_{10}) = 1$. Moreover, if a user belongs to a certain region, their neighborhood membership is either 1 or 0.

In the context of recommendation learning, boundary users comprise one class of sources of recommendation complexity owing to the similarity of their feature values with those of similar users belonging to different decision classes. This, in turn, leads to low recommendation performance. To enable more reasonable decision-making for boundary users and improve the recommendation performance, a novel shadow set model is constructed in this study based on their neighborhood memberships to obtain a pair of thresholds for three-way recommendation.

3.3. Uncertainty invariance-based shadowed set

One of the fundamental problems in three-way recommendation is the determination of recommendation thresholds that enable reasonable decision-making for boundary users [54]. The shadowed set model is an effective tool to deal with uncertain information [37]. The membership function is used to establish the model as well as determine a pair of thresholds using an objective function.

However, in practical applications, the classical shadowed set may suffer from the following problems. First, the classical model measures uncertainty based on the principle of distance, which is incapable of accurately describing the uncertainty of the corresponding approximation set. Second, the classical model is constructed by transforming the membership degrees, $\mu_{A(x)}$ $(\beta < \mu_{A(x)} < \alpha)$, into the interval, [0, 1], which may introduce additional uncertainty. Finally, in this case, if $\mu_{\max}$ was significantly lower than 1 or $\mu_{\min}$ was much greater than 0, elevation of membership degrees from $\mu_{\max}$ to $S_{\mu_A}(x) = 1$ or their reduction from $\mu_{\min}$ to 0 may be unreasonable. To solve these problems, a novel shadowed set is proposed by combining the concepts of a shadowed set [37] and an interval shadowed set [57].

Definition 10. (Uncertainty invariance-based shadowed set, UISS) Given a neighborhood membership $\mu(x)$ and a pair of thresholds α^* and β^* $(0 \le \beta^* < \alpha^* \le 1)$, an uncertainty invariance-based shadowed set S^* on a universe U is defined as a mapping from U to the set $\{\mu_{min}, [\beta^*, \alpha^*], \mu_{max}\}$, that is, $S^* : U \to \{\mu_{min}, [\beta^*, \alpha^*], \mu_{max}\}$. The mapping is defined as follows,

$$S^*_{\mu_X}(x) = \begin{cases} \mu_{\max} & \mu(x) \ge \alpha^*, \\ \mu_{\min} & \mu(x) \le \beta^*, \\ [\beta^*, \alpha^*] & \beta^* < \mu(x) < \alpha^*. \end{cases} \tag{13}$$

$\mu_{\max}$ and $\mu_{\min}$ denote the maximum and minimum values of membership function, respectively.

The mapping on the elevated and reduced areas is more dependent on the actual data distribution under the UISS model compared to the classical shadowed set model. In particular, the UISS model degenerates into the classical shadowed set model if $\mu_{\max} = 1$ and $\mu_{\min} = 0$. Therefore, the UISS model is a natural generalization of the shadowed set. As depicted in Fig. 5, corresponding to any membership function, the UISS model can be constructed to approximately describe the fuzzy concept, as follows:

(1) $\forall x \in U$, the membership degree is elevated from $\mu(x)$ to $S^*_{\mu_X} = \mu_{\max}$ if $\mu(x) \ge \alpha^*$ and the range of alteration is defined to be the elevated area.

(2) $\forall x \in U$, the membership degree is reduced from $\mu(x)$ to $S^*_{\mu_X} = \mu_{\min}$ if $\mu(x) \le \beta^*$ and the range of alteration is defined to be the reduced area.

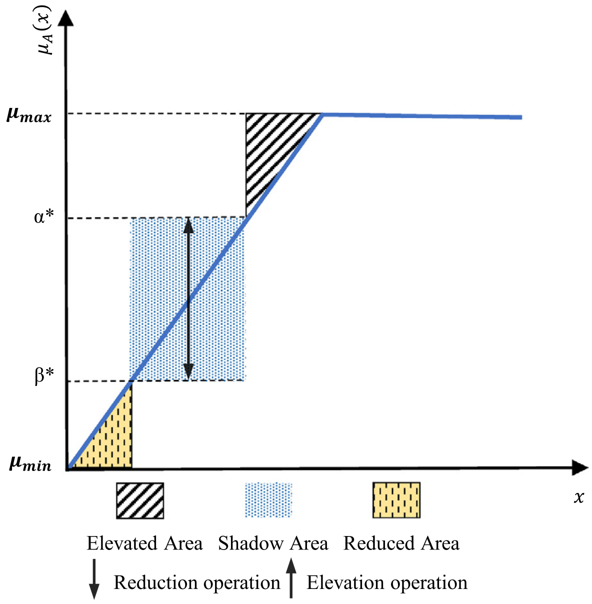


Fig. 5. Uncertainty invariance-based shadowed set.

(3) $\forall x \in U$, the membership degree is transformed from $\mu(x)$ to $S^*_{\mu X} = [\beta^*, \alpha^*]$ if $\beta^* < \mu(x) < \alpha^*$, and the range of alteration is defined to be the shadowed area.

In [47], Yao et al. proposed an optimization principle of uncertainty invariance to quantify the discrepancy between a fuzzy set and its three-way approximation and assumed that lower discrepancy values correspond to better approximations. Following this principle, the following systematic method can be used to calculate the optimal thresholds, (β^*, α^*), to construct a UISS model without any entropy loss.

$$Q_{(\beta^*,\alpha^*)}(N, S^*) = |e(N) - e_{(\beta^*,\alpha^*)}(S^*)| = 0, \tag{14}$$

where $e(N)$ denotes the neighborhood entropy, and $e_{(\beta^*,\alpha^*)}(S^*)$ denotes the entropy of the UISS model, defined as follows:

$$\begin{aligned} e_{(\beta^*,\alpha^*)}(S^*) &= e^*(Elevated\text{Area}) + e^*(Reduced\text{Area}) + e^*(Shadow\text{Area}) \\ &= \sum_{\mu(x)\geq\alpha^*} \mu_{\max}(1-\mu_{\max}) + \sum_{\mu(x)\leq\beta^*} \mu_{\min}(1-\mu_{\min}) + \sum_{x\in Shadow\text{Area}} \overline{e}_{[\beta^*,\alpha^*]}(x), \end{aligned} \tag{15}$$

where $e^*(Elevated\text{Area})$, $e^*(Reduced\text{Area})$ and $e^*(Shadow\text{Area})$ denote the entropy of the elevated, reduced, and shadowed areas, respectively. The interval fuzzy entropy of the shadowed area can be calculated using Definition 7, as follows:

$$\begin{aligned} e^*(Shadow\text{Area}) &= \sum_{x\in Shadow\text{Area}} \overline{e}_{[\beta^*,\alpha^*]}(x) \\ &= \frac{\int_{\beta^*}^{\alpha^*} \mu(x)(1-\mu(x))d\mu(x)}{\alpha^*-\beta^*} \times card(\{x|\mu(x) \in (\beta^*,\alpha^*)\}). \end{aligned} \tag{16}$$

However, in practical applications, the identification of optimal thresholds, β^* and α^*, that satisfy Eq. (14) is difficult. Therefore, the problem is reduced to an optimization problem. Given any membership function, the optimal thresholds, β^* and α^*, can be calculated by solving the following minimization problem:

$$\arg\min_{(\beta^*,\alpha^*)} Q_{(\beta^*,\alpha^*)}(N, S^*), \tag{17}$$

where α^* and β^* are related by $\alpha^* + \beta^* = 1$ as in the case of the classical shadowed set model. Therefore, the objective function, $Q_{(1-\alpha,\alpha)}(N, S^*)$, is given as follows:

$$\begin{aligned} Q_{(1-\alpha^*,\alpha^*)}(N, S^*) = |&\sum_{\mu_A(x)\geq\alpha^*} (\mu(x)(1-\mu(x)) - \mu_{\max}(1-\mu_{\max})) \\ &+ \sum_{\mu(x)\leq 1-\alpha^*} (\mu(x)(1-\mu(x)) - \mu_{\min}(1-\mu_{\min})) \\ &+ \sum_{1-\alpha^*<\mu(x)<\alpha^*} \mu(x)(1-\mu(x)) \\ &- \frac{\int_{1-\alpha^*}^{\alpha^*} \mu(x)(1-\mu(x))d\mu(x)}{1-2\alpha^*} \times card(\{x|\mu(x) \in (1-\alpha^*,\alpha^*)\})|. \end{aligned} \tag{18}$$

Algorithm 1 Constructing neighborhood and determining boundary users.

Input: $S = (U_{train}, C, d, V = \{V_a | a \in C \cup \{d\}\}, f)$
Output: $\bar{\delta}$, BND, $N(x_i, \bar{\delta})$.
1: Initialize $BND = \emptyset$;
2: compute $\Delta(x_i, x_j)$, where $(x_i, x_j) \in U_{train} \times U_{train}$;
3: **for** i from 1 to $Card(U_{train})$ **do**
4:     **for** j from 1 to $Card(U_{train})$ **do**
5:         obtain the optimal neighborhood radius of each object $\delta^*_{x_i}$;
6:     **end for**
7: **end for**
8: compute $\bar{\delta}$;
9: **for** i from 1 to $Card(U_{train})$ **do**
10:     **if** $\delta^*_{x_i} \le \bar{\delta}$ **then**
11:         $BND = BND \cup \{x_i\}$;
12:     **end if**
13:     construct the neighborhood of object $(N(x_i, \bar{\delta}))$;
14: **end for**
15: **return** $\bar{\delta}$, BND, $N(x_i, \bar{\delta})$.

It is evident that besides maintaining a balance between the increment and decrement in uncertainty, $Q_{(1-\alpha^*,\alpha^*)}(N, S^*)$ also negates entropy loss in the model under the best circumstances. The calculation of thresholds and entropy loss under the UISS model is illustrated in detail in Example 4 by comparing it with that in the classical model.

Example 4. $\{\mu_{x_1}, \mu_{x_2}, \mu_{x_3}, \mu_{x_4}, \mu_{x_5}, \mu_{x_6}, \mu_{x_7}, \mu_{x_8}, \mu_{x_9}, \mu_{x_{10}}\} = \{0.11, 0.85, 0.49, 0.25, 0.01, 0.98, 0.5, 0.73, 0.89, 0.95\}$ be an arbitrary set of memberships.

Step 1: The neighborhood entropy is calculated following Definition 5, i.e. $e(N) = \sum_{i=1}^{10} \mu(x_i)(1 - \mu(x_i)) = 1.285$.

Step 2: The UISS model is constructed following Definition 10, and a pair of thresholds is obtained using Eq. (17), i.e. $(1 - \alpha^*, \alpha^*) = (0.11, 0.89)$. Similarly, the classical shadowed set model is constructed following Definition 6, and then a pair of recommendation thresholds is obtained using Eq. (8), i.e. $(1 - \alpha^*, \alpha^*) = (0.51, 0.49)$.

Step 3: The entropy of the UISS model is calculated using Eq. (15), i.e., $e(S^*) = 1.127$. Under the classical shadowed set model, the entropies of the elevated and reduced areas are both 0 and the entropy of the shadowed area is calculated following Definition 7. Thus, $e(S) = \sum_{x \in ShadowArea} \bar{e}_{[0,1]}(x) = 0.167$.

Step 4: The entropy loss between the neighborhood model and its UISS model is $Q_{(0.11,0.89)}(N, S^*) = e(N) - e(S^*) = 0.158$. Similarly, the entropy loss between the neighborhood model and its shadowed set model is $Q_{(0.51,0.49)}(N, S) = e(N) - e(S) = 1.118$.

Example 4 demonstrates that the entropy loss during the construction of the UISS model is much smaller than the entropy loss during the construction of the shadowed set model. Therefore, the UISS model is superior to the shadowed set model according to Yao's optimization principle on uncertainty invariance [47].

In three-way recommendation, the membership degrees of uncertain users can be calculated using the method described in Example 3. Subsequently, a pair of three-way recommendation thresholds can be obtained by mimicking Example 4. The corresponding algorithms are presented in Section 3.4 to discuss these arguments in greater detail.

3.4. Algorithm of the 3WR-UISS model

In this section, Algorithms 1 and 2 are first presented to clarify the two primary issues of three-way recommendation. Finally, the algorithm of the 3WR-UISS model is presented in Algorithm 3.

Algorithm 1 presents the construction of neighborhoods and the identification of boundary users. The Euclidean distance between each pair of users is calculated in Step 2 to obtain the similarity matrix of the training set. Steps 3-7 describe the calculation of the optimal neighborhood radius. Steps 9-14 describe the identification of boundary users and the construction of the neighborhoods.

Algorithm 2 presents the construction of the UISS model used to estimate a pair of recommendation thresholds, $(1 - \alpha^*, \alpha^*)$, for three-way recommendation. Steps 7 -19 describe the process of minimizing the objective function by traversal along α^*.

Algorithm 3 presents a three-way recommendation model based on the UISS model constructed above. Steps 2-10 depict the construction of the neighborhoods of test users. Steps 12-21 implement three-way recommendation. A user, x, is classified within the positive region if their membership degree satisfies $\mu^{\bar{\delta}}_{X_1}(x) \ge \alpha^*$, within the negative region if $\mu^{\bar{\delta}}_{X_1}(x) \le 1 - \alpha^*$, and within the boundary region otherwise. RSs adopt different recommendation strategies for users belonging to different regions: recommendations are made for users lying in the positive region, recommendations are deferred for users lying in the boundary region, and recommendations are withheld for users belonging to the negative region.

Algorithm 2 Constructing the UISS model.

Input: $S = (U_{train}, C, d, V = \{V_a | a \in C \cup \{d\}\}, f)$
Output: $\alpha^*, Q(N, S^*)$.
Obtain $\bar{\delta}, BND, N(x_i, \bar{\delta})$ by Algorithm 1;
Partition U_{trian} according to the decision attribute d ($U_{trian}/\{d\} = \{X_1, X_2\}$), and X_1 is the set of users who like the movie d;
\\\\ In the following description, $\mu_{X_1}^{\bar{\delta}}(x_i)$ denoted as $\mu(x_i)$
for i from 1 to $Card(BND)$ **do**
  Compute $\mu(x_i)$ with Definition 4;
  Select the $\mu_{\min}$ and $\mu_{\max}$ which satisfies:
  $\mu_{\min} = \min_i(\mu(x_i))$, $\mu_{\max} = \max_i(\mu(x_i))$
end for
for α^* from 0.5 to 1 **do**
  Compute $\bar{e}_{[1-\alpha^*,\alpha^*]}(x)$ with Definition 7;
  $Q_{(1-\alpha^*,\alpha^*)} = 0$;
  for i from 1 to $Card(BND)$ **do**
    if $\mu_{x_i} \geq \alpha^*$ **then**
      $Q_{(1-\alpha^*,\alpha^*)} \leftarrow Q_{(1-\alpha^*,\alpha^*)} + \mu(x_i)(1 - \mu(x_i) - \mu_{\max}(1 - \mu_{\max})$;
    else if $\mu_{x_i} \leq 1 - \alpha^*$ **then**
      $Q_{(1-\alpha^*,\alpha^*)} \leftarrow Q_{(1-\alpha^*,\alpha^*)} + \mu(x_i)(1 - \mu(x_i) - \mu_{\min}(1 - \mu_{\min}))$;
    else
      $Q_{(1-\alpha^*,\alpha^*)} \leftarrow Q_{(1-\alpha^*,\alpha^*)} + \mu(x_i)(1 - \mu(x_i) - \bar{e}_{[1-\alpha^*,\alpha^*]}(x)$;
    end if
  end for
end for
for α^* from 0.5 to 1 **do**
  select the optimal recommendation thresholds $1 - \alpha^*$ and α^*, which makes $Q_{(1-\alpha^*,\alpha^*)}$ obtain the minimum value;
end for
return $\alpha^*, Q_{(1-\alpha^*,\alpha^*)}$.

Algorithm 3 The algorithm of 3WR-UISS model.

Input: $S = (U_{train} \bigcup U_{test}, C, d, V = \{V_a | a \in C \cup \{d\}\}, f)$, $N(x_i, \bar{\delta}), \alpha^*, \bar{\delta}$
Output: $POS(X_1), NEG(X_1), BND(X_1)$.
Initialize $POS(X_1) = \emptyset, NEG(X_1) = \emptyset, BND(X_1) = \emptyset$;
for i from 1 to $Card(U_{test})$ **do**
  $N(x_i, \bar{\delta}) = \emptyset$;
  for j from 1 to $Card(U_{test})$ **do**
    compute Euclidean distance $\Delta(x_i, x_j)$, where $(x_i, x_j) \in U_{test} \times U_{train}$;
    if $\Delta(x_i, x_j) \leq \bar{\delta}$ **then**
      $N(x_i, \bar{\delta}) = N(x_i, \bar{\delta}) \cup \{x_j\}$;
    end if
  end for
end for
$U_{trian}/\{d\} = \{X_1, X_2\}$, X_1 is the set of users who like the movie d;
for i from 1 to $Card(U_{test})$ **do**
  Computer $\mu_{X_1}^{\bar{\delta}}(x_i)$;
  if $\mu_{X_1}^{\bar{\delta}}(x_i) \geq \alpha^*$ **then**
    $POS(X_1) = POS(X_1) \cup \{x_i\}$;
  else if $\mu_{X_1}^{\bar{\delta}}(x_i) \leq 1 - \alpha^*$ **then**
    $NEG(X_1) = NEG(X_1) \cup \{x_i\}$;
  else
    $BND(X_1) = BND(X_1) \cup \{x_i\}$;
  end if
end for
return $POS(X_1), NEG(X_1), BND(X_1)$.

4. Experiments

Experiments were performed on two Movielens datasets: *Movielens-100K* and *Movielens-1M*, which are typical datasets used to evaluate RSs [31,53]. The main information of the two datasets is summarized in Table 3. All ratings in Movielens datasets are scores on a 5-star scale, where 1 represents the least favorite and 5 represents the most favorite. In the context of the decision system of recommendation learning, following a common rule used in RSs [4], unrated movies are assigned scores of 0, ratings of 4–5 correspond to "like", denoted by "Y", and ratings of 0–3 correspond to "dislike", denoted by "N". Ten movies were randomly selected from the top 30 most frequently rated movies for the experimental predictions. 5-fold cross validation was used in each experiment. The experimental results are presented in Tables 4–5 and Tables 7–10.

The experiments were performed on a Windows 10 64-bit operating system with 8 GB RAM and Intel® Core™ i7-8565U CPU @1.80 GHz processors. The programming language used was Python.

Table 3
The description of data sets.

Data set	ratings	users	movies	source
Movielens-100K	100000	943	1682	https://grouplens.org/datasets/movielens/
Movielens-1M	1000209	6040	3900	https://grouplens.org/datasets/movielens/

Table 4
Comparative experiments between the SS model and the UISS model in Movielens-100K.

No.	Movie-ID	UISS		SS		$\bar{\delta}$
		$(1-\alpha^*, \alpha^*)$	$Q(N, S^*)$	$(1-\alpha, \alpha)$	$Q(N, S)$	
1	1	(0.10,0.90)	**0.04** $\pm$ **0.01**	(0.27,0.73)	23.63 $\pm$ 0.03	0.769 $\pm$ 0.01
2	50	(0.10,0.90)	**0.01** $\pm$ **0.01**	(0.29,0.71)	16.00 $\pm$ 0.01	0.746 $\pm$ 0.01
3	100	(0.37,0.63)	**2.04** $\pm$ **0.01**	(0.31,0.69)	31.00 $\pm$ 0.01	0.811 $\pm$ 0.02
4	121	(0.44,0.56)	**0.80** $\pm$ **0.01**	(0.45,0.55)	18.99 $\pm$ 0.02	0.719 $\pm$ 0.01
5	151	(0.33,0.67)	**0.09** $\pm$ **0.01**	(0.35,0.65)	1.00 $\pm$ 0.02	0.718 $\pm$ 0.01
6	174	(0.38,0.62)	**1.02** $\pm$ **0.01**	(0.27,0.73)	13.58 $\pm$ 0.01	0.691 $\pm$ 0.01
7	181	(0.28,0.72)	**0.41** $\pm$ **0.02**	(0.37,0.63)	24.25 $\pm$ 0.03	0.756 $\pm$ 0.02
8	258	(0.33,0.67)	**4.00** $\pm$ **0.01**	(0.47,0.53)	41.59 $\pm$ 0.01	0.817 $\pm$ 0.01
9	286	(0.30,0.70)	**0.09** $\pm$ **0.01**	(0.27,0.73	16.31 $\pm$ 0.02	0.747 $\pm$ 0.01
10	405	(0.33,0.67)	**2.12** $\pm$ **0.01**	(0.43,0.57)	13.49 $\pm$ 0.02	0.705 $\pm$ 0.01

Table 5
Comparative experiments between the SS model and the UISS model in Movielens-1M.

No.	Movie-ID	UISS		SS		$\bar{\delta}$
		$(1-\alpha^*, \alpha^*)$	$Q(N, S^*)$	$(1-\alpha, \alpha)$	$Q(N, S)$	
1	1	(0.42,0.58)	**2.59** $\pm$ **0.01**	(0.45,0.55)	20.87 $\pm$ 0.02	0.753 $\pm$ 0.01
2	260	(0.32,0.68)	**1.97** $\pm$ **0.01**	(0.33,0.67)	19.28 $\pm$ 0.04	0.754 $\pm$ 0.01
3	480	(0.43,0.57)	**5.00** $\pm$ **0.01**	(0.45,0.55)	27.68 $\pm$ 0.02	0.771 $\pm$ 0.01
4	608	(0.16,0.84)	**0.59** $\pm$ **0.01**	(0.31,0.69)	21.57 $\pm$ 0.03	0.746 $\pm$ 0.01
5	858	(0.33,0.67)	**0.73** $\pm$ **0.01**	(0.37,0.63)	22.40 $\pm$ 0.03	0.760 $\pm$ 0.01
6	1196	(0.32,0.68)	**1.95** $\pm$ **0.01**	(0.39,0.61)	22.45 $\pm$ 0.04	0.748 $\pm$ 0.01
7	1270	(0.27,0.73)	**1.40** $\pm$ **0.01**	(0.37,0.63)	18.94 $\pm$ 0.01	0.739 $\pm$ 0.01
8	2028	(0.31,0.69)	**0.55** $\pm$ **0.01**	(0.39,0.61)	23.53 $\pm$ 0.02	0.780 $\pm$ 0.01
9	2858	(0.49,0.51)	**4.20** $\pm$ **0.01**	(0.41,0.59)	12.80 $\pm$ 0.03	0.810 $\pm$ 0.01
10	1210	(0.38,0.62)	**2.38** $\pm$ **0.01**	(0.33,0.67)	21.23 $\pm$ 0.04	0.759 $\pm$ 0.01

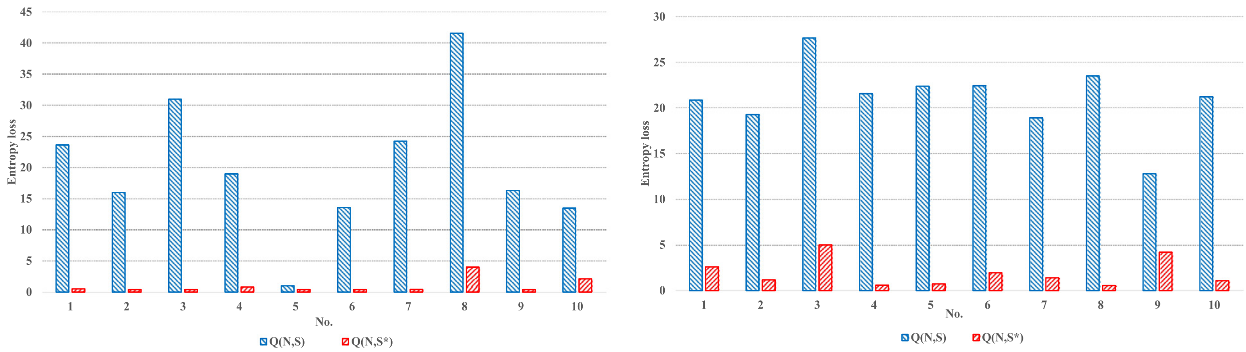
4.1. Validity and rationality of the UISS model

According to Algorithm 2, the validity and rationality of the proposed UISS model can be verified in terms of entropy loss by comparing it with the shadowed set (SS) model. The experimental results on the *Movielens-100K* and *Movielens-1M* datasets are presented in Table 4 and Table 5, respectively. $Q(N, S)$ denotes the discrepancy in the entropies corresponding to the neighborhood rough set model and its SS counterpart, and $Q(N, S^*)$ denotes the discrepancy in entropies corresponding to the neighborhood rough set model and the corresponding UISS model. Thus, Q is inversely proportional to the quality of the model. Two pairs of thresholds, $(1-\alpha, \alpha)$ and $(1-\alpha^*, \alpha^*)$, were calculated under the SS model and the UISS model, respectively. The neighborhood radius, $\bar{\delta}$, obtained via Algorithm 1 was used to construct the neighborhoods of active users belonging to the testing set.

As is evident from Fig. 6, $Q(N, S^*) < Q(N, S)$ in all instances. Theoretical analysis attributes this result to two primary factors. First, the objective function of the UISS model is established by following the principle of uncertainty invariance [47]. Second, according to Definition 10, the uncertainty of the UISS model in the elevated and reduced areas does not completely disappear when the maximum membership function is not 1 or the minimum is not 0. Therefore, the entropy loss under the UISS model is much smaller than that under the SS model. The experimental results corroborate that the UISS model is more reasonable and effective in terms of entropy loss.

4.2. Recommendation quality of 3WR-UISS model

The validity and rationality of the 3WR-UISS model are verified by comparing its performance with those of two classical binary recommendations: item-item collaborative filtering (Item-Item CF) and user-item collaborative filtering (User-Item CF) [14,18,25], and two state-of-the-art three-way recommendations: three-way recommendation based on game-theoretic rough sets (3WR-GTRS) [4] and three-way recommendation based on random forest (3WD-RF) [53]. Euclidean distance is selected as the similarity measure in 3WR-UISS, User-Item CF, and Item-Item CF. As in [53], the optimal thresholds of 3WD-RF set as $(\alpha, \beta) = (0.5, 0.6)$.



(a) Movielens-100K. (b) Movielens-1M

Fig. 6. The comparison between entropy loss of the SS model and the UISS mode.

Table 6
Decision matrix of three-way recommendation.

Decision rules	Real labels	
	Like (P)	Dislike (N)
Recommendation (R)	n_{RP}	n_{RN}
Deferment-recommendation (B)	n_{BP}	n_{BN}
Not-recommendation (N)	n_{NP}	n_{NN}

As recorded in Table 6, a three-way recommender has three available actions: recommendation (R), deferment of recommendation (B), and non-recommendation (N) and two real labels: like (P) and dislike (N) [31]. The cardinalities of available combinations of real labels and actions in their corresponding status are denoted by $n_{RP} = POS(X_1) \cap X_1$, $n_{RN} = POS(X_1) \cap X_1^c$, $n_{BP} = BND(X_1) \cap X_1$, $n_{BN} = BND(X_1) \cap X_1^c$, $n_{NP} = NEG(X_1) \cap X_1$ and $n_{NN} = NEG(X_1) \cap X_1^c$, respectively. $POS(X_1)$, $BND(X_1)$, $NEG(X_1)$, and X_1 can be obtained by Algorithm 3. X_1^c is the set complement of X_1, i.e., $X_1^c = X_2$, as in Algorithm 3.

Compared to binary recommenders, three-way recommenders improve recommendation performance by categorizing uncertain objects into the boundary region. From the perspective of performance, it sacrifices coverage to improve accuracy. Therefore, to evaluate the performance of 3WR-UISS more reasonably in comparison experiments, five metrics: *Precision, Recall, F-measure (F), Accuracy, and Generality* are used. *Precision, Recall*, and *F* [18] are the commonly used metrics for binary recommenders, and these three metrics are also suitable for three-way recommenders. However, *Accuracy and Generality* [4] are more suitable to evaluate three-way recommenders because boundary objects are considered by these two metrics. The definitions of the five metrics are as follows:

$$Precison = \frac{n_{RP}}{n_{RP} + n_{RN}}, \tag{19}$$

$$Recall = \frac{n_{RP}}{n_{RP} + n_{NP}}, \tag{20}$$

$$F = \frac{2 \cdot Precison \cdot Recall}{Precison + Recall}, \tag{21}$$

$$Accuracy = \frac{n_{RP} + n_{NN}}{Card(U_{test}) - n_{BP} - n_{BN}}, \tag{22}$$

$$Generality = \frac{Card(U_{test}) - n_{BP} - n_{BN}}{Card(U_{test})}, \tag{23}$$

where $Card(U_{test}) = n_{RP} + n_{RP} + n_{NP} + n_{NN} + n_{BP} + n_{BN}$. Obviously, for binary recommenders, $Generality$ is always equal to 1.

The *Precision*, *Recall*, and *F* corresponding to the results obtained for the test datasets of Movielens-100K and *Movielens-1M* via the five models are depicted in Tables 7–8. In most cases, the *Recall* of the 3WR-UISS model is observed to be superior to that of the other models. In terms of *Precision*, 3WR-UISS is observed to be slightly weaker than Item-Item CF and User-Item CF. However, the *Recall* of Item-Item CF and User-Item CF are lower than that of the other models.

Evaluation of recommendation performances based on solely *Precision* or *Recall* is flawed. *F* serves as a better metric because it is the harmonic mean of *Precision* and *Recall*. As is evident from Tables 7–8, in most cases, *F* of the 3WR-UISS model is superior to that of other models. In addition, the average of *F* and the average of *Recall* of the 3WR-UISS model on the two datasets are superior to those of other models. Therefore, the 3WR-UISS model can be considered to be most suitable in scenarios where both *Precision* and *Recall* are considered simultaneously.

Table 7
The *Precision*, *Recall*, and *F* of test sets among five models in Movielens-100K.

No.	Movie-ID	3WR-UISS			Item-Item CF			User-Item CF			3WR-GTRS			3WR-RF		
		Precision	Recall	F	Precision	Recall	F	Precision	Recall	F	Precision	Recall	F	Precision	Recall	F
1	1	0.61±0.03	0.45±0.02	**0.52±0.03**	**0.78±0.01**	0.07±0.01	0.12±0.01	0.77±0.01	0.12±0.03	0.21±0.03	0.54±0.02	**0.47±0.01**	0.50±0.03	0.70±0.03	0.40±0.03	0.51±0.01
2	50	0.88±0.03	**0.82±0.03**	**0.85±0.03**	**0.96±0.02**	0.05±0.01	0.09±0.03	0.92±0.02	0.34±0.01	0.50±0.02	0.82±0.03	0.69±0.03	0.75±0.03	0.86±0.03	0.65±0.03	0.74±0.03
3	100	0.55±0.01	**0.57±0.02**	**0.56±0.02**	0.66±0.02	0.05±0.01	0.09±0.02	**0.69±0.01**	0.19±0.01	0.30±0.01	0.52±0.02	0.39±0.02	0.45±0.02	0.67±0.03	0.34±0.02	0.45±0.02
4	151	0.43±0.03	0.34±0.04	0.38±0.04	0.48±0.02	0.08±0.02	0.14±0.02	0.61±0.02	0.04±0.02	0.08±0.02	**0.68±0.02**	**0.49±0.03**	**0.57±0.03**	0.60±0.02	0.21±0.02	0.31±0.02
5	121	0.56±0.01	**0.44±0.02**	**0.49±0.02**	**0.70±0.01**	0.08±0.01	0.15±0.01	0.69±0.01	0.08±0.01	0.14±0.01	0.62±0.01	0.34±0.02	0.44±0.02	0.06±0.01	0.32±0.01	0.42±0.01
6	174	0.89±0.03	**0.70±0.04**	**0.78±0.04**	**0.98±0.01**	0.07±0.01	0.13±0.01	1.00±0.02	0.12±0.02	0.22±0.02	0.63±0.03	0.56±0.03	0.59±0.03	0.5±0.03	0.23±0.03	0.32±0.03
7	181	0.71±0.01	**0.70±0.02**	**0.70±0.02**	**0.85±0.01**	0.06±0.01	0.11±0.01	0.88±0.01	0.19±0.01	0.31±0.01	0.78±0.01	0.49±0.02	0.60 ±0.02	0.73±0.01	0.43±0.01	0.54±0.01
8	258	0.53±0.01	**0.39±0.01**	**0.45±0.01**	**0.82±0.01**	0.08±0.01	0.15±0.03	0.63±0.02	0.18±0.03	0.28±0.01	0.50±0.01	0.27±0.02	0.36±0.02	0.25±0.01	0.29±0.02	0.27±0.02
9	286	0.36±0.04	0.31±0.03	0.33±0.04	0.27±0.02	0.04±0.02	0.07±0.02	0.34±0.02	0.08±0.02	0.13±0.02	0.45±0.02	**0.42±0.03**	**0.43±0.03**	**0.56±0.02**	0.35±0.03	**0.43±0.03**
10	405	0.65±0.04	**0.30±0.03**	**0.41±0.04**	0.76±0.01	0.15±0.02	0.24±0.02	0.82±0.02	0.06±0.01	0.11±0.02	0.60±0.03	0.41±0.02	0.50±0.03	**0.83±0.02**	0.34±0.03	0.48±0.03
Average	–	0.62	**0.50**	**0.55**	**0.73**	0.07	0.13	**0.73**	0.14	0.23	0.61	0.45	0.52	0.63	0.36	0.45

Table 8
The *Precision*, *Recall*, and *F* of test sets among five models in Movielens-1M.

No.	Movie-ID	3WR-UISS			Item-Item CF			User-Item CF			3WR-GTRS			3WR-RF		
		Precision	Recall	F	Precision	Recall	F	Precision	Recall	F	Precision	Recall	F	Precision	Recall	F
1	1	0.59±0.03	**0.40±0.01**	**0.48 ±0.02**	0.69±0.01	0.24±0.01	0.35±0.01	**0.78±0.01**	0.13±0.03	0.23±0.03	0.55±0.01	0.23±0.02	0.33±0.02	0.55±0.02	0.09±0.01	0.16±0.02
2	260	0.76±0.03	0.60±0.03	0.67±0.03	**0.94±0.02**	0.17±0.02	0.28±0.02	0.90±0.02	0.30±0.02	0.45±0.02	0.8±0.02	**0.68±0.03**	**0.73±0.03**	0.74±0.02	0.24±0.03	0.37±0.03
3	480	0.62 ±0.04	**0.57±0.03**	**0.60±0.04**	0.63 ±0.02	0.18±0.02	0.27±0.02	0.63±0.02	0.20±0.03	0.30±0.04	0.55±0.02	0.41±0.02	0.47±0.02	**0.68±0.02**	0.20±0.03	0.31±0.03
4	608	0.67±0.01	**0.54±0.02**	**0.60±0.02**	**0.75±0.01**	0.19±0.01	0.31±0.01	**0.75±0.01**	0.21±0.01	0.35±0.01	0.59±0.01	0.41±0.03	0.48±0.02	0.64±0.02	0.14±0.02	0.23±0.02
5	858	0.67±0.03	**0.43±0.04**	**0.52±0.04**	0.75±0.03	0.18±0.03	0.29±0.03	**0.76±0.03**	0.19±0.03	0.30±0.03	0.56±0.03	0.37±0.032	0.44±0.03	0.64±0.02	0.14±0.03	0.23±0.03
6	1196	0.77±0.03	**0.59±0.02**	**0.67±0.03**	**0.90±0.03**	0.18±0.01	0.30±0.02	0.87±0.02	0.25±0.02	0.39±0.02	0.82±0.02	0.67±0.03	0.74±0.03	0.85±0.02	0.47±0.02	0.61±0.02
7	1270	0.63±0.04	**0.56±0.04**	**0.60±0.04**	0.81±0.02	0.21±0.03	0.34±0.03	**0.84±0.04**	0.22±0.03	0.35±0.03	0.65±0.04	0.42±0.03	0.51±0.04	0.73±0.04	0.39±0.03	0.51±0.04
8	2028	0.62±0.03	**0.54±0.02**	**0.58±0.03**	0.78±0.01	0.18±0.01	0.29±0.01	0.76±0.01	0.22±0.01	0.35±0.01	0.61±0.01	0.38±0.02	0.47±0.02	0.6±0.01	0.28±0.02	0.38±0.02
9	2858	0.62±0.03	**0.58±0.03**	**0.60±0.03**	**0.73±0.03**	0.13±0.03	0.22±0.03	0.69±0.03	0.32±0.02	0.44±0.02	0.58±0.04	0.5±0.03	0.53±0.03	0.62±0.03	0.42±0.04	0.5±0.04
10	1210	0.60±0.04	**0.56±0.04**	0.58±0.04	0.75±0.01	0.18±0.02	0.29±0.02	0.77±0.02	0.22±0.01	0.34±0.02	0.72±0.03	0.61±0.03	0.66±0.03	**0.82±0.03**	0.55±0.02	**0.66±0.03**
Average	–	0.66	**0.54**	**0.59**	**0.77**	0.18	0.29	**0.77**	0.23	0.35	0.64	0.47	0.54	0.69	0.29	0.40

Table 9
The *Accuracy* and *Generality* of test set among three-way recommender models in Movielens-100K.

No.	Movie-ID	3WR-UISS		3WR-GTRS		3WR-RF	
		Accuracy	Generality	Accuracy	Generality	Accuracy	Generality
1	1	**0.72±0.03**	0.83±0.03	0.60±0.03	0.71±0.03	0.55±0.03	**0.86±0.03**
2	50	**0.81±0.04**	**0.93±0.03**	0.74±0.03	0.75±0.03	0.66±0.04	0.80±0.04
3	100	0.50±0.03	**0.78±0.03**	**0.52±0.03**	0.66±0.03	0.45±0.03	0.77±0.03
4	121	**0.79±0.04**	**0.86±0.04**	0.75±0.04	0.79±0.04	0.74±0.04	0.84±0.04
5	151	**0.81±0.03**	0.81±0.03	0.72±0.02	0.74±0.02	0.73±0.03	**0.91±0.03**
6	174	0.62±0.03	**0.91±0.03**	**0.65±0.03**	0.82±0.03	**0.65±0.03**	**0.91±0.03**
7	181	0.81±0.04	**0.88±0.04**	**0.82±0.03**	0.75±0.04	0.65±0.04	0.81±0.04
8	258	**0.58±0.03**	0.68±0.03	0.34±0.03	0.46±0.03	0.50±0.03	**0.80±0.03**
9	286	0.56±0.04	0.65±0.04	0.44±0.03	0.69±0.04	**0.57±0.02**	**0.81±0.02**
10	405	**0.82±0.02**	0.79±0.03	0.65±0.03	0.65±0.02	0.78±0.03	**0.82±0.03**
Average	–	**0.70**	0.81	0.62	0.70	0.63	**0.83**

Table 10
The *Accuracy* and *Generality* of test set among three-way recommender models in Movielens-1M.

No.	Movie-ID	3WR-UISS		3WR-GTRS		3WR-RF	
		Accuracy	Generality	Accuracy	Generality	Accuracy	Generality
1	1	**0.79±0.04**	**0.91±0.03**	0.70±0.03	0.78±0.04	0.69±0.04	**0.91±0.04**
2	260	**0.82±0.04**	0.81±0.04	0.76±0.04	0.88±0.03	0.62±0.03	**0.83±0.03**
3	480	**0.80±0.03**	**0.93±0.04**	0.71±0.04	0.79±0.02	0.66±0.02	0.87±0.04
4	608	**0.83±0.04**	0.60±0.03	0.64±0.04	0.67±0.04	0.66±0.04	**0.89±0.04**
5	858	**0.82±0.04**	0.79±0.04	0.62±0.03	0.69±0.03	0.69±0.04	**0.88±0.03**
6	1196	**0.83±0.03**	0.79±0.04	0.78±0.04	0.81±0.04	0.70±0.03	**0.92±0.04**
7	1270	**0.82±0.04**	0.76±0.03	0.69±0.04	0.75±0.02	0.69±0.04	**0.89±0.04**
8	2028	**0.81±0.03**	0.81±0.04	0.59±0.03	0.65±0.04	0.68±0.04	**0.89±0.02**
9	2825	**0.79±0.04**	**0.97±0.02**	0.58±0.04	0.9±0.04	0.57±0.02	0.81±0.04
10	1210	**0.82±0.04**	0.86±0.04	0.76±0.04	0.86±0.04	0.73±0.04	**0.93±0.03**
Average	–	**0.81**	0.82	0.68	0.78	0.67	**0.88**

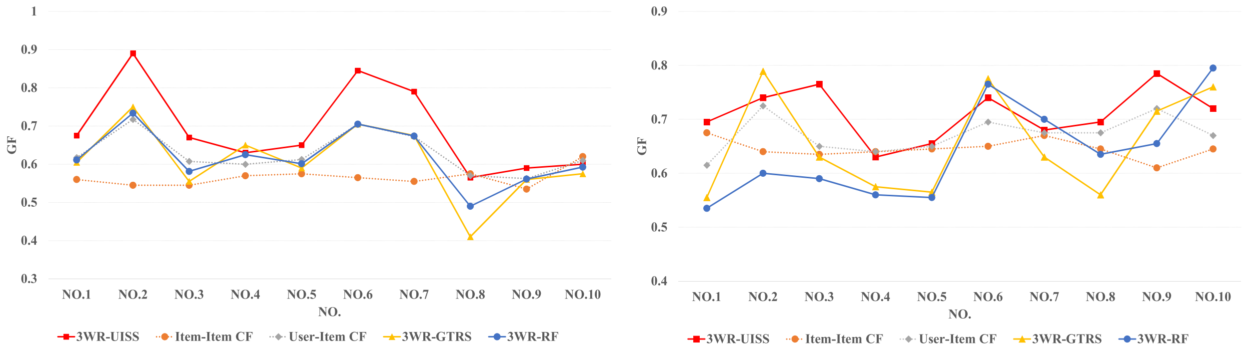
In the cases of Item-Item CF and User-Item CF, *Generality* is always equal to 1 because no objects are categorized into boundary regions. In addition, because *Accuracy* is calculated following the elimination of boundary objects from the testset, it not appropriate to compare a binary recommender with a three-way recommender model. Therefore, these two metrics are only used to verify the performance of three-way recommenders.

The experimentally obtained values of *Accuracy* and *Generality* for *Movielens-100K* and *Movielens-1M* via three three-way recommenders are presented in Table 9 and Table 10, respectively. It is evident that the 3WR-UISS model is more robust for *Movielens-1M* than for *Movielens-100K* because the *Accuracy* and *Generality* in all cases fluctuates within a narrow range. In other words, in the case of a relatively large number of users, it is beneficial to improve the recommendation performance, because similar users of active users are more consistent with their preferences. In addition, the average *Accuracy* of 3WR-UISS is superior to that of other models, while its average *Generality* is slightly lower than that of WR-RF and more than that of 3WR-GTRS. Therefore, the 3WR-UISS model is effective overall. Compared to 3WR-RF, 3WR-UISS is based on a data-driven non-parametric model. In other words, the thresholds of the three-way recommender can be altered to account for changes in the dataset. Compared to 3WR-GTRS, 3WR-UISS can be applied to a wider range of fields because of its ability to process continuous datasets.

Compared to *Precision* and *Recall*, F is a more comprehensive metric to evaluate the performance of the model. It is evident from the experimental results that three-way recommenders sacrifice *Generality* to improve F. To account for the trade-off between binary recommenders and three-way recommenders in terms of performance and coverage more comprehensively, the average of *Generality* and F, $GF = \frac{Generality+F}{2}$, is calculated in this study. A more intuitive visualization of GF for *Movielens-100K* and *Movielens-1M* is depicted in Fig. 7. Overall, the GF of 3WR-UISS is observed to be superior to that of binary recommenders (represented by two dotted lines). In addition, to capture the superiority of the comprehensive performance of 3WR-UISS over those of the other three-way recommenders effectively, the average of *Accuracy*, *Generality*, and F, $GFA = \frac{Accuracy+Generality+F}{3}$, is calculated for the two datasets, as depicted in Fig. 8. The results suggest that the GFA of 3WR-UISS is superior to the other two three-way recommenders in most cases, and that 3WR-UISS is more robust owing to the small range of variation of GFA, especially for *Movielens-1M*.

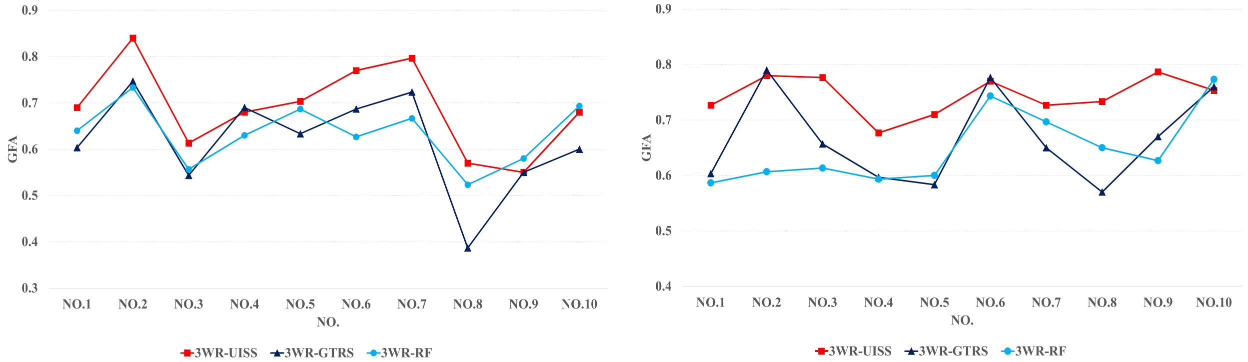
5. Conclusions

RSs have been extensively researched in recent years. In this paper, the 3WR-UISS model is proposed to improve the recommendation quality corresponding to uncertain users. It overcomes the shortcomings of traditional recommendation algorithms arising from the prediction of ratings by incorporating a neighborhood rough set model into a three-way recommendation system, which aids the identification of users similar to active users. In particular, the proposed UISS model



(a) Movielens-100K. (b) Movielens-1M.

Fig. 7. GF comparison among five models. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)



(a) Movielens-100K. (b) Movielens-1M.

Fig. 8. GFA comparison among three-way recommenders.

divides all users into positive, negative, and boundary regions based on their neighborhood memberships. The threshold parameters of the UISS model are optimized following the principle of uncertainty invariance, and the optimization method maintains a balance between the increment and decrement in uncertainty, besides minimizing entropy loss. This enables RSs to adopt different recommendation-making strategies corresponding to users lying in different regions. Experimental results presented in the study verify the effectiveness of the UISS model in terms of entropy loss. Finally, the UISS mode is utilized to implement a three-way recommendation system. The recommendation results suggest that the proposed 3WR-UISS model is more suitable than available alternatives in scenarios where both *Precision* and *Recall* are considered simultaneously.

We intend to explore the following issues in our future works. We wish to provide a novel interpretation of the objective function and expand the scope of applications of the shadowed set model. In addition, we intend to investigate models that can improve the recommendation quality while reducing time complexity.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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