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Three-way decision and granular computing

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ABSTRACT

Based on results from cognitive science, this paper examines the two fields of three-way decision and granular computing, as well as their interplay. The ideas from one field shed new light on the other field. The integration of the two gives rise to three-way granular computing, that is, thinking, problem solving, and information processing in threes. We discuss a wide sense of three-way decision and propose a trisecting–acting–outcome (TAO) model. We explain fundamental notions of granular computing based on the philosophy of three-way decision as thinking in threes. We discuss a model of three-way granular computing by making use of two particular types of granular structures represented, respectively, by three granules and three levels. We use examples across different disciplines to demonstrate the values of the two types. Our investigation suggests that, in many situations, the power of granular computing is indeed the power of three-way decision, i.e., thinking in threes.

1. Introduction

A recently proposed theory of three-way decision (3WD) [74,76,79] promotes thinking, problem solving, and information processing in threes, that is, using three parts, three elements, three components, three perspectives, three views, three levels, three generations, three periods, three stages, three steps, triangles, triads, triplets, and many others. A narrow sense of three-way decision [43,79] was introduced for interpreting the three types of classification rules in rough set theory [49]. In order to apply three-way decision to a wide spectrum of domains, we need to consider a wide sense [74], in which "decision" may be viewed as computing, processing, analysis, and so on. We can take either a literal sense of "three," as being taken in the present study, or a figurative sense of "three" as "a few" or "about three." It is this wide sense that increases the power of three-way decision.

In explaining a philosophy of thinking in threes, Minsky says,[1] "I find most people say, well it's either this or that and I'm always inclined to look for a third thing. ... if somebody says, is it left or right, I'm always looking for a third way. ... of course, most of the time you can't find one, but every now and then I get a new theory ... ." A useful lesson from Minsky is trying to think in three whenever you have two, although it may fail sometimes. This is just one of many examples to be examined to demonstrate the needs for and the benefits of three-way decision as thinking in threes.

In the context of cognitive science, in this paper we study the interplay of three-way decision and granular computing as two specific human-inspired paradigms of cognitive computing. We discuss a wide sense of three-way decision and propose a trisecting–acting–outcome (TAO) model. We explain, in the light of three-way decision, the basic questions, notions and concepts of granular computing, leading to a conception of granular computing in threes or three-way granular computing.

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[1] https://www.youtube.com/watch?v=_POW-geD5no.

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1.1. Three-way decision and granular computing in the cognitive era

Categorization and abstraction are basic cognitive tools for dealing with the complexity of the real world. We group ideas, objects, and events into categories, usually for a specific purpose, by recognizing their similarity, dissimilarity, or functionality. The ideas of categorization and abstraction serve as a cognitive basis for both three-way decision and granular computing. Three-way decision is about thinking, processing, and computing in threes. It is motivated by an observation that using three parts, components, and levels to understand a whole appears to be practiced universally in human problem solving. We can relate three parts of three-way decision to the ideas of categorization and abstraction, namely, three-way decision uses three categories of things. Granular computing concerns theory and practice of structured multilevel approaches, in which each level of granularity or abstraction is represented and interpreted through a family of granules. We can draw some correspondence between granules used in granular computing and categories used in cognition and human understanding. If we further draw a correspondence between three parts used in three-way decision and granules used in granular computing, we can unify the two fields of studies as special types of human-inspired paradigms of computing.

Kelly [34] puts forward a three-era evolution model of computing, consisting of the tabulating era (1900s–1940s), the programming era (1950s–onward), and the cognitive era (2011–). The cognitive era is marked by a shift of attention towards human-machine cooperation, integration, and inter-creation. Studying human ways to problem solving, human-inspired algorithms and systems, and human understandable systems is of significant importance in the cognitive era. As exemplified by Kelly's three-era classification, granular thinking in general and three-way thinking in specific are everyday common human practice in complex problem solving. The view of three-way decision and granular computing as two special models of cognitive computing is the basis of the investigation of the present paper.

1.2. Three basic questions of granular structures

Granular computing explores effective use of granules and granular structures constructed from granules. Granules, having a similar interpretation as categories, are clusters, groups, and sets of ideas and objects. Although basic ideas and principles of granular computing have appeared frequently across many disciplines, a study of granular computing as a new scientific field of investigation [7,20,50,52,53,64,71,78] is much influenced by artificial intelligence [26,77], fuzzy sets [84], and rough sets [49,81].

The triarchic theory of granular computing consists of three main components: philosophy of structured thinking, methodology of structured problem solving, and mechanism of structured information processing [73]. A key underlying notion is a granular structure characterized by granules, levels, and a hierarchy [80]. Granules are units, focal points, and vocabularies for representing and interpreting a problem. An essential feature of granules is their granularity or degree of abstraction, which may be intuitively interpreted in terms of the size of granules. A level of granularity or abstraction is formed by granules of similar nature or granularity. Levels are (partially) ordered according to their granularity, resulting in a hierarchy of multiple levels.

When constructing and using granular structures, one may ask at least the following three fundamental questions:

(1) Are the granules, levels, and hierarchy meaningful for a particular problem?
(2) What are the most appropriate number of levels and the number of granules?
(3) What is the most effective level of granularity?

Some attempts have been made to address these questions. The principle of justifiable granularity, proposed by Pedrycz [51], represents an important initiative in answering question (1). Several authors discuss question (2) by drawing attention to Miller's [47] influential work on limited information processing capacity of humans, which is about 7 ± 2 units of information. Yao [80] considers its implications to constructing granular structures. Chen and Du [13] discuss its uses in economic decision making in an interdisciplinary context. One of the conclusions is that it is effective and sufficient to use only a few granules or a few levels. In many data analysis models, different subsets of attributes or features induce possibly distinct granular views of the same data set. Regarding question (3), attribute reduction in rough set theory [49] and feature selection in pattern recognition and machine learning may be viewed as a search for an appropriate level of granularity.

1.3. Main results and the organization of this paper

A study of three-way decision and granular computing, in the context of results from cognitive science, is by itself an example of thinking in threes. Fig. 1 is a conceptual map of the investigation of this paper. We discuss pertinent ideas, notions and concepts for each of the three fields, as indicated by the rectangular boxes at the three corners of the triangle. The numbers inside the brackets give the sections or subsections in which materials about the respective field are covered. For examples, Sections 2, 3 and 5 contain materials about three-way decisions. We study pair-wise relationships of the three fields. We examine, in particular, three types of relations: a) one field provides a basis of another field as indicated by solid lines labeled by "support," b) one field provides a special model for another field as indicated by dotted lines labeled by "special model," and c) methods and tools in one field can be used to explain ideas of another field as indicated by

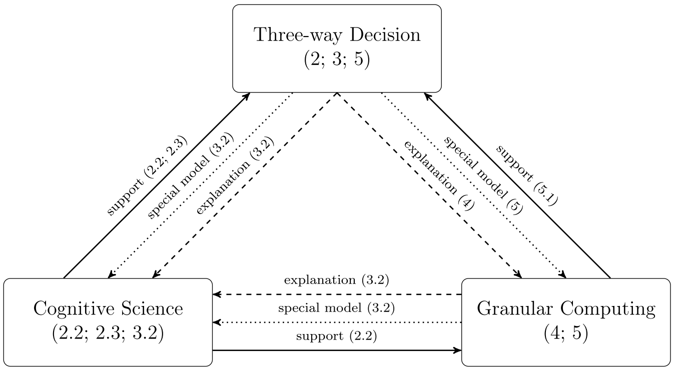

Fig. 1. Relations among cognitive science, three-way decision, and granular computing.

dashed lines labeled by "explanation." The numbers inside the brackets following a label are the sections or subsections in which the particular relation is discussed. For example, cognitive science provides a basis for three-way decision, which is discussed in subsections 2.2 and 2.3. The philosophy and methodology of three-way decision, as thinking in threes, can be used to study granular computing, which is discussed in Section 4.

By using the structure of Fig. 1 as a blueprint, we demonstrate that answers, or at least partial answers, to the three basic questions of granular structures may be found in a theory of three-way decision [74]. To achieve our goal, we organize the rest of the paper into three parts as follows.

Part I consists of Sections 2 and 3. In Section 2, we present a wide sense of three-way decision. The magical number three underlies three-way decision and may hold the key to realize the power of granular computing. We examine three narrow senses of three-way decision. We look at the cognitive basis of thinking in threes, as well as its optimality. In Section 3, we present a trisecting–acting–outcome (TAO) model of three-way decision. We use Parr's captivology [48] as an example to illustrate the working of the TAO model.

Part II consists of Section 4. We explain many notions of granular computing under the light of three-way decision. Three-way decision corresponds to granular computing in threes. We explain granular computing from the three perspectives of philosophy, methodology, and mechanism. We explain granular structures in terms of the three components of granules, levels, and hierarchies. We consider the three modes of top-down, bottom-up, and middle-out computing.

Part III consists of Section 5. We use "granular computing in threes" or "three-way granular thinking" to label a special class of models resulted from an integration of three-way decision and granular computing. It is a model of thinking in threes, problem solving in threes, and computing in threes. In many situations, using three parts, three granules, or three levels plays a crucial role in a problem-solving process. We examine two types of granular structures that make use of threes. One type consists of structures with three levels and multiple granules at each level, which is called the 3LmG type. The other type consists of structures with multiple levels and three granules at each level, which is called the mL3G type. We use examples across many disciplines to demonstrate the value of the two types of structures for thinking, problem solving, and information processing in threes.

2. A wide sense of three-way decision

Three-way decision is thinking in threes and computing in threes, where three is the number of parts, elements, and components under consideration. We examine both a narrow sense and a wide sense of three-way decision, present three specific senses of three-way decision as thinking in threes, and discuss the cognitive basis of thinking in threes.

2.1. Three-way decision as thinking in threes

In rough set theory [49], a set is approximated by three pair-wise disjoint sets called the positive, boundary, and negative regions. The notion of three-way decision (3WD) is introduced to interpret the three types of decision rules obtained from the three regions in a probabilistic rough set model [79], namely, acceptance, non-commitment, and rejection rules. The earlier research mainly focused on this narrow sense of three-way decision [43]. Most recent studies have led to a wide sense of three-way decision [74]. The wide sense of three-way decision is built on a philosophy of thinking in threes. To have a full understanding of the wide sense, we have at least the following three related but distinct specific senses.

A third. We interpret three-way decision as trichotomous thinking, which introduces a third option to dichotomous thinking with two options. With three-way decision, we move from true/false, black/white, and yes/no to true/unsure/false, black/grey/white, and yes/maybe/no. The third option provides the necessary flexibility and universality of thinking in threes.

Trisection. We interpret three-way decision in terms of the notions of parts and whole, as given in a trisecting-and-acting (T&A) framework of three-way decision [74]. The basic ideas are (a) to divide a whole into three parts, and (b) to devise the most effective strategies to act upon the three parts. Dividing a whole into three parts reduces the complexity of the whole and leads to simplicity.

Triad. We interpret three-way decision with reference to a common practice of using a triad or triplet consisting of three elements or three components. A triad offers sufficient complexity in terms of individuals and relationships between them and, at the same time, is simple enough for memorization and processing.

All three specific senses use triplets of three options, parts, or elements. They differ with respect to the semantics of integrative triplets. An example of the first specific sense is three-valued logic in which a third value is used to denote a value different from both true and false. An example of the second specific sense is to divide tax payers into low, middle, and high income classes and devise corresponding strategies for each class. An example of the third specific sense is the use of three words, three phrases, and three sentences in writing and speech. In many cases, the three specific senses may be independent, may introduce each other, and may appear together. For example, a trisection may introduce a third option and a triad of three parts. A triad may consider three options without a clear reference to the process from which the triad is derived. The possible combinations of the three specific senses offer a desirable flexibility of three-way decision. Our interpretations of three-way decision, in terms of three specific senses of "a third," "trisection," and "triad," are indeed an example of thinking in threes. In other words, we divide the whole of three-way decision into three parts represented by the three specific senses.

In a set-theoretic setting, a trisection of a universal set consists of three pair-wise disjoint subsets such that their union is the universal set. There are two other equivalent representations of a trisection. One representation uses a pair of nested sets, such as the pair of lower and upper approximations in rough sets [49] and the pair of lower and upper bounds in interval sets [72]. The lower approximation/bound forms one set, the difference between the upper approximation/bound and the lower approximation/bound forms another set, and the complement of the upper approximation/bound forms the third set. The other representation uses a pair of disjoint subsets called an orthopair, which was proposed and systematically studied by Ciucci [15,16]. The third subset in the trisection can be obtained by the set complement of the union of the two subsets in the orthopair.

The wide sense of three-way decision offers a new understanding. By focusing on "three-way" as the use of threes, we may replace "decision" by other words to introduce new types of three-way approaches, such as three-way thinking, three-way computing, three-way processing, three-way classification [10,37,83,87,88], three-way analysis, three-way clustering [1,82], three-way recommendation [5,85], three-way decision support [67,70], three-way concept analysis [57,59,63,72], three-way concept learning [28,38], temporal and spacial three-way decision [42], three-way attribute reduction [45,86], and many more [27,39,40,44]. Studies up to date indicate that a theory of three-way decision in the wide sense is urgently needed, comes at the right time, and will rise in the future.

2.2. Cognitive basis of thinking in threes

One plausible justification for using threes in three-way decision is our limited capacity of short-term working memory, a fundamental concept in cognitive science and cognitive psychology. We can only process a small number of units of information in short-term memory. The use of the magical number three in three-way decision is consistent with and supported by results from cognitive science.

In 1956, Miller [47] reported an important finding on the capacity of human information processing in short-term memory. We can normally hold 7 ± 2 units of information in short-term memory for immediate recall and processing. A clever solution to this limitation is a technique known as chunking. Through encoding, a large amount of information is chunked into a manageable number of units or chunks. Simon [62] suggested the magical number five as the number of chunks, instead of Miller's magical number seven. Later studies, for example, Cowan [18] and Gobet and Clarkson [24], indicated that the actual number of chunks is smaller than seven, is about or below four, and is three or two. Existing studies show that there does not exist a general agreement on the exact number of chunks. The disagreements may be explained by the fact that the actual magical number, if exists at all, depends largely on the interpretation of chunks and the size of chunks. But we do not have a common agreed understanding of the latter. Nevertheless, there is a general agreement: we can only manage a small number of chunks and this number can be qualitatively described as "a few," which is usually less than seven.

Warfield [69] presented a very interesting argument that relates the number three and the number seven. The argument considers the interaction between chunks. Let A, B, and C denote three chunks. If we consider them as individuals, we have three chunks A, B, and C. If we consider the pair-wise relationships, we have three pairs (A, B), (A, C), and (B, C). Finally, if we consider all three chunks together, we have one triplet (A, B, C). The total number is $3 + 3 + 1 = 7$, which is exactly magical number seven. By following the same argument, if we consider two chunks A and B, we have two individuals and one pair (A, B). It gives rise to a total of 3. That is, we obtain the magical number three from two chunks. If we use four chunks, we would have four individuals, six pairs, four triplets, and one quadruple. The total becomes $4 + 6 + 4 + 1 = 15$, which is far more beyond the capacity as suggested by the magical number seven. An important implication of Warfield's argument is that when relationships between chunks are also considered, the magical number is three, which is the one used in three-way decision and three-way computing.

2.3. Economy of the base three system

An additional piece of evidence in support of thinking and processing in threes is the economy of the base three system and ternary numbers [21,25,55].

Given a number $(a_{n-1}\dots a_0)_b$ of base b in a positional notation, its decimal value is given by:

$$a_{n-1}b^{n-1} + \cdots + a_0 b^0, \tag{1}$$

where $0 \le a_i \le b-1$, $i = 0, 1, \dots, n-1$. The cost of a system of a particular base is determined by two factors. The base b is the depth or radix of a number, which is the number of different symbols in each digit position. The number of digits in a number is the width of the number. As suggested in [21], bn represents "a fair estimate of the number of tubes required in the system." If n digits are used, the number of numbers expressible is b^n, that is, 0, 1, ..., $b^n - 1$. By holding b^n as a constant, that is, the number of numbers to be expressed is a constant, one can minimize bn to find the most economical base b. If one treats both b and n as continuous variables, the problem can be easily solved. The optimal value is e, the base of the natural logarithm [21,25], that is, $b = e = 2.71828\dots$. Hayes [25] shows that for integer values of b and n, 3, the integer closest to e, is "almost always the most economical integer radix."

As an interesting application of ternary numbers, Hayes suggests the use of third-cut file folders, i.e., folders in which "the tabs appear in three positions, *left*, *middle*, and *right*," as an alternative to half-cut folders. With third-cut folders, you can easily insert a new folder that differs from its two neighbors. Furthermore, base three is the smallest base having this property. The third-cut folders are a simple example to demonstrate the power of thinking in threes.

Phythian [55] presents another view on the economy of number bases for the purpose of reading and understanding numbers, instead of representing and implementing numbers by physical devices. With respect to the depth b, we need b symbols, which has the same interpretation as before. With respect to the width n, we need $n-1$ names for $b^1, \dots, b^{n-1}$, which is different from the earlier interpretation. For example, in the decimal system, we have ten for 10^1, hundred for 10^2, thousand for 10^3, and so on. It follows that for a number of n digits, we need a total of $b + (n-1)$ symbols/names, assuming that the different names are used for $b^1, \dots, b^{n-1}$ (as we will see later, this may not be true in general). By holding b^n as a constant, we can minimize $b + (n-1)$ to find the most economical base. By treating b and n as continuous variables, one can find the most economical base by solving the equation $b(\log_e b)^2 = \log_e N$, where N is the number to be expressed. In this case, the optimal base depends on the value of N.

According to Phythian's argument, with integer bases and integer numbers, base ten is one of the most economical bases in the sense that it only requires 34 symbols for numbers from 0 to 10^{24}. For numbers from 0 to 10^{12}, base seven, eight, and nine are the most economical bases that require 21 symbols, while base 10 requires 22 symbols. The results seem to suggest that if we only use numbers up to 10^{12}, which is actually the case in reality, we perhaps should reduce our base. However, the actual practice is very different: we use decimal system rather than base seven system. This seemingly inconsistency needs a closer examination.

In defining the economical index $b + (n-1)$ of base b system with n digits, Phythian assumes that distinct names are required for all $n-1$ positions, leading to a conclusion of base reduction. As an alternative solution, we may choose to name some positions rather than all positions. This is, in fact, the solution offered by a clever way of counting by threes, in which the magical number three plays an important role. We name positions at two levels. At a lower level, we have one for 10^0 (but does not need a new name), ten for 10^1, hundred for 10^2 as the three basic names. At a higher level, we have one for 10^0, thousand for 10^3, million for 10^6, and billion for 10^9, trillion for 10^{12}, and so on with an increment of three in the power of ten. To explicitly express counting by threes, we insert a comma to group digits in a number into groups of threes. For example, 1200000 is written as 1,200,000. This immediately provides a two-level understanding of the magnitude of numbers. The high level, indicated by commas, is given by ones, thousands, millions, and so on. For a specific higher level magnitude, we can refine it into three refined magnitude by using ones, tens, and hundreds to read the three numbers for that magnitude. For example, at the magnitude of millions, we have millions, tens of millions, and hundreds of millions. In this way, 1,200,000 is one million and two hundred thousands. Through counting by threes, for numbers from 0 to 10^{12}, we actually use 16 names (in English, if we also count the irregularity caused by eleven, twelve, -teen, -ty, the number is 20), which is less than 21 names required by the base seven system. That is, the decimal system is actually the most economical system with the technique of counting by threes. This is another example to illustrate the power of thinking in threes.

As an interesting application of Phythian's model, we can justify the use of base three system when a small number of numbers is under consideration. An ancient use of base three system was introduced by Chinese scholar Yang Xiong (53 BCE-18 CE) for coding 81 numbers in Tai Xuan Jing (The Canon of Supreme Mystery). Based on the formulation given by Phythian, for 81 numbers, base three, base four, and base five systems are all the most economical systems that require seven names. Similar to the earlier example of third-cut folders given by Hayes [25], base three is the smallest base for coding 81 numbers most economically from an understanding point of view. This example demonstrates again the power of thinking in threes.

2.4. Further discussions

If we accept the hypothesis that the capacity of short-term memory, measured in chunks independent of the materials for forming a chunk, is three, we should find many examples of thinking in threes in real life. This is indeed the case.

Thinking in threes is a frequently occurring theme in human knowing, understanding and problem solving [9,36,74], for example, three parts of a whole, three components of a system or a theory, three perspectives/angles of an issue, three levels of understanding, etc. Languages, as a basic cognitive tool for thinking, understanding, and communication, provide an excellent example to illustrate that three is a magical number in thinking. Take English as an example. We have three degrees for adjectives and adverbs, that is, positive, comparative, and superlative. We use three basic tenses of verbs, that is, past, present, and future tenses. For representing points of view, we may use any one of three choices, namely, the first person, the second person, and the third person. Furthermore, we frequently use triplets, that is, three words, three phrases, and three sentences, in writing and speech. In general, approaches that use threes appear across many fields and disciplines, including computer science, information science, statistical science, management science, engineering, social science, medical decision-making, and many more. Our limited information processing capacity dictates us to think in threes. Thinking and processing in threes are a kind of our second nature and have been widely practiced in everyday life.

The two interpretations of the economy of numerical bases shed light on the use of the magical number three. The product of the depth and width, bn, is the cost of representing numbers with physical devices. The sum of the depth and width, $b + (n - 1)$, is the cognitive cost in understanding and reading numbers. Although the magical number three appears in both interpretations, it might not be the case in general. In developing a theory of three-way decision, we focus more on the cognitive perspectives. In other words, we explore the economy of three-way decision to reduce cognitive overload. The method of counting by three in reading decimal numbers is an excellent example to illustrate our position. By grouping digits in a number into groups of threes, it is much easier for us to read, interpret, and use decimal numbers. The method does not reduce the number of physical devices in representing a number, which requires the use of base three. On the other hand, using base three system will lead to a cognitive difficulty, as we do not have names for 3^1, 3^2, ... Therefore, thinking in threes with decimal numbers is a much preferred choice.

The wide sense of three-way decision as thinking in threes has a solid cognitive basis and is supported by a vast amount of evidence. We may formulate models of three-way decision based on everyday uses of three parts, three elements, three components, three perspectives, three views, three levels, three generations, three periods, three stages, three steps, triangles, triads, triplets and many others. Thinking in threes avoids the oversimplification of thinking in ones and twos and, at the same time, does not suffer from the complexity of thinking in fours or more. In developing a theory of three-way decision, the magical number is three [74].

3. A TAO model of three-way decision

According to the principle of thinking in threes, we propose a trisecting–acting–outcome (TAO) model by adding a third component to the original trisecting-and-acting model [74]. We use Parr's captivology [48] to explain the working of the TAO model.

3.1. The model

A central idea of three-way decision is thinking in threes. To model the specific "trisection" sense of three-way decision, we consider three related questions:

(1) How to construct a trisection from a whole?
(2) How to devise strategies to process the three parts of the trisection?
(3) How to evaluate the effectiveness of the trisection and strategies?

An earlier trisecting-and-acting model [74] only covers the first two questions. To give a more complete picture, we add a third component about the evaluation. This leads to a trisecting–acting–outcome (TAO) model of three-way decision with three components. Existing studies on three-way decision concentrate on trisecting and acting. Only a few papers touch upon the issue of outcome evaluation [23,31].

Fig. 2 presents a high-level description of the TAO model with three components of trisecting, acting, and outcome evaluation. The function of trisecting, denoted by solid lines, is to divide a whole into three related and relatively independent parts. The resulting three parts are called a trisection of the whole. The function of acting, denoted by dashed lines, is to apply a set of strategies to process the three parts. By trisecting a whole and acting on the resulting trisection, it would produce an expected outcome. The function of outcome evaluation is to measure the effectiveness of the results from a combined effort of trisecting and acting as enclosed by the large dashed rectangular box in the figure. The TAO model provides an architectural framework of three-way decision. When applying the model to a particular application, one needs to use semantically and physically meaningful trisections, profitable actions, and informative measures of effectiveness.

There are many ways to divide a whole into three parts and many strategies to act upon the three parts. We may have strategies for a particular part, strategies for two parts, and strategies for all three parts, respectively. The effectiveness of three-way decision depends on a proper match of trisecting methods and strategies for actions. Several modes of combination are possible. The simplest mode assumes the independence of trisections and actions, as was done in earlier studies of three-way decision. This is usually done by considering the trisecting methods and strategies separately. In Fig. 2, we give two trisection and action interdependent modes. A trisection-driven mode requires that actions and strategies are

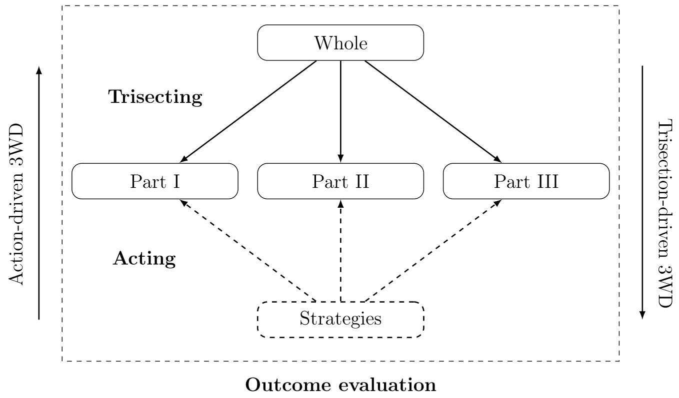

Fig. 2. TAO model of three-way decision.

built based on results of trisections. In other words, a trisecting method guides the search for effective strategies and their applications. Conversely, an action-driven mode starts with a set of strategies and a suitable trisection is sought so that these strategies can be effectively applied. In other words, strategies determine how to trisect a whole. In many situations, a combined mode of trisection-driven and action-driven may be preferred. It can be a co-supportive iterative process in the sense that a trisection provides hints for devising better strategies and a set of strategies may further improve trisecting. The iterative circle continues until we arrive at a right combination of a trisection and a set of strategies.

3.2. An illustration of the model

The high-level description of the TAO model of three-way decision enables us to explain many commonly used three-way approaches in human problem solving. We use a simple practical example to explain the working of the model.

In studying human memory, a three-way classification scheme is typically used, that is, we have three types of memory known as sensory memory, short-term memory, and long-term memory [4,56]. According to Plotnik and Kouyoumdjian [56], sensory memory is a process of receiving and holding environmental raw information for only a brief period of time, short-term memory is a process of holding a limited amount of information for only a short period of time, and long-term memory is the process of holding large amounts of information over long periods of time. Each type of memory serves for specific functions, sensory memory for recording, short-term memory for working, and long-term memory for storing. Furthermore, selective attention transfers sensory memory to short-term memory and encoding transfers short-term memory to long-term memory. To a large extent, human problem solving is constrained by the three types of memory. Corresponding to the three types of memory, there are different cognitive strategies and tools.

Studying human memory in three types is an example of thinking in threes. The three-way classification is meaningful and is supported by ample empirical evidence. The division of the three types gives a clear and simple understanding of memory in terms of both time periods and the associated functions. It might be possible to build another model that has more than three types. The model would be unnecessarily complicated in the light of the hypothesis of a limited capacity of short-term memory. Given a trisection of memory, an important task is to examine strategies that make effective use of the three types of memory.

In developing a theory and practice of captivology, Parr [48] considers three types of attention, namely, immediate attention, short attention, and long attention. We can easily establish a correspondence between the three types of memory and three types of attention. Two essential tasks of captivology are capturing the three types of attention and facilitating the transfer of immediate attention to short attention and short attention to long attention. In the light of the TAO model of three-way decision, captivology focuses on action and outcome according to the trisection of attention.

Based on features and functions of the three types of attention, Parr examines a set of strategies and gives real world examples to demonstrate their effectiveness. As examples for illustration, we briefly mention some of these strategies. Immediate attention is an automatic response, working in a similar way as sensory memory. To capture immediate attention, one may explore strategies of creating contrast and association. In other words, one's immediate attention will be drawn by something that is very different from its surroundings. Building associations is a basic subconscious cognitive activity and associations lead to attention. To transfer immediate attention to short and long attention and to retain long attention, many different strategies may be used. For example, people pay attention to things that fit their frame. It is effective to adapt to the audiences' frame of reference to capture their attention. It is also possible to explore the scarcity effects, bizarreness effects, and mystery effects, as all of them are closely related to gaining attention. Using rewards, including, extrinsic and surprising rewards, as well as acknowledgments, proves to be effective attention grasping strategies. Finally, in order to have long attention, it is crucial to create values and to build a reputation. People will remember things or people by their values and reputation.

Many of the strategies given by Parr [48] are applicable to two or three types of attention, rather than one particular type. Some strategies facilitate moves from one type to another type. This is related to a specific model of three-way decision known as actionable three-way decision [23].

4. Granular computing explained in threes

By applying the principles of thinking in threes, we explain the basic concepts and notions in granular computing in threes, namely, the three elements of granular structures, the triarchic theory of granular computing, and the three modes of granular information processing.

4.1. Three elements of granular structures: granules, levels, and hierarchies

Granular computing concerns structured approaches to thinking, problem solving, and information processing [7,8,20,22,26,52,53,64,68,71,73,77,78]. The power of granular computing is the power of useful structures. Granular structures should reflect three important aspects of structuredness. They are called the three M's of granular structures, namely, multilevel, multiview, and multipurpose. Granular structures provide a multilevel understanding from a particular view, offer a multiview vision covering various angles, and have multiple functions to serve different purposes. While the first two M's have been extensively discussed [78], the third M is introduced in this paper to emphasize the role and effectiveness of granular structures.

We explain granular structures based on three main ingredients: granules, levels, and hierarchies [73,77,80,81].

Granules. In granular computing, granules may be interpreted as the units, elements, concepts, and notions that we use for representing and describing a complex system or a complex problem as a whole. For example, in systems theory we may interpret granules as the parts of a whole [11]. In human information processing, information granules correspond to the notion of information chunks. That is, a chunk is a granule representing a meaningful whole that binds together individual pieces of information. In text processing, granules at various levels are words, sentences, paragraphs, and articles, or alternatively, titles, section headings, and subsection headings. In different contexts, we use different languages and vocabularies to define and explain granules. There are two important characteristics of granules. One is the granularity of granules, which may be considered as the size of granules. The other is the dual roles played by granules. A granule may be a whole consisting of many granules as its parts. At the same time, the same granule may serve as one part of another granule. According to these two features, we can order granules according to their granularity and we can also connect granules to form a web of granules. This enables us to construct very general granular structures in a graph-theoretic setting. In the web of granules, regions of the web give rise to another type of granules.

Levels. Although a web of granules has the desired generality and expressive power, in many practical situations we only use relatively simple granular structures. A hierarchical granular structure consists of multiple levels. Each level is represented and characterized by a family of granules of similar nature and similar granularity. The levels are ordered according to the granularity of granules involved. A higher level controls and determines its next lower level. We add more detailed information and more concrete ideas as we move top-down. A lower level supports its next upper level. We remove particular information and abstract the essential ideas as we move bottom-up. With a hierarchy, we have a multilevel understanding, which also leads naturally to multilevel processing.

Hierarchies. A hierarchy is a representation of reality from a particular angle, rather than the reality itself. An advantage, and at the same time a disadvantage, of a representation is that it makes certain aspects explicit and clear at the expense of pushing some other aspects to the background [46]. Furthermore, one representation may only serve the purpose of its intended applications and will not serve the purpose of other applications. To compensate for the shortcomings of a particular hierarchy and to serve multiple purposes, we use many hierarchies for a multiview vision.

A granule at each level provides a partial, fragmentary understanding. A family of granules forms a level and provides a full, complete understanding at a particular level of granularity. A family of levels forms a hierarchy and provides a multilevel understanding from a specific viewpoint. Finally, a family of hierarchies provides a multiview understanding from many viewpoints. That is, granular computing is both a multilevel and a multiview approach. While multilevel accounts for different levels of abstraction and details, multiview avoids the potential pitfalls caused by bias of a single view.

The first two M's of granular structures, i.e., multilevel and multiview, state the desired structural properties, as captured by the notions of granules, levels, and hierarchies. The third M of granular structures, i.e., multipurpose, states the desired functional property. As a matter of fact, the first two M's naturally offer the third M, that is, the multiplicity of levels and views leads to the multiplicity of functions to serve many purposes.

Consider first the issue of supporting multipurpose through multilevel. Each level in a multilevel hierarchy represents a particular abstraction and description. In general, we may use different vocabularies and languages to describe different levels [80]. In this way, the same thing is described and represented in multiple ways. As a result, we can ask the right questions at the right level by using the right languages and for the right purposes.

The point may be illustrated by using software system development as an example. Normally, a software system is developed in multiple stages in which a stage corresponds to a particular level of description. At the initial stage, the requirements and high-level specifications may be given by using a natural language, possibly aided by figures and tables.

We ask the question of what is the system. The main purpose is to specify the functions of the system. Once we are satisfied with specifications, we may move to a stage of designing a system architecture. We ask the question of what are the components of the system and their functions. The purpose is to provide a blueprint for building the system. At a final stage, we focus on the implementation by using a programming language. We ask the question of how to implement the functions of the various components. The main purpose is to build an operational system. Although such a three-stage or three-level description of software system development is an oversimplification of a complex process involving many stages or levels, it is sufficient for the purpose of illustrating our point. The idea of dividing a complex process into multiple stages or levels for serving multiple purposes is our main concern, and the actual number of stages or levels is less crucial and less relevant.

When explaining how language works, Crystal [19] considers a five-level hierarchy for syntactic investigation, consisting of sentences, clauses, phrases, words, and morphemes. Rules for constructing the five types of elements of a language are not entirely the same. As a result, it may be necessary to have different methods for representation, description, and investigation. The purposes of investigations at different levels are different and we ask different questions at the different levels. The example again supports the argument that multilevel leads naturally to multipurpose.

Similarly, we can consider the issue of supporting multipurpose through multiview. This can be easily done by drawing a correspondence between levels and views. Different views describe the same thing from different angles and possibly by different languages. A particular view typically makes a certain aspect become more apparent for a specific purpose and pushes other aspects into the background. To illustrate this point, we use a multiple hierarchy model of social stratification proposed by Jeffries and Ransford [30]. They consider social inequality hierarchies formed, respectively, by class, ethnicity, gender, and age. While each hierarchy serves the purpose of enabling us to identify one type of social inequality, it fails to identify other types. Moreover, they argue that a study of social stratification should not focus on isolated and fragmented views, but holistic and unified views by using multiple hierarchies. In other words, while each hierarchy may serve one purpose, an integration of many hierarchies may serve multiple purposes. We can easily find many more examples to support the argument that multiview leads naturally to multipurpose [14].

4.2. Triarchic theory of granular computing: philosophy, methodology, and mechanism

A motivation for introducing the triarchic theory is a plea for a holistic study of granular computing in response to an imbalanced research effort [73]. Existing studies have been focused on machine-oriented and process-based approaches to granular computing. We concentrate on specific models and methods by making use of information granules. We pay less attention to the philosophical foundations and the methodological questions of granular computing. The triarchic theory unifies the three aspects of granular computing: philosophy, methodology, and mechanism. In this way, granular computing is both human-oriented and machine-oriented [78]. Results from studies on the philosophy and methodology aspects empower us with strategies, principles, methods, and cognitive tools, so that we become better at solving problems. Results from studies on the mechanism aspect will play a valuable role in the design and implementation of intelligent systems [73,78].

Philosophy of structured thinking. Granular computing views the world through a lens of structures. We represent, interpret, and study a complex system or a complex problem with the aid of a web of granules called a granular structure. By taking a structured view of the world, we turn complexity into simplicity. The philosophy of granular computing is granular thinking. It is about thinking structurally by making use of granules, levels, and hierarchies to form useful structures. Granular computing adopts ideas from other philosophical thinkings, including reductionist thinking, systems thinking, and levelist thinking. Granular computing also combines analytical thinking and synthetical thinking.

Methodology of structured problem solving. Granular computing is about using granular structures in the process of problem solving, in which granular structures can be either explicitly built or implicitly embodied in the process. Methodology of granular computing concerns about general principles and application-independent ways to problem solving. For example, an ordering of levels according to their granularity suggests several approaches, such as top-down, bottom-up, and middle-out methods. These approaches have proved to be effective in computer programming. The underlying principles can be equally applicable to other types of problem solving, for example, structured scientific investigation and structured writing.

Mechanism of structured information processing. Both the philosophy and methodology guide us designing structured approaches to information processing in machines. Studies on mechanism turn attention to architectures of granular computing for structured information processing. In other words, we explore concrete ways to implement structured information processing. To make effective use of granular structures, we may consider graph-theoretic representations of architectures of granular computing. In a graphic model, a graph itself gives a topological architecture, in which nodes and components of a graph represent granules and edges represent relationships among granules.

In many situations, the three aspects are interwoven together and they are integrated into a comprehensive whole. Their separation is for the purpose of understanding, in line with a general principle of three-way decision. The three components of the triarchic theory mutually support each other and any one of them is indispensable. The triarchic theory studies both the separation and integration of the three aspects.

4.3. Three modes of multilevel computing: top-down, bottom-up, and middle-out

A hierarchy consists of multiple levels of differing granularity. The ordering of levels according to their granularity represents a coarsening-refinement relation, which may be interpreted in terms of control-support, generalization-specialization, and abstraction-concretization. According to the ordering, we have at least three modes of multilevel information processing [73].

Top-down mode. Top-down approaches are analytic thinking and may be considered as conceptually driven methods of information processing [41]. We work from higher levels of granularity downwards to lower levels of granularity. As we move towards lower levels, we gradually add more details. This makes an abstract understanding more concrete. The correctness of a higher level ensures the correctness of subsequent lower levels. Thus, for a top-down approach to work, we must have a global view and a conceptual understanding of the whole problem.

Bottom-up mode. Bottom-up approaches are synthetical thinking and may be interpreted as data-driven methods [41]. We work from lower levels of granularity upwards to higher levels of granularity. By extracting the most common and useful features, we build a higher level abstraction supported by its lower details. Details at a lower level normally suggest many possible abstractions at a higher level. For a bottom-up approach to work, we must construct a best abstraction at each level based on the evidence from its lower level.

Middle-out mode. Middle-out approaches combine both top-down and bottom-up thinkings. We start at a level that we have sufficient information and a good understanding. This level may be viewed as a basic level that provides goals for guiding top-down investigations and evidence for supporting bottom-up explorations. Based on an understanding and goals formed at the basic level, we move downwards to develop more details. At the same time, we move upwards for further abstraction.

Top-down approaches require a global understanding for guiding successive divisions. However, we may not have a global view unless we know some details of its components. That is, an understanding of a whole requires some understanding of its parts. On the other hand, bottom-up approaches may suffer from a lack of a clearly stated overall goal for successive combination. That is, an understanding of parts requires some understanding of the whole. Middle-out approaches are based on a partial understanding of the whole and the parts. For effective problem solving, any one of the three modes may not be sufficient. We may use different modes at different stages of a problem solving process [2,35,61]. In a bottom-up manner, we can form some goals based on available information. Based on the formed goals, we can have further top-down investigations. At the same time, we can take middle-out approaches to help us to attain a better understanding at a particular level. It is expected that we may use the three modes iteratively and alternatively.

5. Three-way granular thinking

Granular computing is granular thinking at multiple levels of granularity and with multiple granules at each level. By applying the principles of three-way decision to granular thinking, we have a conception of granular computing in threes. This specific type of granular computing in threes or three-way granular computing uses three levels or three granules. Three granules and three levels may be further interpreted based on other notions that have a more concrete physical meaning, as explained in the last section.

5.1. The use of threes

Granules may be viewed as the basic units of information and knowledge, vocabulary of discussion, or focal points of investigation. Depending on particular domains and applications, we have different ways to interpret granules. For example, granules may represent types, classes, elements, components, stages, periods, and many others. To support the use of three levels and three granules for modeling granular computing, we first collect some statistical data about the use of different numbers.

Table 1 summarizes the results of Google searches that we did on February 4, 2018 by using exact phrases of the form "n g of", where n takes values given in the first column of the table and g takes values given in the first row of the table. Each cell is the number of results reported by Google. For example, the number reported for a search of the exact phrase "three levels of" is 11,800,000. Each column of the table, except for the first one, gives a frequency distribution with respect to numbers from two to seven. For example, the second column of the table, labeled by "levels," summarizes the frequencies of

Table 1
Statistics on the use of numbers two to seven.

	levels	types	classes	elements	components	stages	periods
two	12,300,000	54,700,000	9,550,000	7,090,000	9,570,000	4,970,000	3,320,000
three	11,800,000	27,000,000	4,940,000	4,500,000	5,710,000	6,970,000	1,660,000
four	2,370,000	9,480,000	2,130,000	1,310,000	2,100,000	3,220,000	233,000
five	920,000	3,170,000	493,000	636,000	418,000	596,000	94,500
six	575,000	739,000	264,000	444,000	247,000	388,000	77,600
seven	514,000	456,000	117,000	199,000	66,000	619,000	45,800

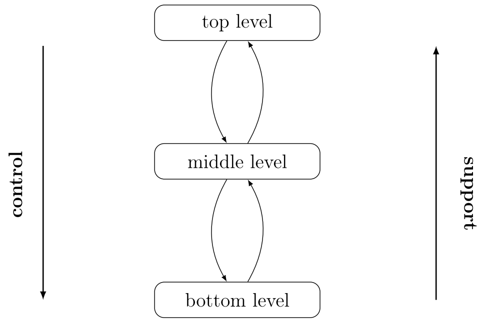

Fig. 3. 3LmG structure: Three levels and multiple granules in each level.

the use of different numbers of levels. Levels of granularity are a fundamental concept to describe granular structures [80]. In general, levels may represent layers, extends, or degrees of something. The next six columns represent three different scenarios: "types" and "classes" for classification and categorization, "elements" and "components" for three parts of a whole, and "stages" and "periods" for progressive development and evolution processes. The four representative categories were chosen so that both three ordered and unordered things are considered, which provides a reasonable coverage of things in reality. From Table 1, we have the following three observations and plausible explanations.

Dominance of two, three and four. In all cases, two, three, and four play dominant roles, while five, six, and seven play limited roles. These results are consistent with findings regarding the capacity of short-term memory. For the purpose of cognitive simplicity, it is effective to consider only a small number of entities or things. In the light of Table 1, the number is 3 ± 1 with 3 as the middle point.

Fast decreasing speed. Except for "stages," thinking in twos seems to be the most common theme, followed by thinking in threes and thinking in fours. Except for "levels" and "stages," thinking in threes is about half of thinking in twos, and thinking in fours is less than half of thinking in threes. There is a large drop when moving from four to five. Let us consider only the first two levels in the framework given by Warfield [69], that is, given n things we only count the numbers of individuals and pairs. We have $3 + 3 = 6$ for three, $4 + 6 = 10$ for four, and $5 + 10 = 15$ for five. They suggest that, if pair-wise connections are considered, five things may be too many to process effectively.

"Stages" as an outlier. The number three is the magical number for "stages." When it comes to "stages," a two-stage thinking may be too simple and cannot provide a good explanation of real phenomena. A three-stage thinking may bring additional insights. It is also interesting to note that seven-stage thinking is often used. A plausible explanation is that there exist some well-accepted three-stage and seven-stage theories and models. For example, a Google search of "three stages of" produces top-ranked topics of law of three stages, three stages of inflammation, and three stages of marriage. A Google search of "seven stages of" produces top-ranked topics of seven stages of grief, seven stages of a man's life, and seven stages of action.

Although our statistical data is relatively small, it still provides some important implications. Compared with thinking in twos, thinking in threes is not as popular and natural. This observation motivates a turn of attention to thinking in threes in order to avoid problems caused by polarized and dualistic thinking [54]. If we think in threes as naturally as we think in twos, we will be more effective in solving many problems. In some situations, although thinking in twos is a preferred choice, we are forced to think in threes in order to cope with uncertainty or a lack of information. When we are confident and have sufficient information, we choose to think in twos, otherwise, we think in threes as an approximation. The main idea of sequential three-way decision is to construct a sequence of approximations in the search of a final two-way decision [37,58,75].

One of the goals of three-way granular computing is to articulate thinking in three levels and thinking in three granules. Thinking in threes is widely practiced, which suggests that granular computing in threes is a promising direction of research.

5.2. Thinking in three levels

Three-level, three-layer, and three-tier structures may provide a simple and yet comprehensive description of a complicated issue or a complex system. Thinking in three levels divides a whole into three levels and explores a natural ordering of the three levels. At each level, we may use a family of granules. A granular structure with three levels and multiple granules at each level is called a 3LmG structure, which is depicted in Fig. 3. With a 3LmG structure, it is assumed that granules at each level are relatively independent and the three levels are interdependent. Under these assumptions, a large number of granules at each level may not increase the complexity, as we can consider the granules individually. The focus of attention will be on the relationships of levels.

In Fig. 3, a top-down reading of the three levels shows a relation of control and a bottom-up reading shows a relation of support. The top level and the bottom level are indirectly related by the middle level, rather than directly related. In this way, we have three individual levels and four transitions between levels (i.e., two between the top level and the middle level, and another two between the middle level and the bottom level). There are a total of seven items to be considered (i.e., three levels plus four transitions). However, in most cases we do not need to consider all seven items at the same time. In fact, for a particular transition, we only need to consider two adjacent levels and the transition itself, which is a total of three items. Each level has its own specific focus and purpose, and may be represented and studied by means of many granules of a similar nature or granularity. It is normally required that granules in each level are relatively independent or at least nearly independent.

One example of 3LmG structures is the three levels of government of Canada, consisting of the federal level, the provincial level, and the municipal level. Another example is a three-stratum model of cognitive abilities proposed by Carroll [12], which consists of the Stratum I of narrow specific abilities, Stratum II of broad abilities, and Stratum III of general abilities. We briefly describe three additional three-level models that are related to computing.

Marr's three-level understanding of information processing systems. Marr [46] argues that a full understanding of an information processing system requires understanding at multiple levels. We explore different kinds of explanations by using different levels of descriptions. Marr suggests in particular a framework based on three levels:

1. The computational theory level deals with computation in the abstract. Computation is understood as mappings between different kinds of information. Their abstract properties, appropriateness, and adequacy are some of the main concerns.
2. The representation and algorithm level addresses the problem of implementing computation through algorithmic processes, based on a proper representation of information and associated manipulating processes.
3. The hardware implementation level is about the realization of representation and algorithms by using physical devices.

A series of explanations, from abstract to representation and algorithms, and to physical devices, enables us to have a full understanding of an information system in its fullest sense.

Three-level database architecture. ANSI-SPARC architecture of database management systems consists of three levels, namely, the external level, conceptual level, and internal level [29]. The three levels address three different types of questions and issues:

1. The external view describes the data from the user's perspective, that is, a user view of the data.
2. The conceptual view describes how the data is represented and processed conceptually, as well as how the data is inter-related, that is, a logic view of the data.
3. The internal level focuses on how the data is physically stored by using the computer hardware, that is, a physical view of the data.

The separation of the three levels allows data independence so that changes made at one level do not require extensive changes at other levels.

Weaver's three levels of communications problems. In presenting a general framework for communications, Weaver [60] suggests to ask serially questions at three levels:

1. Level A of the technical problem: How accurately can the symbols of communication be transmitted?
2. Level B of the semantic problem: How precisely do the transmitted symbols convey the desired meaning?
3. Level C of the effectiveness problem: How effectively does the received meaning affect conduct in the desired way?

The three levels concern questions of progressive difficulty. Level A focuses on the engineering aspects of communication, which is beautifully addressed by Shannon's mathematical theory of communication [60], also known as Shannon's information theory. The other two levels involve the semantics and the success in realizing the desired value of the transmitted information, which have not been satisfactorily solved. An insight from Weaver's three-level model is dividing a hard problem into simpler problems and solving the simplest problem first.

As demonstrated by the three examples, there are several advantages of thinking in multiple levels. First, a totally ordered sequence of levels provides a natural and simple way for a progressive investigation, from simple and concrete levels to complex and abstract levels. In a top-down manner, we can gradually develop and concretize ideas. In a bottom-up manner, we can abstract essential ideas by removing unnecessary details. Second, a separation of levels allows us to focus on different types of issues at different levels. This normally leads to clarification and simplification, so that the right questions are asked at the right level, as well as avoiding distractions and confusions from asking the wrong questions. Third, a sequence of levels offers efficiency in processing. At each level, we typically consider a subset of all problems and questions, which reduces the complexity. Furthermore, the ordering of levels reflects the dependencies of problems and questions, which leads to structured approaches and solutions.

As a special case of multilevel thinking, thinking in three levels avoids an over-simplification of thinking in two levels and, at the same time, does not suffer from a complexity of thinking in four or more levels. It is not surprising to observe examples of thinking in three levels across many different disciplines. In philosophy, we study ethics at three levels of meta-ethics, normative ethics, and applied ethics. In physics, we have three levels of micro-approaches, meso-approaches, and macro-approaches. In economics, we have microeconomics, mesoeconomics, and macroeconomics. In biology, we have three levels of biodiversity, that is, genetic diversity, species diversity, and ecosystem diversity. In management, we have

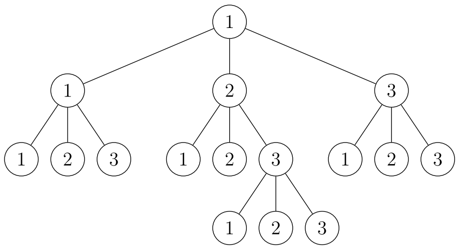

Fig. 4. mL3G structure: Multiple levels and three granules.

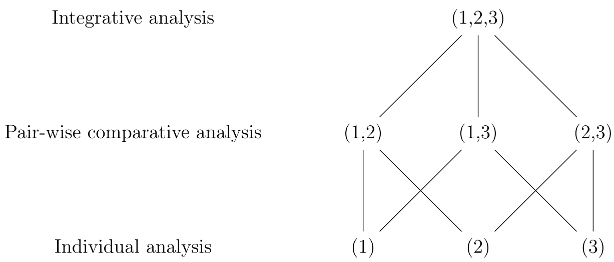

Fig. 5. Computing in three granules.

three levels of low-level management, middle-level management, and top-level management. These examples may be sufficient for illustrating the value and effectiveness of thinking in three levels and we can easily expand this list of examples.

5.3. Thinking in three granules

Dividing a whole into three components or three parts and looking at the whole in terms of three dimensions or three perspectives are another type of common practice and heuristic of human problem solving. We may consider each component, part, dimension, or perspective as a granule. Furthermore, we can divide a granule into three smaller granules and obtain a multilevel granular structure. Such a granular structure with multiple levels and three granules is called an mL3G structure, as depicted in Fig. 4. With an mL3G structure, it is assumed that three granules are interdependent, mutually support each other, and are individually indispensable. Moreover, granules at different levels are less dependent and can be investigated in separation. The focus of attention will be on the relationships of the three granules.

Following the argument of Warfield [69] discussed earlier, we give a framework of three-level analysis for granular thinking in three granules in Fig. 5. The bottom level of individual analysis focuses on one of the three granules without considering the other two granules. That is, we concentrate on problems that are locally related to one granule. The middle level of pair-wise comparative analysis examines one granule in relation to another granule. This normally involves a comparative study of two granules in a search for commonalities and differences. The pairs in the middle level correspond to the notion of orthopairs introduced by Ciucci [15,16]. Finally, the top integrative analysis examines three granules based on results from individual analysis and pair-wise comparison. Through three-level analysis, we may arrive at a full understanding of the whole.

The triarchic theory of granular computing, the three elements of granular structures (i.e., granules, levels, and hierarchies), and the three modes of computing (i.e., top-down, bottom-up, and middle-out), as discussed in Section 4, are examples of thinking in threes. We briefly mention three more examples.

Sternberg's triarchic theory of intelligence. Sternberg [65,66] develops a triarchic theory of human intelligence by considering three aspects of intelligence, namely, creative, analytical, and practical:

1. Creative abilities are related to the creation, invention, and generation of novel ideas or the design of new things.
2. Analytical abilities are related to analysis, comparison, and evaluation of ideas, in order to ascertain whether the ideas are good ones.
3. Practical abilities are related to the implementation of ideas, concerning the realization of the value of ideas through application and utilization.

It is interesting to note that Sternberg [65] describes the triarchic theory by using a tree-structure similar to Fig. 4. He divides the theory into three subtheories, consisting of an experiential subtheory, a componential subtheory, and a contextual subtheory. Each subtheory is further divided. For example, the componential subtheory is divided into a theory of fluid abilities and a theory of crystallized abilities. The theory of fluid abilities is again further divided into a theory of induction and a theory of deduction. The first level of Sternberg's structure is the same as given by Fig. 4. However, in subsequent levels, Sternberg uses two parts or one part. Using the terms of granular computing, this is equivalent to saying that a granule is not necessarily divided into exactly three granules.

Clayton's three components of influence. Clayton [17] gives three components of influence that determine the amount of one's influence:

1. The influencer: We play an essential role in the influencing process. Our confidence and presence are related to the effectiveness of our influence.
2. The message: How we craft a structured and compelling message is a prerequisite for achieving influence.
3. The ways of delivering: The way to deliver our message is equally important, if not more, as the message. We need to be able to negotiate, to persuade, and to take others' perspectives.

The trisection of influence is the first, and relatively the simpler, step. Similar to Parr's [48] theory of captivology, Clayton's [17] theory of influence is more about building strategies with respect to the three components of influence. In other words, the TAO model of three-way decision may also shed some light on the three-component based theory of influence.

Keidel's triadic pattern. In an attempt to organize and digest the vast amount of business literature, Keidel [33] puts forward a triangular framework by studying organizational strategies with respect to the following three types of relationships:

1. Disjunction (i.e., non-overlapping) and autonomy: In set-theoretic terms, Keidel's notion of disjunction is non-overlapping. That is, two parts are separated and independent, leading to autonomy of each.
2. Intersection (i.e., overlapping and non-containment) and cooperation: Keidel's notion of intersection corresponds to the case in which two sets have a non-empty overlap and anyone is not a subset of the other. The two parts are related and dependent, but not in a boss/subordinate relation. In this case, the cooperation of the two parts is needed.
3. Containment and control: The containment may be interpreted as a boss/subordinate relation, showing a controlling and following relationship.

Organizational strategies and designs can be explained in terms of tradeoff of three variables corresponding to autonomy, cooperation, and control. In particular, Keidel uses a triangle to describe mutual supports of the three, leading to an easy-to-understand and simple-to-use structure for organizing and understanding business literature and practice.

As demonstrated by these examples, thinking in threes turns complexity into simplicity. Similar to the case of thinking in three levels, we can also observe plenty of examples of thinking in three parts across different disciplines. In understanding Peirce's theory of signs, one may use a three-part basic sign structure consisting of the signifying element of signs, the object, and the interpretant [3]. Ball [6] considers nature's patterns as "a trilogy composed of Shapes, Flow, and Branches." Kagan [32] looks at the modern academy according to the three cultures of natural sciences, social sciences, and the humanities.

One may apply the three-level framework, as given by Fig. 5, to systematically study a whole through the analysis of the three parts and their interdependencies.

6. Concluding remarks

In this paper, we explain three-way decision and three-way granular computing as examples of thinking in threes. The use of threes is predetermined by our limited capacity of information processing, as constrained by short-term working memory. We present three different types of argument to demonstrate the needs for and the benefits of thinking in threes. First, we show mathematically that, under some assumptions, the number three is the optimal choice for reducing complexity in a search for a cognitive simplicity. The argument is based on the optimality of the base three system. In one sense, thinking in threes may be viewed as a natural law, although further investigation is needed for supporting such a strong claim. Second, we compile everyday uses of numbers by using the Google search engine. The results show that the number three plays a dominating role. In another sense, thinking in threes may be viewed as an useful empirical law. As future work, it is necessary to collect more statistical data in support of thinking in threes. Third, throughout the paper, we present many examples of thinking in threes, for example, Marr's [46] three-level view of information processing systems, Sternberg's [65] triarchic theory of human intelligence, Yao's [73] triarchic theory of granular computing, Parr's [48] theory of captivology, Clayton's [17] theory of influence, and others. In a third sense, thinking in threes is a type of practical cognitive tools to deal with the complexity of real world problems. Further research may be focused on collecting and analyzing these cognitive tools, in order to have a general theory of three-way decision as thinking in threes.

In a 2015 white paper entitled "Computing, cognition and the future of knowing," Kelly [34] suggests a three-era evolution model of computing: the tabulating era (1900s–1940s), the programming era (1950s-onward), and the cognitive era (2011–). The model by itself is another example of thinking in threes. To build theory and models of cognitive computing, as a prerequisite we must first understand fully human ways to think. To make computer systems effective, machines should explain their results in human understandable terms. Three-way decision, as a special class of human ways to think, is timely and fits well in the cognitive era of computing [74]. The discussions of this paper are aimed at drawing a reader's attention to three-way decision and three-way granular computing in the cognitive era of computing. The discussions are therefore presented at a more abstract and conceptual level by omitting many technical details. The paper provides a long-term and, perhaps personal, vision for future research. The value of thinking in threes must be fully explored for future human and machine problem solving.

By applying the principles of thinking in threes, we propose a TAO model of three-way decision. The TAO indicates three main components of three-way decision, that is, trisecting (T), acting (A), and outcome (O). It is crucial to combine the right trisection with the right actions to achieve the right outcome. Existing studies on three-way decision concentrate mainly on the task of trisecting. It is equally important to investigate the two tasks of acting and outcome evaluation.

We use the principles of thinking in threes to explain granular computing. According to the triarchic theory, we investigate granular computing from three aspects: philosophy of structured thinking, methodology of structured problem solving, and mechanism of structured information processing [73]. Granular structures result in structuredness in granular computing. We explain granular structures based on three main ingredients: granules, levels, and hierarchies. Levels in one hierarchy give a multilevel understanding. Many hierarchies offer a multiview understanding. The ordering of levels according to their granularity suggests three modes of information processing: top-down, bottom-up, and middle-out approaches. An integration of three types of approaches is useful for investigating the mechanism of granular computing.

Three important points that have not been discussed but need commenting.

"Three" as "a few." In the discussion of three-way decision and three-way granular computing, we make solely and explicitly use of the magical number three. We consider the use of three levels and three granules. For example, in Fig. 4 we divide a whole into exactly three granules and further divide a granule into exactly three smaller granules. Although there is overwhelming evidence in support of the use of threes, we should not take too literal sense of three. Sometimes, we should consider more than three levels or more than three granules. Other times we may only need to consider two or one. In a more practical sense, we may view "three" as "a few" when interpreting the principles of three-way decision and three-way granular computing. Nevertheless, as shown by Table 1, three plays a dominating role and should be taken as a first choice. The multiple level tree structure of Sternberg's [65] triarchic theory of human intelligence may be considered as an example of thinking in threes, if we interpret three as a few.

Strengthening our cognitive toolbox through "thinking in threes." A reviewer of the paper states, "while reading the paper it seems sometimes that we are obliged, whatever the cost, to think in three." The statement correctly points out an unintended bias of our arguments and examples in support of thinking in threes. Table 1 shows that thinking in twos are so popular and natural. We are more inclined to thinking in twos. The table also shows that thinking in other numbers is possible. By interpreting "three" figuratively as "a few," we correct the bias to a certain degree. In order to put thinking in threes in its right perspective, we must acknowledge that a) thinking in threes is only one of the cognitive tools, b) thinking in threes should not be considered as a denial or replacement of other ways to think, and c) thinking in threes is complementary to, rather than competitive with, other ways to think. Our arguments in this paper may be taken as a plea for attention to thinking in threes. By integrating thinking in threes with other ways to think, we may be able to strengthen our cognitive toolbox.

"Levels" and "hierarchies" as "granules." Granules are parts of a whole. The whole determines the interpretation of parts and parts assign meaning to the whole. This part-whole setting for studying granules allows us to use different semantical interpretations of the notion of granules. What are parts and what is the whole depend on a standpoint. A part may be viewed as a whole if we focus on the part and look at its internal structure. A whole may be a part when forming another whole. In our discussion, we treat granules, levels, and hierarchies as different types of elements. In the part-whole setting, they may all be treated as different types of granules. For example, we consider granules as parts when forming a level as the whole of a family of granules. When considering a hierarchy, levels are parts and a hierarchy of a family of levels is a whole. In this case, levels may also be viewed as a different type of granules. When considering multiple hierarchies, hierarchies are parts and a family of hierarchies is a whole. That is, hierarchies may be viewed as another type of granules. The abstract notion of granules may be attached with different semantics and, consequently, is labeled differently. Nevertheless, some general principles remain to be the same, namely, a whole is a family of parts called granules and relationships of parts give rise to granular structures. This understanding of granules and granular structures allows the required flexibility of granular computing.

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