PyPHS Documentation
Version 0.1.5

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11 novembre 2016

1 Package structure

Below is a list of each PyPHS module of practical use, along with a short description.

`symbs` Container for all the SymPy symbolic variables.

Attributes are ordered *list of symbols* associated with the system's vectors components :

- `x` : state vector symbols $\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}}$,
- `w` : dissipative vector variable symbols $\mathbf{w} \in \mathbb{R}^{n_\mathbf{w}}$,
- `u` : input vector symbols $\mathbf{u} \in \mathbb{R}^{n_\mathbf{y}}$,
- `y` : output vector symbols $\mathbf{y} \in \mathbb{R}^{n_\mathbf{y}}$,
- `cu` : input vector symbols for connectors $\mathbf{c_u} \in \mathbb{R}^{n_\mathbf{y}}$,
- `cy` : output vector symbols for connectors $\mathbf{c_y} \in \mathbb{R}^{n_\mathbf{y}}$,
- `p` : Time-varying parameters symbols $\mathbf{p} \in \mathbb{R}^{n_\mathbf{y}}$.

Methods :

- `dx()` : Returns the symbols associated with the state differential $\mathrm{d}\mathbf{x}$ formed by appending the prefix d to each symbol in `x`.
- `args()` : Return the list of symbols associated with the vector of all arguments of the symbolic expressions (`expr` module).

`exprs` Container for all the SymPy symbolic expressions associated with the system's functions.

Attributes : For scalar function (e.g. the storage function `H`), arguments are SymPy expressions ; for vector functions (e.g. the disipative function $\mathbf{z}$), arguments are ordered lists of SymPy expressions ; for matrix functions (e.g. the Jacobian matrix of disipative function $\mathbf{z}$), arguments are `sympy.Matrix` objects. Notice the expression arguments (`sympy.Symbols`) must belong to the following lists of the `pyphs.PorthamiltonianObject : symbs.x`, `symbs.dx()`, `symbs.w`, `symbs.u`, `symbs.p`, and the keys of the dictionary `symbs.subs`.

- `H` : storage function $\mathrm{H} \in \mathbb{R}$,
- `z` : dissipative function $\mathbf{z} \in \mathbb{R}^{n_\mathbf{w}}$,
- `g` : input/output gains vector function $\mathbf{g} \in \mathbb{R}^{n_\mathbf{g}}$,

The following expression are computed from the `exprs.build()` metho (see below) :

`dxH` : the continuous gradient vector of storage scalar function $\nabla\mathrm{H}(\mathbf{x})$,

`hessH` : the continuous hessian matrix of storage scalar function (computed as $\nabla\nabla\mathrm{H}(\mathbf{x})$),

`y` : the expression of the continuous output vector function $\mathbf{y}(\nabla\mathrm{H}, \mathbf{z}, \mathbf{u})$,

`dxHd` : the discrete gradient vector of storage scalar function $\overline{\nabla}\mathrm{H}(\mathbf{x}, \delta\mathbf{x})$,

`yd` : the expression of the discrete output vector function $\mathbf{y}(\overline{\nabla}\mathrm{H}, \mathbf{z}, \mathbf{u})$,

`jacz` : the continuous jacobian matrix of dissipative vector function $\nabla\mathbf{z}(\mathbf{w})$.

Methods :

`build()` : Build the following system functions as SymPy expressions and append them as attributes to the `exprs` module :

- `dxH` : the continuous gradient vector of storage scalar function $\nabla\mathrm{H}(\mathbf{x}) \in \mathbb{R}^{n_{\mathrm{x}}}$,
- `dxHd` : the discrete gradient vector of storage scalar function $\overline{\nabla}\mathrm{H}(\mathbf{x}, \delta\mathbf{x}) \in \mathbb{R}^{n_{\mathrm{x}}}$,
- `hessH` : the continuous hessian matrix of storage scalar function (computed as $\nabla\nabla\mathrm{H}(\mathbf{x}) \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$),
- `jacz` : the continuous jacobian matrix of dissipative vector function $\nabla\mathbf{z}(\mathbf{w}) \in \mathbb{R}^{n_{\mathrm{w}} \times n_{\mathrm{w}}}$.
- `y` : the expression of the continuous output vector function $\mathbf{y}(\nabla\mathrm{H}, \mathbf{z}, \mathbf{u}) \in \mathbb{R}^{n_{\mathrm{y}}}$,
- `yd` : the expression of the discrete output vector function $\overline{\mathbf{y}}(\overline{\nabla}\mathrm{H}, \mathbf{z}, \mathbf{u}) \in \mathbb{R}^{n_{\mathrm{y}}}$,

`setexpr(name, expr)` : Add the SymPy expression `expr` to the `exprs` module, with argument `name`, and add `name` to the set of `exprs._names`.

`freesymbols()` : Retrun a python set of all the free symbols (`sympy.symbols`) that appear at least once in all expressions with names in `exprs._names`.

`dims`

`inds`

`struc`

`exprs`

`funcs`

`simu`

`data`

`graph`

— —

2 Port-Hamiltonian system

$$\underbrace{\begin{pmatrix} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \\ \mathbf{w} \\ \mathbf{y} \end{pmatrix}}_{\mathbf{b}} = \underbrace{\Big(\mathbf{J} - \mathbf{R}\Big)}_{\mathbf{M}} \underbrace{\begin{pmatrix} \nabla \mathrm{H}(\mathbf{x}) \\ \mathbf{z}(\mathbf{w}) \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}} \tag{1}$$

with $\mathrm{P} = \mathbf{u}^\intercal\, \mathbf{y}$ the power received *by* the sources *from* the system.

3 Split in linear and nonlinear parts

The state is split according to $\mathbf{x} = (\mathbf{x}_{\mathtt{l}}^\intercal,\, \mathbf{x}_{\mathtt{nl}}^\intercal)^\intercal$ with

$\mathbf{x}_{\mathtt{l}} = (x_1, \cdots, x_{n_{\mathtt{xl}}})^\intercal$ the states associated with the quadratic components of the Hamilotnian $\mathrm{H}_{\mathtt{l}}(\mathbf{x}_{\mathtt{l}}) = \mathbf{x}_{\mathtt{l}}^\intercal\, \mathbf{Q}\, \mathbf{x}_{\mathtt{l}}/2$

$\mathbf{x}_{\mathtt{nl}} = (x_{n_{\mathtt{xl}}+1}, \cdots, x_{n_{\mathtt{x}}})^\intercal$ the states associated with the non-quadratic components of the Hamiltonian with $n_{\mathtt{x}} = n_{\mathtt{xl}} + n_{\mathtt{xnl}}$.

The set of dissipative variables is split according to $\mathbf{w} = (\mathbf{x}_{\mathtt{l}}^\intercal,\, \mathbf{w}_{\mathtt{nl}}^\intercal)^\intercal$ with

$\mathbf{w}_{\mathtt{l}} = (w_1, \cdots, w_{n_{\mathtt{wl}}})^\intercal$ the variables associated with the linear components of the dissipative relation $\mathbf{z}_{\mathtt{l}}(\mathbf{w}_{\mathtt{l}}) = \mathbf{Z}_{\mathtt{l}}\, \mathbf{w}_{\mathtt{l}}$

$\mathbf{w}_{\mathtt{nl}} = (w_{n_{\mathtt{wl}}+1}, \cdots, w_{n_{\mathtt{w}}})^\intercal$ the variables associated with the nonlinear components of the dissipative relation $\mathbf{z}_{\mathtt{nl}}(\mathbf{w}_{\mathtt{nl}})$ with $n_{\mathtt{w}} = n_{\mathtt{wl}} + n_{\mathtt{wnl}}$.

$$\underbrace{\begin{pmatrix} \frac{\mathrm{d}\mathbf{x}_{\mathtt{l}}}{\mathrm{d}t} \\ \frac{\mathrm{d}\mathbf{x}_{\mathtt{nl}}}{\mathrm{d}t} \\ \mathbf{w}_{\mathtt{l}} \\ \mathbf{w}_{\mathtt{nl}} \\ \mathbf{y} \end{pmatrix}}_{\mathbf{b}} = \underbrace{\left(\begin{array}{cc|cc|c} \mathbf{M}_{\mathtt{xlxl}} & \mathbf{M}_{\mathtt{xlxnl}} & \mathbf{M}_{\mathtt{xlwl}} & \mathbf{M}_{\mathtt{xlwnl}} & \mathbf{M}_{\mathtt{xly}} \\ \mathbf{M}_{\mathtt{xnlxl}} & \mathbf{M}_{\mathtt{xnlxnl}} & \mathbf{M}_{\mathtt{xnlwl}} & \mathbf{M}_{\mathtt{xnlwnl}} & \mathbf{M}_{\mathtt{xnly}} \\ \hline \mathbf{M}_{\mathtt{wlxl}} & \mathbf{M}_{\mathtt{wlxnl}} & \mathbf{M}_{\mathtt{wlwl}} & \mathbf{M}_{\mathtt{wlwnl}} & \mathbf{M}_{\mathtt{wly}} \\ \mathbf{M}_{\mathtt{wnlwl}} & \mathbf{M}_{\mathtt{wnlxnl}} & \mathbf{M}_{\mathtt{wnlwl}} & \mathbf{M}_{\mathtt{wnlwnl}} & \mathbf{M}_{\mathtt{wnly}} \\ \hline \mathbf{M}_{\mathtt{yxl}} & \mathbf{M}_{\mathtt{yxnl}} & \mathbf{M}_{\mathtt{ywl}} & \mathbf{M}_{\mathtt{ywnl}} & \mathbf{M}_{\mathtt{yy}} \end{array}\right)}_{\mathbf{M}} \underbrace{\begin{pmatrix} \nabla \mathrm{H}_{\mathtt{l}} \\ \nabla \mathrm{H}_{\mathtt{nl}} \\ \mathbf{Z}_{\mathtt{l}}\, \mathbf{w}_{\mathtt{l}} \\ \mathbf{z}_{\mathtt{nl}} \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}} \tag{2}$$

3.1 Linear subsystem

$$\underbrace{\begin{pmatrix} \frac{\mathrm{d}\mathbf{x}_{\mathtt{l}}}{\mathrm{d}t} \\ \mathbf{w}_{\mathtt{l}} \end{pmatrix}}_{\mathbf{b}_{\mathtt{l}}} = \underbrace{\left(\begin{array}{cc|cc|c} \mathbf{M}_{\mathtt{xlxl}} & \mathbf{M}_{\mathtt{xlxnl}} & \mathbf{M}_{\mathtt{xlwl}} & \mathbf{M}_{\mathtt{xlwnl}} & \mathbf{M}_{\mathtt{xly}} \\ \mathbf{M}_{\mathtt{wlxl}} & \mathbf{M}_{\mathtt{wlxnl}} & \mathbf{M}_{\mathtt{wlwl}} & \mathbf{M}_{\mathtt{wlwnl}} & \mathbf{M}_{\mathtt{wly}} \end{array}\right)}_{\mathbf{M}_{\mathtt{l}}} \underbrace{\begin{pmatrix} \nabla \mathrm{H}_{\mathtt{l}} \\ \nabla \mathrm{H}_{\mathtt{nl}} \\ \mathbf{Z}_{\mathtt{l}}\, \mathbf{w}_{\mathtt{l}} \\ \mathbf{z}_{\mathtt{nl}} \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}_{\mathtt{l}}} \tag{3}$$

3.2 Nonlinear subsystem

$$\underbrace{\begin{pmatrix}\frac{\mathrm{d}\mathbf{x}_{\mathtt{nl}}}{\mathrm{d}t}\\ \mathbf{w}_{\mathtt{nl}}\end{pmatrix}}_{\mathbf{b}_{\mathtt{nl}}} = \underbrace{\left(\begin{array}{cc|cc|c} \mathbf{M}_{\mathtt{xnlxl}} & \mathbf{M}_{\mathtt{xnlxnl}} & \mathbf{M}_{\mathtt{xnlwl}} & \mathbf{M}_{\mathtt{xnlwnl}} & \mathbf{M}_{\mathtt{xnly}} \\ \hline \mathbf{M}_{\mathtt{wnlwl}} & \mathbf{M}_{\mathtt{wnlxnl}} & \mathbf{M}_{\mathtt{wnlwl}} & \mathbf{M}_{\mathtt{wnlwnl}} & \mathbf{M}_{\mathtt{wnly}} \end{array}\right)}_{\mathbf{M}_{\mathtt{nl}}} \underbrace{\begin{pmatrix}\nabla \mathrm{H}_{\mathtt{l}}\\ \nabla \mathrm{H}_{\mathtt{nl}}\\ \mathbf{Z}_{\mathtt{l}}\,\mathbf{w}_{\mathtt{l}}\\ \mathbf{z}_{\mathtt{nl}}\\ \mathbf{u}\end{pmatrix}}_{\mathbf{a}_{\mathtt{nl}}} \tag{4}$$

4 Presolve linear part

4.1 Numerical linear subsystem

In the sequel, quantities are defined on the current time step $\mathbf{x} \equiv \mathbf{x}(t_k)$, with $k \in \mathbb{N}_+^*$. The dicrete gradient for the quadratic part of the Hamiltonian is $\nabla \mathrm{H}_{\mathtt{l}} = \frac{1}{2}\,\mathbf{Q}\,(2\mathbf{x}_{\mathtt{l}} + \delta\mathbf{x}_{\mathtt{l}})$ and the discret linear subsystem is

$$\begin{aligned}
\mathbf{D}_{\mathtt{l}}^{-1} = \mathbf{iD}_{\mathtt{l}} \quad &= \quad \begin{pmatrix}\frac{\mathbf{I_d}}{\delta t} & 0\\ 0 & \mathbf{I_d}\end{pmatrix} - \begin{pmatrix}\mathbf{M}_{\mathtt{xlxl}} & \mathbf{M}_{\mathtt{xlwl}}\\ \mathbf{M}_{\mathtt{wlxl}} & \mathbf{M}_{\mathtt{wlwl}}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\,\mathbf{Q} & 0\\ 0 & \mathbf{Z}_{\mathtt{l}}\end{pmatrix},\\
\underbrace{\begin{pmatrix}\delta\mathbf{x}_{\mathtt{l}}\\ \mathbf{w}_{\mathtt{l}}\end{pmatrix}}_{\mathbf{v}_{\mathtt{l}}} \quad &= \quad \underbrace{\mathbf{D}_{\mathtt{l}}\underbrace{\begin{pmatrix}\mathbf{M}_{\mathtt{xlxl}}\\ \mathbf{M}_{\mathtt{wlxl}}\end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{lxl}}}}}_{\mathbf{N}_{\mathtt{lxl}}}\mathbf{Q}\;\mathbf{x}_{\mathtt{l}} + \underbrace{\mathbf{D}_{\mathtt{l}}\underbrace{\begin{pmatrix}\mathbf{M}_{\mathtt{xlxnl}} & \mathbf{M}_{\mathtt{xlwnl}}\\ \mathbf{M}_{\mathtt{wlxnl}} & \mathbf{M}_{\mathtt{wlwnl}}\end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{lnl}}}}}_{\mathbf{N}_{\mathtt{lnl}}}\underbrace{\begin{pmatrix}\nabla \mathrm{H}_{\mathtt{nl}}\\ \mathbf{z}_{\mathtt{nl}}\end{pmatrix}}_{\mathbf{f}_{\mathtt{nl}}} + \underbrace{\mathbf{D}_{\mathtt{l}}\underbrace{\begin{pmatrix}\mathbf{M}_{\mathtt{xly}}\\ \mathbf{M}_{\mathtt{wly}}\end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{ly}}}}}_{\mathbf{N}_{\mathtt{ly}}}\mathbf{u}
\end{aligned} \tag{5}$$

5 Implicite nonlinear function

5.1 Numerical nonlinear subsystem

$$\begin{aligned}
\begin{pmatrix}\frac{\mathbf{I_d}}{\delta t} & 0\\ 0 & \mathbf{I_d}\end{pmatrix}\underbrace{\begin{pmatrix}\delta\mathbf{x}_{\mathtt{nl}}\\ \mathbf{w}_{\mathtt{nl}}\end{pmatrix}}_{\mathbf{v}_{\mathtt{nl}}} \quad &= \quad \underbrace{\begin{pmatrix}\mathbf{M}_{\mathtt{xnlxnl}} & \mathbf{M}_{\mathtt{xnlwnl}}\\ \mathbf{M}_{\mathtt{wnlxnl}} & \mathbf{M}_{\mathtt{wnlwnl}}\end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{nlnl}}}}\mathbf{f}_{\mathtt{nl}} + \underbrace{\begin{pmatrix}\mathbf{M}_{\mathtt{xnly}}\\ \mathbf{M}_{\mathtt{wnly}}\end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{nly}}}}\mathbf{u}\\
&\quad + \underbrace{\begin{pmatrix}\mathbf{M}_{\mathtt{xnlxl}} & \mathbf{M}_{\mathtt{xnlwl}}\\ \mathbf{M}_{\mathtt{wnlxl}} & \mathbf{M}_{\mathtt{wnlwl}}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\,\mathbf{Q} & 0\\ 0 & \mathbf{Z}_{\mathtt{l}}\end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{nll}}}}\mathbf{v}_{\mathtt{l}} + \underbrace{\begin{pmatrix}\mathbf{M}_{\mathtt{xnlxl}}\\ \mathbf{M}_{\mathtt{wnlxl}}\end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{nlxl}}}}\mathbf{Q}\;\mathbf{x}_{\mathtt{l}}
\end{aligned} \tag{6}$$

5.2 Presolve numerical nonlinear subsystem

$$
\begin{pmatrix} \frac{\mathbf{I_d}}{\delta t} & 0 \\ 0 & \mathbf{I_d} \end{pmatrix} \mathbf{v}_{\mathtt{nl}} = \underbrace{\left(\overline{\mathbf{N}_{\mathtt{nlxl}}} + \overline{\mathbf{N}_{\mathtt{nll}}}\,\mathbf{N}_{\mathtt{lxl}}\right)}_{\mathbf{N}_{\mathtt{nlxl}}} \mathbf{x}_{\mathtt{l}} + \underbrace{\left(\overline{\mathbf{N}_{\mathtt{nlnl}}} + \overline{\mathbf{N}_{\mathtt{nll}}}\,\mathbf{N}_{\mathtt{lnl}}\right)}_{\mathbf{N}_{\mathtt{nlnl}}} \mathbf{f}_{\mathtt{nl}} \\
\underbrace{\left(\overline{\mathbf{N}_{\mathtt{nly}}} + \overline{\mathbf{N}_{\mathtt{nll}}}\,\mathbf{N}_{\mathtt{ly}}\right)}_{\mathbf{N}_{\mathtt{nly}}} \mathbf{u} \tag{7}
$$

6 Algorithm

6.1 Inputs

$$
\begin{aligned}
\mathbf{iD}_{\mathtt{l}} &= \begin{pmatrix} \frac{\mathbf{I_d}}{\delta t} & 0 \\ 0 & \mathbf{I_d} \end{pmatrix} - \begin{pmatrix} \mathbf{M}_{\mathtt{xlxl}} & \mathbf{M}_{\mathtt{xlwl}} \\ \mathbf{M}_{\mathtt{wlxl}} & \mathbf{M}_{\mathtt{wlwl}} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathbf{Q} & 0 \\ 0 & \mathbf{Z}_{\mathtt{l}} \end{pmatrix} \\
\overline{\mathbf{N}_{\mathtt{lxl}}} &= \begin{pmatrix} \mathbf{M}_{\mathtt{xlxl}} \\ \mathbf{M}_{\mathtt{wlxl}} \end{pmatrix} \mathbf{Q} \\
\overline{\mathbf{N}_{\mathtt{lnl}}} &= \begin{pmatrix} \mathbf{M}_{\mathtt{xlxnl}} & \mathbf{M}_{\mathtt{xlwnl}} \\ \mathbf{M}_{\mathtt{wlxnl}} & \mathbf{M}_{\mathtt{wlwnl}} \end{pmatrix} \\
\overline{\mathbf{N}_{\mathtt{ly}}} &= \begin{pmatrix} \mathbf{M}_{\mathtt{xly}} \\ \mathbf{M}_{\mathtt{wly}} \end{pmatrix} \\
\overline{\mathbf{N}_{\mathtt{nlnl}}} &= \begin{pmatrix} \mathbf{M}_{\mathtt{xnlxnl}} & \mathbf{M}_{\mathtt{xnlwnl}} \\ \mathbf{M}_{\mathtt{wnlxnl}} & \mathbf{M}_{\mathtt{wnlwnl}} \end{pmatrix} \\
\overline{\mathbf{N}_{\mathtt{nll}}} &= \begin{pmatrix} \mathbf{M}_{\mathtt{xnlxl}} & \mathbf{M}_{\mathtt{xnlwl}} \\ \mathbf{M}_{\mathtt{wnlxl}} & \mathbf{M}_{\mathtt{wnlwl}} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathbf{Q} & 0 \\ 0 & \mathbf{Z}_{\mathtt{l}} \end{pmatrix} \\
\overline{\mathbf{N}_{\mathtt{nlxl}}} &= \begin{pmatrix} \mathbf{M}_{\mathtt{xnlxl}} \\ \mathbf{M}_{\mathtt{wnlxl}} \end{pmatrix} \mathbf{Q} \\
\overline{\mathbf{N}_{\mathtt{nly}}} &= \begin{pmatrix} \mathbf{M}_{\mathtt{xnly}} \\ \mathbf{M}_{\mathtt{wnly}} \end{pmatrix} \\
\mathcal{J}_{\mathbf{f}_{\mathtt{nl}}}(\mathbf{v}_{\mathtt{nl}}) &= \begin{pmatrix} \mathcal{J}_{\nabla \mathrm{H}_{\mathtt{nl}}} & 0 \\ 0 & \mathcal{J}_{\mathbf{z}_{\mathtt{nl}}} \end{pmatrix} \\
\mathbf{I}_{\mathtt{nl}} &= \begin{pmatrix} \frac{\mathbf{I_d}}{\delta t} & 0 \\ 0 & \mathbf{I_d} \end{pmatrix}
\end{aligned} \tag{8}
$$

6.2 Process

$$
\begin{array}{rcl}
\mathbf{D}_{\mathtt{l}} &=& \mathbf{iD}_{\mathtt{l}}^{-1} \\
\mathbf{N}_{\mathtt{lxl}} &=& \mathbf{D}_{\mathtt{l}}\,\overline{\mathbf{N}_{\mathtt{lxl}}} \\
\mathbf{N}_{\mathtt{lnl}} &=& \mathbf{D}_{\mathtt{l}}\,\overline{\mathbf{N}_{\mathtt{lnl}}} \\
\mathbf{N}_{\mathtt{ly}} &=& \mathbf{D}_{\mathtt{l}}\,\overline{\mathbf{N}_{\mathtt{ly}}} \\
\mathbf{N}_{\mathtt{nlxl}} &=& \overline{\mathbf{N}_{\mathtt{nlxl}}} + \overline{\mathbf{N}_{\mathtt{nll}}}\,\mathbf{N}_{\mathtt{lxl}} \\
\mathbf{N}_{\mathtt{nlnl}} &=& \overline{\mathbf{N}_{\mathtt{nlnl}}} + \overline{\mathbf{N}_{\mathtt{nll}}}\,\mathbf{N}_{\mathtt{lnl}} \\
\mathbf{N}_{\mathtt{nly}} &=& \overline{\mathbf{N}_{\mathtt{nly}}} + \overline{\mathbf{N}_{\mathtt{nll}}}\,\mathbf{N}_{\mathtt{ly}} \\
\mathbf{c} &=& \mathbf{N}_{\mathtt{nlxl}}\,\mathbf{x}_{\mathtt{l}} + \mathbf{N}_{\mathtt{nly}}\,\mathbf{u} \\
\text{Iterate} &:& \mathbf{F}_{\mathtt{nl}}(\mathbf{v}_{\mathtt{nl}}) = \mathbf{I}_{\mathtt{nl}}\,\mathbf{v}_{\mathtt{nl}} - \mathbf{N}_{\mathtt{nlnl}}\,\mathbf{f}_{\mathtt{nl}} - \mathbf{c} \\
&& \mathcal{J}_{\mathbf{F}_{\mathtt{nl}}}(\mathbf{v}_{\mathtt{nl}}) = \mathbf{I}_{\mathtt{nl}} - \mathbf{N}_{\mathtt{nlnl}}\,\mathcal{J}_{\mathbf{f}\mathtt{nl}}(\mathbf{v}_{\mathtt{nl}}) \\
&& \mathbf{v}_{\mathtt{nl}} = \mathbf{v}_{\mathtt{nl}} - \mathcal{J}_{\mathbf{F}_{\mathtt{nl}}}^{-1}(\mathbf{v}_{\mathtt{nl}})\,\mathbf{F}_{\mathtt{nl}}(\mathbf{v}_{\mathtt{nl}}) \\
\mathbf{v}_{\mathtt{l}} &=& \mathbf{N}_{\mathtt{lxl}}\,\mathbf{x}_{\mathtt{l}} + \mathbf{N}_{\mathtt{lnl}}\,\mathbf{f}_{\mathtt{nl}} + \mathbf{N}_{\mathtt{ly}}\,\mathbf{u} \\
\mathbf{y} &=& \mathbf{M}_{\mathtt{yxl}}\,\nabla \mathrm{H}_{\mathtt{l}} + \mathbf{M}_{\mathtt{yxnl}}\,\nabla \mathrm{H}_{\mathtt{nl}}\mathbf{M}_{\mathtt{ywl}}\,\mathbf{Z}_{\mathtt{l}}\,\mathbf{w}_{\mathtt{l}} + \mathbf{M}_{\mathtt{ywnl}}\,\mathbf{z}_{\mathtt{nl}} + \mathbf{M}_{\mathtt{yy}}\,\mathbf{u} \\
\mathbf{x} &=& \mathbf{x} + \delta\mathbf{x}
\end{array} \tag{9}
$$

$$
\begin{aligned}
\mathbf{y} &= \mathrm{M}_{\mathtt{yxl}}\,\nabla \mathrm{H}_{\mathtt{l}} + \mathbf{M}_{\mathtt{yxnl}}\,\nabla \mathrm{H}_{\mathtt{nl}}\mathbf{M}_{\mathtt{ywl}}\,\mathbf{Z}_{\mathtt{l}}\,\mathbf{w}_{\mathtt{l}} + \mathbf{M}_{\mathtt{ywnl}}\,\mathbf{z}_{\mathtt{nl}} + \mathbf{M}_{\mathtt{yy}}\,\mathbf{u} && (10)\\
&= \mathrm{M}_{\mathtt{yxl}}\,\nabla \mathrm{H}_{\mathtt{l}} + \mathbf{M}_{\mathtt{yxnl}}\,\nabla \mathrm{H}_{\mathtt{nl}}\mathbf{M}_{\mathtt{ywl}}\,\mathbf{Z}_{\mathtt{l}}\,\mathbf{w}_{\mathtt{l}} + \mathbf{M}_{\mathtt{ywnl}}\,\mathbf{z}_{\mathtt{nl}} + \mathbf{M}_{\mathtt{yy}}\,\mathbf{u} && (11)\\
& && (12)
\end{aligned}
$$