

1 Port-Hamiltonian system

$$\underbrace{\begin{pmatrix} \frac{d\mathbf{x}}{dt} \\ \mathbf{w} \\ \mathbf{y} \end{pmatrix}}_{\mathbf{b}} = \underbrace{(\mathbf{J} - \mathbf{R})}_{\mathbf{M}} \underbrace{\begin{pmatrix} \nabla H(\mathbf{x}) \\ \mathbf{z}(\mathbf{w}) \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}} \quad (1)$$

with $\mathbf{P} = \mathbf{u}^\top \mathbf{y}$ the power received *by* the sources *from* the system.

2 Split in linear and nonlinear parts

The state is split according to $\mathbf{x} = (\mathbf{x}_1^\top, \mathbf{x}_{n1}^\top)^\top$ with

$\mathbf{x}_1 = (x_1, \dots, x_{n_{x1}})^\top$ the states associated with the quadratic components of the Hamiltonian $H_1(\mathbf{x}_1) = \mathbf{x}_1^\top \mathbf{Q} \mathbf{x}_1 / 2$

$\mathbf{x}_{n1} = (x_{n_{x1}+1}, \dots, x_{n_x})^\top$ the states associated with the non-quadratic components of the Hamiltonian with $n_x = n_{x1} + n_{xn1}$.

The set of dissipative variables is split according to $\mathbf{w} = (\mathbf{w}_1^\top, \mathbf{w}_{n1}^\top)^\top$ with

$\mathbf{w}_1 = (w_1, \dots, w_{n_{w1}})^\top$ the variables associated with the linear components of the dissipative relation $\mathbf{z}_1(\mathbf{w}_1) = \mathbf{Z}_1 \mathbf{w}_1$

$\mathbf{w}_{n1} = (w_{n_{w1}+1}, \dots, w_{n_w})^\top$ the variables associated with the nonlinear components of the dissipative relation $\mathbf{z}_{n1}(\mathbf{w}_{n1})$ with $n_w = n_{w1} + n_{wn1}$.

$$\underbrace{\begin{pmatrix} \frac{d\mathbf{x}_1}{dt} \\ \frac{d\mathbf{x}_{n1}}{dt} \\ \mathbf{w}_1 \\ \mathbf{w}_{n1} \\ \mathbf{y} \end{pmatrix}}_{\mathbf{b}} = \underbrace{\begin{pmatrix} \mathbf{M}_{x1x1} & \mathbf{M}_{x1xn1} & \mathbf{M}_{x1w1} & \mathbf{M}_{x1wn1} & \mathbf{M}_{x1y} \\ \mathbf{M}_{xn1x1} & \mathbf{M}_{xn1xn1} & \mathbf{M}_{xn1w1} & \mathbf{M}_{xn1wn1} & \mathbf{M}_{xn1y} \\ \mathbf{M}_{w1x1} & \mathbf{M}_{w1xn1} & \mathbf{M}_{w1w1} & \mathbf{M}_{w1wn1} & \mathbf{M}_{w1y} \\ \mathbf{M}_{wn1x1} & \mathbf{M}_{wn1xn1} & \mathbf{M}_{wn1w1} & \mathbf{M}_{wn1wn1} & \mathbf{M}_{wn1y} \\ \mathbf{M}_{yx1} & \mathbf{M}_{yxn1} & \mathbf{M}_{yw1} & \mathbf{M}_{ywn1} & \mathbf{M}_{yy} \end{pmatrix}}_{\mathbf{M}} \underbrace{\begin{pmatrix} \nabla H_1 \\ \nabla H_{n1} \\ \mathbf{Z}_1 \mathbf{w}_1 \\ \mathbf{z}_{n1} \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}} \quad (2)$$

2.1 Linear subsystem

$$\underbrace{\begin{pmatrix} \frac{d\mathbf{x}_1}{dt} \\ \mathbf{w}_1 \end{pmatrix}}_{\mathbf{b}_1} = \underbrace{\begin{pmatrix} \mathbf{M}_{x1x1} & \mathbf{M}_{x1xn1} & \mathbf{M}_{x1w1} & \mathbf{M}_{x1wn1} & \mathbf{M}_{x1y} \\ \mathbf{M}_{w1x1} & \mathbf{M}_{w1xn1} & \mathbf{M}_{w1w1} & \mathbf{M}_{w1wn1} & \mathbf{M}_{w1y} \end{pmatrix}}_{\mathbf{M}_1} \underbrace{\begin{pmatrix} \nabla H_1 \\ \nabla H_{n1} \\ \mathbf{Z}_1 \mathbf{w}_1 \\ \mathbf{z}_{n1} \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}_1} \quad (3)$$

2.2 Nonlinear subsystem

$$\underbrace{\begin{pmatrix} \frac{d\mathbf{x}_{n1}}{dt} \\ \mathbf{w}_{n1} \end{pmatrix}}_{\mathbf{b}_{n1}} = \underbrace{\begin{pmatrix} \mathbf{M}_{\mathbf{x}n1\mathbf{x}1} & \mathbf{M}_{\mathbf{x}n1\mathbf{x}n1} & \mathbf{M}_{\mathbf{x}n1\mathbf{w}1} & \mathbf{M}_{\mathbf{x}n1\mathbf{w}n1} & \mathbf{M}_{\mathbf{x}n1\mathbf{y}} \\ \mathbf{M}_{\mathbf{w}n1\mathbf{w}1} & \mathbf{M}_{\mathbf{w}n1\mathbf{x}n1} & \mathbf{M}_{\mathbf{w}n1\mathbf{w}1} & \mathbf{M}_{\mathbf{w}n1\mathbf{w}n1} & \mathbf{M}_{\mathbf{w}n1\mathbf{y}} \end{pmatrix}}_{\mathbf{M}_{n1}} \underbrace{\begin{pmatrix} \nabla H_1 \\ \nabla H_{n1} \\ \mathbf{Z}_1 \mathbf{w}_1 \\ \mathbf{z}_{n1} \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}_{n1}} \quad (4)$$

3 Presolve linear part

3.1 Numerical linear subsystem

In the sequel, quantities are defined on the current time step $\mathbf{x} \equiv \mathbf{x}(t_k)$, with $k \in \mathbb{N}_+^*$. The discrete gradient for the quadratic part of the Hamiltonian is $\nabla H_1 = \frac{1}{2} \mathbf{Q} (2\mathbf{x}_1 + \delta\mathbf{x}_1)$ and the discret linear subsystem is

$$\begin{aligned} \mathbf{D}_1^{-1} = i\mathbf{D}_1 &= \begin{pmatrix} \frac{\mathbf{I}_d}{\delta t} & 0 \\ 0 & \mathbf{I}_d \end{pmatrix} - \begin{pmatrix} \mathbf{M}_{\mathbf{x}1\mathbf{x}1} & \mathbf{M}_{\mathbf{x}1\mathbf{w}1} \\ \mathbf{M}_{\mathbf{w}1\mathbf{x}1} & \mathbf{M}_{\mathbf{w}1\mathbf{w}1} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \mathbf{Q} & 0 \\ 0 & \mathbf{Z}_1 \end{pmatrix}, \\ \underbrace{\begin{pmatrix} \delta\mathbf{x}_1 \\ \mathbf{w}_1 \end{pmatrix}}_{\mathbf{v}_1} &= \underbrace{\mathbf{D}_1 \begin{pmatrix} \mathbf{M}_{\mathbf{x}1\mathbf{x}1} \\ \mathbf{M}_{\mathbf{w}1\mathbf{x}1} \end{pmatrix}}_{\underbrace{\overline{\mathbf{N}_{1\mathbf{x}1}}}_{\mathbf{N}_{1\mathbf{x}1}}} \mathbf{Q} \mathbf{x}_1 + \underbrace{\mathbf{D}_1 \begin{pmatrix} \mathbf{M}_{\mathbf{x}1\mathbf{x}n1} & \mathbf{M}_{\mathbf{x}1\mathbf{w}n1} \\ \mathbf{M}_{\mathbf{w}1\mathbf{x}n1} & \mathbf{M}_{\mathbf{w}1\mathbf{w}n1} \end{pmatrix}}_{\underbrace{\overline{\mathbf{N}_{1n1}}}_{\mathbf{N}_{1n1}}} \underbrace{\begin{pmatrix} \nabla H_{n1} \\ \mathbf{z}_{n1} \end{pmatrix}}_{\mathbf{f}_{n1}} + \underbrace{\mathbf{D}_1 \begin{pmatrix} \mathbf{M}_{\mathbf{x}1\mathbf{y}} \\ \mathbf{M}_{\mathbf{w}1\mathbf{y}} \end{pmatrix}}_{\underbrace{\overline{\mathbf{N}_{1y}}}_{\mathbf{N}_{1y}}} \mathbf{u} \end{aligned} \quad (5)$$

4 Implicite nonlinear function

4.1 Numerical nonlinear subsystem

$$\begin{aligned} \begin{pmatrix} \frac{\mathbf{I}_d}{\delta t} & 0 \\ 0 & \mathbf{I}_d \end{pmatrix} \underbrace{\begin{pmatrix} \delta\mathbf{x}_{n1} \\ \mathbf{w}_{n1} \end{pmatrix}}_{\mathbf{v}_{n1}} &= \underbrace{\begin{pmatrix} \mathbf{M}_{\mathbf{x}n1\mathbf{x}n1} & \mathbf{M}_{\mathbf{x}n1\mathbf{w}n1} \\ \mathbf{M}_{\mathbf{w}n1\mathbf{x}n1} & \mathbf{M}_{\mathbf{w}n1\mathbf{w}n1} \end{pmatrix}}_{\overline{\mathbf{N}_{n1n1}}} \mathbf{f}_{n1} + \underbrace{\begin{pmatrix} \mathbf{M}_{\mathbf{x}n1\mathbf{y}} \\ \mathbf{M}_{\mathbf{w}n1\mathbf{y}} \end{pmatrix}}_{\overline{\mathbf{N}_{n1y}}} \mathbf{u} \\ &+ \underbrace{\begin{pmatrix} \mathbf{M}_{\mathbf{x}n1\mathbf{x}1} & \mathbf{M}_{\mathbf{x}n1\mathbf{w}1} \\ \mathbf{M}_{\mathbf{w}n1\mathbf{x}1} & \mathbf{M}_{\mathbf{w}n1\mathbf{w}1} \end{pmatrix}}_{\overline{\mathbf{N}_{n11}}} \begin{pmatrix} \frac{1}{2} \mathbf{Q} & 0 \\ 0 & \mathbf{Z}_1 \end{pmatrix} \mathbf{v}_1 + \underbrace{\begin{pmatrix} \mathbf{M}_{\mathbf{x}n1\mathbf{x}1} \\ \mathbf{M}_{\mathbf{w}n1\mathbf{x}1} \end{pmatrix}}_{\overline{\mathbf{N}_{n1x1}}} \mathbf{Q} \mathbf{x}_1 \end{aligned} \quad (6)$$

4.2 Presolve numerical nonlinear subsystem

$$\begin{pmatrix} \mathbf{I}_d & 0 \\ 0 & \mathbf{I}_d \end{pmatrix} \mathbf{v}_{nl} = \underbrace{(\overline{\mathbf{N}_{nlxl}} + \overline{\mathbf{N}_{nll}} \mathbf{N}_{lxl})}_{\mathbf{N}_{nlxl}} \mathbf{x}_1 + \underbrace{(\overline{\mathbf{N}_{nl nl}} + \overline{\mathbf{N}_{nll}} \mathbf{N}_{lnl})}_{\mathbf{N}_{nl nl}} \mathbf{f}_{nl} \\ \underbrace{(\overline{\mathbf{N}_{nly}} + \overline{\mathbf{N}_{nll}} \mathbf{N}_{ly})}_{\mathbf{N}_{nly}} \mathbf{u}$$
(7)

5 Algorithm

5.1 Inputs

$$\begin{aligned} \mathbf{iD}_1 &= \begin{pmatrix} \mathbf{I}_d & 0 \\ 0 & \mathbf{I}_d \end{pmatrix} - \begin{pmatrix} \mathbf{M}_{xlxl} & \mathbf{M}_{xlwl} \\ \mathbf{M}_{wlxl} & \mathbf{M}_{wlwl} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \mathbf{Q} & 0 \\ 0 & \mathbf{Z}_1 \end{pmatrix} \\ \overline{\mathbf{N}_{1xl}} &= \begin{pmatrix} \mathbf{M}_{xlxl} \\ \mathbf{M}_{wlxl} \end{pmatrix} \\ \overline{\mathbf{N}_{1nl}} &= \begin{pmatrix} \mathbf{M}_{xlxnl} & \mathbf{M}_{xlnl} \\ \mathbf{M}_{wlnl} & \mathbf{M}_{wnlwl} \end{pmatrix} \\ \overline{\mathbf{N}_{ly}} &= \begin{pmatrix} \mathbf{M}_{xly} \\ \mathbf{M}_{wly} \end{pmatrix} \\ \overline{\mathbf{N}_{nl nl}} &= \begin{pmatrix} \mathbf{M}_{xnlxnl} & \mathbf{M}_{xnlwnl} \\ \mathbf{M}_{wnlxl} & \mathbf{M}_{wnlwnl} \end{pmatrix} \\ \overline{\mathbf{N}_{nll}} &= \begin{pmatrix} \mathbf{M}_{xnlxl} & \mathbf{M}_{xnlwl} \\ \mathbf{M}_{wnlxl} & \mathbf{M}_{wnlwl} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \mathbf{Q} & 0 \\ 0 & \mathbf{Z}_1 \end{pmatrix} \\ \overline{\mathbf{N}_{nlxl}} &= \begin{pmatrix} \mathbf{M}_{xnlxl} \\ \mathbf{M}_{wnlxl} \end{pmatrix} \mathbf{Q} \\ \overline{\mathbf{N}_{nly}} &= \begin{pmatrix} \mathbf{M}_{xnlly} \\ \mathbf{M}_{wnlly} \end{pmatrix} \\ \mathcal{J}_{f_{nl}}(\mathbf{v}_{nl}) &= \begin{pmatrix} \mathcal{J}_{\nabla H_{nl}} & 0 \\ 0 & \mathcal{J}_{z_{nl}} \end{pmatrix} \\ \mathbf{I}_{nl} &= \begin{pmatrix} \mathbf{I}_d & 0 \\ 0 & \mathbf{I}_d \end{pmatrix} \end{aligned}$$
(8)

5.2 Process

$$\begin{aligned}
\mathbf{D}_1 &= \mathbf{iD}_1^{-1} \\
\mathbf{N}_{1x1} &= \mathbf{D}_1 \overline{\mathbf{N}_{1x1}} \\
\mathbf{N}_{1n1} &= \mathbf{D}_1 \overline{\mathbf{N}_{1n1}} \\
\mathbf{N}_{1y} &= \mathbf{D}_1 \overline{\mathbf{N}_{1y}} \\
\mathbf{N}_{n1x1} &= \overline{\mathbf{N}_{n1x1}} + \overline{\mathbf{N}_{n11}} \mathbf{N}_{1x1} \\
\mathbf{N}_{n1n1} &= \overline{\mathbf{N}_{n1n1}} + \overline{\mathbf{N}_{n11}} \mathbf{N}_{1n1} \\
\mathbf{N}_{n1y} &= \overline{\mathbf{N}_{n1y}} + \overline{\mathbf{N}_{n11}} \mathbf{N}_{1y} \\
\mathbf{c} &= \mathbf{N}_{n1x1} \mathbf{x}_1 + \mathbf{N}_{n1y} \mathbf{u} \\
\text{Iterate} : \quad &\mathbf{F}_{n1}(\mathbf{v}_{n1}) = \mathbf{I}_{n1} \mathbf{v}_{n1} - \mathbf{N}_{n1n1} \mathbf{f}_{n1} - \mathbf{c} \\
&\mathcal{J}_{\mathbf{F}_{n1}}(\mathbf{v}_{n1}) = \mathbf{I}_{n1} - \mathbf{N}_{n1n1} \mathcal{J}_{\mathbf{f}_{n1}}(\mathbf{v}_{n1}) \\
&\mathbf{v}_{n1} = \mathbf{v}_{n1} - \mathcal{J}_{\mathbf{F}_{n1}}^{-1}(\mathbf{v}_{n1}) \mathbf{F}_{n1}(\mathbf{v}_{n1}) \\
\mathbf{v}_1 &= \mathbf{N}_{1x1} \mathbf{x}_1 + \mathbf{N}_{1n1} \mathbf{f}_{n1} + \mathbf{N}_{1y} \mathbf{u} \\
\mathbf{y} &= \mathbf{M}_{yx1} \nabla H_1 + \mathbf{M}_{yxn1} \nabla H_{n1} \mathbf{M}_{yw1} \mathbf{Z}_1 \mathbf{w}_1 + \mathbf{M}_{ywn1} \mathbf{z}_{n1} + \mathbf{M}_{yy} \mathbf{u} \\
\mathbf{x} &= \mathbf{x} + \delta \mathbf{x}
\end{aligned} \tag{9}$$