1 Port-Hamiltonian system

$$\underbrace{\begin{pmatrix} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \\ \mathbf{w} \\ \mathbf{y} \end{pmatrix}}_{\mathbf{b}} = \underbrace{\left(\mathbf{J} - \mathbf{R}\right)}_{\mathbf{M}} \underbrace{\begin{pmatrix} \nabla \mathrm{H}(\mathbf{x}) \\ \mathbf{z}(\mathbf{w}) \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}} \tag{1}$$

with $\mathrm{P} = \mathbf{u}^\intercal \, \mathbf{y}$ the power received *by* the sources *from* the system.

2 Split in linear and nonlinear parts

The state is split according to $\mathbf{x} = (\mathbf{x}_{\mathtt{l}}^\intercal, \, \mathbf{x}_{\mathtt{nl}}^\intercal)^\intercal$ with

$\mathbf{x}_{\mathtt{l}} = (x_1, \cdots, x_{n_{\mathtt{xl}}})^\intercal$ the states associated with the quadratic components of the Hamilotnian $\mathrm{H}_{\mathtt{l}}(\mathbf{x}_{\mathtt{l}}) = \mathbf{x}_{\mathtt{l}}^\intercal \, \mathbf{Q} \, \mathbf{x}_{\mathtt{l}}/2$

$\mathbf{x}_{\mathtt{nl}} = (x_{n_{\mathtt{xl}}+1}, \cdots, x_{n_{\mathtt{x}}})^\intercal$ the states associated with the non-quadratic components of the Hamiltonian with $n_{\mathtt{x}} = n_{\mathtt{xl}} + n_{\mathtt{xnl}}$.

The set of dissipative variables is split according to $\mathbf{w} = (\mathbf{x}_{\mathtt{l}}^\intercal, \, \mathbf{w}_{\mathtt{nl}}^\intercal)^\intercal$ with

$\mathbf{w}_{\mathtt{l}} = (w_1, \cdots, w_{n_{\mathtt{wl}}})^\intercal$ the variables associated with the linear components of the dissipative relation $\mathbf{z}_{\mathtt{l}}(\mathbf{w}_{\mathtt{l}}) = \mathbf{Z}_{\mathtt{l}} \, \mathbf{w}_{\mathtt{l}}$

$\mathbf{w}_{\mathtt{nl}} = (w_{n_{\mathtt{wl}}+1}, \cdots, w_{n_{\mathtt{w}}})^\intercal$ the variables associated with the nonlinear components of the dissipative relation $\mathbf{z}_{\mathtt{nl}}(\mathbf{w}_{\mathtt{nl}})$ with $n_{\mathtt{w}} = n_{\mathtt{wl}} + n_{\mathtt{wnl}}$.

$$\underbrace{\begin{pmatrix} \frac{\mathrm{d}\mathbf{x}_{\mathtt{l}}}{\mathrm{d}t} \\ \frac{\mathrm{d}\mathbf{x}_{\mathtt{nl}}}{\mathrm{d}t} \\ \mathbf{w}_{\mathtt{l}} \\ \mathbf{w}_{\mathtt{nl}} \\ \mathbf{y} \end{pmatrix}}_{\mathbf{b}} = \underbrace{\left(\begin{array}{cc|cc|c} \mathbf{M}_{\mathtt{xlxl}} & \mathbf{M}_{\mathtt{xlxnl}} & \mathbf{M}_{\mathtt{xlwl}} & \mathbf{M}_{\mathtt{xlwnl}} & \mathbf{M}_{\mathtt{xly}} \\ \mathbf{M}_{\mathtt{xnlxl}} & \mathbf{M}_{\mathtt{xnlxnl}} & \mathbf{M}_{\mathtt{xnlwl}} & \mathbf{M}_{\mathtt{xnlwnl}} & \mathbf{M}_{\mathtt{xnly}} \\ \hline \mathbf{M}_{\mathtt{wlxl}} & \mathbf{M}_{\mathtt{wlxnl}} & \mathbf{M}_{\mathtt{wlwl}} & \mathbf{M}_{\mathtt{wlwnl}} & \mathbf{M}_{\mathtt{wly}} \\ \mathbf{M}_{\mathtt{wnlwl}} & \mathbf{M}_{\mathtt{wnlxnl}} & \mathbf{M}_{\mathtt{wnlwl}} & \mathbf{M}_{\mathtt{wnlwnl}} & \mathbf{M}_{\mathtt{wnly}} \\ \hline \mathbf{M}_{\mathtt{yxl}} & \mathbf{M}_{\mathtt{yxnl}} & \mathbf{M}_{\mathtt{ywl}} & \mathbf{M}_{\mathtt{ywnl}} & \mathbf{M}_{\mathtt{yy}} \end{array}\right)}_{\mathbf{M}} \underbrace{\begin{pmatrix} \nabla \mathrm{H}_{\mathtt{l}} \\ \nabla \mathrm{H}_{\mathtt{nl}} \\ \mathbf{Z}_{\mathtt{l}} \, \mathbf{w}_{\mathtt{l}} \\ \mathbf{z}_{\mathtt{nl}} \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}} \tag{2}$$

2.1 Linear subsystem

$$\underbrace{\begin{pmatrix} \frac{\mathrm{d}\mathbf{x}_{\mathtt{l}}}{\mathrm{d}t} \\ \mathbf{w}_{\mathtt{l}} \end{pmatrix}}_{\mathbf{b}_{\mathtt{l}}} = \underbrace{\left(\begin{array}{cc|cc|c} \mathbf{M}_{\mathtt{xlxl}} & \mathbf{M}_{\mathtt{xlxnl}} & \mathbf{M}_{\mathtt{xlwl}} & \mathbf{M}_{\mathtt{xlwnl}} & \mathbf{M}_{\mathtt{xly}} \\ \mathbf{M}_{\mathtt{wlxl}} & \mathbf{M}_{\mathtt{wlxnl}} & \mathbf{M}_{\mathtt{wlwl}} & \mathbf{M}_{\mathtt{wlwnl}} & \mathbf{M}_{\mathtt{wly}} \end{array}\right)}_{\mathbf{M}_{\mathtt{l}}} \underbrace{\begin{pmatrix} \nabla \mathrm{H}_{\mathtt{l}} \\ \nabla \mathrm{H}_{\mathtt{nl}} \\ \mathbf{Z}_{\mathtt{l}} \, \mathbf{w}_{\mathtt{l}} \\ \mathbf{z}_{\mathtt{nl}} \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}_{\mathtt{l}}} \tag{3}$$

2.2 Nonlinear subsystem

$$\underbrace{\begin{pmatrix} \frac{\mathrm{d}\mathbf{x}_{\mathtt{nl}}}{\mathrm{d}t} \\ \mathbf{w}_{\mathtt{nl}} \end{pmatrix}}_{\mathbf{b}_{\mathtt{nl}}} = \underbrace{\left(\begin{array}{cc|cc|c} \mathbf{M}_{\mathtt{xnlxl}} & \mathbf{M}_{\mathtt{xnlxnl}} & \mathbf{M}_{\mathtt{xnlwl}} & \mathbf{M}_{\mathtt{xnlwnl}} & \mathbf{M}_{\mathtt{xnly}} \\ \hline \mathbf{M}_{\mathtt{wnlwl}} & \mathbf{M}_{\mathtt{wnlxnl}} & \mathbf{M}_{\mathtt{wnlwl}} & \mathbf{M}_{\mathtt{wnlwnl}} & \mathbf{M}_{\mathtt{wnly}} \end{array}\right)}_{\mathbf{M}_{\mathtt{nl}}} \underbrace{\begin{pmatrix} \nabla \mathrm{H}_{\mathtt{l}} \\ \nabla \mathrm{H}_{\mathtt{nl}} \\ \mathbf{Z}_{\mathtt{l}}\, \mathbf{w}_{\mathtt{l}} \\ \mathbf{z}_{\mathtt{nl}} \\ \mathbf{u} \end{pmatrix}}_{\mathbf{a}_{\mathtt{nl}}} \tag{4}$$

3 Presolve linear part

3.1 Numerical linear subsystem

In the sequel, quantities are defined on the current time step $\mathbf{x} \equiv \mathbf{x}(t_k)$, with $k \in \mathbb{N}_+^*$. The dicrete gradient for the quadratic part of the Hamiltonian is $\nabla \mathrm{H}_{\mathtt{l}} = \frac{1}{2}\, \mathbf{Q}\, (2\mathbf{x}_{\mathtt{l}} + \delta \mathbf{x}_{\mathtt{l}})$ and the discret linear subsystem is

$$\begin{aligned} \mathbf{D}_{\mathtt{l}}^{-1} = \mathbf{iD}_{\mathtt{l}} \quad &= \begin{pmatrix} \frac{\mathbf{I_d}}{\delta t} & 0 \\ 0 & \mathbf{I_d} \end{pmatrix} - \begin{pmatrix} \mathbf{M}_{\mathtt{xlxl}} & \mathbf{M}_{\mathtt{xlwl}} \\ \mathbf{M}_{\mathtt{wlxl}} & \mathbf{M}_{\mathtt{wlwl}} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathbf{Q} & 0 \\ 0 & \mathbf{Z}_{\mathtt{l}} \end{pmatrix}, \\ \underbrace{\begin{pmatrix} \delta \mathbf{x}_{\mathtt{l}} \\ \mathbf{w}_{\mathtt{l}} \end{pmatrix}}_{\mathbf{v}_{\mathtt{l}}} \quad &= \underbrace{\mathbf{D}_{\mathtt{l}} \underbrace{\begin{pmatrix} \mathbf{M}_{\mathtt{xlxl}} \\ \mathbf{M}_{\mathtt{wlxl}} \end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{lxl}}}} \mathbf{Q}}_{\mathbf{N}_{\mathtt{lxl}}}\, \mathbf{x}_{\mathtt{l}} + \underbrace{\mathbf{D}_{\mathtt{l}} \underbrace{\begin{pmatrix} \mathbf{M}_{\mathtt{xlxnl}} & \mathbf{M}_{\mathtt{xlwnl}} \\ \mathbf{M}_{\mathtt{wlxnl}} & \mathbf{M}_{\mathtt{wlwnl}} \end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{lnl}}}}}_{\mathbf{N}_{\mathtt{lnl}}} \underbrace{\begin{pmatrix} \nabla \mathrm{H}_{\mathtt{nl}} \\ \mathbf{z}_{\mathtt{nl}} \end{pmatrix}}_{\mathbf{f}_{\mathtt{nl}}} + \underbrace{\mathbf{D}_{\mathtt{l}} \underbrace{\begin{pmatrix} \mathbf{M}_{\mathtt{xly}} \\ \mathbf{M}_{\mathtt{wly}} \end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{ly}}}}}_{\mathbf{N}_{\mathtt{ly}}} \mathbf{u} \end{aligned} \tag{5}$$

4 Implicite nonlinear function

4.1 Numerical nonlinear subsystem

$$\begin{aligned} \begin{pmatrix} \frac{\mathbf{I_d}}{\delta t} & 0 \\ 0 & \mathbf{I_d} \end{pmatrix} \underbrace{\begin{pmatrix} \delta \mathbf{x}_{\mathtt{nl}} \\ \mathbf{w}_{\mathtt{nl}} \end{pmatrix}}_{\mathbf{v}_{\mathtt{nl}}} \quad &= \underbrace{\begin{pmatrix} \mathbf{M}_{\mathtt{xnlxnl}} & \mathbf{M}_{\mathtt{xnlwnl}} \\ \mathbf{M}_{\mathtt{wnlxnl}} & \mathbf{M}_{\mathtt{wnlwnl}} \end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{nlnl}}}} \mathbf{f}_{\mathtt{nl}} + \underbrace{\begin{pmatrix} \mathbf{M}_{\mathtt{xnly}} \\ \mathbf{M}_{\mathtt{wnly}} \end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{nly}}}} \mathbf{u} \\ &+ \underbrace{\begin{pmatrix} \mathbf{M}_{\mathtt{xnlxl}} & \mathbf{M}_{\mathtt{xnlwl}} \\ \mathbf{M}_{\mathtt{wnlxl}} & \mathbf{M}_{\mathtt{wnlwl}} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathbf{Q} & 0 \\ 0 & \mathbf{Z}_{\mathtt{l}} \end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{nll}}}} \mathbf{v}_{\mathtt{l}} + \underbrace{\begin{pmatrix} \mathbf{M}_{\mathtt{xnlxl}} \\ \mathbf{M}_{\mathtt{wnlxl}} \end{pmatrix}}_{\overline{\mathbf{N}_{\mathtt{nlxl}}}} \mathbf{Q}\, \mathbf{x}_{\mathtt{l}} \end{aligned} \tag{6}$$

4.2 Presolve numerical nonlinear subsystem

$$\begin{pmatrix} \frac{\mathbf{I_d}}{\delta t} & 0 \\ 0 & \mathbf{I_d} \end{pmatrix} \mathbf{v_{nl}} = \underbrace{\left(\overline{\mathbf{N_{nlxl}}} + \overline{\mathbf{N_{nll}}}\,\mathbf{N_{lxl}}\right)}_{\mathbf{N_{nlxl}}} \mathbf{x_l} + \underbrace{\left(\overline{\mathbf{N_{nlnl}}} + \overline{\mathbf{N_{nll}}}\,\mathbf{N_{lnl}}\right)}_{\mathbf{N_{nlnl}}} \mathbf{f_{nl}} \\ \underbrace{\left(\overline{\mathbf{N_{nly}}} + \overline{\mathbf{N_{nll}}}\,\mathbf{N_{ly}}\right)}_{\mathbf{N_{nly}}} \mathbf{u} \tag{7}$$

5 Algorithm

5.1 Inputs

$$\begin{aligned}
\mathbf{iD_l} &= \begin{pmatrix} \frac{\mathbf{I_d}}{\delta t} & 0 \\ 0 & \mathbf{I_d} \end{pmatrix} - \begin{pmatrix} \mathbf{M_{xlxl}} & \mathbf{M_{xlwl}} \\ \mathbf{M_{wlxl}} & \mathbf{M_{wlwl}} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathbf{Q} & 0 \\ 0 & \mathbf{Z_l} \end{pmatrix} \\
\overline{\mathbf{N_{lxl}}} &= \begin{pmatrix} \mathbf{M_{xlxl}} \\ \mathbf{M_{wlxl}} \end{pmatrix} \\
\overline{\mathbf{N_{lnl}}} &= \begin{pmatrix} \mathbf{M_{xlxnl}} & \mathbf{M_{xlwnl}} \\ \mathbf{M_{wlxnl}} & \mathbf{M_{wlwnl}} \end{pmatrix} \\
\overline{\mathbf{N_{ly}}} &= \begin{pmatrix} \mathbf{M_{xly}} \\ \mathbf{M_{wly}} \end{pmatrix} \\
\overline{\mathbf{N_{nlnl}}} &= \begin{pmatrix} \mathbf{M_{xnlxnl}} & \mathbf{M_{xnlwnl}} \\ \mathbf{M_{wnlxnl}} & \mathbf{M_{wnlwnl}} \end{pmatrix} \\
\overline{\mathbf{N_{nll}}} &= \begin{pmatrix} \mathbf{M_{xnlxl}} & \mathbf{M_{xnlwl}} \\ \mathbf{M_{wnlxl}} & \mathbf{M_{wnlwl}} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathbf{Q} & 0 \\ 0 & \mathbf{Z_l} \end{pmatrix} \\
\overline{\mathbf{N_{nlxl}}} &= \begin{pmatrix} \mathbf{M_{xnlxl}} \\ \mathbf{M_{wnlxl}} \end{pmatrix} \mathbf{Q} \\
\overline{\mathbf{N_{nly}}} &= \begin{pmatrix} \mathbf{M_{xnly}} \\ \mathbf{M_{wnly}} \end{pmatrix} \\
\mathcal{J}_{\mathbf{fnl}}(\mathbf{v_{nl}}) &= \begin{pmatrix} \mathcal{J}_{\nabla \mathrm{H_{nl}}} & 0 \\ 0 & \mathcal{J}_{\mathbf{z_{nl}}} \end{pmatrix} \\
\mathbf{I_{nl}} &= \begin{pmatrix} \frac{\mathbf{I_d}}{\delta t} & 0 \\ 0 & \mathbf{I_d} \end{pmatrix}
\end{aligned} \tag{8}$$

5.2 Process

$$
\begin{array}{rcl}
\mathbf{N}_{\mathtt{lxl}} & = & \mathbf{D}_{\mathtt{l}}\,\overline{\mathbf{N}_{\mathtt{lxl}}} \\
\mathbf{N}_{\mathtt{lnl}} & = & \mathbf{D}_{\mathtt{l}}\,\overline{\mathbf{N}_{\mathtt{lnl}}} \\
\mathbf{N}_{\mathtt{ly}} & = & \mathbf{D}_{\mathtt{l}}\,\overline{\mathbf{N}_{\mathtt{ly}}} \\
\mathbf{N}_{\mathtt{nlxl}} & = & \overline{\mathbf{N}_{\mathtt{nlxl}}} + \overline{\mathbf{N}_{\mathtt{nll}}}\,\mathbf{N}_{\mathtt{lxl}} \\
\mathbf{N}_{\mathtt{nlnl}} & = & \overline{\mathbf{N}_{\mathtt{nlnl}}} + \overline{\mathbf{N}_{\mathtt{nll}}}\,\mathbf{N}_{\mathtt{lnl}} \\
\mathbf{N}_{\mathtt{nly}} & = & \overline{\mathbf{N}_{\mathtt{nly}}} + \overline{\mathbf{N}_{\mathtt{nll}}}\,\mathbf{N}_{\mathtt{ly}} \\
\mathbf{c} & = & \mathbf{N}_{\mathtt{nlxl}}\,\mathbf{x}_{\mathtt{l}} + \mathbf{N}_{\mathtt{nly}}\,\mathbf{u} \\
\text{Iterate} & : & \mathbf{F}_{\mathtt{nl}}(\mathbf{v}_{\mathtt{nl}}) = \mathbf{I}_{\mathtt{nl}}\,\mathbf{v}_{\mathtt{nl}} - \mathbf{N}_{\mathtt{nlnl}}\,\mathbf{f}_{\mathtt{nl}} - \mathbf{c} \\
 & & \mathcal{J}_{\mathbf{F}_{\mathtt{nl}}}(\mathbf{v}_{\mathtt{nl}}) = \mathbf{I}_{\mathtt{nl}} - \mathbf{N}_{\mathtt{nlnl}}\,\mathcal{J}_{\mathbf{f}\mathtt{nl}}(\mathbf{v}_{\mathtt{nl}}) \\
 & & \mathbf{v}_{\mathtt{nl}} = \mathbf{v}_{\mathtt{nl}} - \mathcal{J}_{\mathbf{F}_{\mathtt{nl}}}^{-1}(\mathbf{v}_{\mathtt{nl}})\,\mathbf{F}_{\mathtt{nl}}(\mathbf{v}_{\mathtt{nl}}) \\
\mathbf{v}_{\mathtt{l}} & = & \mathbf{N}_{\mathtt{lxl}}\,\mathbf{x}_{\mathtt{l}} + \mathbf{N}_{\mathtt{lnl}}\,\mathbf{f}_{\mathtt{nl}} + \mathbf{N}_{\mathtt{ly}}\,\mathbf{u} \\
\mathbf{y} & = & \mathbf{M}_{\mathtt{yxl}}\,\nabla \mathrm{H}_{\mathtt{l}} + \mathbf{M}_{\mathtt{yxnl}}\,\nabla \mathrm{H}_{\mathtt{nl}} \mathbf{M}_{\mathtt{ywl}}\,\mathbf{Z}_{\mathtt{l}}\,\mathbf{w}_{\mathtt{l}} + \mathbf{M}_{\mathtt{ywnl}}\,\mathbf{z}_{\mathtt{nl}} + \mathbf{M}_{\mathtt{yy}}\,\mathbf{u} \\
\mathbf{x} & = & \mathbf{x} + \delta \mathbf{x}
\end{array}
\tag{9}
$$