Appendix

Einstein or tensor notation

In mathematics, especially in applications of linear algebra to fluid mechanics, the **Einstein notation** or **Einstein summation convention** is a notational convention that is useful when dealing with coordinate equations or formulas.

According to this convention, when an index variable appears twice in a single term, it implies that we are summing over all of its possible values. In typical applications, these are 1, 2 and 3 (for calculations in Euclidean space).

Definitions

In the traditional usage, one has in mind a vector space V with finite dimension n, and a specific basis of V. We can write the basis vectors as $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Then, if $\mathbf{v}$ is a vector in V, it has coordinates $v_1, \ldots, v_n$ relative to this basis.

The basic rule is

$$(\mathbf{v})_i = v_i \mathbf{e}_i .$$

In this expression, it is assumed that the term on the right-hand side is to be summed as i goes from 1 to n, because the index i appears twice. The index i is known as a *dummy index* since the result is not dependent on it; thus we could also write, for example,

$$(\mathbf{v})_j = v_j \mathbf{e}_j .$$

An index that is not summed over is a *free index*, and should be found in each term of the equation or formula.

If $\boldsymbol{H}$ is a matrix and $\mathbf{v}$ is a column vector, then $\boldsymbol{H}\mathbf{v}$ is another column vector. To define $\mathbf{w} = \boldsymbol{H}\mathbf{v}$, we can write

$$w_i = H_{ij} v_j .$$

The dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ can be written

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i .$$

There are two useful symbols that simplify multiplication rules, the

Kronecker delta,

$$\delta_{ij} = (\mathbf{I})_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and the Levi-Civita symbol e (or ε),

$$e_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is a positive permutation of } (1, 2, 3) \\ -1 & \text{if } (ijk) \text{ is a negative permutation of } (1, 2, 3) \\ 0 & \text{if } (ijk) \text{ is a not a permutation of } (1, 2, 3) \text{ at all;} \end{cases}$$

(1, 2, 3), (3, 1, 2) and (2, 3, 1) are positive, (3, 2, 1), (1, 3, 2) and (2, 1, 3) are negative and (1, 2, 2) etc. are not permutations of (1, 2, 3).

If $n = 3$, we can write the cross product, using the Levi-Civita symbol. Specifically, if $\mathbf{w}$ is $\mathbf{u} \times \mathbf{v}$, then

$$w_i = e_{ijk} u_j v_k. \qquad .$$

Operators

For general operations on scalars, vectors and matrices in fluid mechanics, ϕ is any scalar having the rank 0, $\mathbf{u}$ is a velocity vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

having the rank 1 and $\boldsymbol{\tau}$ is a (3×3) tensor

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

having the rank 2.

Various operators will change the rank of the tensor:

- gradients will increase the rank by 1,
- $\times$ product decreases the rank by 1,
- $\bullet$ product decreases the rank by 2,
- : product decreases the rank by 4.

The gradient of a scalar is

$$\nabla\phi = \begin{bmatrix} \dfrac{\partial\phi}{\partial x_1} \\ \dfrac{\partial\phi}{\partial x_2} \\ \dfrac{\partial\phi}{\partial x_3} \end{bmatrix}$$

or

$$(\nabla\phi)_i = \frac{\partial}{\partial x_i}\phi.$$

The rank is $0 + 1 = 1$.

The Laplacian is

$$\nabla\cdot\nabla\phi = \nabla^2\phi = \Delta\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\phi$$

or

$$\Delta\phi = \frac{\partial^2\phi}{\partial x_i\,\partial x_i}.$$

The rank is $0 + 1 - 1 = 0$.

For **u**

$$\nabla^2\mathbf{u} = \begin{bmatrix} \dfrac{\partial^2 u_1}{\partial x_1^2} + \dfrac{\partial^2 u_1}{\partial x_2^2} + \dfrac{\partial^2 u_1}{\partial x_1^2} \\ \dfrac{\partial^2 u_2}{\partial x_1^2} + \dfrac{\partial^2 u_2}{\partial x_2^2} + \dfrac{\partial^2 u_2}{\partial x_3^2} \\ \dfrac{\partial^2 u_3}{\partial x_1^2} + \dfrac{\partial^2 u_3}{\partial x_2^2} + \dfrac{\partial^2 u_3}{\partial x_3^2} \end{bmatrix}$$

or

$$(\Delta\mathbf{u})_i = \frac{\partial^2 u_i}{\partial x_j\,\partial x_j}.$$

The rank is $1 + 1 - 1 = 1$.

For the dot product, the divergence,

$$\nabla\cdot\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

or

$$\nabla \cdot \mathbf{u} = \frac{\partial}{\partial x_i} u_i;$$

$$\mathbf{u} \cdot \nabla \mathbf{u} = \begin{bmatrix} u_1 \dfrac{\partial u_1}{\partial x_1} + u_2 \dfrac{\partial u_1}{\partial x_2} + u_3 \dfrac{\partial u_1}{\partial x_3} \\ u_1 \dfrac{\partial u_2}{\partial x_1} + u_2 \dfrac{\partial u_2}{\partial x_2} + u_3 \dfrac{\partial u_2}{\partial x_3} \\ u_1 \dfrac{\partial u_3}{\partial x_1} + u_2 \dfrac{\partial u_3}{\partial x_2} + u_3 \dfrac{\partial u_3}{\partial x_3} \end{bmatrix}$$

or

$$(\mathbf{u} \cdot \nabla \mathbf{u})_i = u_j \frac{\partial u_i}{\partial x_j}.$$

The rank is $1 + (1 + 1) - 2 = 1$.

For the cross product, the curl,

$$\nabla \times \mathbf{u} = \begin{bmatrix} \dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} \end{bmatrix}$$

or

$$(\nabla \times \mathbf{u})_i = e_{ijk} \frac{\partial}{\partial x_j} u_k.$$

The rank is $1 + 1 - 1 = 1$.

For the Frobenius inner product

$$\begin{aligned} \boldsymbol{\tau} : \nabla \boldsymbol{U} = \tau_{11} \frac{\partial U_1}{\partial x_1} + \tau_{12} \frac{\partial U_1}{\partial x_2} + \tau_{13} \frac{\partial U_1}{\partial x_3} \\ + \tau_{21} \frac{\partial U_2}{\partial x_1} + \tau_{22} \frac{\partial U_2}{\partial x_2} + \tau_{23} \frac{\partial U_2}{\partial x_3} \\ + \tau_{31} \frac{\partial U_3}{\partial x_1} + \tau_{32} \frac{\partial U_3}{\partial x_2} + \tau_{33} \frac{\partial U_3}{\partial x_3} \end{aligned}$$

or

$$\boldsymbol{\tau} : \nabla \boldsymbol{U} = \tau_{ij} \frac{\partial U_i}{\partial x_j}.$$

The rank is $2 + (1 + 1) - 4 = 0$.