Bitte setzt euch in den vordersten drei Reihen!

Lineare Algebra

Übung 1, 25 September 2025

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Programm

- Nachbesprechung Serie 0
- Theorie-Input
- In-class Exercise

Nachbesprechung Assignment 0

Bitte schreibt nicht in Rot oder Grün

- Mit diesen Farben werde ich korrigieren

Was ist ein Beweis?

- Die logische Erklärung, warum eine Aussage wahr oder falsch ist

The "things" that can be true or false are called *statements*, and the logical arguments are called *proofs*. When speaking about statements and proofs, mathematical language is very precise, typically much more precise and less "forgiving" than natural language. The mathematician needs a very clear understanding of what a statement really means, in order to argue about it in a proof.

Mehr dazu:
https://ti.inf.ethz.ch/ew/courses/LA24/mathe-vorschau/language-and-logic.html
https://www.logik.uni-jena.de/logikmedia/75/einfuehrung-in-das-mathematische-beweisen.pdf?nonactive=1&suffix=pdf

Wie schreibt man einen Beweis?

- Stellt euch vor, ihr erklärt einem Mitstudierenden, wieso die Aussage stimmt. Kann diese Person verstehen, was ihr meint?
- Erklärt, was ihr macht
- Benutzt Wörter!

Beispiel für einen Beweis

a) Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$.

We want $a, b \in \mathbb{R}$ so that $av + bw = u$.

$$a \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\rightarrow \left| \begin{array}{l} a - b = u_1 \\ a + b = u_2 \end{array} \right| \rightarrow \begin{array}{l} a = u_1 + b \\ u_1 + 2b = u_2 \end{array} \rightarrow \begin{array}{l} b = \frac{u_2 - u_1}{2} \\ a = \frac{u_2 + u_1}{2} \end{array}$$

This means that we can write any vector $u \in \mathbb{R}^2$ as a linear combination $a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ with $a = \frac{u_1 + u_2}{2}$, $b = \frac{u_2 - u_1}{2}$.

Fragen?

Theorie

Skalarprodukt

Definition 1.9 (Scalar product). *Let*

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} \in \mathbb{R}^m.$$

The scalar product of $\mathbf{v}$ *and* $\mathbf{w}$ *is the number*

$$\mathbf{v} \cdot \mathbf{w} := v_1 w_1 + v_2 w_2 + \cdots + v_m w_m = \sum_{i=1}^{m} v_i w_i.$$

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Observation 1.10. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^m$ *be vectors and* $\lambda \in \mathbb{R}$ *a scalar. Then*

(i) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$; (symmetry)

(ii) $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (\lambda \mathbf{w})$; (taking out scalars)

(iii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ *and* $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$; (distributivity)

(iv) $\mathbf{v} \cdot \mathbf{v} \geq 0$, *with equality exactly if* $\mathbf{v} = \mathbf{0}$. (positive-definiteness)

Euklidische Norm

Definition 1.11 (Euclidean norm). *Let* $\mathbf{v} \in \mathbb{R}^m$. *The Euclidean norm of* $\mathbf{v}$ *is the number*

$$\|\mathbf{v}\| := \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

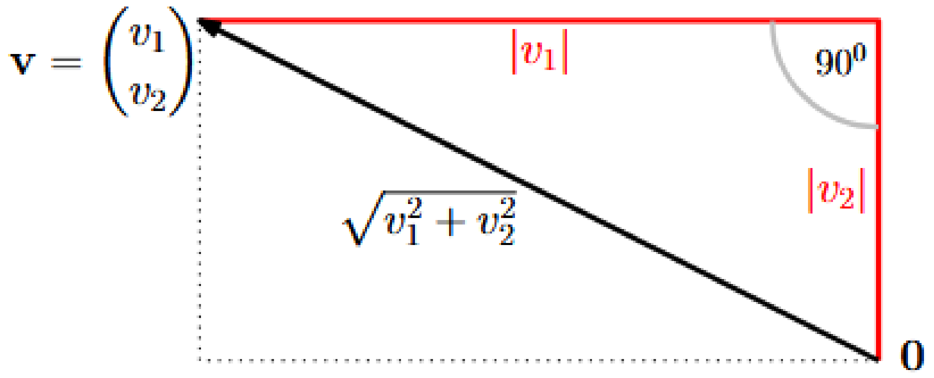

Figure 1.10: The Euclidean norm measures the length of a vector in $\mathbb{R}^2$.

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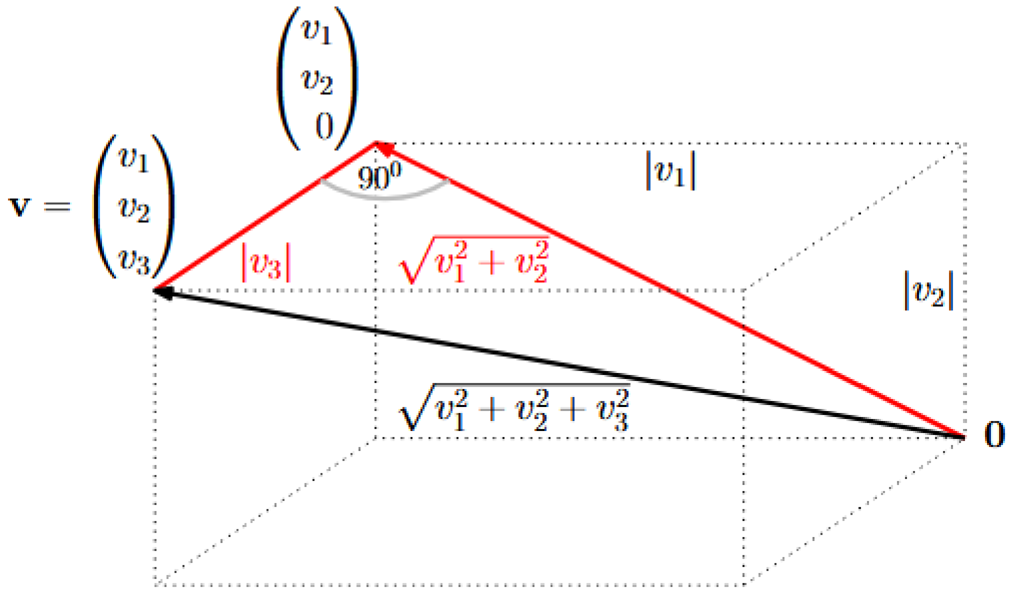
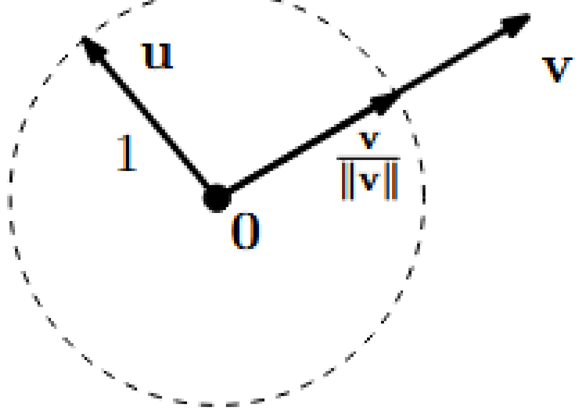


Figure 1.11: The Euclidean norm measures the length of a vector in $\mathbb{R}^3$.

Einheitsvektoren

Unit vectors. A *unit vector* is a vector $\mathbf{u}$ such that $\|\mathbf{u}\| = 1$.



Einheitsvektoren

For $\mathbf{v} \neq \mathbf{0}$, $\frac{\mathbf{v}}{\|\mathbf{v}\|} := \frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector.

- Wieso?
- Sei $\mathbf{w} = \frac{\mathrm{v}}{\|\mathrm{v}\|}$.
- Wir berechnen $\mathbf{w} \cdot \mathbf{w} = \left(\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right) \cdot \left(\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right) = \frac{1}{\|\mathbf{v}\|} \cdot \frac{1}{\|\mathbf{v}\|}(\mathbf{v} \cdot \mathbf{v})$
- Also ist $\|\mathbf{w}\| = \sqrt{\mathbf{w} \cdot \mathbf{w}} = \sqrt{\frac{1}{\|\mathbf{v}\|} \cdot \frac{1}{\|\mathbf{v}\|}(\mathbf{v} \cdot \mathbf{v})} = \sqrt{\frac{1}{\|\mathbf{v}\|^2} \cdot \|\mathbf{v}\|^2} = 1$

Cauchy-Schwarz Ungleichung

Lemma 1.12 (Cauchy-Schwarz inequality). *For any two vectors* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$*,*

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

Moreover, equality holds exactly if one vector is a scalar multiple of the other.

Winkel

Definition 1.14 (Angle). *Let* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$ *be two nonzero vectors. The angle between them is the unique* α *between* 0 *and* π *(180 degrees) such that*

$$\cos(\alpha) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \in [-1, 1].$$

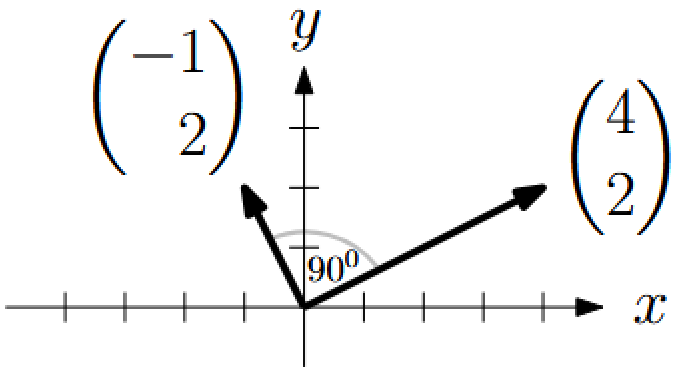
(Here, the interval $[-1, 1] = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ *comes from the Cauchy Schwarz inequality, Lemma 1.12). In other words,*

$$\alpha = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right).$$

Orthogonalität

Orthogonal vectors:

Definition 1.15: Vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are *orthogonal* (or *perpendicular*) if $\mathbf{v} \cdot \mathbf{w} = 0$ (same as $\cos(\alpha) = 0$, or 90 degrees).



$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -4 \cdot 1 + 2 \cdot 2 = 0$$

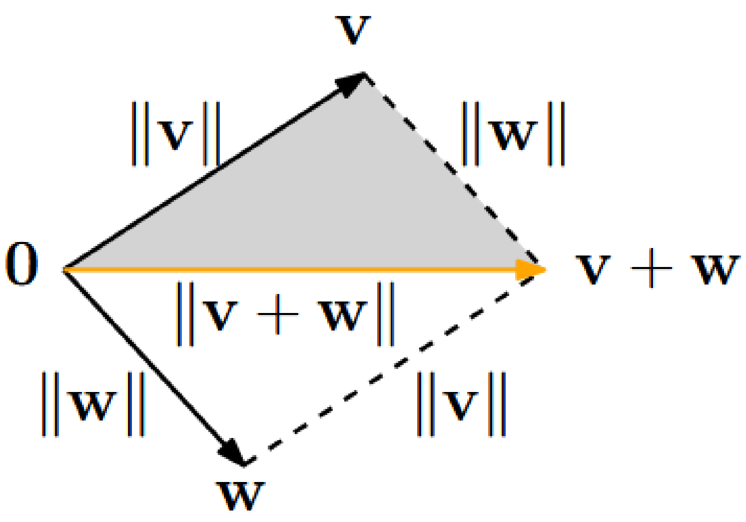
Figure 1.18: Orthogonal vectors: the scalar product equals 0.

Dreiecksungleichung

Lemma 1.17: Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$. Then

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

In $\mathbb{R}^2$: From $\mathbf{0}$ directly to $\mathbf{v} + \mathbf{w}$ is shorter than via $\mathbf{v}$ or $\mathbf{w}$:



Lineare Unabhängigkeit

Linear (in)dependence:
Definition 1.21: Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are *linearly dependent* if at least one of them is a linear combination of the others, i.e. there exists an index $k \in [n]$ and scalars λ_j such that

$$\mathbf{v}_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j \mathbf{v}_j.$$

Otherwise, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are *linearly independent*.

Linear Unabhängig?

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

✓

Linear Unabhängig?

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Linear Unabhängig?

$$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$$

Linear Unabhängig?

$$\mathbf{v} \neq \mathbf{0} \quad ✓$$

Linear Unabhängig?

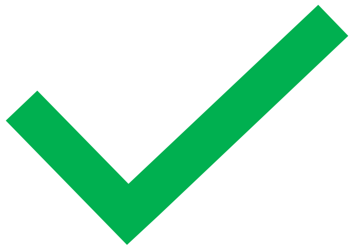
$$\mathbf{v} = \mathbf{0}$$

Linear Unabhängig?

$$\ldots, \mathbf{v}, \ldots, \mathbf{v}, \ldots$$

Linear Unabhängig?

Leere Sequenz ()



Äquivalente Definitionen

Lemma 1.22: Let $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n \in \mathbb{R}^m$. The following statements are *equivalent* (all true, or all false).

(i) At least one of the vectors is a linear combination of the other ones (linearly dependent by Definition 1.21).

(ii) There are scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ besides $0, 0, \ldots, 0$ such that $\sum_{j=1}^{n} \lambda_j \mathbf{v}_j = \mathbf{0}$. Math jargon: $\mathbf{0}$ is a *nontrivial linear combination* of the vectors.

(iii) At least one of the vectors is a linear combination of the previous ones.

Wieso mehrere Definitionen?

- Ist $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ linear unabhängig?

(iii) At least one of the vectors is a linear combination of the previous ones.

Eindeutigkeit von Linearkombinationen

Lemma 1.24. *Let* $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n \in \mathbb{R}^m$ *be linearly independent,* $\mathbf{v} \in \mathbb{R}^m$*. Let*

$$\mathbf{v} = \sum_{j=1}^{n} \lambda_j \mathbf{v}_j = \sum_{j=1}^{n} \mu_j \mathbf{v}_j$$

be two ways of writing $\mathbf{v}$ *as a linear combination of* $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$*. Then* $\lambda_j = \mu_j$ *for all* $j \in [n]$.

Span

Definition 1.25 (Span). *Let* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^m$. *Their* span *is the set of all linear combinations. In formulas,*

$$\mathbf{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) := \left\{ \sum_{j=1}^{n} \lambda_j \mathbf{v}_j : \lambda_j \in \mathbb{R} \textit{ for all } j \in [n] \right\}.$$

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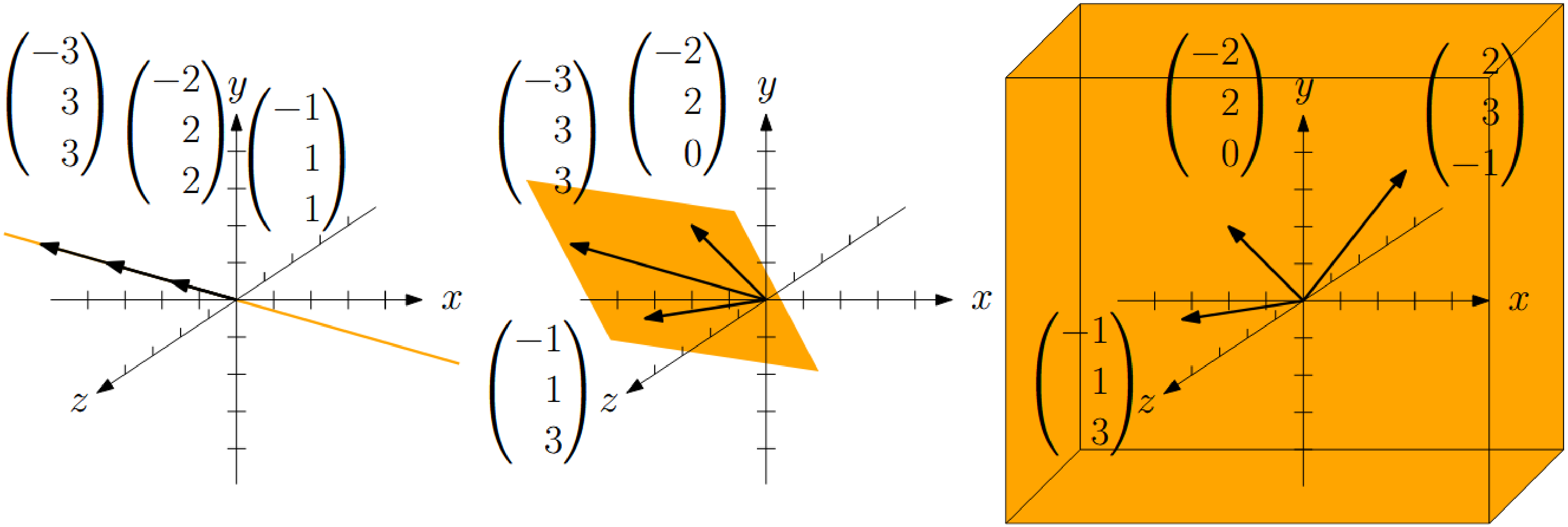

Figure 1.22: The span of three vectors in $\mathbb{R}^3$: a line, a plane, or the whole space

Fragen?

Feedback

Übungen