Peergrade #5

Exercise 1

$Let\ a_n\ denote\ a\ sequence\ such\ that: a_0 = 1, a_1 = 1, a_n = 2 * a_{n-1} + a_{n-2}\ for\ n\ even\ and\ a_n = 3 * a_{n-1}\ for\ n\ odd. Show\ that\ a_i \leq 3^{i-1}\ for\ all\ i \in \mathbb{Z}^+.$

Predicate:

$$P(i) := \begin{cases} 1 & i < 2 \\ 2 * P(i-1) + P(i-2) & i\ is\ even \\ 3 * P(i-1) & i\ is\ odd \end{cases}$$

Base cases:

$$P(1) = 1 \leq 3^{1-0} = 3$$
$$P(2) = 2 * P(1) + P(0) = 2 * 1 + 1 = 3 \leq 3^{2-1} = 3$$

We see that both P(1) and P(2) holds.

Inductive case:

Assume $P(k)$ holds for all $k < i$ to prove that $P(i)$ holds.

To prove $P(i)$, we have to split the proof into two different cases: When I is even and when I is odd.

Let us start with the even case:

We assume that $P(i-1)$ and $P(i-2)$ holds to prove that $P(i) = 2 * P(i-1) + P(i-2) \leq 3^{i-1}$ holds.

Since i is even we know that $i-1$ is odd and thus we can substitute $P(i-1)$ with the definition for $P(i)$ where i is odd:

$$P(i) = 2 * \left(3 * P(i-1-1)\right) + P(i-2)$$

This is equivalent to $P(i) = 7 * P(i-2)$ by multiplying 2 into the parenthesis.

We know that $P(i-2) \leq 3^{i-3}$ according to the definition of $P(i)$. Using exponentiation rules we can get that $3^{i-3} = \frac{3^{i-1}}{3^2}$.

$$P(i-2) \leq \frac{3^{i-1}}{3^2}$$

Let us multiply by 9 on both sides of the equation:

$$9 * P(i-2) \leq 3^{i-1}$$

Since $7 * P(i-2) < 9 * P(i-2)$, and $9 * P(i-2) \leq 3^{i-1}$, we have:

$$7 * P(i-2) < 3^{i-1}$$

Since we showed earlier that $P(i) = 7 * P(i-2)$ we can substitute $7 * P(i-2)$ with $P(i)$:

$$P(i) < 3^{i-1}$$

Thus we have proven the inductive hypothesis for even integers.

Let us do the odd case:

We assume that $P(i-1) \leq 3^{i-1-1}$ holds to prove $3 * P(i-1) \leq 3^{i-1}$.

Let us use exponentiation rules:

$$3^{i-1-1} = \frac{3^{i-1}}{3^1}$$

Thus we have:

$$P(i-1) \leq \frac{3^{i-1}}{3^1}$$

And thus:

$$P(i-1) * 3 \leq 3^{i-1}$$

Since the order of multiplication does not matter, we have that $3 * P(i-1) \leq 3^{i-1}$ which was what we wanted.

Exercise 2

Fifteen women and twelve men are attending a conference, and need to form the committee to decide the next venue.

1. How many ways are there to select a committee of five conference attendees if at least two women must be on the committee?

We can use our combinatorics formula to determine the combination of attendees possible where we choose 5 among the 27 in total:

$$C(n, r) = \frac{n!}{r! * (n-r)!}$$

$$C(27,5) = \frac{27!}{5! * (27-5)!} = \frac{27!}{5! * 22!} = 80.730 \text{ } combinations$$

Now let's consider the combinations that are invalid. That is all the combinations where we have 4 men (thus only having one woman) or having 5 men (thus having no women).
For the 4 men, the combinations formula is as follows:

$$C(12,4) = \frac{12!}{4! * (12-4)!} = \frac{12!}{4! * 8!} = 495$$

In this case we also have one woman on the committee (such that we will all 5 spots). Thus we have $C(4,12) * 15$ different combinations.

$$C(12,4) * 15 = 495 * 15 = 7.425 \text{ } combiantions$$

Lastly, let's consider the case for 5 men:

$$C(12,5) = \frac{12!}{5! * (12-5)!} = \frac{12!}{5! * 7!} = 792 \text{ } combinations$$

Thus we have 80.730 combinations to create this committee of which 7.425 plus 792 are invalid:

$$80.730 - 7.425 - 792 = 72.513 \text{ } combinations$$

Thus we have 72.513 different ways to create a committee with at least 2 women.

2. How many ways are there to select a committee of five attendees if at least one woman and at least one man must be on the committee?

Like above, we will calculate the number of ways to select a committee of five attendees and then subtract the invalid committees from this set.
We again have the total number of committees:

$$C(27,5) = 80.730 \text{ } combinations$$

We also have the number of committees where it consists of 5 men:

$$C(12,5) = 792 \text{ } combinations$$

Now let's calculate the number of committees that consists of 5 women:

$$C(15,5) = \frac{15!}{5! * (15-5)!} = \frac{15!}{5! * 10!} = 3.003 \text{ } combinations$$

Thus we have a number of valid committees:

$$80.730 - 792 - 3.003 = 76.935 \text{ } combinations$$

Thus we have 76.935 different ways to create a committee with at least one man and one woman.

Exercise 3

A manufacturing facility for PCBWay produces Printed Circuit Boards (PCBs) with an accuracy rate of 92%: that is, 8% of the PCBs are faulty and need to be discarded. PCBWay's quality control is in charge of discarding faulty PCBs. It detects 95% of the faulty PCBs, and also erroneously detects as faulty 2% of the good PCBs. What is the probability that

- *a discarded PCB is indeed faulty?*

In terms of probability, we can calculate this using conditional probability: What is the probability that a PCB is faulty given that it is discarded?
Let A be the event that a PCB is faulty and B be the event that a PCB is discarded.
Thus $A \cap B$ is the event that a PCB is both faulty and discarded.
Then conditional probability is calculated using the formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
We know that $P(A \cap B\ ) = 0{,}08 * 0{,}95 = 0{,}076$ and $P(B) = 0{,}02 * 0{,}92 + 0{,}08 * 0{,}95 = 0{,}0944$. Thus we have:

$$P(A|B) = \frac{0{,}076}{0{,}0944} = 0{,}805 \approx 80{,}5\%$$

I.e. the probability that a discarded PCB is indeed faulty is 80,5%.

- *a discarded PCB is not faulty?*

Consider the above calculation. Since a PCB can only be either faulty or not faulty, we have that the probability must be $P(A^c|B) = 1 - P(A|B) = 1 - 0{,}805 = 0{,}195 \approx 19{,}5\%$.
Thus the probability that a discarded PCB is not faulty is 19,5%.

- *a PCB that passed quality control is faulty?*

Again, we can describe this question using conditional probability: What is the probability that a PCB is faulty given it has passed quality control?
Using the same definitions for events as above, this is written $P(A|B^c)$.
Thus the formula is:

$$P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)}$$

To calculate $P(A \cap B^c)$, that is, the probability that a PCB is faulty and not discarded, we can do the following:

$$P(A \cap B^c) = (1 - 0{,}95) * 0{,}08 = 0{,}004 \approx 0{,}4\%$$

To calculate $P(B^c)$, that is, the probability that a PCB is not discarded, we can do the following:

$$P(B^c) = 1 - P(B) = 1 - 0{,}0944 = 0{,}9056 \approx 90{,}6\%$$

Then, the formula is as follows:

$$P(A|B^c) = \frac{P(A \cap b^c)}{P(B^c)} = \frac{0{,}004}{0{,}9056} = 0{,}0044 \approx 0{,}44\%$$

I.e. the probability that a PCB is faulty given it has passed quality control is 0,44%.

- *a PCB that passed quality control is not faulty?*

Again, using the probability of complements of events we can do the following:

$$P(A^c|B^c) = 1 - P(A|B^c) = 1 - 0{,}0044 = 0{,}9956 \approx 99{,}56\%$$

I.e. the probability of a PCB is not faulty given that it has passed quality control is 99,56%.