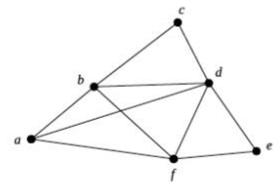
Peergrade #3

Exercise 1

1.1

a) The graph is connected because there is a path from any point x to any other point y through the edges between the vertices.

1. 

(a) The graph is connected
(b) The graph has an Euler circuit
(c) The graph has a Hamiltonian circuit

b) We see that the points have the following number of edges:

a. 3
b. 4
c. 2
d. 5
e. 2
f. 4

For an Euler circuit we would have to start a path that uses all edges from one vertex and see this path end at the same vertex (meaning it would then be able to start at any vertex and create the same cycle). We see that two vertices have an odd number of edges whereas the rest have an even number.
For an Euler circuit, we would then require an edge to get to a vertex and another edge to move away from the vertex again, hence an even number of vertices.
We do however see an Eulerian path because there are only two vertices with an odd number of vertices. An Eulerian path could be as follows:

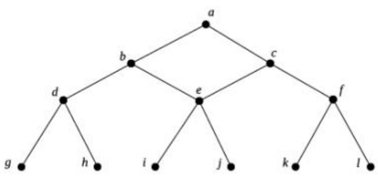
$$a \to b \to c \to d \to e \to f \to a \to d - b \to f \to d$$

Because it is an undirected graph, the Eulerian path could also go in the opposite direction. We have that all edges are used exactly once but as the path starts in vertex *a* and ends in vertex *d* we do not have a circuit.

c) For a Hamiltonian circuit we would have to find a path which goes through all vertices only once and starts and ends in the same point.
There is one along the outer edges as follows:

$$a \to b \to c \to d \to e \to f \to a$$

Here we see that all vertices are used and that it ends up again at the same point as the start. Since it is an undirected graph, we can once again do this in both directions. Since it is a circuit, we could also start at any other point and use the same circuit to find a Hamiltonian circuit.

2. 

(a) The graph represents a tree
(b) There is an unique simple path between any two vertices
(c) The graph represents a full binary tree

1.2

a) For the graph to represent a tree, we would for all pairs of vertices a single, unique path.
However because e is connected to both b and c, this is not true.
We see that a path from j to a can be both through e & b or through e & c. Hence we do not have a tree.

b) As described above, we do not have unique simple paths between all set of two vertices.

c) As we do not have a tree as discussed above, we do not have a binary tree either.
Had there not been a simple circuit in the tree, we would see that all levels but the last would satisfy, that the vertices have exactly two children (also called the internal vertices) and as such would be a binary tree.

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Exercise 2

2.1

a) $a_j = j^2$

The sequence will be calculated as follows for $1 < j < 10$:

$$a_2 = 2^2, a_3 = 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2$$

This would give the following result:

$$\{a_j\} = \{4, 9, 16, 25, 36, 49, 64, 81\}$$

b) $b_j = \frac{1}{j}$

The sequence is as follows:

$$b_2 = \frac{1}{2}, b_3 = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}$$

The sequence would then be as follows:

$$\{b_j\} = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\right\}$$

c) $c_j = \sum_{i=1}^{j} a_i * b_i$

The sequence is then as follows:

$$c_2 = a_1 * b_1 + a_2 * b_2 = 1^2 * \frac{1}{1} + 2^2 * \frac{1}{2} = 3$$

$$c_3 = a_1 * b_1 + a_2 * b_2 + a_3 * b_3 = 1^2 * \frac{1}{1} + 2^2 * \frac{1}{2} + 3^2 * \frac{1}{3} = 6$$

$$c_4 = \frac{1^2}{1} + \frac{2^2}{2} + \frac{3^2}{3} + \frac{4^2}{4} = 10$$

$$c_5 = 15$$

$$c_6 = 21$$

$$c_7 = 28$$

$$c_8 = 36$$

$$c_9 = 45$$

Hence the sequence is:

$$\{c_j\} = \{3, 6, 10, 15, 21, 28, 36, 45\}$$

2.2

a) $(37^{300})\ mod\ 111$

We see that $(37^1)\ /\ 111 = 0 * 111 + 37$ and hence the remainder is 37, i.e. $(37^1)\ mod\ 111 = 37$.

We also see that $(37^2)\ /\ 111 = 1369\ /\ 111 = 12 * 111 + 37$ and hence the remainder is again 37. We then now that $(37^n)\ /\ 111 = a * 111 + 37$ where $a \in \mathbb{Z}$. Following this we have that $(37^n)\ mod\ 111 = 37 \rightarrow (37^{300})\ mod\ 111 = 37$.

b) $\big(9987421 * (32413 + 43256)\big)\ mod\ 57$

We have the following two theorems from class:

$$(a + b)\ mod\ m = \big((a\ mod\ m) + (b\ mod\ m)\big)\ mod\ m$$

$$(a * b)\ mod\ m = \big((a\ mod\ m) * (b\ mod\ m)\big)\ mod\ m$$

We can use the second theorem to rewrite the equation to the following:

$$(9987421\ mod\ 57 * (32413 + 43256)\ mod\ 57)\ mod\ 57$$

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Consider $(32413 + 43256)\ mod\ 57$ in the above equation. We can rewrite that to the following using the first theorem:

$$(32413\ mod\ 57 + 43256\ mod\ 57)\ mod\ 57$$

And combined that gives the following:

$$\big(9987421\ mod\ 57 * \big((32413\ mod\ 57 + 43256\ mod\ 57)\ mod\ 57\big)\big)\ mod\ 57$$

Let's calculate each simple step:

$$9987421 = 175217 * 57 + 10 \rightarrow 987421\ mod\ 57 = 52$$
$$32413 = 568 * 57 + 37 \rightarrow 32413\ mod\ 57 = 37$$
$$43256 = 758 * 57 + 50 \rightarrow 43256\ mod\ 57 = 50$$

Hence we have

$$\big(52 * \big((37 + 50)\ mod\ 57\big)\big) mod\ 57$$

We can then calculate the following $(37 + 50)\ mod\ 57$:

$$37 + 50 = 87$$
$$87 = 1 * 57 + 30 \rightarrow 87\ mod\ 57 = 30$$

Hence we have left:

$$(52 * 30)\ mod\ 57$$
$$1560 = 27 * 57 + 21 \rightarrow 300\ mod\ 57 = 21$$

Hence we have that $\big(9987421 * (32413 + 43256)\big)\ mod\ 57 = 21$

2.3

a) Convert $(35710)_8$ to its binary and hexadecimal expansion

Let's start converting the octal number to a decimal number.

That is done as follows, where we consider each digit in the octal number by itself:

$$0 * 8^0 = 0$$
$$1 * 8^1 = 8$$
$$7 * 8^2 = 448$$
$$5 * 8^3 = 2560$$
$$3 * 8^4 = 12288$$

We then have that $(35710)_8 = (0 + 8 + 448 + 2560 + 12288)_{10} = (15304)_{10}$

We can then convert this decimal number to the binary and hexadecimal expansions. Let's start with the binary expansion:

$$15304 = 7652 * 2 + 0$$
$$7652 = 3826 * 2 + 0$$
$$3826 = 1913 * 2 + 0$$
$$1913 = 956 * 2 + 1$$
$$956 = 478 * 2 + 0$$
$$478 = 239 * 2 + 0$$
$$239 = 119 * 2 + 1$$
$$119 = 59 * 2 + 1$$
$$59 = 29 * 2 + 1$$
$$29 = 14 * 2 + 1$$
$$14 = 7 * 2 + 0$$
$$7 = 3 * 2 + 1$$
$$3 = 1 * 2 + 1$$
$$1 = 0 * 2 + 1$$

We then have the decimal number as this combination of 1's and 0's read from behind:

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$$(35710)_8 = (15304)_{10} = (11101111001000)_2$$

For the hexadecimal we use the same method but using 16 instead of 2:

$$15304 = 956 * 16 + 8$$
$$956 = 59 * 16 + 12$$
$$59 = 3 * 16 + 11$$
$$3 = 0 * 16 + 3$$

As we are using the hexadecimal number system, we will substitute 11 with B and 12 with C so we get the hexadecimal number $(3BC8)_{16}$

We then have the expansions:

$$(35710)_8 = (11101111001000)_2 = (3BC8)_{16}$$

b) Convert $(91038)_{10}$ to its octal and hexadecimal expansion.

Using the same method as above:

$$91038 = 11379 * 8 + 6$$
$$11379 = 1422 * 8 + 3$$
$$1422 = 177 * 8 + 6$$
$$177 = 22 * 8 + 1$$
$$22 = 2 * 8 + 6$$
$$2 = 0 * 8 + 2$$

Hence we have the octal expansion $(261636)_8$.

For the hexadecimal expansion:

$$91038 = 5689 * 16 + 14$$
$$5689 = 355 * 16 + 9$$
$$355 = 22 * 16 + 3$$
$$22 = 1 * 16 + 6$$
$$1 = 0 * 16 + 1$$

Substituting 14 with the hexadecimal equivalent E, we have the expansion $(1639E)_{16}$

Hence the expansions combined $(91038)_{10} = (261636)_8 = (1639E)_{16}$

2.4

a) $\gcd(2574{,}1976)$

We'll use the Euclidian Algorithm to calculate the greatest common denominator.

$$2574 = 1976 * 1 + 598$$
$$1976 = 598 * 3 + 182$$
$$598 = 182 * 3 + 52$$
$$182 = 52 * 3 + 26$$
$$52 = 26 * 2$$

Hence 26 is the greatest common divisor for 2574 and 1976 because 26 is the last nonzero remainder.

b) $lcm(1525, 4405)$

First we'll factorize the two numbers by finding prime numbers that divides the numbers.

$$5|1525, yields\ 305$$
$$5|305\ yields\ 61$$

Since 61 is a prime, we have that $1525 = 5^2 * 61^1$

For 4405 we'll do the same.

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$$5|4405, yields\ 881$$

881 is also a prime so we have that $4405 = 5^1 * 881^1$

The $lcm(5^2 * 61^1, 5^1 * 881^1) = 5^{\max(2,1)} * 61^{\max(1,0)} * 881^{\max(0,1)} = 5^2 * 61^1 * 881^1 =$ 1.343.525

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Exercise 3

Let x, y and z be integers. Assume that $x \equiv y \ (mod\ m)$ and that $m|z$.
Prove that $x + 2z \equiv y \ (mod\ m)$.

$$x + 2z \equiv y \ (mod\ m) \Leftrightarrow (x + 2z)\ mod\ m = y\ mod\ m$$

We will use the following theorem on the left side of the equation:

$$(a + b)\ mod\ m = \big((a\ mod\ m) + (b\ mod\ m)\big)\ mod\ m$$

$$\big((x\ mod\ m) + (2z\ mod\ m)\big)\ mod\ m = y\ mod\ m$$

We know that m divides z meaning that $z = am$ for some integer a.
The modulo function is the remainder *r* in $a = dq + r$, where $a, d, q, r \in \mathbb{Z}$ and $0 \leq r < d$.
We can then derive that if $z = am$ then the remainder in a divison $m|z$ must be 0.
In our equation we get:

$$\big((x\ mod\ m) + 0\big)\ mod\ m = y\ mod\ m$$

The value zero has no effect on the modulo function and hence we have:

$$(x\ mod\ m)\ mod\ m = y\ mod\ m$$

Since we know $r < d$, doing the modulo function on the same number twice will give us the same result as just using the modulo function once. We then have that:

$$x\ mod\ m = y\ mod\ m$$

Which is an assumption of this proof and hence we have proven that $x + 2z \equiv y \ (mod\ m)$.