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# Speaking Mathematically

## 1.1 Variables

1.1.2 Fill in the blanks using variables: Is there an integer that has a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6?

1. Is there an integer *n* such that *n* has \_\_\_?  
   (1): The remainder 2 when it is divided by 5 and a remainder of 3 when it is divided by 6?
2. Does there exist \_\_\_ such that if *n* is divided by 5 the remainder is 2 and if \_\_\_?  
   (1): An integer *n*  
   (2): *n* is divided by 6, the remainder is 3?

Such a number could be 27:

27 / 5 = 5 with the remainder 2

27 / 6 = 4 with the remainder 3

1.1.4 Fill in the blanks using variables: Given any real number, there is a real number that is greater.

1. Given any real number *r*, there is \_\_\_\_ *s* such that *s* is \_\_\_\_.  
   (1): a real number  
   (2): greater than r
2. For any \_\_\_\_, \_\_\_\_ such that *s* > *r*.   
   (1): real number *r*  
   (2): there is a real number *s*

1.1.9 Fill in the blanks: For every equation E, if E is quadratic then E has at most two real solutions.

1. All quadratic equations \_\_\_.  
   (1): has at most two real solutions
2. Every quadratic equation \_\_\_.  
   (1): has at most two real solutions
3. If an equation is quadratic, then it \_\_\_.  
   (1): has at most two real solutions
4. If *E* \_\_\_, then *E* \_\_\_.  
   (1): is a quadratic equation  
   (2): has at most two real solutions
5. For every quadratic equation *E*, \_\_\_.  
   (1): *E* has at most two real solutions

## 1.4 The Language of Graphs

1.4.8 Consider the below graph.

Radar chart

Description automatically generated

1. Find all edges that are incident on .  
   incidents of
2. Find all vertices that are adjacent to .  
   adjacent to
3. Find all edges that are adjacent to .  
   adjacent to
4. Find all loops.  
   loops:
5. Find all parallel edges.  
   parallel:
6. Find all isolated vertices.  
   isolated:
7. Find the degree of .  
   degree

1.4.9 Consider the below graph.

Diagram, schematic

Description automatically generated

1. Find all edges that are incident on .  
   incident of
2. Find all vertices that are adjacent to .  
   adjacent to
3. Find all edges that are adjacent to .  
   adjacent to
4. Find all loops.  
   loops:
5. Find all parallel edges.  
   parallel:
6. Find all isolated vertices.  
   isolated:
7. Find the degree of .  
   Degree

# The Logic of Compound Statements

## 2.1 Logical Form and Logical Equivalence

2.1.5 Indicate which of the following sentences are statements.

1. 1,024 is the smallest four-digit number that is a perfect square.  
   By the definition of statements, a statement must be either true or false and not both. The above statement cannot be both true and false.
2. She is a mathematics major.  
   Same as above, this is either true or false and thus must be a statement.

2.1.8 Write in symbolic form using . Let h = “John is healthy,” w = “John is wealthy,” and s = “John is wise.”

1. John is healthy and wealthy but not wise.
2. John is not wealthy but he is healthy and wise.
3. John is neither healthy, wealthy, nor wise.
4. John is neither wealthy nor wise, but he is healthy.
5. John is wealthy, but he is not both healthy and wise.

2.1.10 Let p be the statement “DATAENDFLAG is off,” q the statement “ERROR equals 0,” and r the statement “SUM is less than 1,000.” Express the following sentences in symbolic notation.

1. DATAENDFLAG is off, ERROR equals 0, and SUM is less than 1,000.
2. DATAENDFLAG is off but ERROR is not equal to 0.
3. DATAENDFLAG is off; however, ERROR is not 0 or SUM is greater than or equal to 1,000.
4. DATAENDFLAG is on and ERROR equals 0 but SUM is greater than or equal to 1,000.
5. Either DATAENDFLAG is on or it is the case that both ERROR equals 0 and SUM is less than 1,000.

2.1.17 Construct a truth table and determine if the following two statements are logically equivalent:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| p | q |  |  |  |
| T | T | T | F | F |
| T | F | F | T | F |
| F | T | F | T | F |
| F | F | F | T | T |

Since our two statements (last two columns) do not match, the statements are not logically equivalent.

2.1.19 Construct a truth table and determine if the following two statements are logically equivalent:

|  |  |  |
| --- | --- | --- |
| t | p |  |
| T | T | T |
| T | F | F |

Since our two statements (last two columns) match, the statements are logically equivalent.

2.1.29 Use De Morgan’s laws to write a negation for the statement “This computer program has a logical error in the first ten lines or it is being run with an incomplete data set.”

This computer program does not have a logical error in the first ten lines and it is being run with a complete data set.

2.1.42 Use a truth table to determine if the following statement is a tautology or a contradiction:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| p | q | r |  |  |  | Statement |
| T | T | T | F | T | F | F |
| T | T | F | F | F | F | F |
| T | F | T | F | F | T | F |
| T | F | F | F | F | T | F |
| F | T | T | T | T | F | F |
| F | T | F | T | F | F | F |
| F | F | T | F | F | T | F |
| F | F | F | F | F | T | F |

Since the statement is always false it must be a contradiction.

2.1.43 Use a truth table to determine if the following statement is a tautology or a contradiction:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| p | q |  |  | Statement |
| T | T | T | F | T |
| T | F | F | T | T |
| F | T | T | F | T |
| F | F | T | F | T |

Since the statement is always true it must be a tautology.

## 2.2 Conditional Statements

2.2.2 Rewrite to the if-then form: “I am on time for work if I catch the 8:05 bus.”

If I catch the 8:05 bus then I am on time for work.

2.2.4 Rewrite to the if-then form: “Fix my ceiling or I won’t pay my rent.”

If you don’t fix my ceiling then I won’t pay my rent.

2.2.17 Write the following two statements in symbolic notation and determine if they are logically equivalent using a truth table.

If 2 is a factor of n and 3 is a factor of n, then 6 is a factor of n.

2 is not a factor of n or 3 is not a factor of n or 6 is a factor of n.

2.2.30 Use to convert the following logical equivalence to a tautology and use a truth table to verify the tautology:

Tautology:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| p | q | r |  | Left |  |  | Right | Statement |
| T | T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | F | T | T |
| T | F | T | T | T | F | T | T | T |
| T | F | F | F | F | F | F | F | T |
| F | T | T | T | F | F | F | F | T |
| F | T | F | T | F | F | F | F | T |
| F | F | T | T | F | F | F | F | T |
| F | F | F | F | F | F | F | F | T |

Since the statement is always true it is a tautology.

## 2.3 Valid and Invalid Arguments

2.3.9 Create a truth table highlighting premises and conclusions of the following argument and determine the validity:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| p | q | r | Premise: | Premise: | Premise: | Conclusion |
| T | T | T | F | T | T |  |
| T | T | F | T | T | T | T |
| T | F | T | T | T | T | F |
| T | F | F | T | T | T | T |
| F | T | T | T | F | T |  |
| F | T | F | T | F | T |  |
| F | F | T | T | T | F |  |
| F | F | F | T | T | F |  |

Since the conclusion is not a tautology the argument is invalid.

In the following exercises, use symbols to write the logical form of each argument. If the argument is valid identify the rule of inference that guarantees it’s validity, otherwise state whether the converse or inverse error is made.

2.3.28 If there are as many rational numbers as there are irrational numbers, then the set of all irrational numbers is infinite.

The set of all irrational numbers is infinite.

There are as many rational numbers as there are irrational numbers.

Let p = “There are as many rational numbers as there are irrational numbers”

Let q = “The set of all irrational numbers is infinite”

Converse error. Just because the “then” part of an “if, then” statement is true, doesn’t require the “if” part of the statement to be true.

2.3.29 If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.

Neither of these two numbers is divisible by 6.

The product of these two numbers is not divisible by 6.

Let p = “At least one of these two numbers is divisible by 6”

Let q = “The product of these two numbers is divisible by 6”

Inverse error. Just because the “if” part of a statement is false doesn’t mean the “then” part of the statement is also false.

## Exercises from LearnIT

Prove, in natural deduction, the "exercises on propositional logic" in this week's section.

A piece of paper with writing on it

Description automatically generated with medium confidence

A piece of paper with writing on it

Description automatically generated

A picture containing text, shoji, whiteboard

Description automatically generated

A picture containing shoji, whiteboard, text

Description automatically generated

A picture containing text, shoji, whiteboard

Description automatically generated

A piece of paper with writing on it

Description automatically generated

# The Logic of Quantified Statements

## 3.1 Predicates and Quantified Statements I

3.1.16 Rewrite each of the following statements in the form “\_\_\_ x, \_\_\_.”

1. All dinosaurs are extinct.
2. Every real number is positive, negative, or zero.
3. No irrational numbers are integers.
4. No logicians are lazy.
5. The number 2,147,581,953 is not equal to the square of any integer.
6. The number 21 is not equal to the square of any real number.

3.1.17 Rewrite each of the following statements in the form “\_\_\_ x such that \_\_\_\_.”

1. Some exercises have answers.
2. Some real numbers are rational.

3.1.30 Rewrite the following statements without quanitifers and indicate which are true and which are false.

Let the domain of x be , the set of integers, and let Odd(x) be “x is odd,” Prime(x) be “x is prime,” and Square(x) be “x is a perfect square.” (An integer n is said to be a perfect square if, and only if, it equals the square of some integer. For example, 25 is a perfect square because .)

1. There exists an integer which is a prime number and even number.   
   This is true, there is one such number (2).
2. All primes are not perfect squares.  
   This is true because if it is a perfect square, the root of the number is an integer which is not 1 nor the number itself and thus the number is not a prime.
3. There exists an integer which is odd and a perfect square.  
   This is true, an example could be 81 which is a perfect square and odd

## 3.3 Statements with Multiple Quantifiers

3.3.11 Let S be the set of students at your school, let M be the set of movies that have ever been released, and let V(s, m) be “student s has seen movie m.” Rewrite each of the following statements without using the symbol , the symbol , or variables.

1. There exist a student who has seen the movie Casablanca.
2. All students have seen Star Wars.
3. All students has seen at least one movie.
4. There exist a movie that all students have seen.
5. There are two distinct students who have seen the same movie.
6. There are two distinct where one student has seen at least all the movies that the second student has seen.

In 33–39, (a) rewrite the statement formally using quantifiers and variables, and (b) write a negation for the statement.

3.3.33 Everybody loves somebody.

3.3.34 Somebody loves everybody.

3.3.35 Everybody trusts somebody.

3.3.36 Somebody trusts everybody.

3.3.37 Any even integer equals twice some integer.

3.3.38 Every action has an equal and opposite reaction.

3.3.39 There is a program that gives the correct answer to every question that is posed to it.

# Elementary Number Theory and Methods of Proof

## 4.1 Direct Proof and Counterexample I: Introduction

**4.1.2** **Assume that c is a particular integer.**

1. Is -6c an even integer?  
   We want to see if we can rewrite -6c to the form where such that it must be an even integer.   
      
   Since c and -3 are both integers, they must be an integer when multiplied. Thus we can write k as -3c:

and thus -6c is an even integer.

1. Is 8c+5 an odd integer?  
      
   Let (which is an integer for the same reason as above plus the fact that addition also keeps it an integer)  
    and thus it is an odd integer.
2. Is an even integer?   
   We’ll rewrite:  
      
   Since 0 is even, it must be true. It can be rewritten as:

**4.1.13 For every integer n, if n is odd then is odd. (Negation of statement + Counterproof)**

Negation:

Counterproof:

Let which is odd because we can write it as .

Now, according to the statement is odd, but we can evaluate this to 4. 4 can be written as and is thus even.

**4.1.20 Determine if the following statement is always, sometimes or never true: The average of any two odd integers is odd.**

That is, .

Let us simplify the numerator:

Thus we can divide the two out from the denominator and numerator:

We can remove the 1 from either side of the equation:

Now we can see if it is true for any examples:

i.e. it is possible to find examples where it is true. But we can also find a counterexample such that it is not always true:

i.e. 12,5 is not an integer and we found a counterexample.

The numbers 12 and 14 would translate, in the example, to 2\*12+1=25 and 2\*14+1=29 which would have the average 27. That is an odd number. Thus proof of existence.

The numbers 12 and 13 would translate, in the example, to 2\*12+1=25 and 2\*13+1=27 which would have the average 26. That is an even number. Thus counterpoof.

## 4.2 Direct Proof and Counterexample II: Writing Advice

**4.2.5 If a and b are any odd integers, then is even.**

Write out the multiplication of the exponents, including the definition of odd numbers

Let us multiply the parentheses:

Let

Now let us factorize with 2:

Since we, in the parenthesis, only have multiplication and addition, we can rewrite this to a new integer:

then:

and thus is an even number.

QED.

**4.2.11 If n is any odd integer, then (Proof)**

Let us substitute n with the definition of odd integers:

Using rules of exponentiation, we can write this as:

Let us evaluate . Also, we know that .

Let us write out the as multiplication (outer exponent):

Since , then we know that then:

and we can show this k times such that we have:

Now we know that :

QED.

**4.2.23 Determine if the product of any even integer and any integer is even.**

Let us use the definition of an even integer:

to show

Let us factorize with 2:

Since multiplication of two integers gives a new integer, we can substitute .

Thus follows the definition of an even integer.

QED.

**4.2.25 Determine if the difference of any two even integers is even.**

We want to know if .

Let us replace a and b with the definition for even integers:

Let us factorize with 2:

Since are both integers, the subtraction will generate a new integer, let us call that :

Since this follows the definition of an even integer (2k), we can conclude that the difference of two even integers is even.

QED.

**4.2.26 Determine if for all integers a, b, and c, if a, b, and c are consecutive, then is even.**

Let us disprove this statement using a counterexample:

Let such that the three numbers are consecutive.

Then we have .

Since 9 can be rewritten as , we know that it must be an odd number and thus this statement is not true for all consecutive integers a, b, c.

**4.2.35 If m and n are any positive integers and mn is a perfect square, then m and n are perfect squares.**

Let us disprove this statement using a counterexample:

Let such that which is a perfect square .

However 2 and 8 are not perfect squares:

Thus this statement is false.

**4.2.39 Suppose that integers m and n are perfect squares.**

**Then is also a perfect square.**

**Why?**

Since m and n are perfect squares, their square root must be a positive integer.

Thus are positive integers. If we add these together we will get a new integer such that this integer squared is also a perfect square. Let k be the perfect square that the statement creates.

So we have that:

The parenthesis is squared so we can multiply it by itself:

If we multiply these two parentheses:

A square root to the power of 2 cancels each other out. For the two square roots multiplied we know that this is equal to the numbers multiplied by eachother before taking the square root. Thus we have:

QED.

## 4.3 Direct Proof and Counterexample III: Rational Numbers

**4.3.10 Assume that m and n are both integers and that . Explain why must be a rational number.**

The definition of a rational number is that it can be written as a fraction: where n is the numerator and d is the denominator and both are integers. We know that multiplication of two integers produces a new integer, so 5\*m must be a new integer, 12\*n must be a new integer and 4\*n must be a new integer. Let us call these .

Now we also know that subtraction of two integers produces a new integer, so .

Thus we have that can be written as where .

We then know that this is a fraction.

QED.

**4.3.14 Consider the statement: The cube of any rational number is a rational number.**

1. Write the statement formally using a quantifier and a variable.
2. Determine whether the statement is true or false and justify your answer.  
   Let us construct a rational number:  
      
      
   We can also write the parentheses exponentiation as follows:  
      
   We multiplying two fractions we multiply the numerators with each other and the denominators with each other:  
      
   Since , then the multiplication of these numbers will produce new integers:  
      
   We see that follows the definition of a rational number.  
   QED.

**4.3.22 True or false? If a is any odd integer, then is even. Explain.**

True.

Let us substitute a with the definition of an odd integer.

Multiplying the first parenthesis with itself, we get:

We can further simplify:

Let us factorize with 2:

We know that multiplication and addition for integers produces new integers, such that :

Since is an integer, we now see that it follows the definition of an even integer.

QED.

**4.3.28 Suppose a, b, c, and d are integers and . Suppose also that x is a real number that satisfies the equation . Must x be rational?**

Let us multiply the denominator on both sides of the equation:

Let us subtract b and cx on either side of the equation:

Let us factorize x:

Let us divide by :

Since subtraction of two integers produce a new integer and , then we can deduce the following:

Thus x has the form of a rational number.

QED.

## 4.7 Indirect Argument: Contradiction and Contraposition

**4.7.7 Formulate the negation and then prove by contradiction the following statement: There is no least positive rational number.**

The negation:

In natural language: There exist a rational number x which is positive but less than all other rational numbers.

Since x is a rational number, we can describe it as .

Now consider a rational number such that . Since is an integer and the numerator is the same for x and y, we have that and thus we have a contradiction of the negation. Thus the original statement was true.

**4.7.22 Consider the statement “For every real number r, if is irrational then r is irrational.”**

1. Write what you would suppose and what you would need to show to prove this statement by contradiction.  
   To prove by contradiction, I would have to show that the negation of the statement is logically false (that is, meets a contradiction).   
   So the negation would be:  
      
   In other words we are saying there exist a number r which is rational (i.e. not irrational) whose square is irrational.   
   We can write r as a rational number:  
      
   Thus :  
      
   By rules of exponentiation we can rewrite this to the following:  
      
   Since multiplication of two integers creates a new integer, we can deduce that . We thus have:  
   .  
   Since is defined as irrational, but we see that it follows the form of a rational number, we have a contradiction of the negation and thus the statement must be true.
2. Write what you would suppose and what you would need to show to prove this statement by contraposition.  
   To prove by contraposition, I would assume that for all rational numbers , it’s square would also be rational . I.e. showing that if a number is rational it cannot be irrational, because that would be a counterproof for the statement.   
      
   We thus have:  
      
      
   Using rules of exponentiation, we can rewrite this to the following:  
      
   Since and since the multiplication of two integers produces a new integer, we have that:  
      
   Thus we see that for a rational number , it’s square is also rational and thus we have proven the statement by contraposition.

**4.7.24 The reciprocal of any irrational number is irrational. (The reciprocal of a nonzero real number x is ).**

1. Prove the above statement by contraposition.  
   To prove by contraposition, we will prove that the negation is true. That is, the reciprocal of any rational number is rational.  
   Assume r is a rational number to prove is rational.  
      
   Then the reciprocal will be written:  
      
   Since we can write:  
      
   We know that dividing two fractions is the same as swapping the numerator and denominator in the divisor and multiply instead:  
      
   We can now multiply these and get:  
      
   Thus we can see that the reciprocal is a rational number.  
   QED.
2. Prove the above statement by contradiction.  
   Assume there exists an irrational number r whose reciprocal is rational.   
      
   Multiply either side with the rational number:  
      
   Multiply either side with the irrational number.  
      
   We thus see that r is a rational number but that is a contradiction with the statement and thus the negation is false and the statement is true.

**4.7.27 Prove by any method that for all positive real numbers r and s, .**

By contradiction:

Assume there exists two positive real numbers r and s such that .   
We will multiply either side with itself (putting it to the power of 2):  
   
On the left side the square root and the square cancel each other out. On the right side we use the square theorem .

We see that again the square roots and the squares cancel each other out and we thus have:

We subtract r and s on either side:

Since r and s are positive, real numbers we have a contradiction and thus the negation must be false while the original statement is true.

QED.

## 4.9 Application: The Handshake Theorem

4.9.1 Find the degree of each vertex and the total degree of the graph. Check that the number of edges equals one-half of the total degree.

A picture containing schematic

Description automatically generated

Degrees:

Total degree of the graph:

The number of vertices, 6, is exactly one half of the total degree of the graph.

4.9.5 Draw the graph with five vertices of degrees 1, 2, 3, 3, and 5 or explain why it is not possible.

A piece of paper with writing on it

Description automatically generated

4.9.6 Draw the graph with five vertices of degrees 1, 2, 3, and 3 or explain why it is not possible.

The total degree of the graph is which is odd and thus not possible.

4.9.7 Draw the graph with five vertices of degrees 1, 1, 1 and 4 or explain why it is not possible.

The total degree of the graph is which is odd and thus not possible.

4.9.8 Draw the graph with five vertices of degrees 1, 2, 3 and 4 or explain why it is not possible.

A piece of paper with writing on it

Description automatically generated with medium confidence

4.9.20

1. Draw , a complete graph on six vertices.  
   A picture containing shoji, text, whiteboard

   Description automatically generated
2. Use the result of Example 4.9.9 to show that the number of edges of a simple graph with n vertices is less than or equal to .

Since the number of edges in a complete graph can be determined to be the number of vertices (n) multiplied by the number of remaining vertices (n-1) and then divided by two, such that each edge is only counted once and not twice (not for each vertex incident on the edge).

Then we have the formula for the number of edges. Since a simple graph contains no parallel edges or loops, the max number of edges will then be the number of edges in the complete graph. That is, all vertices are connected to all other vertices.

4.9.21

1. In a simple graph, must every vertex have degree that is less than the number of vertices in the graph? Why?  
   Yes, since there are no loops or parallel edges in a simple graph, all edges incident on a vertex v can only connect v with the any other vertex once and never itself. Thus in a simple graph, the highest degree a vertex v can have is .
2. Can there be a simple graph that has four vertices all of different degrees? Why?  
   No. For to have distinct degrees, they would have to have 0, 1, 2 and 3 incident edges. Consider . Then it must be adjacent to all other vertices, that is . But since exactly one of these should have 0 incident edges, then and we have thus found a contradiction that disproves the given statement.   
   Alternatively, if no vertex had 0 incident edges then they would have 1, 2, 3 and 4 incident edges but since a vertex can only have incident edges this is not possible.
3. For any integer , can there be a simple graph that has n vertices all of different degrees? Why?  
   No, for the same reason as in b).

4.9.24 Find which of the following graphs are bipartite.

Chart, radar chart

Description automatically generated

A graph is bipartite if we can put the vertices in two groups with distinct elements and with no edge between two vertices in the same group.

1. is bipartite if we put in one group and in another.
2. is not bipartite.
3. is bipartite if we put in one group and in another.
4. is not bipartite.
5. is bipartite if we put in one group and in another.
6. is not bipartite.

# Number Theory and Cryptography (Rosen)

## 4.1 Divisibility and Modular Arithmetic

4.1.13 Suppose that a and b are integers, . Find the integer c with such that

4.1.21 Evaluate these quantities.

4.1.29 Decide whether each of these integers is congruent to 5 modulo 17.

1. 80

so no

1. 103

so no

1. -29

so yes

1. -122

so no

4.1.38 Show that if *n* is an integer then

That means that . We have even and odd integers on the forms .

Even case:

and since is a new integer, we have that and thus follows the above form .

Odd case:

and since is a new integer, we have that it follows the above form .

QED.

4.1.39 Use Exercise 38 to show that if is a positive integer of the form for some nonnegative integer , then is not the sum of the squares of two integers.

Since k is a nonnegative integer, we have that . We have from exercise 38 that an integer squared mod 4 will give either 0 or 1. Thus we need either , none of which are true and thus it is not possible.

## 4.2 Integer Representations and Algorithms

4.2.1 Convert the decimal expansion of each of these integers to a binary expansion.

1. 231
2. 4532
3. 97644

4.2.7 Convert the hexadecimal expansion of each of these integers to a binary expansion.

A screenshot of a computer

Description automatically generated with medium confidence

4.2.9 Convert from its hexadecimal expansion to its binary expansion.

Using same table as above.

4.2.21 Find the sum and the product of each of these pairs of numbers. Express your answers as a binary expression.

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Letter

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A piece of paper with writing on it

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Description automatically generated

4.2.24 Find the sum and products of each of these pairs of numbers. Express your answers as a hexadecimal expansion.

A picture containing text, shoji, crossword puzzle, whiteboard

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A picture containing text, shoji, crossword puzzle

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## 4.3 Primes and Greatest Common Divisors

4.3.1 Determine whether each of these integers is prime.

1. 21

No, divided by 3.

1. 29

Yes.

1. 71

Yes.

1. 97

Yes.

1. 111

No, divided by 3.

1. 143

No, divided by 11.

4.3.21 Find these values of the Euler -function.

4.3.22 Show that *n* is prime if and only if

Assume

That means there exists a number k such that and a number such that . If such a number exists, is by definition not prime.

4.3.25 What are the greatest common divisor of these pairs of integers?

4.3.33 Use the Euclidian Algorithm to find

4.3.39 Express the greatest common divisor of each of these pairs of integers as a linear combination of these integers.

1. 10, 11
2. 21, 44
3. 36, 48
4. 34, 55
5. 117, 213
6. 0, 223
7. 123, 2347
8. 3454, 4666
9. 9999, 11111

4.3.41 Use the extended Euclidian algorithm to express as a linear combination of 26 and 91.

## 4.4 Solving Congruences

4.4.5 Find an inverse of modulo for each of these pairs of relatively prime integers using the method followed in Example 2.

Thus the inverse is -2.

Thus the inverse are -89 and 12.

4.4.11 Solve 4.4.5 first

4.4.21

# Sequences, Mathematical Induction and Recursion

## 5.1 Sequences

**5.1.1 Write the first 4 terms of for .**

**5.1.2 Write the first 4 terms of for .**

**5.1.3 Write the first 4 terms of for .**

**5.1.4 Write the first 4 terms of for .**

**5.1.19 Compute the summation**

**5.1.20 Compute the product**

**5.1.21 Compute the summation**

**5.1.22 Compute the product**

**5.1.43 Write using summation or product notation.**

**5.1.44 Write using summation or product notation.**

**5.1.45 Write using summation or product notation.**

## 5.2 Mathematical Induction I: Proving Formulas

**5.2.6 Prove: For every integer**

Inductive Hypothesis:

Base case:

Thus the basecase holds.

Inductive case:

Assume holds to prove . That is, assume to prove .

Add 2\*(k+1) on either side.

Multiply the right hand side parenthesis:

Note that .

QED.

**5.2.7 Prove: For every integer**

Base case (1):

We assume that holds to show that holds.

We have that

To prove that

We have that

Let us remove the parentheses:

We will turn into a fraction by multiplying and dividing by 2:

Since the two fractions we have left have a common divisor, we can simplify it into one fraction instead. We’ll also multiply the k and the 2 into the parenthesis:

We can rewrite to the following:

Let’s factorize 5:

Using inside parentheses:

Let’s use that :

Let’s factorize k+1:

QED.

**5.2.10 Prove:**

Inductive hypothesis:

Base case:

Thus the base case holds.

Inductive case:

Assume holds to prove holds.

to prove

Add on either side.

Calculate the parenthesis.

Make into a fraction with 6 as denominator.

Combine the fractions.

Multiply the last parenthesis.

Multiply the first two parenthesis together.

Now, we want to assume that the right hand side equals .

Let us calculate the first two parentheses on that side and the inner parenthesis.

Now let us calculate the remaining two parentheses.

Note that this is the same as we found earlier and thus we have proven this by mathematical induction.

**5.2.11 Prove:**

Base case(1):

If then

QED.

5.2.13 Prove:

Inductive hypothesis:

Base case:

Thus the base case holds.

Inductive case:

Assume holds to prove holds.

Add on either side.

Reduce on left side. Create a fraction on the right hand side.

Multiply the numerators of both fractions and combine them.

Factorize with k.

Note that . Thus we can rewrite.

QED.

5.2.14 Prove:

Inductive hypothesis:

Base case:

Thus the base case holds.

Inductive case:

Assume P(k) to prove P(k+1), that is

Add to either side.

Reduce the left side. Multiply the parenthesis on the right side.

Note that and reduce.

Factorize

QED.

5.2.36 Find the mistake in the following proof:

Theorem: For any integer ,

Proof by mathematical induction: Certainly the theorem is true for because and . So the basis step is true. For the inductive step, suppose that *k* is any integer with . We must show that .

The mistake lies in the fact that the theorem is summation, that is the inductive step should not be but .

**5.2.37 Prove:**

Base case:

Hence P(0) is true.

Assume to prove .

Hence:

QED.

## 5.3 Mathematical Induction II: Applications

**5.3.1 Use mathematical induction (and the proof of proposition 5.3.1 as a model) to show that any amount of money of at least 14¢ can be made up using 3¢ and 8¢ coins.**

Inductive hypothesis:

Base case:

Inductive case:  
Assume P(i) holds for all i<k to prove P(k) holds. Let

QED.

**5.3.3 Stamps are sold in packages containing either 5 stamps or 8 stamps.**

1. Show that a person can obtain 5, 8, 10, 13, 15, 16, 20, 21, 24, or 25 stamps by buying a collection of 5-stamp packages and 8-stamp packages.

1. Use mathematical induction to show that any quantity of at least 28 stamps can be obtained by buying a collection of 5-stamp packages and 8-stamp packages.

Prove that

Basecases:

Inductive case:

Let

QED.

5.3.4 For each positive integer n, let P(n) be the sentence that describes the following divisibility property:

1. Write P(0). Is P(0) true?

Since 4 divides 0 , the base case holds.

1. Write P(k).
2. Write P(k+1).
3. In a proof by mathematical induction that this divisibility property holds for every integer , what must be shown in the inductive step?

That you can go from P(k) to P(k+1).

5.3.8 Prove:

Inductive hypothesis:

Base case:

Thus the base case holds.

Inductive case:

Let such that .

QED.

5.3.9 Prove: .

Inductive hypothesis:

Base case:

Thus the base case holds.

Inductive case:

Let such that .

QED.

5.3.10 Prove by mathematical induction:

Inductive hypothesis

Base case:

Inductive case

Multiply the cubed parenthesis once with itself.

Multiply the two parentheses together.

Reduce.

Put by itself, still divided by the same denominator.

Subtract the second fraction from either sides of the equal sign.

We know that and thus is an integer and must also be an integer.

QED.

5.3.15 Prove: .

Inductive Hypothesis:

Base case:

Thus the base case holds.

Inductive case:

From our inductive hypothesis, is divisible by 6. So is the standalone 6.

Lastly, must also be divisible by 6, because is always divisible by 2 and thus 3\*2 is divisible by 6. being divisible by 2 will be proven in two cases:

k is odd:

Thus if k is odd, will be even.

k is even:

Thus if k is even, will be even.

QED.

5.3.19 Prove by mathematical induction:

Inductive hypothesis:

Base case:

Thus the base case holds

Inductive case:

Let us add 2k+1 on either side of the inequality. Note that .

Since , we have that

Thus we have:

Note that

Thus we have:

QED.

## 5.4 Strong Mathematical Induction and the Well-Ordering Principle for the Integers

5.4.1 Suppose is a sequence defined as

Prove that is odd for every integer .

Base cases:

Thus is odd for .

Assume is odd for all integers i such that to prove is odd.

We know from our inductive hypothesis that and are both odd so we can substitute these values with the definition of odd integers.

Let us factorize:

Let such that

Thus we have that such that must be an odd integer.

QED.

5.4.6 Suppose that is a sequence defined as follows:

Prove that for each integer .

Base cases:

Thus the base cases holds.

Assume holds for all to prove .

From the inductive hypothesis we can substitute with

QED.

5.4.7 Suppose that is a sequence defined as follows:

Prove that for every integer .

Base cases:

Thus the base cases holds.

Assume holds for every positive integer to prove

From our inductive hypothesis we can substitute and .

QED.

# Set Theory

## 6.1 – Definitions and the element method of proof

**6.1.1. In each of (a)-(f), answer the following questions: Is ? Is ? Is either *A* or *B* a proper subset of the other?**

a.

A is a proper subset of B because {{2}} is in B but not in A.

b.

Both sets have 3 as the only distinct element so they are equal to each other and hence a subset of each other without being a proper subset.

c.

Are not subsets of each other and .

d.

Are not subsets of each other because .

e.

B is a proper subset of A because and

f.

Set A and B are equal and thus subsets but not proper.

**6.1.4. Let and . Prove or disprove each of the following statements.**

a.

Let’s use a witness to disprove this statement. Let the witness .

Let such that because .

However there are no integer *s* such that 5 is in B.

We can show this because which implies such that , however 0,25 is not an integer and hence 5 is in the set A but not in the set B and A is therefore not a subset of B.

b.

Since , we can rewrite this to .

Since *s* is an integer we know that *4s* must also be an integer that we can call *r*. Hence we have that B is a subset of A.

**6.1.8. Write in words how to read each of the following out loud. Then write each set using symbols for union, intersection, set difference, or set complement.**

a.

U is the set of all numbers x where x is in A and x is in B.

b.

U is the set of all numbers x where x is in A or x is in B.

c.

U is the set of all numbers x where x is in A but x is not in B.

d.

U is the set of all numbers x where x is not in A.

**6.1.10. Let** . Find each of the following:

a.

{1, 3, 5, 6, 7, 9}

b.

{3, 9}

c.

d.

e.

{1, 5, 7}

f.

{6}

g.

{2, 3, 4, 6, 8, 9}

h.

{6}

**6.1.31 Suppose A = {1, 2} and B = {2, 3}. Find each of the following:**

1. Let us first decide :  
      
   Thus the powerset must be .
2. The powerset must be .
3. Let us first decide :  
      
   Thus the powerset must be
4. Let us first decide :  
      
   Thus the powerset must be

## 6.2 Properties of sets

**6.2.1**

**a. To say that an element is in means that it is in (1) and in (2).**

(1): *A*

(2):

**b. To say that an element is in means that it is in (1) or in (2).**

(1):

(2): *C*

**c. To say that an element is in means that it is in (1) and not in (2).**

(1): *A*

(2):

**d. To prove that , we suppose that *x* is any element in (1). Then we must show that (2).**

(1):

(2):

**e. If *A, B* and *C* are any sets such that ,to prove that , we suppose that that *x* is any element in (1). Then we must show that (2).**

(1):

(2):

**6.2.2 The following are two proofs that for all sets A and B, . The first is less formal, and the second is more formal. Fill in the blanks.**

1. Proof: Suppose A and B are any sets. To show that , we must show that every element in (1) is in (2). But any element in is in (3) and not in (4) (by definition of ). In particular, such an element is in A.

(1):

(2):

(3):

(4):

1. Proof: Suppose A and B are any sets and . [We must show that (1)] By definition of set difference, (2) and (3). In particular, (4) [which is what was to be shown].

(1):

(2):

(3):

(4):

**6.2.6 Let and stand for the words “intersection” and “union,” respectively. Fill in the blanks in the following proof that for all sets A, B, and C, .**

|  |  |  |
| --- | --- | --- |
| Proof: Suppose A, B, and C are any sets. |  | |
| (1) Proof that : |  | |
| Let . [We must show that (a)]. | (a): | |
| By definition of , (b) and . | (b): | |
| Thus and, by definition of (c) | (c): | |
| **Case 1**: : In this case, by definition of |  | |
| **Case 2**: : In this case, by definition of |  | |
| By cases 1 and 2, or , and so, by definition of , (d). | (d): | |
| [So by definition of subset.] |  | |
| (2) Proof that : |  | |
| Let . [We must show that .] |  | |
| By definition of (a) . | (a): or | |
| **Case 1**: : In this case, by definition of , and . |  | |
| Since , then by definition of . |  | |
| **Case 2**: : In this case, by definition of , (b) | (b): and | |
| Since , then by definition of . |  | |
| In both cases and , and so, by definition of , (c) | (c): | |
| [So by definition of (d)] | (d): subset | |
| (3) Conclusion: [Since both subset relations have been proved, it follows, by definition of set equality, that (a)] | | (a): |

**6.2.10 Prove for all sets A, B and C, by element proof.**

Consider an element such that . We must prove that .

By definition of intersection, and . By definition of union, or .

Thus we have that or . I.e. .

Following from this, we see that either the element is in and thus is in since the definition of says x is in both A and C and thus we know that it is in the superset A. Since , then .

Otherwise the element is in and thus is in by definition of union.

Thus by definition of subset, .

QED.

**6.2.11 Prove for all sets A, B and C, by element proof.**

Consider an element x such that . By definition of intersection, and . By definition of set difference, .

From this we can deduce that .

Since from definition of intersection.

Since

Since, by the definition of intersection, we only consider elements that are in both and , we do not need to subtract any elements but those in . Thus, in this case, in set difference.

Thus any element in .

By definition of subset, .

QED.

**6.2.12 Prove for all sets A, B and C, by element proof.**

Consider an element . By definition of union, or . By definition of set difference, .

Thus, either , or .

By definition of union, if the element is in just one of these, it is in their union:

Thus, any element in is also in . By definition of subset, .

QED.

**6.2.23 Find the mistake in the following “proof” for all sets *A, B* and *C*, if and then .**

“Proof: Suppose *A, B* and *C* are any sets such that and . Since , there is an element *x* such that , and since , there is an element *x* such that . Hence there is an element *x* such that and so .

We cannot use *x* for both statements and follow from that that it must be the same element.

## 6.3 Disproofs and algebraic proofs

**For each of 1-4 find a counterexample to show that the statement is false. Assume all sets are subsets of a universal set *U*.**

6.3.1 For all sets *A, B* and *C*,

Let the sets be as follows:

Then where so

6.3.2 For all sets *A* and *B*,

Let the sets be as follows:

Then but

6.3.3 For all sets *A, B* and *C*, if

Let the sets be as follows:

Then *A* is not a subset of *B* and *B* is not a subset of *C*, but *A* is a subset of *C*.

6.3.4 For all sets *A, B* and *C*, if

Let the sets be as follows:

Then and the union of these is

**6.3.5 Prove or show a counterproof that for all sets A, B and C, .**

Consider

From definition of set difference, and thus while and thus these are not equal.

**6.3.6 Prove or show a counterproof that for all sets A and B, .**

Case 1:

Consider an element

By definition of union, . By definition of intersection, . Thus any element in is in both the union of A and B and in A itself. Any element in B that is not in A is not in both sets and thus not considered. Thus by definition of subset, .

Case 2:

Consider an element .

By definition of union, if . By definition of intersection, iff x is in A and the union of A and B. Thus by definition of subset, . By definition of subset equality, .

**6.3.7 Prove or show a counterproof that for all sets A, B and C, .**

Consider .

Then we have that and and thus .

We also have that and thus .

Since , the two sides are not equal.

**6.3.8 Prove or show a counterproof that for all sets A and B, if .**

By definition of set complement, , also written as .

By definition of subset, then all elements in are also in . That means, on the other hand, that there are no elements of that are not in B and thus must at least contain all elements that are not in B. By definition of subset, then is the set of all elements in both A and B and since A contains at least all elements not in B, then combined they contain all elements of U.

**6.3.9 Prove or show a counterproof that for all sets A, B and C, if .**

By definition of subset, then all elements in is also in C. Same goes for all elements in B are in C.

From definition of union, all elements in A and all elements in B are in . Thus all elements in A and B are in C and thus .

6.3.10 Prove or show a counterproof that for all sets A and B, if .

By subset definition, any element in A is also in B. By definition of set complement, any element in is not in . That is, . Since all elements of A are in B, no elements of A are in . And then by definition of intersection, contains no element and is thus equivalent to the emptyset:

.

6.3.17 Prove or show a counterproof that for all sets A and B, if .

Consider an element in . By definition of powersets, this element is a set of zero or more elements in A. Thus, by definition of subset, if these elements are in A they are also in B. By definition of subsets, when these elements are all present in B there is a subset consisting of exactly these in the powerset of B.

QED.

6.3.18 Prove or show a counterproof that for all sets A and B, .

Consider then by definition of union, . Thus by definition of subsets, contains, amongst others, the set . By definition of subsets this set should be in . However by definition of powersets, and and thus and thus doesn’t contain . Hence the statement is false.

6.3.19 Prove or show a counterproof that for all sets A and B, .

By definition of powerset, any set of 0 or more elements in A is an element of the powerset of A. Same goes by the powerset for B for 0 or more elements in B.

By definition of union, all elements in either or are in . Then, by definition of subset, any of these elements must be present in .

By definition of this union, any elements in A or B are in .

By definition of powerset, contains all possible subsets that can be made of the elements in . Thus, if an element in then it consists of elements from the set A. Since all these elements are also in they must also be present in . Same goes for any element in .

QED.

6.3.20 Prove or show a counterproof that for all sets A and B, .

By definition of intersection, only elements both in A and in B are part of . By definition of powersets, all elements in will thus be in .

By definition of subsets, all of these elements must be in .

If an element is in then it is in both A and B and thus 0 or more elements from the sets will be present as a set in and since they’re in both A and B they must also be in both . Thus, by definition of intersection, any elements that are in both powersets will be in .

QED.

6.3.21 Prove or show a counterproof that for all sets A and B, .

Let then by definition of set multiplication we have that . Then by definition of powersets we have these powersets:

We also see that if we multiply these sets, we can only get tuples of sets, i.e. etc.

Since the powerset of is a set containing sets of tuples we have it the other way around and as such there are no overlapping elements.

# Properties of functions

## 7.1 - Functions defined on general sets

**7.1.2 Let . Define by the following arrow diagram.**

*Diagram

Description automatically generated*

a. Write the domain of *g* and the co-domain of *g*.

Domain:

Co-domain:

b. Find .

c. What is the range of *g*?

d. Is 3 an inverse image of *a*? Is 1 an inverse image of *b*?

No and yes.

e. What is the inverse image of *b*?of *c*?

f. Represent *g* as a set of ordered pairs.

**7.1.3 Indicate whether the statements in pairs (a)-(d) are true or false for all functions. Justify your answers.**

a. If two elements in the domain of a function are equal, then their images in the co-domain are equal.

Yes, if , then and their images are likewise equal.

b. If two elements in the co-domain of a function are equal, then their reimages in the domain are also equal.

Yes, for the same reason as above.

c. A function can have the same output for more than one input.

Yes, because two inputs can map to the same output. As long as one input does not map to two different outputs.

d. A function can have the same input for more than one output.

No, then it is not a well defined function as it does not satisfy both of the two conditions for a function. An element in the input set cannot map to more than one element in the output set.

7.1.7 Let , and define a function as follows: For each set X in ,

Find the following:

Since has 3, an odd number of, elements,

Since has 0, an even number of, elements,

Since has 2, an even number of, elements,

Since has 4, an even number of, elements,

7.1.11 Define as follows: For every ordered pair (a, b) of integers, .

Find the following:

1. F(4,4)
2. F(2, 1)
3. F(3, 2)
4. F(1, 5)

7.1.25 Let and . Let p1 and p2 be the projections of onto the first and second coordinates. That is, for each pair

.

1. Find . What is the range of ?

and the range of is .

1. Find . What is the range of ?

and the range of is .

In 41–49 let X and Y be sets, let A and B be any subsets of X, and let C and D be any subsets of Y. Determine which of the properties are true for every function F from X to Y and which are false for at least one function F from X to Y. Justify your answers.

7.1.41 If then

Yes, because all elements of A are in B and thus all elements map to some element of . In other words, all elements of A maps to a subset from and since all these elements from A are also in B, B must map to at least the same elements from as A did.

7.1.42

Yes, all elements of will map to a subset of . All elements of A or B will map to some subset of Y and if we only keep those elements of the co-domain that are seen in both and , then we have which will contain the same elements as .

7.1.43

Yes, whether we find the intersection of the sets before or after they are mapped to the co-domain won’t change whether they are equal to each other.

7.1.44 For all subsets A and B of X,

No, let’s come up with a counterexample:

Let and and

Then

And

This is true if the function is not one-to-one.

## 7.2 – One-to-one, onto and inverse functions

**7.2.1 The definition of one-to-one is stated in two ways: and . Why are these two logically equivalent?**

These are equivalent because either you’re defining it where all equivalent images must be from the same element of the domain, or you’re doing it in reverse saying all different images must be from a distinct element of the domain. This is basically a Boolean function determining whether or not two numbers are equivalent or not.

**7.2.7 Let . Define functions *F* and *G* by the arrow diagrams below.**

Diagram

Description automatically generated

**a. Is *F* one-to-one? Why or why not? Is it onto? Why or why not?**

It is not one-to-one because elements *c* and *d* both map to the same element of *Y*. It is however onto because all elements of *y* is mapped to from at least one element of the domain *X*.

**b. Is *G* one-to-one? Why or why not? Is it onto? Why or why not?**

Again, all elements in the co-domain is mapped to from at least one element in the domain, so it is onto. However once again two elements of the domain maps to the same element in the co-domain such that it is not one-to-one.

**7.2.8 Let . Define functions *H* and *K* by the arrow diagrams below.  
Diagram

Description automatically generated**

**a. Is *H* one-to-one? Why or why not? Is it onto? Why or why not?**

*H* is not one-to-one because elements *b* and *c* from the domain both maps to *f* in the co-domain.

It is neither onto because *e* is not mapped to from any elements of the domain.

**b. Is *G* one-to-one? Why or why not? Is it onto? Why or why not?**

*G* is one-to-one because each element of the domain maps to a distinct element of the co-domain.

It is not onto because *g* is not mapped to from any elements of the domain.

**7.2.12 a. Define by the rule for each integer *n*.**

**(i) Is *F* one-to-one? Prove or give a counterexample.**

To prove the functions one-to-oneness we will try to show that .

Suppose and are integers such that .

Subtract 2 from either side

Divide by -3 on either side

Hence *F* is one-to-one.

**(ii) Is *F* onto? Prove or give a counter-example.**

Suppose *y* is an integer. Let’s show that there is an integer *n* such that .

Since this looks more like a rational number than an integer, let’s try to find a counterexample.

Suppose *y* is 3. Then we have that:

Since is not an integer, this is a counterexample of *F* being onto.

7.2.23 Define as follows: For every A in ,

1. Is F one-to-one? Prove or give a counterexample.

No. We see that . Also, we see that and thus the function is not one-to-one.

1. Is F onto? Prove or give a counterexample.

No, the co-domain is infinite (All integers), but since the elements of the powerset can only contain between 0 and 3 elements, it is not possible to find an element such that .

7.2.29 Define as follows:

1. Is H one-to-one? Prove or give a counterexample.

Yes.

For it not to be one-to-one we would have to find two distinct elements such that but .

Let us say

We then know that . If then and we have that

1. Is H onto? Prove or give a counterexample.

Yes. Consider any . Let us show that there exists such that .

Let and . Then we have that and since the set of is infinite, .

## 7.3 – Composition of functions

**7.3.1** **Functions *f* and *g* are defined by arrow diagrams. Find and determine whether .**

Diagram, shape, circle

Description automatically generated

We see that .

Hence

**7.3.8 Let be the set of all strings in a´s and b´s and let be the length function:**

**For all strings**

**Let**

a.

b.

c.

7.3.9 Define and by the following formulas:

We have that

7.3.16 Prove Theorem 7.3.1(b): If f is any function from a set X to a set Y, then , where is the identity function on Y.

From the definition of the composition of functions, . Since maps to elements of the co-domain Y, we will always have that .

The identity function just returns the same value and since . Thus we have that and thus .

7.3.17 Prove Theorem 7.3.2(b): If is a one-to-one and onto function with inverse function , then , where is the identity function on Y.

From the definition of an onto function and one-to-one function, that is, a bijective function, we know that all elements of X maps to exactly one element of Y and that no two elements of X maps to the same element of Y. Likewise, all elements of Y is mapped to from exactly one element of X.

From the definition of the composition of functions we have that for some element . Since , we have that for some element . Since the function is bijective, we know that . We thus have that . From the definition of the identity function, we have that or in other words . Since .

7.3.19 If and are functions and is one-to-one, must g be one-to-one? Prove or give a counterexample.

No, consider the sets:

Consider the functions:

We see that since , g is not one-to-one. But since , we have these possible outcomes:

Since no two elements of X maps to the same element of Y, we have a one-to-one function though g is not one-to-one.

# Properties of Relations

## 8.1 Relations on Sets

8.1.5 Let . Recall that is the powerset of X. Define a S relation on as follows: For all sets A and B in .

1. Is   
   Yes, because |A| == |B|
2. Is   
   No, because |A| != |B|
3. Is

Yes, because |A| == |B|

8.1.6 Let . Define a relation J on as follows: For all sets A and B in ,

1. Is   
   No, the intersections is empty.
2. Is   
   Yes, both elements contains b.
3. Is   
   Yes, both elements contains a and b.

8.1.8 Let A be the set of all strings of a’s and b’s of

length 4. Define a relation R on A as follows: For

every ,

1. Is abaa R abba?

Yes, both start with “ab”.

1. Is aabb R bbaa?

No, they do not start with the same first two characters.

1. Is aaaa R aaab?

Yes, both start with “aa”.

1. Is baaa R abaa?

No, they do not start with the same first two characters.

8.1.10 Let and and let R be the “less than” relation. That is, for every ordered pair ,

State explicitly which ordered pairs are in .

8.1.13 Draw the digraph: Define a relation R on .

A piece of paper with writing on it

Description automatically generated

8.1.22 Define relations R and S on R as follows:

Graph R, S, , and in the Cartesian plane.

Diagram, engineering drawing

Description automatically generated

The graph for R is the circle while the graph for S is the diagonal line. The graph for is all you see above while the graph for is the two intersections between the two relations marked by points A and B.

## 8.2 Reflexivity, Symmetry and Transitivity

In 1–8, a number of relations are defined on the set . For each relation:

a. Draw the directed graph.

b. Determine whether the relation is reflexive.

c. Determine whether the relation is symmetric.

d. Determine whether the relation is transitive.

Give a counterexample in each case in which the relation does not satisfy one of the properties.

8.2.1

A picture containing table

Description automatically generated

8.2.2

Diagram

Description automatically generated with medium confidence

8.2.3

Letter

Description automatically generated with medium confidence

8.2.4

Letter

Description automatically generated with medium confidence

8.2.5

Text, letter

Description automatically generated

8.2.6

A picture containing text, chest of drawers

Description automatically generated

8.2.7

Text, letter

Description automatically generated with medium confidence

8.2.8

Text, letter

Description automatically generated

In 9–33, determine whether the given relation is reflexive, symmetric, transitive, or none of these. Justify your answers.

8.2.9

Reflexive, because of the equal operator ( x = y means reflexive)

Transitive because if

Not symmetric because then

8.2.10

Not reflexive because there exists (x,y) in the circle but not (x,x)

Symmetric because (addition is symmetric)

Not transitive because if but the circle equation means but we already know that it is not reflexive.

8.2.11

Reflexive because any number multiplied by itself will always give a positive number (or 0 if the number is 0).

Symmetric because multiplication is symmetric

Not transitive (Counterexample:

8.2.12

Symmetric because: and if then

Reflexive because if n = m then n – m = 0 and 4 | 0.

Transitive because if two numbers are equivalent modulo 4 and one number is equivalent with a third number modulo 4 then all three numbers are equivalent modulo 4.

8.2.13

Same as 8.2.12 but with 5

8.2.14

Not reflexive because if both numbers are even or both numbers are odd then subtraction gives an even number.

Subtraction is symmetric as shown above.

Not transitive because 1 – 2 and 2 – 3 but not 1 – 3.

8.2.21 Let and be the power set of X.

A relation is defined on as follows: For every the number of elements in A is less than the number of elements in B.

The same as the less than operator.

Not symmetric because if a < b then b is not < a

Not reflexive because a is not < a.

Transitive because if a < b and b < c then a < c.

8.2.34 Prove or disprove: If R is reflexive, then is reflexive.

R is reflexive if and only if for all elements and thus . Thus is not reflexive.

8.2.35 Prove or disprove: If R is symmetric, then is symmetric.

If (a,b) is in R, then (b,a) is in R. And (a,b) and (b,a) is then not in . If (a,b) is not in R then (b,a) is not in R and then (a,b) and (b,a) is in and thus will always be symmetric if R is symmetric.

8.2.36 Prove or disprove: If R is transitive then is transitive.

Consider A = {0, 1, 2, 3} and such that R is transitive.

Then contains but not .

8.2.40 If R and S are reflexive, is reflexive? Why?

Yes, because then

## 8.3 Equivalence Relations

8.3.1 Suppose that and R is a relation on S such that a R b, b R c, and d R e. List all of the following that must be true if R is (a) reflexive (but not symmetric or transitive), (b) symmetric (but not reflexive or transitive), (c) transitive (but not reflexive or symmetric), and (d) an equivalence relation.

c R b - c R c - a R c - b R a

a R d - e R a - e R d - c R a

1. Reflexive  
   c R c
2. Symmetric  
   c R b, b R a, e R d
3. Transitive  
   a R c
4. Equivalence relation  
   c R b, c R c, a R c, b R a, e R d, c R a  
   Aka all of the above + c R a to make it symmetric (after adding a R c from the transitive part)

8.3.2 Each of the following partitions of {0, 1, 2, 3, 4} induces a relation R on {0, 1, 2, 3, 4}. In each case, find the ordered pairs in R.

1. {0, 2}, {1}, {3, 4}  
   (0,0), (0,2), (2,0), (2,2), (1,1), (3,3), (3,4), (4,3), (4,4)
2. {0}, {1, 3, 4}, {2}  
   (0,0), (1,1), (1,3), (1,4), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4), (2,2)
3. {0}, {1, 2, 3, 4}  
   (0,0), (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)

8.3.4 R is an equivalence relation on A. Find the specified equivalence classes and then state the number of distinct equivalence classes for R and list them.

Equivalence classes: [a], [b], [c], [d]

There are 3 distinct equivalence classes since [a] = [d].

In 8.3.9 and 8.3.13, find the distinct equivalence classes.

8.3.9 and . R is defined on

as follows: For all sets s and t in ,

Let us find the powerset of X:

The sum can range from -1 to 1. Thus we have three equivalence classes:

8.3.13 A is the set of all strings of length 4 in a’s and b’s. R is defined on A as follows: For all strings s and t in A,

There will be a total of 16 pairs in R: . There are different equivalence classes, as there are 4 combinations of starting the string with the same two characters.

8.3.19 Prove that the following relation is an equivalence relation and describe the distinct equivalence classes.

A is the set of all students at your college.

1. R is the relation defined on A as follows: For every x and y in A,

For every two students with the same major, they will both relate to each other. Thus the relation is symmetric. Any student will have the same major as him- or herself and thus the relation is reflexive. Any student with the same major as another student, who has the same major as a third student, means that the first and the third student also have the same major. Thus the relation is transitive and an equivalence relation.   
There are an equivalence class per major at the college which consists of all students with said major.

1. S is the relation defined on A as follows: For every ,

All students are the same age as themselves and thus the relation is reflexive.   
If one student has the same age as a second student, then the second student has the same age as the first student and thus both relate to each other and the relation is symmetric.  
If one student has the same age as a second student, and the second student has the same age as a third student, then the first and third student also have the same age and the relation is transitive and thus an equivalent relation.  
There is one equivalence class per age that the students have, plus one for all ages where there are no students – thus being all empty and sharing the same equivalence class.

8.3.34 The documentation for the computer language Java recommends that when an “equals method” is defined for an object, it be an equivalence relation. That is, if R is defined as follows:

for all objects in the class, then R should be an equivalence relation. Suppose that in trying to optimize some of the mathematics of a graphics application, a programmer creates an object called a point, consisting of two coordinates in the plane. The programmer defines an equals method as follows: If p and q are any points, then

where c is a small positive number that depends on the resolution of the computer display. Is the programmer’s equals method an equivalence relation? Justify your answer.

The distance between points p and q is the same as the distance between points q and p and thus the relation is symmetric.

The distance between two exactly equal points are 0 and thus less than c meaning that all points relate to themselves and the relation is thus reflexive.

We could however imagine two points, p and q, with the distance of exactly c. Then a third point r with the distance of c to q on the opposite side of p, meaning that the distance between p and r will be between c and 2c. Since p relates to q, and q relates to r, for the relation to be transitive, p should also relate to r, but since the distance may be up to 2c, it doesn’t always and thus this method is not an equivalence relation.

Let R be an equivalence relation on a set A. prove each of the statements in 36–41 directly from the definitions of equivalence relation and equivalence class without using the results of Lemma 8.3.2, Lemma 8.3.3, or theorem 8.3.4.

8.3.36

Since R is an equivalence relation on A, R must be reflexive meaning that all elements must relate to themselves. I.e. . From the definition of equivalence classes we have that and thus we know that all elements must be in their own equivalence class.

8.3.37

We know from the definition of an equivalence relation that it must be symmetric and we know from the definition of equivalence classes, that . So for the symmetric property we have that and thus .

8.3.38

We know that an equivalence relation is transitive. Thus .

We know that an element in an equivalence class means that all the elements relate to that element, i.e. if . We also know then, that . We thus have that because . Again from the definition of an equivalence class, because .

8.3.39 .

We know from the symmetric property that . We know that and thus . From the definition of equivalence classes, if .

8.3.40

We know from the definition of an equivalence class that if . From the definition of an equivalence relation we know that it is transitive property. Thus we know that if .

## 8.5 Partial Order Relations

8.5.3 Let S be the set of all strings of a’s and b’s. Define a relation R on S as follows: For every ,

where L(x) denotes the length of a string x. Is R antisymmetric? Prove or give a counterexample.

Not antisymmetric because L(a) and L(b) is equal and thus (a,b) and (b,a) is in the relation but since a != b then it is not antisymmetric.

8.5.5 Let R be the set of all real numbers and define a relation as follows: For every (a, b) and (c, d) in ,

(a, b) R (c, d) either or both and

Is R a partial order relation? Prove or give a counterexample.

Reflexive because if a = a and b = b then it follows that relation.

Antisymmetric because for a relation to go both ways, a < c and c < a which is not possible or b <= d and d <= b where b != d which is not possible either (because a = c already).

Transitive because if a < c and c < k then a < k or a = c and then b <= d and d <= l and then b <= l. Thus partial order relation.

8.5.6 Let P be the set of all people who have ever lived and define a relation R on P as follows: For every ,

Is R a partial order relation? Prove or give a counterexample.

Assuming no incest, R is a partial order relation because it is reflexive (r = s), it is transitive because r’s father is an ancestor, and r’s fathers father is both r’s and r’s father’s ancestor and thus transitive. It is also antisymmetric because it is not possible to be the ancestor of your own ancestor (assuming no time-travel).

8.5.10 Suppose R and S are antisymmetric relations on a set A. Must also be antisymmetric? Explain.

No, because and such that both are antisymmetric but which is not antisymmetric.

8.5.14 Let

1. Describe all partial order relations on A for which a is a maximal element.

8.5.17 Consider the “subset” relation on for each of the following sets S. Draw the Hasse diagram for each relation.

A piece of paper with writing on it

Description automatically generated with medium confidence

See left side graph for a. and right side graph for b.

8.5.20 Let and consider the partial order relation R defined on as follows: For all ordered triples (a, b, c) and (d, e, f) in ,

where denotes the usual “less than or equal to” relation for real numbers. Draw the Hasse diagram for R.

Text, letter

Description automatically generated

8.5.40 Prove that a nonempty, finite, partially ordered set has

1. At least one minimal element.

Proof by contradiction:

There is no least minimal element and thus there must be a <= b and b <= a where a != b and this is not possible and thus there must be at least one minimal element.

1. At least one maximal element.

Same as above.

# Counting and Probability

## 9.2 Possibility Trees and the Multiplication Rule

9.2.5 In a competition between players X and Y, the first player to win three games in a row or a total of four games wins. How many ways can the competition be played if X wins the first game and Y wins the second and third games? (Draw a tree.)

7 ways. 3 of which X wins (XXX, XXYX, XYXX) and 4 of which Y wins (Y, XXYY, XYY, XYXY).

A picture containing player, net

Description automatically generated

9.2.10 Suppose there are three routes from North Point to Boulder Creek, two routes from Boulder Creek to Beaver Dam, two routes from Beaver Dam to Star Lake, and four routes directly from Boulder Creek to Star Lake. (Draw a sketch.)

A picture containing shoji

Description automatically generated

1. How many routes from North Point to Star Lake pass through Beaver Dam?

There are 24 routes from NP to SL. Of these we see that 3\*4=12 doesn’t pass BD and thus we have that 24-12 = 12 routes pass through BD.

1. How many routes from North Point to Star Lake bypass Beaver Dam?

As mentioned above, 12.

9.2.15 A combination lock requires three selections of numbers, each from 1 through 30.

1. How many different combinations are possible?

combinations

1. Suppose the locks are constructed in such a way that no number may be used twice. How many different combinations are possible?

combinations

9.2.21a+b

Suppose A is a set with m elements and B is a set with n elements.

1. How many relations are there from A to B? Explain.

There are elements in the cartesian product . Since a relation is a subset of , there are different possible relations from A to B.

1. How many functions are there from A to B? Explain.

There are since can only map to one element , but several elements can map to from , let us look at the elements n. We see that there are ways to have functions from A to B, that is .

9.2.33 Six people attend the theater together and sit in a row with exactly six seats.

1. How many ways can they be seated together in the row?

They can sit in different ways.

1. Suppose one of the six is a doctor who must sit on the aisle in case she is paged. How many ways can the people be seated together in the row with the doctor in an aisle seat?

Assuming there are 1 aisle seat (the opposite side is against a wall):

different ways to seat the 6 people. That is because the doctor sits at the predecided seat, and then there are to seat the rest.

Assuming there are 2 aisle seats (one on each side):  
 different ways to seat the 6 people. Like above, the doctor has two seats and then the rest has ways to be seated.

1. Suppose the six people consist of three married couples and each couple wants to sit together with the older partner on the left. How many ways can the six be seated together in the row?

There are three pairs of seats for the three couples, so there are ways to choose how the couples sit. Knowing the oldest partner is always on the left, and assuming all couples have one distinct older partner, there are just the 6 ways to choose how the couples sit.

## 9.3 Counting Elements of Disjoint Sets: The Addition Rule

9.3.7 At a certain company, passwords must be from 3–5 symbols long and composed from the 26 uppercase letters of the Roman alphabet, the ten digits 0–9, and the 14 symbols !, @, #, $, %, ^, &, \*, (, ), -, +, {, and }.

1. How many passwords are possible if repetition of symbols is allowed?

There are different symbols to use.

That means there are .

1. How many passwords contain no repeated symbols?

There are:

1. How many passwords have at least one repeated symbol?

That would be .

9.3.12

1. How many ways can the letters of the word THEORY be arranged in a row?

There are 6 letters in the word so

1. How many ways can the letters of the word THEORY be arranged in a row if T and H must remain next to each other as either TH or HT?

Consider TH as one letter so there are 5 letters of the word:

However since we can have TH represented as either TH or HT, that is in two different ways, we have different ways.

9.3.17

1. How many strings of four hexadecimal digits do not have any repeated digits?

That would be permutations if we allow the string to begin with 0.

Otherwise we would have permutations (15 different to choose from for the first letter, then the remaining 14 plus 0 for the next etc.)

1. How many strings of four hexadecimal digits have at least one repeated digit?

There are strings of four hexadecimal digits to subtract those with no repeating digits:

If we again assume 0 can’t be leading the string, we have combinations of valid four letter hexadecimal strings.

## 9.4 The Pigeonhole Principle

9.4.2

1. If 13 cards are selected from a standard 52-card deck, must at least 2 be of the same denomination? Why?

No, there are 13 different card values (1-10 + jack, queen and king) and thus we can match one card to each card value with no repetition.

1. If 20 cards are selected from a standard 52-card deck, must at least 2 be of the same denomination? Why?

Yes, according to the pigeonhole principle, if we attempt to distribute 13 card values onto 20 cards, there will be some repetition.

9.4.11 If integers are chosen from the set

where n is a positive integer, must at least one of them be even? Why?

Yes, since we have even numbers, and we are attempting to choose , we know from the pigeonhole principle that at least one will be a duplicate of another.

9.4.15 If is a positive integer, how many integers from 0 through must you pick in order to be sure of getting at least one that is odd? at least one that is even?

Assume we pick all odd numbers first, we have odd numbers to pick. Thus the pick will be even. Same goes for even numbers.

9.4.28 A programmer writes 500 lines of computer code in 17 days. Must there have been at least 1 day when the programmer wrote 30 or more lines of code? Why?

The programmer has written on average. Assuming the programmer doesn’t write half lines of codes, then he could at max write lines of code in 17 days without reaching the limit of 30 lines of code. Thus he must have written at least 30 lines of code one of the 17 days.

9.4.30 A penny collection contains twelve 1967 pennies, seven 1968 pennies, and eleven 1971 pennies. If you are to pick some pennies without looking at the dates, how many must you pick to be sure of getting at least five pennies from the same year?

It will be possible to pick first 4 from all collections before you pick the 5th of one the sets. Thus you will need to pick at least coins before you certainly picked 5 from the same year.

## 9.5 Counting Subsets of a Set: Combinations

9.5.7 A computer programming team has 13 members.

1. How many ways can a group of seven be chosen to work on a project?
2. Suppose seven team members are women and six are men.
   1. How many groups of seven can be chosen that contain four women and three men?
   2. How many groups of seven can be chosen that contain at least one man?

(Only one group possible that consists of exactly all 7 women)

* 1. How many groups of seven can be chosen that contain at most three women?

Consider the set of groups with 1 woman, with 2 women and with 3 women separately. Since there are only 6 men, it is not possible to create a group with 0 women.

So in total we have with at most three women (since the sets are disjoint).

1. Suppose two team members refuse to work together on projects. How many groups of seven can be chosen to work on a project?
2. Suppose two team members insist on either working together or not at all on projects. How many groups of seven can be chosen to work on a project?

Consider first the groups without these two members.

Consider the groups where these are in thus we need another 5 members from the set of remaining 11 members.

Thus we have in total .

9.5.14

1. How many 16-bit strings contain exactly seven 1’s?
2. How many 16-bit strings contain at least thirteen 1’s?
3. How many 16-bit strings contain at least one 1?

It is easier to consider the reverse, the bitstrings containing a maximum of one 0.

1. How many 16-bit strings contain at most one 1?

Same as above.

9.5.21 In Morse code, symbols are represented by variable-length sequences of dots and dashes. (For example, .) How many different symbols can be represented by sequences of seven or fewer dots and dashes?

If you consider the empty string of dots and dashes invalid, then:

Otherwise:

## 9.8 Probability Axioms and Expected Value

9.8.5 Suppose A and B are events in sample space S and suppose and and . What is ?

We know and

Then

9.8.10 Let A and B be events in a sample space S, and let . Suppose . Find each of the following:

6. because

9.8.17 An urn contains five balls numbered 1, 2, 2, 8, and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls?

Expected value of 1 ball =

That is, we expect the first ball to hold the value 4,2.

The second ball will be drawn from one of the following five sets:

That is, the sets where a given ball is drawn already. Notice how two sets are repeated once each.

The expected value of

The expected value of

The expected value of

Hence we have the expected value of the second ball:

Hence the expected value of the sum of two randomly chosen balls will be .

## 9.9 Conditional Probability, Bayes’ Formula, and Independent Events

9.9.2 Sup pose What is ?

9.9.7 An urn contains 30 red balls and 40 blue balls. Two are chosen at random, one after the other, without replacement.

1. Calculate the following:
   1. The probability that both balls are red  
      The are combinations of choosing two balls.   
      Of the 30 red balls, there are combinations of choosing two red balls.   
      Hence we have
   2. The probability that the first ball is red and the second is not  
      The probability that the first ball is red is   
      The probability that the second ball is not is   
      Hence the probability of
   3. The probability that the first ball is not red and the second is red  
      Probability of first ball not being red   
      Probability of second ball being red   
      Hence the probability of
   4. The probability that neither balls are red  
      Combinations of choosing two balls:   
      Combinations of choosing two non-red balls:   
      Hence we have
2. What is the probability that the second ball is red?
3. What is the probability that at least one of the balls is red?

9.9.12 One urn contains 4 blue balls and 16 white balls, and a second urn contains 10 blue balls and 9 white balls. An urn is selected at random, and a ball is chosen from the urn.

1. What is the probability that the chosen ball is blue?
2. If the chosen ball is blue, what is the probability that it came from the first urn?  
    such that A is the first urn and B is a blue ball.   
   Using Bayes’ Theorem we can rewrite to .  
   We have .  
   Hence

9.9.15 Two different factories both produce a certain automobile part. The probability that a component from the first factory is defective is 2%, and the probability that a component from the second factory is defective is 5%. In a supply of 180 of the parts, 100 were obtained from the first factory and 80 from the second factory.

1. What is the probability that a part chosen at random from the 180 is from the first factory?  
   Let A be the event that a part is from the first factory.
2. What is the probability that a part chosen at random from the 180 is from the second factory?  
   Let B be the event that a part is from the second factory.
3. What is the probability that a part chosen at random from the 180 is defective?  
   Let D be the event that a part is defective.
4. If the chosen part is defective, what is the probability that it came from the first factory?  
   Using Bayes’ Theorem we have:

9.9.20 Suppose a fair coin is tossed three times. Let A be the event that a head appears on the first toss, and let B be the event that an even number of heads is obtained. Show that .

Let S be the sample space. Then

The different possibilities in the sample space are:

(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT).

We see that the possibility of B is satisfied with (HHT), (HTH), (THH), (TTT) of the sample space so .

The probability of a head on the first toss, A, is .

Of these four possibilities, the probability that the first toss is head is satisfied in (HHT) and (HTH) and thus the probability of .

Lastly, given the event A, the probability of B is satisfied with (HHT) and (HTH) and thus .

I.e. .

9.9.22 Prove that if A and B are independent events in a sample space S, then and B are also independent, and so are .

The definition of independent events are that A and B are independent if and only if:

The definition of an events complement in a sample space S is and thus .

One way to calculate the probability of the intersection of two areas are with Bayes’ Theorem:

Thus we have that, if A and B are independent,

We can shorten this to .

Since we have that

9.9.28 A coin is loaded so that the probability of heads is 0.7 and the probability of tails is 0.3. Suppose that the coin is tossed ten times and that the results of the tosses are mutually independent.

1. What is the probability of obtaining exactly seven heads?  
   Bernoulli Trial:
2. What is the probability of obtaining exactly ten heads?
3. What is the probability of obtaining no heads?  
   This is equivalent to obtaining exactly 10 tails:

1. What is the probability of obtaining at least one head?  
   Consider the before question. This is the reverse, so 1 – P(10T).

9.9.32 A person takes a multiple-choice exam in which each question has four possible answers. Suppose that the person has no idea about the answers to three of the questions and simply chooses randomly for each one.

1. What is the probability that the person will answer all three questions correctly?
2. What is the probability that the person will answer exactly two questions correctly?
3. What is the probability that the person will answer exactly one question correctly?
4. What is the probability that the person will answer no questions correctly?  
   Note that this is equivalent to all questions being incorrect and hence:
5. Suppose that the person gets one point of credit for each correct question and point is deducted for each incorrect answer. What is the expected value of the person’s score for the three questions?

## Exercises from LearnIT

**E1 (from Rosen, chapter 7)**: The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, and 25 multiple-choice questions, each worth four points. The probability that Linda answers a true/false question correctly is 0.9, and the probability that she answers a multiple-choice question correctly is 0.8. What is her expected score on the final?

**E2 (from Rosen, chapter 7)**: Suppose that we flip a fair coin until either it comes up tails twice or we have flipped it six times. What is the expected number of times we flip the coin?

Let us calculate the possibility that we stop after n rolls (.

Since there are possible ways that flips can look, and we have possible ways to reach two tails within 2-5 flips, we have the following calculations:

(TT)

(HTT, THT)

(HHTT, HTHT, THHT)

(HHHTT, HHTHT, HTHHT, THHHT)

The probability that we stop after 6 rolls must then be:

We thus have the expected number of flips:

**E3**: Suppose you flip a fair coin 6 times. Let X be the random variable counting the total number of heads that comes up.

1. What is the distribution of X?

We have that X can be any integer between 0 and 6 (minimum of 0 heads and maximum of 6 heads).

There are distinct combinations of 6 flips that we can make. Since the coin is fair, all combinations are equally likely to occur and thus we can use the formula . We thus have the following:

|  |  |  |
| --- | --- | --- |
| Number of heads | Probability of the random variable is r | Distribution |
| 0 |  | 1 |
| 1 |  | 6 |
| 2 |  | 15 |
| 3 |  | 20 |
| 4 |  | 15 |
| 5 |  | 6 |
| 6 |  | 1 |

1. What is the expected value of X?

We multiply r with for all r in the above table.

1. What is the variance of X?

The formula for the variance is:

**E4**: Suppose you roll a fair die 5 times. Let X be the random variable that counts the number of times a 1 shows up.

1. What is the distribution of X?

We have that X can be any integer between 0 and 5 (minimum and maximum number of 1’s)

There are different combinations of 5 dice rolls. Since the dice has  of getting a 1 and  of getting something else, let us then use the formula:

|  |  |  |
| --- | --- | --- |
| Total 1s |  | Distribution  Total=7.776 |
| 0 |  | 3.125 |
| 1 |  | 3.125 |
| 2 |  | 1.250 |
| 3 |  | 250 |
| 4 |  | 25 |
| 5 |  | 1 |

1. What is the expected value of X?

We multiply r with for all r in the above table.

1. What is the variance of X?

The formula for the variance is

# Theory of Graphs and Trees

## 10.1 Trails, Paths, and Circuits

10.1.1 In the graph below, determine whether the following walks are trails, paths, closed walks, circuits, simple circuits, or just walks.

Chart, radar chart

Description automatically generated

Trail (Contains repeated vertex so not a path and doesn’t end at the same as it starts with).

Walk (Contains the repeated edge ).

Closed trivial walk from to . Trail (no repeated vertex).

Circuit (No repeated edges but repeated vertex )

Closed walk (Repeated edges )

Path from to

10.1.4 Consider the following graph.

A picture containing watch, clock

Description automatically generated

1. How many paths are there from to ?

3:

1. How many trails are there from to ?

9:

1. How many walks are there from to ?

Unlimited since repeated vertices and edges are allowed.

10.1.8 Find the number of connected components for each of the following graphs.

Chart, line chart

Description automatically generated

3,

Chart, line chart

Description automatically generated

2,

Chart

Description automatically generated

3,

Chart, scatter chart

Description automatically generated

2,

10.1.12 Find an Euler circuit or explain why there is not one.

Chart, radar chart

Description automatically generated

is an Euler circuit.

10.1.13 Find an Euler circuit or explain why there is not one.

Chart, radar chart

Description automatically generated

There are vertices with an odd degree of edges.

Examples of odd degrees are

10.1.29 Find a Hamiltonian circuit.

Chart, radar chart

Description automatically generated

is a Hamiltonian circuit.

10.1.30 Find a Hamiltonian circuit.

Chart, radar chart

Description automatically generated

is a Hamiltonian circuit.

## 10.4 Trees: Examples and Basic Properties

10.4.3 What is the total degree of a tree with *n* vertices? Why?

That is because any vertex can only have one parent and thus contributes with one out-edge and one in-edge. Minus one because the root does not have a parent node.

10.4.7 Find all leaves (or terminal vertices) and all internal (or branch) vertices for the following trees.

Chart, line chart

Description automatically generated

Leaves:

Internal vertices:

Chart, radar chart, line chart

Description automatically generated

Leaves:

Internal vertices:

10.4.8 Draw or explain why the following graph is not possible: Tree, nine vertices, nine edges.

Not possible, since a tree has no circuits and thus have at max the degree (n-1) \* 2. One edge contributes with 2 to the total degree so 9 edges would give a degree of 18, while 9 vertices allow for a maximum degree of (9-1) \* 2 = 16.

10.4.11 Draw or explain why the following graph is not possible: Tree, six vertices, total degree 14.

Not possible, since a tree has no circuits and thus have at max the degree (n-1) \* 2. 6 vertices allow for a maximum degree of (6-1) \* 2 = 10 which is less than the required 14.

## 10.5 Rooted Trees

10.5.1 Consider the tree shown below with root a.

Chart, radar chart

Description automatically generated

1. What is the level of n?

3, because it goes from a->d->h->n

1. What is the level of a?

0 because it is the root.

1. What is the height of this rooted tree?

5, seen at x, y and z.

1. What are the children of n?

1. What is the parent of g?

1. What are the siblings of j?

1. What are the descendants of f?

1. How many leaves (terminal vertices) are on the tree?

12:

10.5.4 Draw or explain why it is not possible to draw: Full binary tree, five internal vertices.

Chart, line chart

Description automatically generated

10.5.5 Draw or explain why it is not possible to draw: Full binary tree, five internal vertices, seven leaves

Not possible since a full binary tree would require there to be 2 children for every parent and thus an even number of terminal vertices (leaves).

# Modeling Computation (Rosen)

## 13.1 Languages and Grammar

13.1.1 Use the set of productions to show that each of these sentences is a valid sentence.

Set of productions:

1. the happy hare runs

Let noun phrase -> article adjective noun.

Let article -> the

Let adjective -> happy

Let noun -> hare

Let intransitive verb phrase -> intransitive verb

Let intransitive verb -> runs

1. the sleepy tortoise runs quickly

Let noun phrase -> article adjective noun

Let article -> the

Let adjective -> sleepy

Let noun -> tortoise

Let intransitive verb phrase -> intransitive verb adverb

Let intransitive verb -> runs

Let adverb -> quickly

1. the tortoise passes the hare  
   Let noun phrase -> article noun

Let article -> the, noun -> tortoise/hare

Let transitive verb phrase -> transitive verb -> passes

1. the sleepy hare passes the happy tortoise

Let noun phrase -> article adjective noun

Let article -> the, adjective -> sleepy/happy, noun -> hare/tortoise

Let transitive verb phrase -> transitive verb -> passes

13.1.2 Find five other valid sentences, besides those given in 13.1.1

1. the hare runs
2. the tortoise runs
3. the happy tortoise runs
4. the happy hare passes the sleepy tortoise
5. the sleepy tortoise passes the happy hare

13.1.5 Let G = (V, T, S, P) be the phrase-structure grammar with V = {0, 1,A,B, S}, T = {0, 1}, and set of productions P consisting of S → 0A, S → 1A, A → 0B, B → 1A, B → 1.

1. Show that 10101 belongs to the language generated by G.
2. Show that 10110 does not belong to the language generated by G.  
   The only way to start is with 0A or 1A. A will always produce 0B thus we have [1/0]0B. B will always produce a 1 and then can also produce an A at the end. Thus [1/0]01A which again will always give a 0B thus not following the format (1011)0.
3. What is the language generated by G?  
   Where [0/1] is read as either 0 or 1

13.1.7 Construct a derivation of using the grammar given in Example 5.

Grammar from Example 5:

Using, in order,

13.1.11 Construct a derivation of in the grammar given in Example 7.

Grammar from Example 5:

Using, in order,

13.1.17 Construct phrase-structure grammars to generate each of these sets.

For all exercises, the phrase grammar is:



## 13.2 Finite-State Machines with Output

13.2.1 Draw the state diagrams for the finite-state machines with these tables:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | f | | g | |
| 0 | 1 | 0 | 1 |
|  |  |  | 0 | 1 |
|  |  |  | 0 | 1 |
|  |  |  | 0 | 0 |

A piece of paper with writing on it

Description automatically generated with medium confidence

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | f | | g | |
| 0 | 1 | 0 | 1 |
|  |  |  | 0 | 0 |
|  |  |  | 1 | 1 |
|  |  |  | 0 | 1 |
|  |  |  | 1 | 0 |

Diagram

Description automatically generated

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | f | | g | |
| 0 | 1 | 0 | 1 |
|  |  |  | 1 | 1 |
|  |  |  | 0 | 1 |
|  |  |  | 0 | 0 |
|  |  |  | 1 | 1 |
|  |  |  | 1 | 0 |

A drawing on a piece of paper

Description automatically generated with low confidence

13.2.3 Find the output generated from the input string 01110 for the finite-state machine with the state table in

1. Exercise 1(a).

1. Exercise 1(b).
2. Exercise 1(c).

13.2.11 Construct a finite-state machine for the log-on procedure for a computer, where the user logs on by entering a user identification number, which is considered to be a single input, and then a password, which is considered to be a single input. If the password is incorrect, the user is asked for the user identification number again.

Chart

Description automatically generated with medium confidence

## 13.3 Finite-State Machines with No Output

13.3.5 Describe the elements of the set for these values of A.

1. Either the emptystring or any number of repetitions of 1s and 0s.
2. 1 repeated any number divisible by 3 times.
3. I don’t know ☹   
   Any word consisting of 1’s and 0’s but for every 1 there must be a leading 0.
4. I don’t know ☹  
   Any word consisting of 1’s and 0’s but for every 0 there must be both a leading and a trailing 1.

13.3.9 Determine whether the string 11101 is in each of these sets.

1. Yes
2. Yes
3. No. At max two leading 1’s.
4. No. Only an even number of leading 1’s.
5. Yes.
6. Yes. 11 from the first, 101 from the second.

13.3.11 Determine whether each of these strings is recognized by the deterministic finite-state automaton in Figure 1.

Figure 1 automaton:

Diagram

Description automatically generated

1. 111  
   Yes. End state
2. 0011  
   No. End state
3. 1010111  
   Yes. End state
4. 011011011  
   No. End state

13.3.13 Determine whether all the strings in each of these sets are recognized by the deterministic finite-state automaton in Figure 1.

Figure 1 automaton:

Diagram

Description automatically generated

1. Yes. Will always end at
2. Yes. Will always end at
3. No. Can construct 1 which will end at , however if at least one 0 is used it would be possible.
4. No. If at least one (01) is used, it will end in
5. No. For example a single 1 will end at
6. No. Can for example construct 1 which will end at

13.3.17 Find the language recognized by the given deterministic finite-state automaton.

Diagram

Description automatically generated

## 13.4 Language Recognition

13.4.1 Describe in words the strings in each of these regular sets.

1. Any number of 1’s trailed by a 0.
2. Any number of 1’s trailed by at least one 0.
3. Either 111 or 001.
4. Any number of either 1s or double 0s or a combination.
5. Any number of double 0’s (or none) followed by a 1 which is done any number of times (including none).
6. At least one 0 or 1 followed by any number of 0s, 1s or a combination of the two and lastly ending with two 0s.

13.4.3 Determine whether 0101 belongs to each of these regular sets.

1. No, cannot have any trailing 1’s if there are at least two 0’s.
2. No. Has to start with a 0 and then either double 1’s or any number of 01s.
3. Yes, can be done by the leading 0 followed by one 10 and one 1.
4. Yes, using one 0 from the and then the 1 from the union.
5. Yes, using the 01 twice and not the 11.
6. No, if it ends with 1 it has an even number of 1’s.
7. No, ends with two 1’s.
8. Yes, using the 01 from the concatenation and then no 1’s from the or alternatively using the 0 from the concatenation and then a single 1 from the .

13.4.5 Express each of these sets using a regular expression.

1. The set consisting of the strings 0, 11, and 010
2. The set of strings of three 0s followed by two or more 0s

(six 0’s with a star at the end meaning 5 plus zero or more)

1. The set of strings of odd length
2. The set of strings that contain exactly one 1
3. The set of strings ending in 1 and not containing 000

13.4.7 Express each of these sets using a regular expression.

1. The set of strings of one or more 0s followed by a 1
2. The set of strings of two or more symbols followed by three or more 0s
3. The set of strings with either no 1 preceding a 0 or no 0 preceding a 1
4. The set of strings containing a string of 1s such that the number of 1s equals 2 modulo 3, followed by an even number of 0s

13.4.14 Construct a nondeterministic finite-state automaton that recognizes the language generated by the regular grammar , where , , S is the start symbol, and the set of productions is

A picture containing shoji

Description automatically generated

A picture containing shoji

Description automatically generated

A picture containing shoji, whiteboard, text

Description automatically generated

Note that in this grammar, it is not possible to create an A and thus the automaton looks a lot simpler than the productions.

13.4.15 Construct a regular grammar that generates the language recognized by the finite-state machine.

Diagram

Description automatically generated

S is the starting symbol

## 13.5 Turing Machines

3.5.1 Let T be the Turing machine defined by the five-tuples:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

For each of these initial tapes, determine the final tape when T halts, assuming that T begins in initial position.

1. …BB0011BB…   
      
   T halts.
2. …BB101BB…  
      
   T halts.
3. …BB11B01BB…  
      
   T halts.
4. …BBBBB…   
      
   T halts.

13.5.3 What does the Turing machine described by the five-tuples

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

Do when given

1. 11 as input?  
      
   T halts.   
   I.e. the Turing machine changes the first 1 into a 0.
2. An arbitrary bit string as input?  
   If the first bit is 1, it will be converted to a 0. Then any 1 preceded by another 1 will be converted to a 0.

13.5.5 What does the Turing machine described by the five-tuples

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

Do when given

1. 11 as input?  
      
   T halts.  
   I.e. 11 is changed to 01.
2. A bit string consisting entirely of 1s as input?  
   The first one read will be read in and thus be replaced by a 0. Then any subsequent 1 will stay the same until the machine halts when reaching the end of the string.

13.5.7 Construct a Turing machine with tape symbols 0, 1, and B that, when given a bit string as input, replaces the first 0 with a 1 and does not change any of the other symbols on the tape.

This will stay in until it hits the first occurrence of 0 and replaces it with a 1. This will change the Turing machine to which is a final state and the program will halt.

13.5.9 Construct a Turing machine with tape symbols 0, 1, and B that, given a bit string as input, replaces all but the leftmost 1 on the tape with 0s and does not change any of the other symbols on the tape.

This will stay in until it hits the first 1 or the end of the bit string (B) without replacing anything. In the case it hits 1 it will change to and continue. If it hits B it will change to which is effectively a final state.

When it is in and reads a 1 it will convert it to a 0. If it reaches a blank character it will halt the program by changing to . Whenever 0 is read it changes neither state nor bit and continues reading.

13.5.17 Construct a Turing machine that recognizes the set .

# Missing exercises

# Lecture 1

* Prove, in natural deduction, the "exercises on propositional logic" in this week's section.
* Prove, in natural deduction, the equivalences of Exercise 3 (page 9) of the notes,

# Lecture 6

(Rosen chapter 4)

4.3.25, 4.3.41

4.4.5, 4.4.11, 4.4.21

# Lecture 9

* Prove The Multiplication Rule (Epp Theorem 9.2.1) by induction on the number of sets.
* Prove The Addition Rule (Epp Theorem 9.3.1) by induction on the number of sets.

# Lecture 10

9.9.22