Peergrade #4

Exercise 1

$Let\ n \in \mathbb{N}\ and\ n > 6. Prove\ by\ simple\ induction\ that\ 3^n < n!.$

Let us first write the predicate *P*:

$$P(n) \equiv 3^n < n!$$

Let us compute the base case:

$$P(7) = 3^7 = 2187 < 7! = 7 * 6 * 5 * 4 * 3 * 2 * 1 = 5040$$

We hence have that $2187 < 5040$ which is true.

Our inductive hypothesis is as follows:

We assume $P(k)$ to prove $P(k+1)$, that is:

$$\frac{P(k) \quad P(k) \to P(k+1)}{P(k+1)}$$

Since $3^{n+1} = 3^n * 3^1$ and $(n+1)! = n! * (n+1)$, our inductive case will be as follows:

$$3^{k+1} < k! * (k+1)$$

That is equivalent to the following:

$$3^k * 3 < k! * (k+1)$$

If we divide by 3^k on both sides of the inequality, we have only 3 left on the left side. On the right side we know that $\frac{k!}{3^k} > 1$ since our inductive hypothesis tells us that $k! > 3^k$. From the definition of *n* we know that *k* will be at least 7 and hence we have that $3 < 7$ which is true and we have thus proven that if $P(k)$ then $P(k+1)$.

Exercise 2

$Prove\ that\ f_0^2 + f_1^2 + f_2^2 + \cdots + f_n^2 = f_n * f_{n+1}\ for\ n \in \mathbb{N}$

We recall the definition of f_n:

$$f_n = \begin{cases} 1 & if\ n < 2 \\ f_{n-1} + f_{n\_2} & otherwise \end{cases}$$

Our predicate P is as follows:

$$P(n) \equiv \sum_{i=0}^{n} f_i^2 = f_n * f_{n+1}$$

Let us compute the base case:

$$P(0) = \sum_{i=0}^{n} f_i^2 = 1 = f_0 * f_1 = 1$$

Hence we have that the base case $P(0)$ is true.

Our inductive hypothesis is that $P(n) \to P(n+1)$, i.e.:

$$\sum_{i=0}^{n+1} f_i^2 = f_{n+1} * f_{n+2}\ iff \sum_{i=0}^{n} f_i^2 = f_n * f_{n+1}$$

For our inductive case we will assume that $P(n)$ to prove $P(n+1)$:

$$\sum_{i=0}^{n+1} f_i^2 = f_{n+1} * f_{n+2}$$

We will use the definition for the Fibonacci sequence to rewrite the right side of the equation while the left side of the equation we will take f_{n+1}^2 out of the summation:

$$\sum_{i=0}^{n}(f_i^2) + f_{n+1}^2 = f_{n+1} * (f_n + f_{n+1})$$

On the right side of the equation, we multiply the f_{n+1} into the parenthesis:

$$\sum_{(i=0)}^{n}(f_i^2) + f_{n+1}^2 = f_{n+1} * f_n + f_{n+1}^2$$

From our inductive hypothesis, we have that $\sum_{i=0}^{n} f_i^2 = f_n * f_{n+1}$ and hence we can reduce to the following:

$$f_{n+1}^2 = f_{n+1}^2$$

And thus we have proved our inductive case $P(n) \to P(n+1)$.

Exercise 3

$Prove\ the\ division\ theorem\ using\ strong\ induction.$

Our division theorem states that $a = q * b + r$ where $a, q, r \in \mathbb{N}$ and $b \in \mathbb{Z}^+$ and $r < b$. In other words, we want to prove that for any natural number a, we can divide it by a positive integer b such that we get a quotient and a remainder where the remainder is strictly less than b.

Our predicate is as follows: $P(a) = a = q * b + r$.

I will divide the proof into two cases:

$$a < b$$
$$a \geq b$$

First, the case where $a < b$:

Our base case is as follows:

$$P(n) = n = 0 * b + n$$

Because if $q > 0$ such that $P(n) = n = q * b + r$ where $r \in \mathbb{N}$, then $b \geq n$ and thus not in this case. That is because r is only defined for any integer that is at least 0 and hence the right side of the equation $q * b + r$ will at least be b or greater.

In other words, if and only if a is strictly less than b then the remainder will always be equal to a and the quotient will be 0.

In our second case $a \geq b$:

We will consider the above as our base cases, i.e. $\forall n, n < b, \to P(n) = 0 * b + n$.

Inductive Hypothesis: To prove $P(a) = q * b + r$, we assume $P(i)$ is true for all $i < a$.

In particular, we assume $P(a - b)$ is true. That is possible because $(a - b)$ is a natural number if and only if $a \geq b$ and $a - b < a$ and thus following our inductive hypothesis.

Then $P(a - b) = a - b = k * b + r$ to prove $P(a) = a = q * b + r$.

We will then add b to either side of our equation sign:

$$a = k * b + r + b$$

We will factorize b:

$$a = (k + 1) * b + r$$

We know that $k \in \mathbb{N}$ and thus $k + 1 \in \mathbb{N}$. Then let $q = k + 1$ and we have:

$$a = q * b + r$$

We have thus proven that any natural number dividend a can be divided by any positive integer divisor b such that it produces a quotient q and a remainder r.