Exercise 1

1. Let $\boldsymbol{A = \{x \in \mathbb{N} \mid x\ is\ even\}, B = \{x \in \mathbb{N} \mid x\ is\ odd\}, C = \{x \in \mathbb{Z} \mid x \leq 0\}, and\ D = \{1, 2, 3, \dots, 10\}}$

(a) What is $A \cap C, B \cap D, \mathcal{P}(\mathcal{P}(\emptyset))$?

The intersection of two sets are all elements that can be found in both sets.
For the first intersection, $A \cap C$, we have in one set all even numbers such that all numbers is the product of two and some integer while the other set contains all integers less than or equal to zero.
We can combine that to $A \cap C = \{x \in \mathbb{N} \mid x = 2k \land x \leq 0\}$. Since $\mathbb{N}$ is only defined for nonnegative numbers, and C is a set of non-positive numbers, what we have left is thus a set containing the only nonnegative and nonpositive element 0:

$$\{0\}$$

The second intersection $B \cap D$ is the intersection of all odd numbers and a number range from one to ten, including. We can check what numbers in the set D meet the definition of an odd number such that $x = 2k + 1, k \in \mathbb{Z}$.
That will leave us with the finite set:

$$B \cap D = \{1, 3, 5, 7, 9\}$$

For the last we have the powerset of the powerset of an emptyset.
A powerset contains 2^n elements, where n notes the number of elements in the original set.
We get that a powerset of an emptyset will have $2^0 = 1\ element$.
We will calculate the powerset of the emptyset first such that

$$\mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\})$$

Now we need to take the powerset of a set containing the emptyset. The set containing the emptyset has one element so the powerset of the set containing the emptyset will have $2^1 = 2\ elements$.

$$P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}$$

(b) Is A, B, C a partition of $\mathbb{Z}$? Explain why.

A partition of a set is when a set is split into subsets such that all subsets are mutually disjoint. In other words, the subsets cannot contain the same elements.
While A and B are mutually disjoint and where the union of those will be the set $\mathbb{N}$, we also have the set C which includes zero. So zero is both to be found in A and C and hence they do not live up to the definition of being disjoint: $A \cap C \neq \emptyset$.

(c) Let the universe $U = \mathbb{N}$. Is $\overline{A} = B$?

The definition of a complement is $A^c = \{x \in U \mid x \notin A\}$
In order for $A^c \neq B$, we would have to either have an element that is present in both sets such that $A \cap B \neq \emptyset$ or that $A \cup B \neq U$.
We know that an odd number can be written as $x = 2k + 1$ for some integer *k* and likewise an even number can be written as $x = 2k$ for some integer *k*.
For a number to be both odd and even, we would have that $2k + 1 = 2m$ where $k, m \in \mathbb{Z}$.
We could divide both sides by two such that $k + \frac{1}{2} = m$, but as we see, in this case *m* would be some integer plus a rational number such that *m* is no longer in the set of integers.

Hence we have a contradiction and know that there are no number that is both odd and even and the sets must therefore be disjoint.
Now we need to show that $A \cup B = U$ to show that $A^c = B$.
Let's assume a natural number *n* from the set *A*. This number will then be an even number such that $n = 2k$ for an integer k. The next number, $n + 1$ will then have to either be in the set A or B.

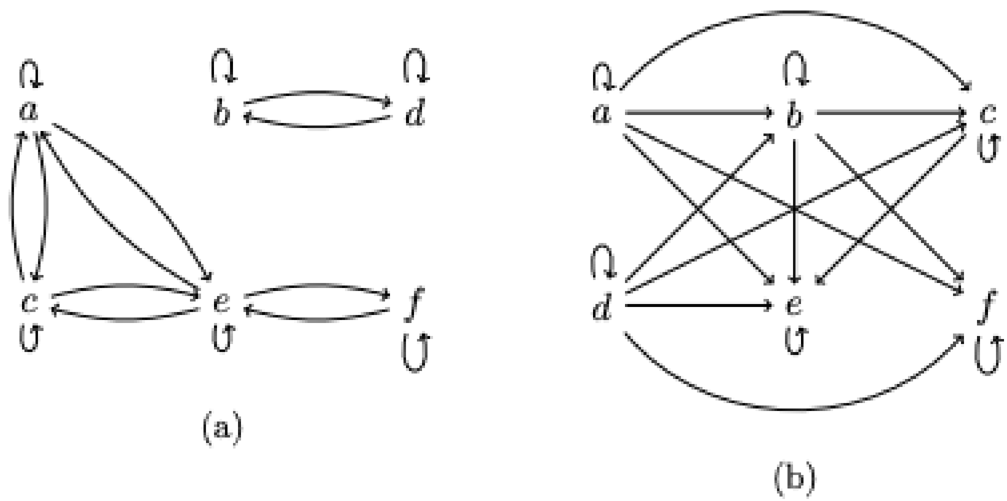
$$n + 1 = 2k + 1$$

We see that this number lives up to the definition of being odd and hence in the set B.
If we likewise assume a number *n* in B that follows the definition of odd numbers such that $n = 2k + 1$, we can see that the next number, $n + 1$, will also be in either set A or B:

$$n + 1 = 2k + 1 + 1$$
$$n + 1 = 2(k + 1)$$

We know that two integers, *k* and 1, added together will give a new integer, and hence we can let $k + 1 = m$ where *m* is an integer and we see that $n + 1 = 2m$ and follows the definition of being even.
Thus shown that any number in the set *U* will be in either the odd or the even numbers set A or B.
We have thus also demonstrated that $A^c = B$.

2. Which of the following graphs defines a partial order or an equivalence relation, or none?
In case of a partial order, show a Hasse diagram representing the relation, and find the maximal, minimal, greatest and least elements, if they exist.
In case of an equivalence relations, find the partitions of the elements that are in the same equivalence class.



For a relation to be an equivalence relation it must have the three properties of being reflexive, symmetric and transitive.
For a relation to be a partial order it must be reflexive, antisymmetric and transitive.
Let's look at the relation depicted in (a) first.

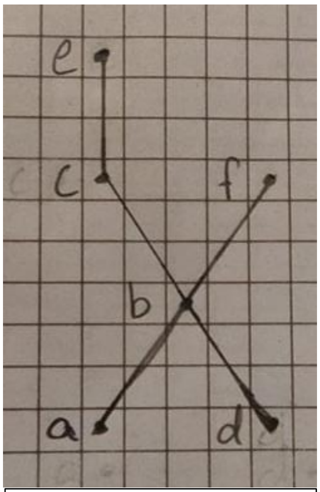
For a relation to be reflexive, it must be that $\forall x \in A, x \, \boldsymbol{R} \, x$ for the set A and the relation $\boldsymbol{R}$.
That is depicted in (a) by the loops from all points $\{a, b, c, d, e, f\}$ that loops back to itself.

For a relation to be symmetric, we must have that $\forall x, y \in A, x\ \boldsymbol{R}\ y \Leftrightarrow y\ \boldsymbol{R}\ x$. We see that depicted in (a) because for all arrows between two distinct elements, there is also an arrow in the opposite direction.
For a relation to be transitive, we must have that $\forall x, y, z \in A, x\ \boldsymbol{R}\ y \wedge y\ \boldsymbol{R}\ z \Leftrightarrow x\ \boldsymbol{R}\ z$.
We see such an example from (a, c, e) where an arrow goes from a to c and from c to e and finally from a to e. However this statement is not true for all of the relations. For example we have that f relates to e and e relates to a but f does not relate to a and hence (a) is not transitive and therefor neither an equivalence relation or a partial order.

For (b) we quickly see that it is not symmetric as we do have a relation from a to b but not from b to a. We do however see that the relation is reflexive as all elements relates to itself. It is also transitive, we see that ordered pairs like $(a, b), (b, c)(a, c)$ exists. So does the following few examples:

$$(d, b), (b, f), (d, f)$$
$$(b, c), (c, e), (b, c)$$

We could continue to show that for all elements, if an element relates to a second, and the second element relates to a third, then the first element will relate to the third element as well.
For the relation to be a partial order it must finally also be antisymmetric which is defined such that for all elements, if an element is related to a distinct second element, then the second element is not related to the first. For the graph, that means no arrow must go "both ways" (i.e. two arrows in opposite directions).
We also see that this is true and hence (b) must be a partial order. We shall then draw a Hasse diagram and find the maximal, minimal, greatest and least elements if they exist.
The Hasse diagram is created by removing all reflexive relations and for all transitive relations where $x\boldsymbol{R}y \wedge y\boldsymbol{R}z \rightarrow x\boldsymbol{R}z$ we will remove the relation $x\boldsymbol{R}z$. That creates the following diagram:



Maximals: e & f
Minimals: a & d
Greatest: None
Least: None

The maximal elements are all elements $x \in A$ such that $\nexists y \in A, x\boldsymbol{R}y$ where A is the set and $\boldsymbol{R}$ is the relation composing the the poset: $(A, \boldsymbol{R})$.
Hence we have the maximals e & f.
The minimal elements are likewise elements $x \in A$ such that $\nexists y \in A, y\boldsymbol{R}x$.
Hence we have the minimals a & d.
The greatest element is an element such that all elements relate to the element. Because f does not relate to e, and all other elements relate to e, we have no greatest element.
Likewise, the least element is an element such that it relates to all other elements. We both have that a and d relates to b but neither a nor d relates to the other and hence we have no least either.

Exercise 2

Provide a proof by contradiction of this statement: $Let\ A = \{x \in \mathbb{N} \mid x\ is\ even\}, B = \{x \in \mathbb{N} \mid x\ is\ odd\}$. Prove that $A \cap B \subset \mathbb{Z}$.

To prove the statement using contradiction, we will try to find an element x in $A \cap B$ such that $x \notin \mathbb{Z}$ and hence the statement is false.
Let's insert the definition of intersection.

$$\forall x(x \in A \cap B \leftrightarrow x \in A \land x \in B)$$

We have that x must be in both sets A and B and hence be both an odd number and an even number. Let's insert the definitions for both.

$$x = 2k$$
$$x = 2m + 1$$
$$k, m \in \mathbb{Z}$$

If x can be constructed using these two definitions, then we have the following:

$$2k = 2m + 1$$
$$k = m + \frac{1}{2}$$

We now see that k is equal to an integer m plus a rational number $\frac{1}{2}$ such that k no longer follows the definition of an integer. Likewise,

$$m = k - \frac{1}{2}$$

Thus either m or k has to be a rational number and we have a contradiction as x is only defined for integers m and k.

Exercise 3

Provide a direct proof of this statement: Let A and B be two arbitrary sets such that $B \subseteq A$, and let $\boldsymbol{R}$ be an equivalence relation on A. Consider the relation $\boldsymbol{S} = \{(a, b) \in R \mid a, b \in B\}$ on the set B. Then $\boldsymbol{S}$ is an equivalence relation on B (that is, reflexive, symmetric and transitive).

Since $\boldsymbol{R}$ is reflexive, such that any element x in A relates to itself: $(x, x) \in \boldsymbol{R}$.
The definition of $\boldsymbol{S}$ states, that any ordered pair $(a, b) \in \boldsymbol{R}$, where the two elements $a, b \in B$, that is also an ordered pair in $\boldsymbol{S}$. Hence $\boldsymbol{S}$ is also reflexive.

We know that $\boldsymbol{R}$ is symmetric such that any ordered pair $(x, y) \in R$ there will also be $(y, x) \in \boldsymbol{R}$. Let $a, b \in B$. If $(a, b) \in \boldsymbol{R}$, then from the definition of $\boldsymbol{S}$, then (a, b) must also be in $\boldsymbol{S}$. Likewise, because $\boldsymbol{R}$ is symmetric, the pair (b, a) is in $\boldsymbol{R}$ such that the same pair must also be in $\boldsymbol{S}$ and hence $\boldsymbol{S}$ is also symmetric.

We also know $\boldsymbol{R}$ is transitive such that $\forall (x, y), (y, z) \in \boldsymbol{R}, (x, z) \in \boldsymbol{R}$.
Similarly, for any three elements $a, b, c \in B$, they must be in A because $B \subseteq A$. If $(a, b) \in \boldsymbol{R}$ and $(b, c) \in \boldsymbol{R}$ then $(a, c) \in \boldsymbol{R}$.
From the definition of $\boldsymbol{S}$, if a relation $(a, b) \in \boldsymbol{R}$, and $a, b \in B$, then $(a, b) \in \boldsymbol{S}$. Hence we have that for the three elements in B, if they have a transitive relation in $\boldsymbol{R}$, then they must also have the same relations and thus a transitive relation in $\boldsymbol{S}$.

We have thus shown that by the definition of $\boldsymbol{S}$, it must be an equivalence relation on B.