Introduction to Information Security

7. Public-key Cryptography

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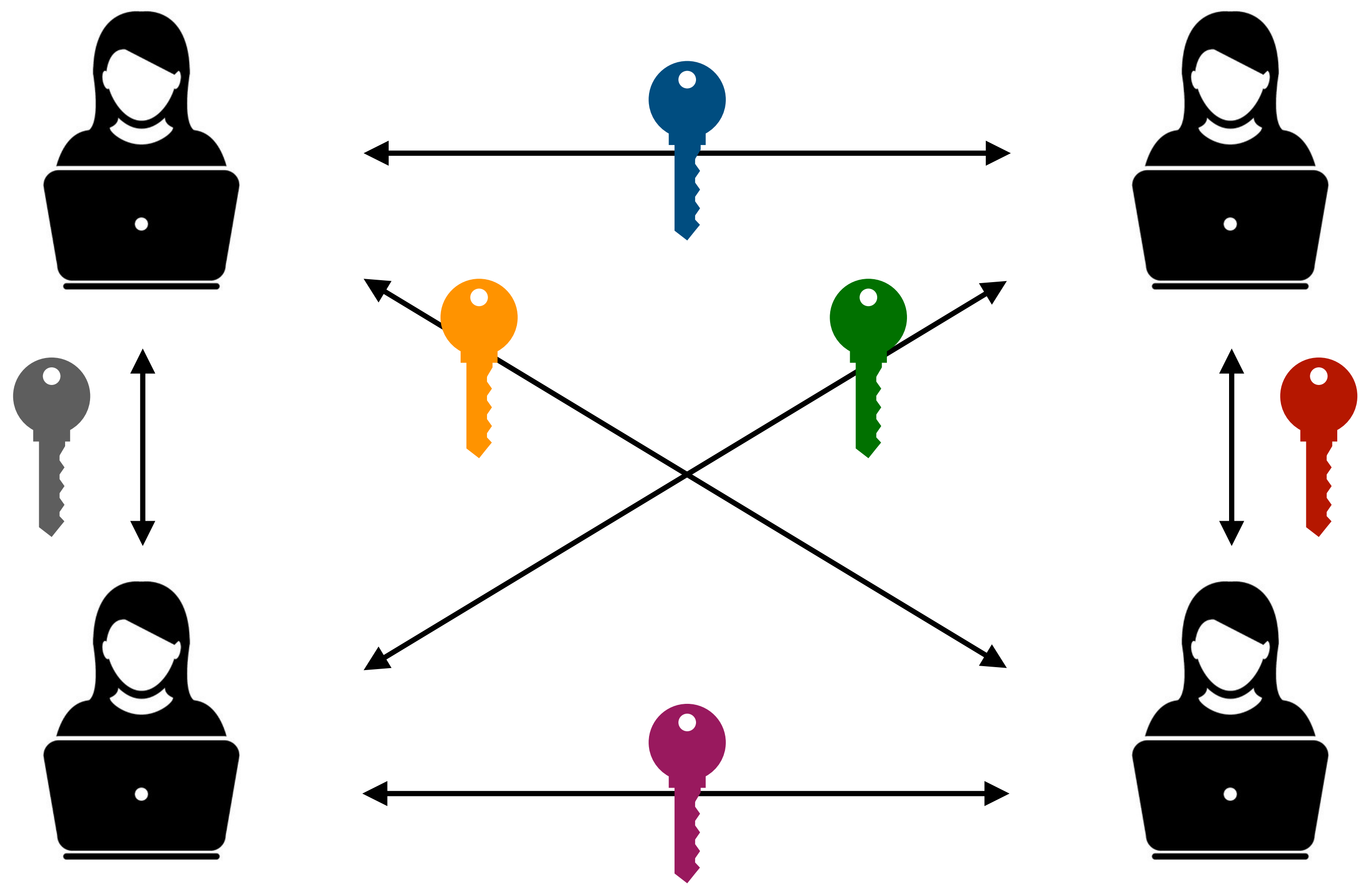
Symmetric-Key Encryption

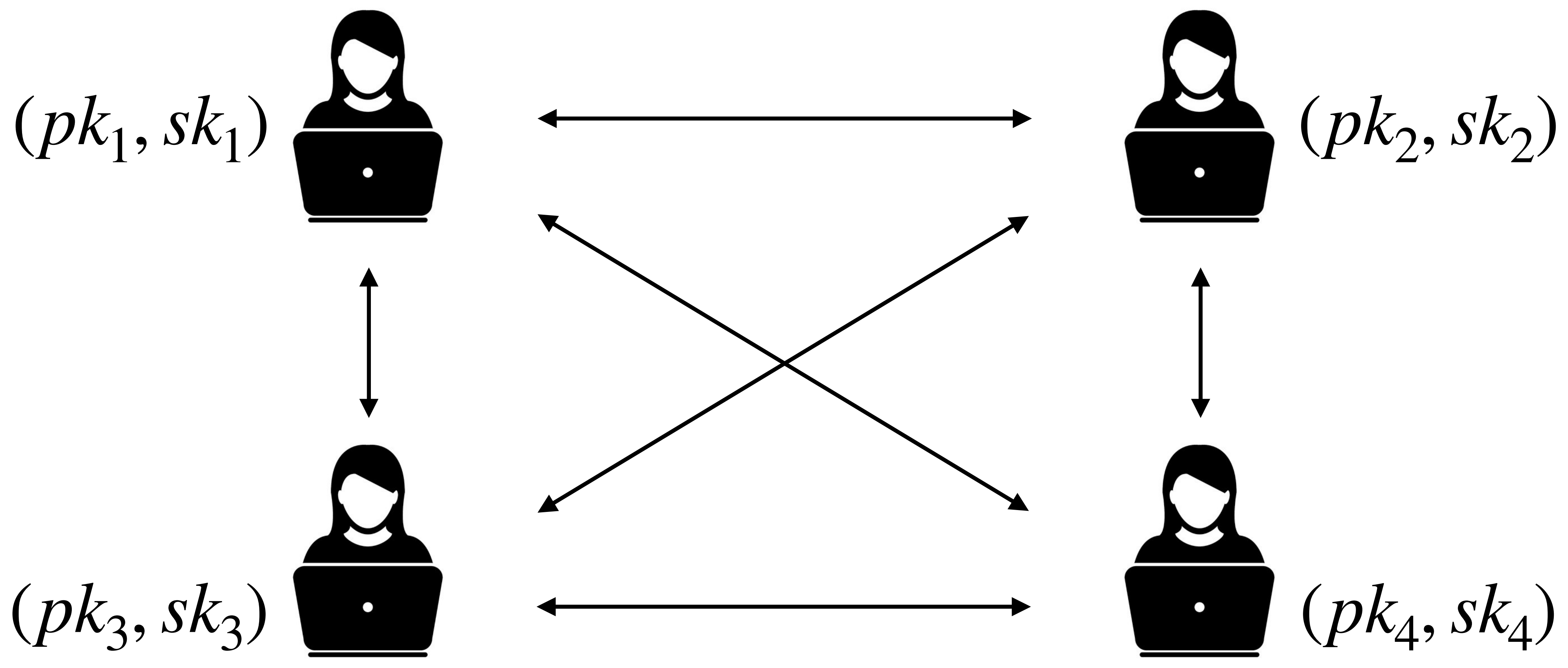
- Recap: the same key shared between two parties
- What happens if there are many users?
  - n users: $\binom{n}{2} = n(n-1)/2$
  - Example: 4950 keys / 100 users
- Key distribution and maintenance problem

How to solve this issue?



Public-Key Revolution

- Invented in 1976 by Diffie and Hellman (ACM Turing Award 2015)
- Problem
  - pk : public key, widely disseminated, used for encryption
  - sk : private key, kept secretly, used for decryption
  - n users: $2n$ keys



Notation

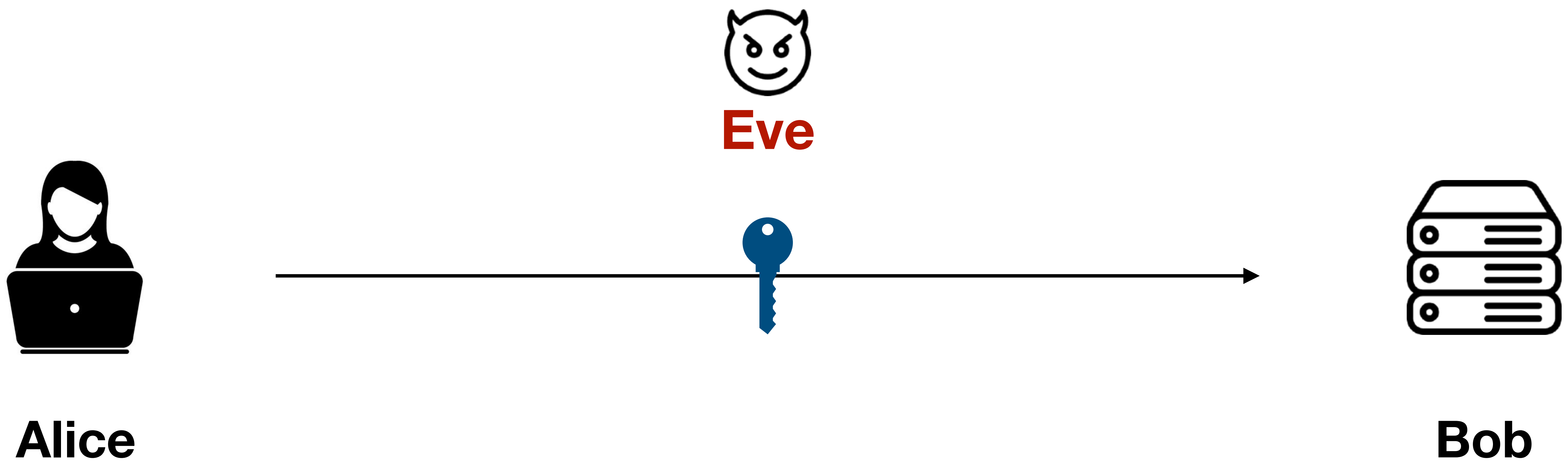
- Modular operation: $x \bmod y = z \iff x = ay + z$
- Modular congruence: $x \equiv y \mod z \iff (x \bmod z) = (y \bmod z)$

Instances

- Secret-key exchange (Diffie-Hellman key exchange)
- Confidentiality: public-key encryption (RSA)
- Integrity: digital signature

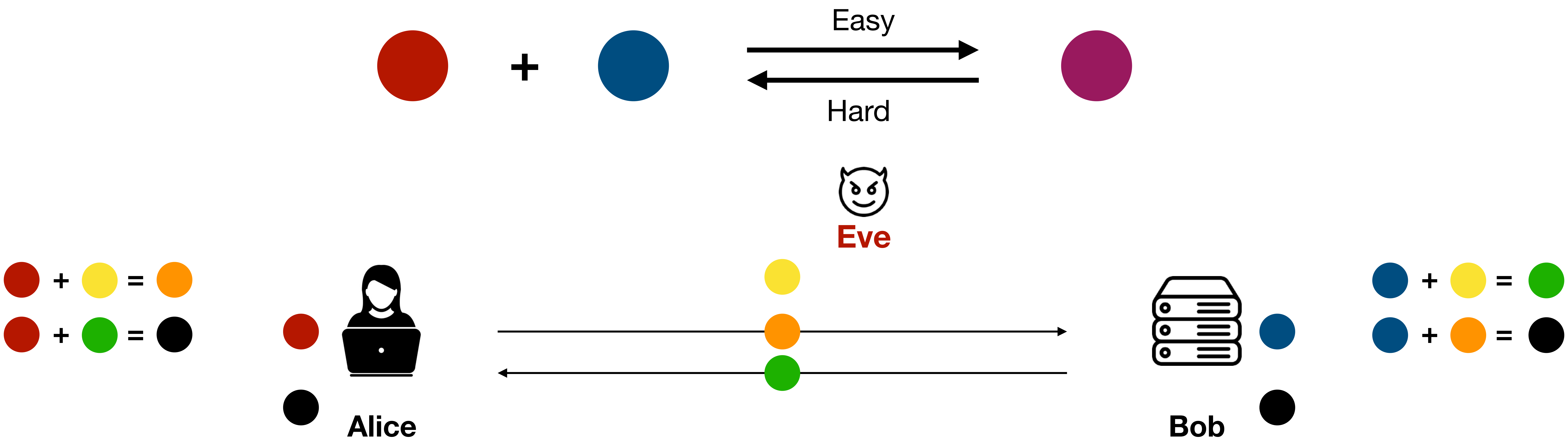
Secret Key Exchange

- Setting: Alice and Bob want to share a secret key using an insecure channel
- Problem: How can two people (who have never met) agree on a secret key?



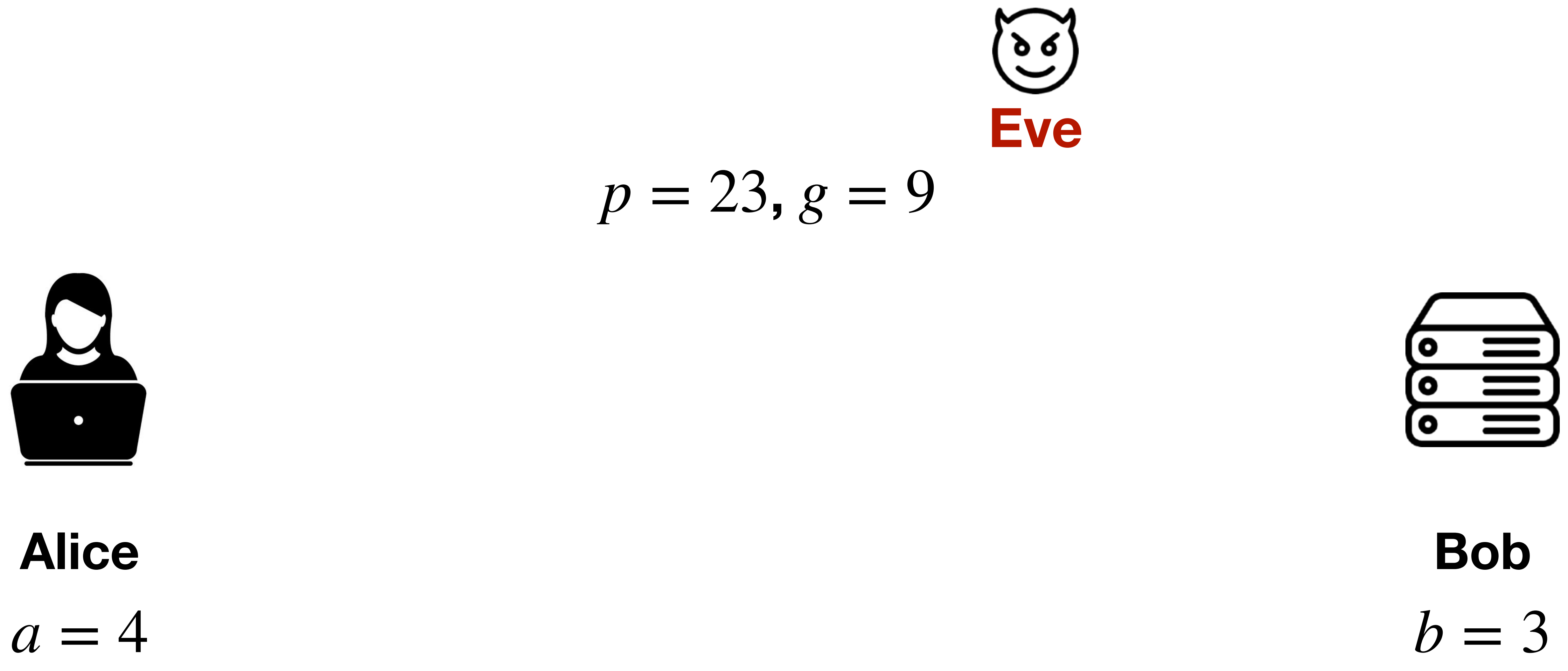
Idea: One-way Function

- Easy in one direction but hard in the reverse direction
  - E.g., discrete logarithm (math), integer factorization (math), color mixing (painting), 비빔밥



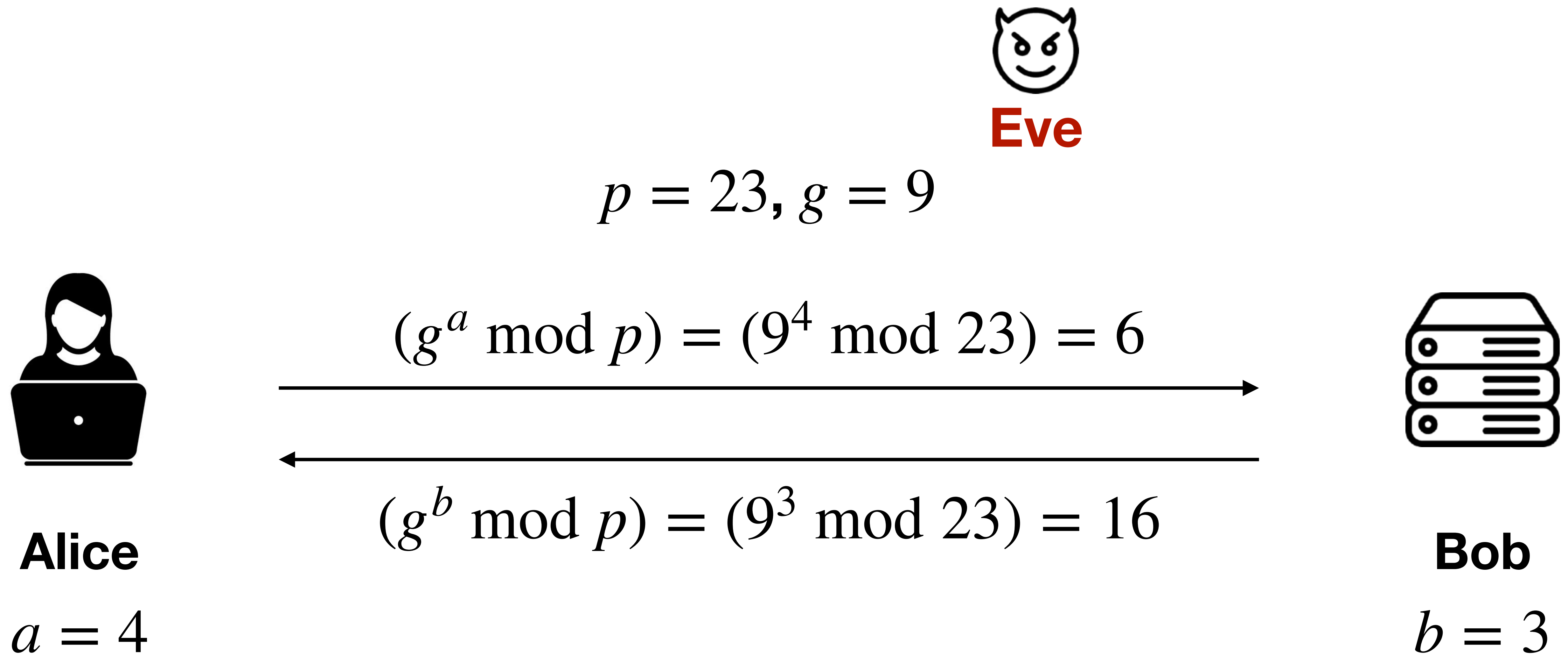
Diffie-Hellman Key Exchange (1)

- Pick two public values: large prime p and generator g
- Alice has secret value a
- Bob has secret value b



Diffie-Hellman Key Exchange (2)

- Alice sends $A = (g^a \bmod p)$ to Bob
- Bob sends $B = (g^b \bmod p)$ to Alice



Diffie-Hellman Key Exchange (3)

- Alice computes $(B^a \bmod p) = \left((g^b \bmod p)^a \bmod p\right)$
- Bob computes $(A^b \bmod p) = \left((g^a \bmod p)^b \bmod p\right)$
- Secret key: $g^{ab} \bmod p$

Eve

$p = 23, g = 9$

$K = (16^4 \bmod 23) = 9$

$(g^a \bmod p) = (9^4 \bmod 23) = 6$

$K = (6^3 \bmod 23) = 9$

$(g^b \bmod p) = (9^3 \bmod 23) = 16$

Alice

$a = 4$

Bob

$b = 3$

Correctness

- Correctness: Is $K_{Alice} = (B^a \bmod p)$ equal to $K_{Bob} = (A^b \bmod p)$?

$$(B^a \bmod p) = \left((g^b \bmod p)^a \bmod p\right) = (g^{ab} \bmod p)$$

$$(A^b \bmod p) = \left((g^a \bmod p)^b \bmod p\right) = (g^{ab} \bmod p)$$

Theorem. Given natural numbers X, Y, p and k,

$$((X \mod p)^k \mod p) = (X^k \mod p)$$

Security

- Eve cannot efficiently compute $(g^{ab} \mod p)$ without knowing a and b
  - Eve can observe p, g, ($g^a \mod p$), and ($g^b \mod p$)
- Discrete logarithm problem: given m, n, and p, find x s.t. $(m^x \equiv n \mod p)$
  - No efficient algorithms (no polynomial time algorithm)
- Not secure against quantum computers
  - An efficient algorithm exists (Shor's algorithm)

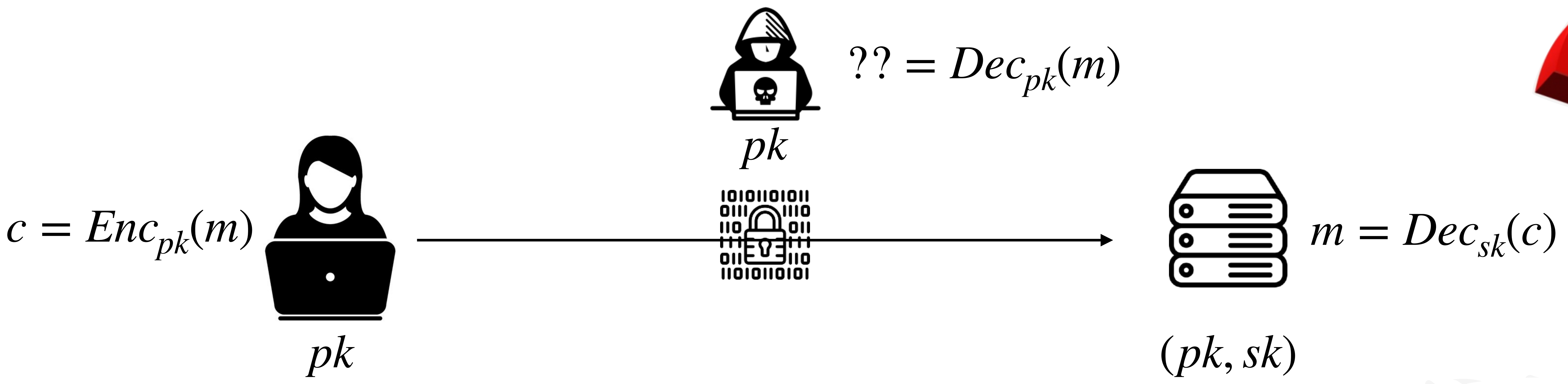
Instances

- Secret-key exchange (Diffie-Hellman key exchange)
- Confidentiality: public-key encryption (RSA)
- Integrity: digital signature

Confidentiality with Public-Key

- Generate (pk, sk) and publicize pk
- Anyone with pk can encrypt message
- Only the one with sk can decrypt message
    - Cannot decrypt with pk

How is it possible?

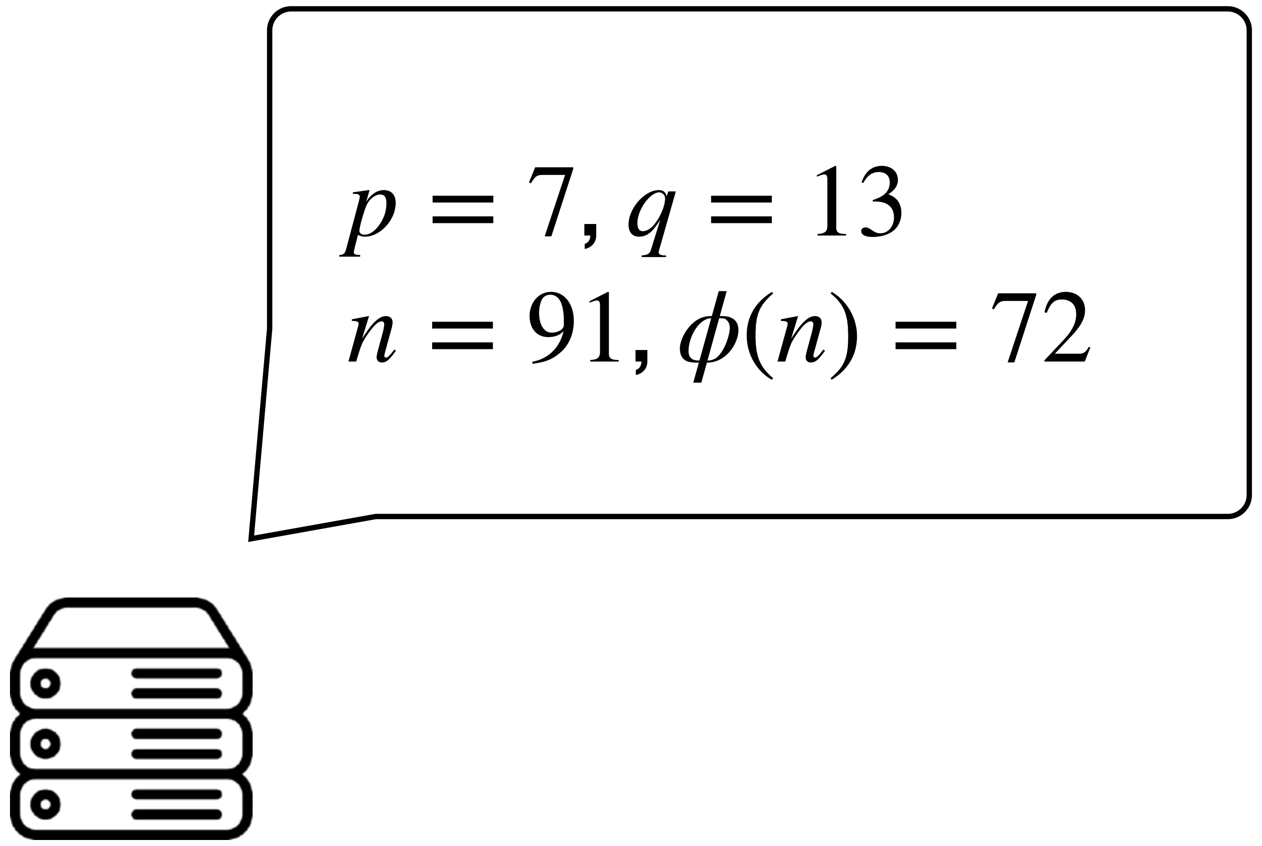


RSA Cryptosystem

- Invented by Rivest, Shamir, and Adleman in 1977
  - ACM Turing award in 2002
- Rely on the practical difficulty of factoring the product of two large prime numbers
  - But efficiently solvable by quantum computers

RSA Algorithm (1)

- Select two large primes p and q
- Compute $n = pq$ and $\phi(n) = (p-1)(q-1)$



RSA Algorithm (2)

- Choose e s.t. $1 < e < \phi(n)$ and $\gcd(e, \phi(n)) = 1$
- Choose d s.t. $1 < d < \phi(n)$ and $ed \equiv 1 \mod \phi(n)$
  - d exists if $\gcd(e, \phi(n)) = 1$

$p = 7, q = 13$
$n = 91, \phi(n) = 72$
$e = 5, d = 29$

Why Relative Prime?

Theorem [Bézout's Identity].
For all $a, b \in \mathbb{Z}$**,if** $\gcd(a, b) = d$ **then**
$\exists x, y \in \mathbb{Z} \,.\, ax + by = d$

$$
\begin{aligned}
& \gcd(e, \phi(n)) = 1 \\
\iff\ & \exists x, y \in \mathbb{Z} \,.\, ex + \phi(n) \cdot y = 1 \\
\iff\ & ex \equiv 1 \mod \phi(n)
\end{aligned}
$$

RSA Algorithm (3)

- Public key: (n, e)
- Private key: (n, d)

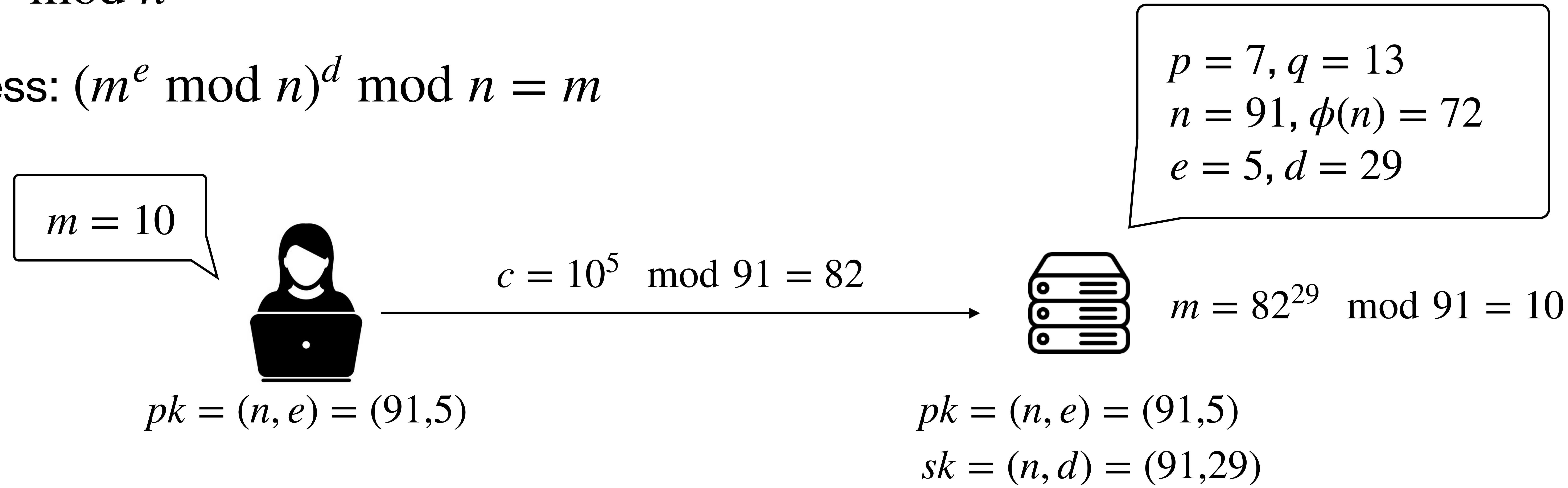
$p = 7, q = 13$
$n = 91, \phi(n) = 72$
$e = 5, d = 29$

$pk = (n, e) = (91,5)$

$pk = (n, e) = (91,5)$
$sk = (n, d) = (91,29)$

RSA Algorithm (4)

- Encryption
  - For plaintext $m < n$, $c = m^e \bmod n$
- Decryption
  - $m = c^d \bmod n$
- Correctness: $(m^e \bmod n)^d \bmod n = m$



Correctness

- Correctness: $(m^e \bmod n)^d \bmod n = m$

$$(m^e \bmod n)^d \bmod n = (m^e)^d \bmod n$$

Theorem

$$((X \bmod p)^k \bmod p) = (X^k \bmod p)$$

$$(m^e)^d = m^{ed} = m^{1+k\cdot\phi(n)}$$

"Choose d s.t. $1 < d < \phi(n)$ and $ed \equiv 1 \mod \phi(n)$"

$$m^{1+k\cdot\phi(n)} \equiv m \mod n$$

Euler's Theorem

If p and q are primes, $n = pq$, then $\forall a \in \mathbb{Z}_n .\, a^{k\cdot\phi(n)+1} \equiv a \mod n$

Security

- Adversary cannot efficiently compute p and q from n
  - $n = pq$ and p, q: large prime numbers
- Adversary can observe n and e (public key) but cannot efficiently compute d (private key)
  - d: $1 < d < \phi(n)$ and $ed \equiv 1 \mod \phi(n)$
- Integer factorization problem: given n, find prime number p and q s.t. $n = pq$

Comparison to Private-Key Encryption

- Pros
  - Does not need any secure key distribution
  - Enable multiple senders to communicate privately with a single receiver
- Cons
  - Roughly 2-3 orders of magnitude slower

Instances

- Secret-key exchange (Diffie-Hellman key exchange)
- Confidentiality: public-key encryption (RSA)
- Integrity: digital signature

Digital Signature (1)

- Only the one with sk can generate signature σ for message m
- Anyone with pk can verify the signature (i.e., integrity)
- Example: SW patch distribution

$p = 7, q = 13$
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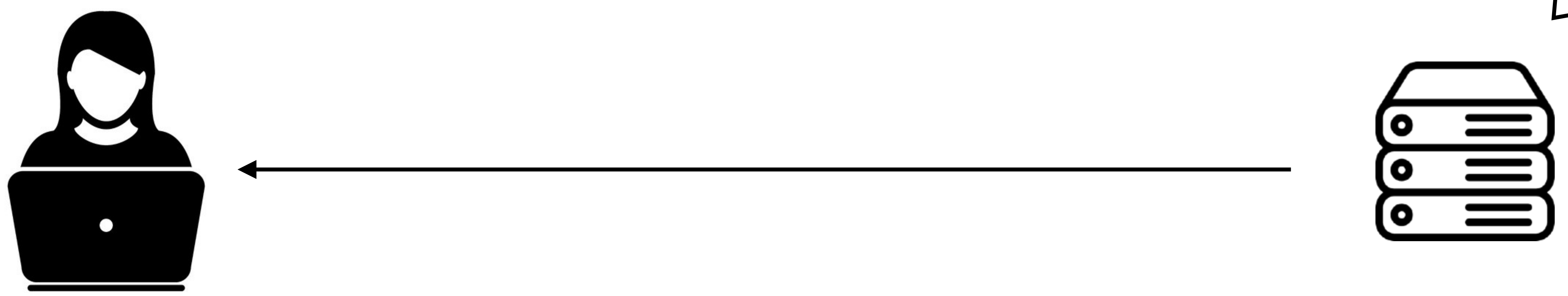

$pk = (n, e) = (91,5)$

$pk = (n, e) = (91,5)$
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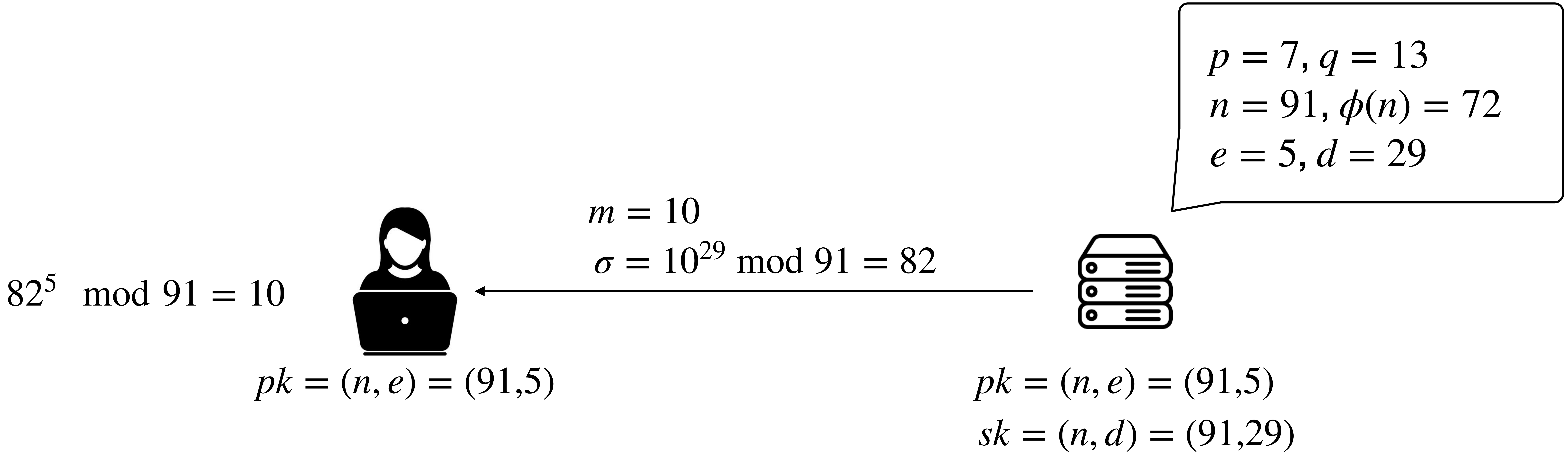
Digital Signature (2)

- Send message m and signature $\sigma = m^d \bmod n$
- Verify the integrity by checking m is equal to $\sigma^e \bmod n$
- Correctness: $(m^d \bmod n)^e \bmod n = m$



Correctness

- Correctness: $(m^d \mod n)^e \mod n = m$

$$(m^e \mod n)^d \mod n = (m^e)^d \mod n$$

Theorem

$$((X \mod p)^k \mod p) = (X^k \mod p)$$

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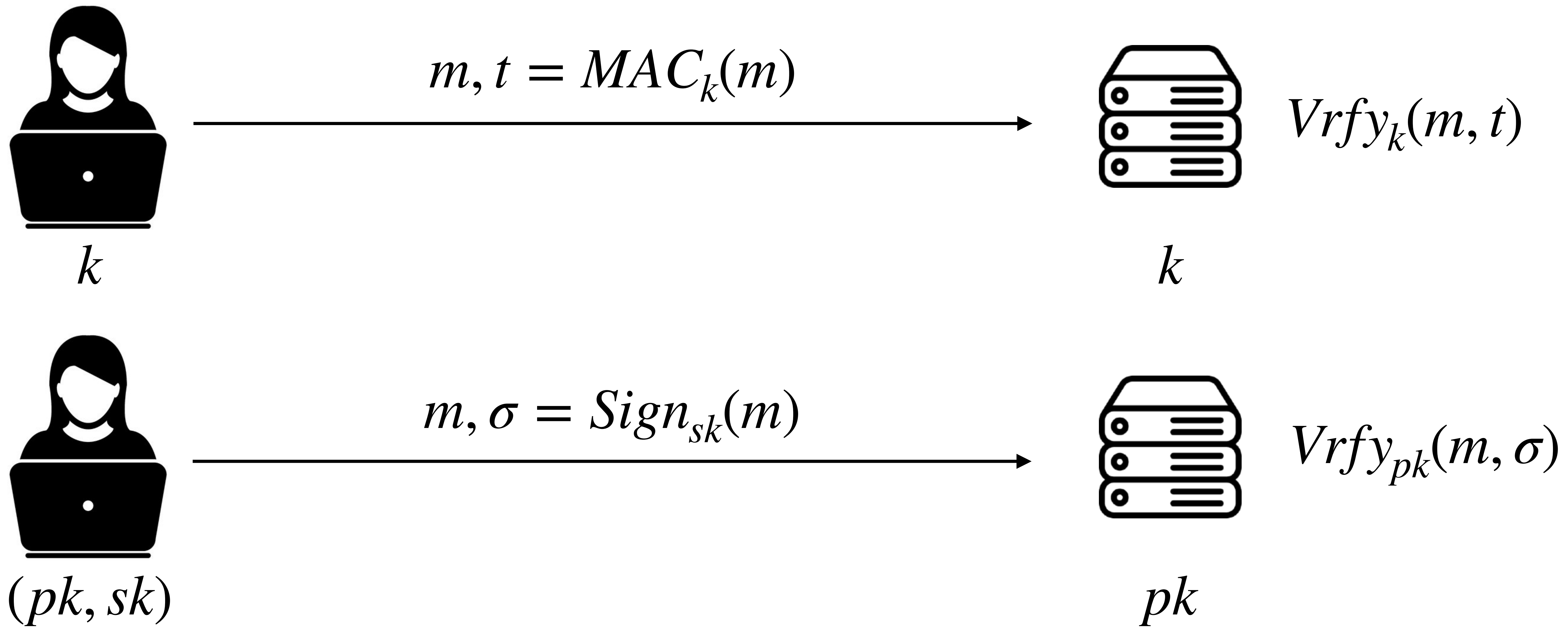
Digital Signature in Practice

- Authentication: proof that you or something you created is legitimate
- Non-repudiation: signed document becomes proof that Alice indeed signed the document
    - Only Alice can generate (m, σ) and
    - Cannot deny having created the signature

Comparison to MAC

- Both: ensure the integrity of transmitted messages
- Pros: public verifiability
    - Multiple receivers can verify the signature
- Cons: efficiency

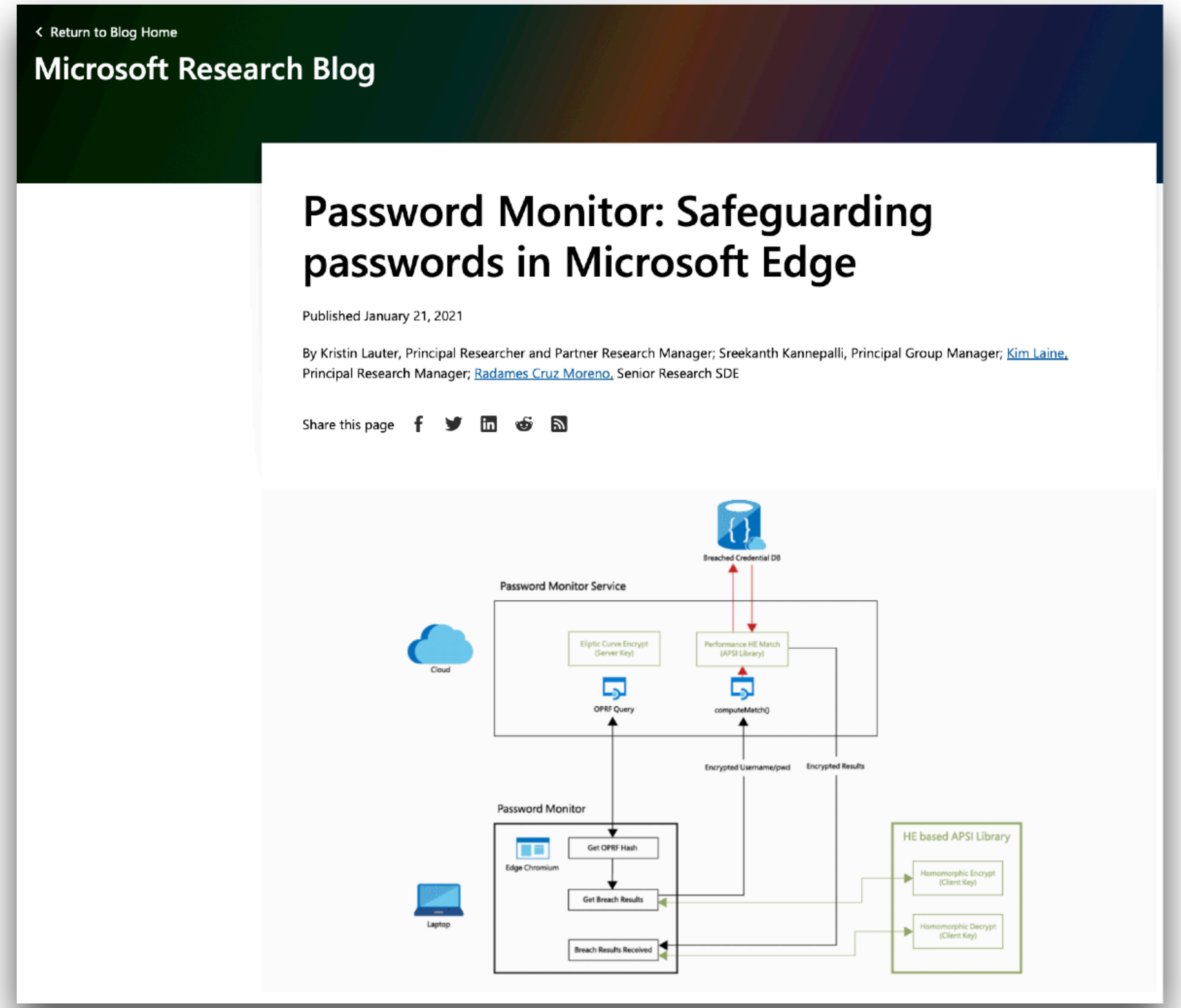


Holy Grail of Cryptography

- Is it possible to provide a secure public service? (i.e., computations on encrypted data)
- Example
  - Average GPA in the class with encrypted individual GPAs
  - Searchable cloud storage with encrypted data
  - Election with encrypted votes
  - Privacy-preserving machine learning
- Necessary property: homomorphism
  - $Dec(c_1 \oplus c_2) = Dec(c_1) \oplus Dec(c_2)$

Homomorphic Encryption (동형 암호)

- Allows computations on encrypted data
- “A Fully Homomorphic Encryption Scheme”, C. Gentry, 2009
- Applications



A Simplified HE Scheme

- Plaintext space: $\{0,1\}$
- Public key: pq, secret key: p,
- Random noise: ϵ
- Encryption: $Enc(m) = m + pq + 2\epsilon$
- Decryption: $Dec(c) = (c \bmod p) \bmod 2$
- Homomorphism
  - $Dec(Enc(m_1) + Enc(m_2)) = Dec(Enc(m_1 + m_2)) = m_1 + m_2$
  - $Dec(Enc(m_1) \times Enc(m_2)) = Dec(Enc(m_1 \times m_2)) = m_1 \times m_2$

Summary

- Public-key revolution: solve key distribution and maintenance problem
  - Diffie-Hellman key exchange
  - Public-key encryption
  - Digital signature
- New emerging technology: homomorphic encryption
  - Computation on encrypted data
  - Application: privacy-preserving services