LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

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RECALL (Evaluation of Line Integral)

(A) Without Parameterize the Curve: Let the path C be given by $y = y(x)$; $a \leq x \leq b$

$$\int_C f(z)\,dz = \int_a^b \{u(x, y(x)) - v(x, y(x))\, y'(x)\}\, dx + i \int_a^b \{v(x, y(x)) + u(x, y(x))\, y'(x)\}\, dx$$

(B) Parameterize the Curve: Let the path C be represented by $z = z(t)$ where $a \leq t \leq b$.

$$\int_C f(z)dz = \int_a^b f(z(t))\, \dot{z}(t)\, dt$$

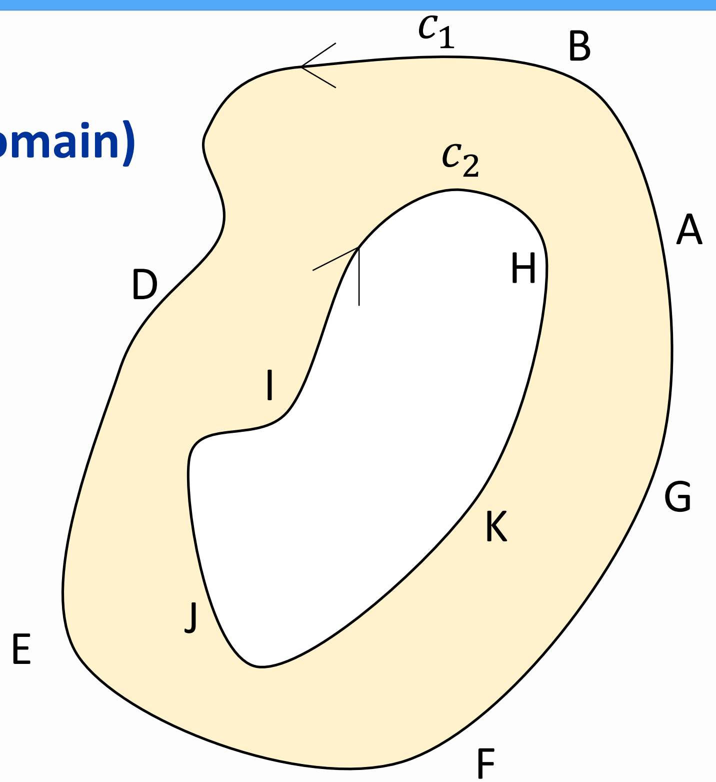
(C) If $f(z)$ is Analytic in a simply connect domain D: (i) For every simple closed C in D, $\oint_C f(z)\, dz = 0$

(ii) There exists an analytic function $F(z)$ with $F'(z) = f(z)$ in D then along any path joining z_1 and z_2 in D

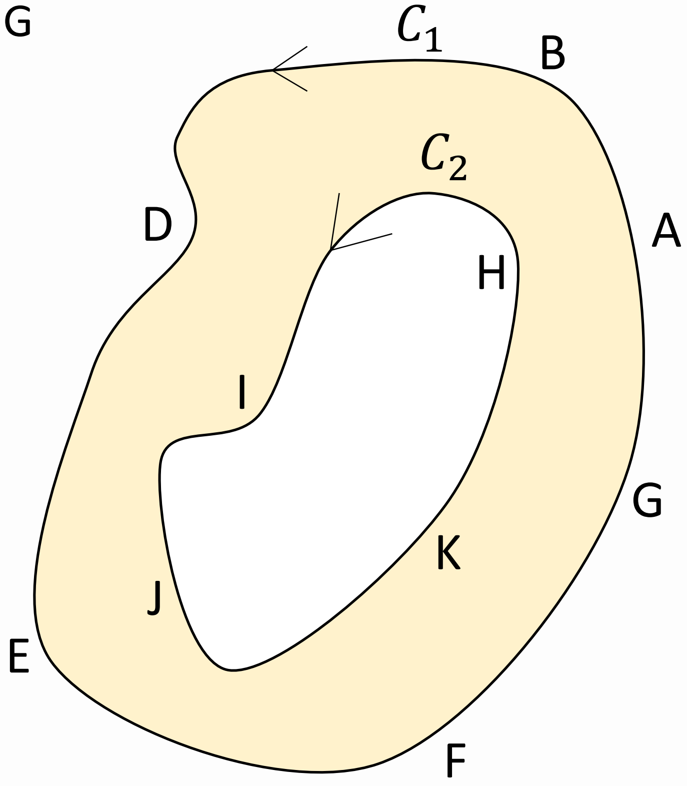
$$\boxed{F(z_1) - F(z_0) = \int_{z_0}^{z_1} f(z)\, dz}$$

Cauchy Integral Theorem (Multiply Connected Domain)

$$\oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = 0$$



Deformation of Path

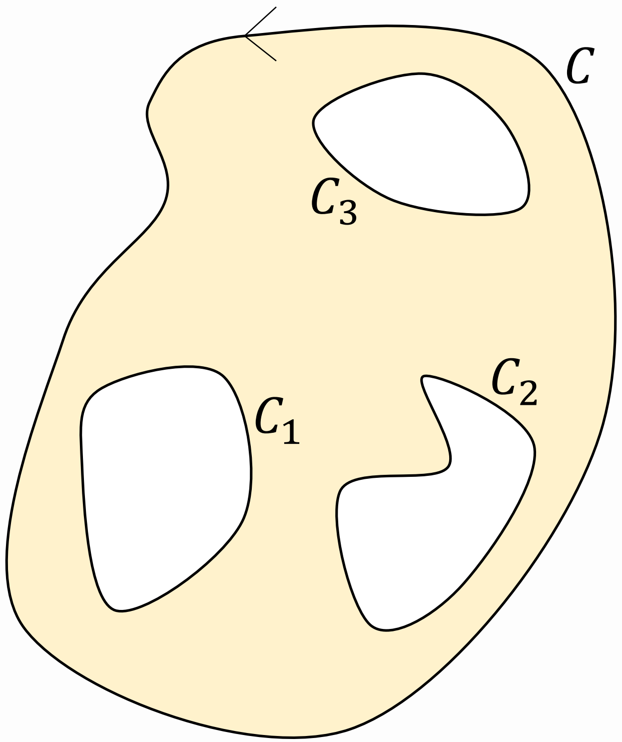


Let $f(z)$ be analytic in a domain D bounded by two simple closed curve C_1 and C_2 and also on C_1 and C_2. Then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

When C_1 and C_2 are both traversed counter clockwise.

Cauchy Integral Theorem (Generalization)



If C_1, C_2, C_3 are traversed clockwise

$$\boxed{\oint_C f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz = 0}$$

If C_1, C_2, C_3 are traversed counter - clockwise

$$\boxed{\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz}$$

Recall : $\oint_C (z - z_0)^m dz = \begin{cases} 2\pi i, & m = -1, \\ 0, & m \neq 0, \end{cases}$ m is an integer

C : circle of radius ρ and center z_0

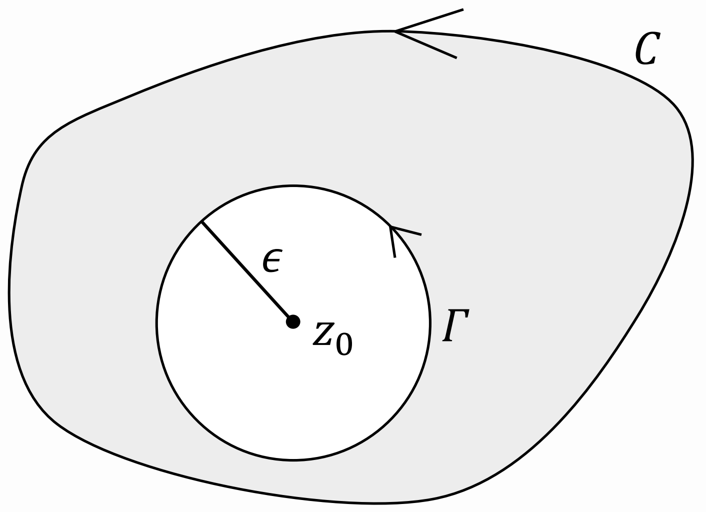
Above results can be generalized for any simple closed curve C due to the REMARK on previous slide

If z_0 is outside the C then $f(z)$ is analytic everywhere inside and on C.

Hence by Cauchy's theorem, we get: $\oint_C (z - z_0)^m dz = 0$

If z_0 is inside C then let Γ be a circle of radius ϵ with center $z = z_0$ so that Γ is inside C.

By **REMARK-3**



$$\oint_C f(z)dz = \oint_\Gamma f(z)dz$$

$$\Rightarrow \oint_c (z-z_0)^m dz = \oint_\Gamma (z-z_0)^m dz = \begin{cases} 2\pi i, & m=-1 \\ 0, & m \neq -1 \text{ \& } m \text{ is an integer} \end{cases}$$

Let C be any simple closed curve C then the counter clockwise integer

$$\oint_C (z-z_0)^m\, dz = \begin{cases} 0, & z_0 \text{ is outside } C \\ 2\pi i, & m=-1 \text{ \& } z_0 \text{ inside } C \\ 0, & m \neq -1 \text{ \& } m \text{ is an integer \& } z_0 \text{ inside } C \end{cases}$$

Note : Some important results from above general result :

$$\oint_C \frac{1}{z-z_0}\,dz = 2\pi i \qquad \text{if } z_0 \text{ is inside } C$$

$$\oint_C \frac{1}{(z-z_0)^m}\,dz = 0 \qquad m = 2,3,\ldots,\ z_0 \text{ is inside } C$$

The result $\oint_C \frac{1}{(z-z_0)^m}\,dz = 0$ does not follow from Cauchy′s theorem as $\frac{1}{(z-z_0)^m}$ is not analytic in D

Remark: Hence, the condition that $f(z)$ is analytic in D is sufficient for $\oint_C f(z)dz = 0$ rather than necessary.

Example: Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where C is the circle $|z+1| = 1$

Let $f(z) = \frac{z+4}{z^2+2z+5}$ Singularities of $f(z)$ are given by $z^2+2z+5=0$

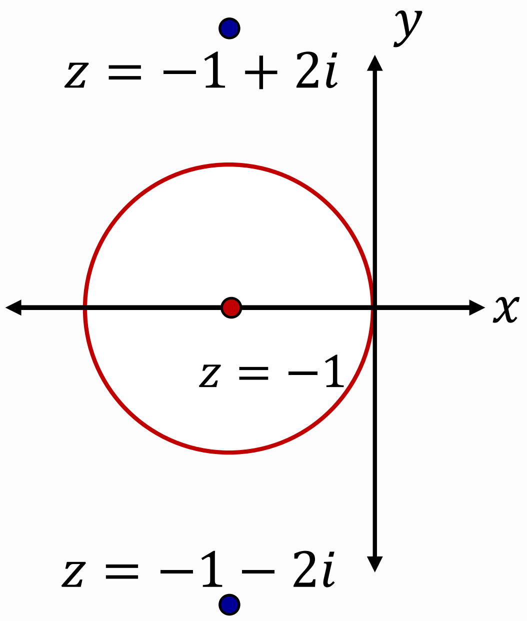
$$\Rightarrow z_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2} \quad \Rightarrow z = -1 \pm 2i$$

Both singularities outside the circle $|z+1| = 1$.

Hence $f(z)$ is analytic everywhere within and on C;

hence by Cauchy's theorem, we get

$$\oint_C f(z)dz = 0 \;\Rightarrow\; \oint_C \frac{z+4}{z^2+2z+5}dz = 0$$



- Cauchy Integral Formula
  - Simply & Multiply Connected Domains
- Derivative of Analytic Functions
- Evaluation of Complex Line Integrals

CAUCHY’S INTEGRAL THEOREM (RECALL)

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed curve C in D, we have

$$\oint_C f(z)\, dz = 0$$

MORERA’s THEOREM (Converse of Cauchy’s Theorem)

Let f be continuous in a simply connected domain D. If

$$\oint_C f(z)\, dz = 0$$

for every closed path C in D, then f is analytic in D.

CAUCHY INTEGRAL FORMULA

Let $f(z)$ be analytic in a simply connected domain D . Then for any point z_0 in D and any simple closed path C in D that encloses z_0, we have

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0) \qquad \text{or} \qquad f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$

Proof : $$\oint_C \frac{f(z)}{z-z_0}dz = \oint_C \frac{f(z_0)+f(z)-f(z_0)}{z-z_0}dz$$

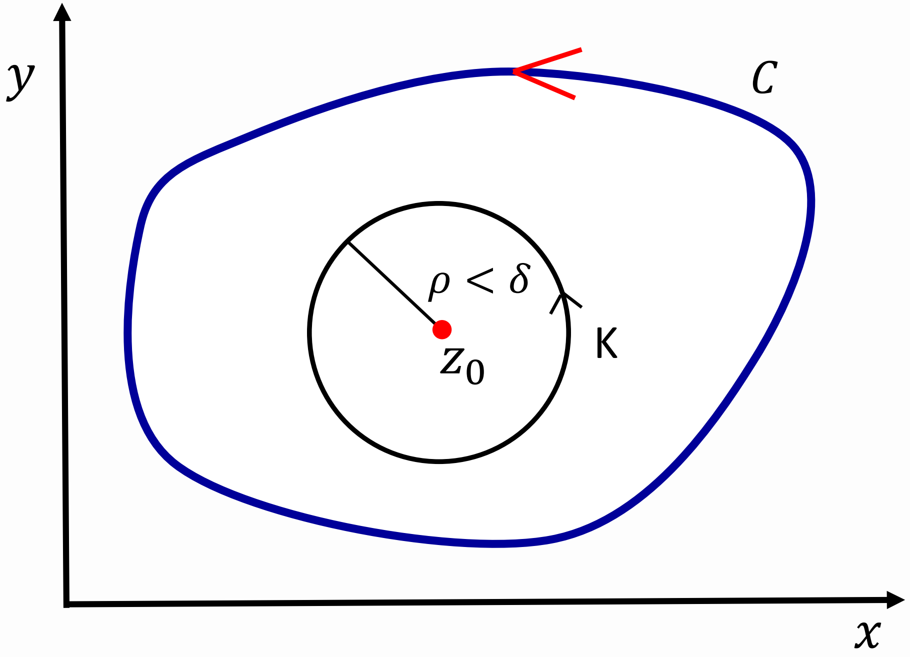
$$= f(z_0)\oint_C \frac{1}{z-z_0}dz + \oint_C \frac{f(z)-f(z_0)}{z-z_0}dz$$

$$= f(z_0)2\pi i + \oint_C \frac{f(z)-f(z_0)}{z-z_0}dz$$

$$\oint_C \frac{f(z) - f(z_0)}{z - z_0} dz$$

Since $f(z)$ is analytic and therefore continuous. Hence for given $\epsilon > 0$ we can find a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{for all} \quad |z - z_0| < \delta$$



Using principle of deformation

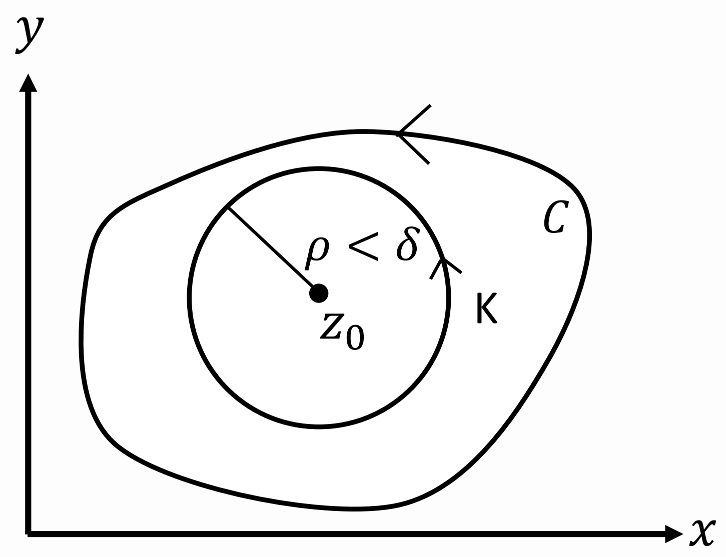
$$\oint_C \frac{f(z) - f(z_0)}{z - z_0} dz = \oint_K \frac{f(z) - f(z_0)}{z - z_0} dz$$

K is a circle of radius ρ, $\rho < \delta$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \frac{\epsilon}{\rho}$$

as $|f(z) - f(z_0)| < \epsilon$ and $|z - z_0| = \rho$

We have $\left|\dfrac{f(z)-f(z_0)}{z-z_0}\right| < \dfrac{\epsilon}{\rho}$



Using M-L inequality $\left|\oint_C f(z)dz\right| \leq ML$

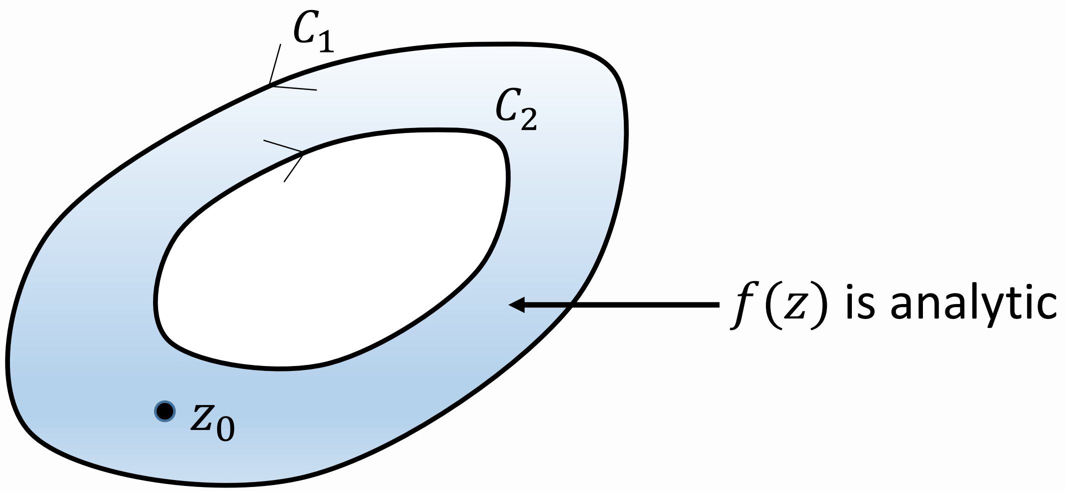
$$\oint_C \frac{f(z)-f(z_0)}{z-z_0}dz < \frac{\epsilon}{\rho}\cdot 2\pi\rho = 2\pi\epsilon$$

Since, ϵ can be chosen arbitrary small, we have

$$\oint_C \frac{f(z)-f(z_0)}{z-z_0}dz = 0$$

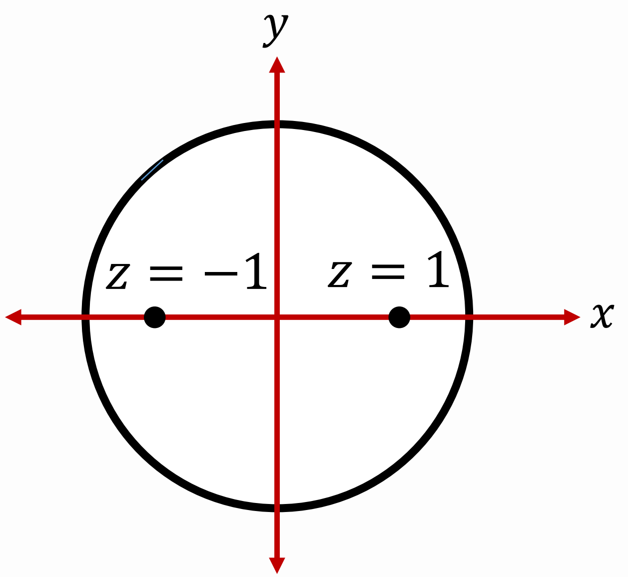
$$\oint_C \frac{f(z)}{z-z_0}dz = f(z_0)2\pi i + \oint_C \frac{f(z)-f(z_0)}{z-z_0}dz \quad \Rightarrow \oint_C \frac{f(z)}{z-z_0}dz = f(z_0)\,2\pi i$$

CAUCHY INTEGRAL FORMULA FOR MULTIPLY CONNECTED DOMAIN



$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z - z_0}dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z - z_0}dz$$

Example: Evaluate $\oint_C \frac{\tan z}{(z^2-1)} dz \qquad C : |z| = \frac{3}{2}$



Singularities of $f(z)$: $\quad z = 1, -1, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Points $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ does not lie inside $|z| = \frac{3}{2}$

$$\oint_C \frac{\tan z}{(z^2-1)} dz = \oint_C \frac{\tan z}{(z-1)(z+1)} dz = \oint_C \frac{\tan z}{2}\left[\frac{1}{(z-1)} - \frac{1}{(z+1)}\right] dz$$

$$= \frac{1}{2}\oint_C \frac{\tan z}{(z-1)} dz - \frac{1}{2}\oint_C \frac{\tan z}{(z+1)} dz = \frac{1}{2} 2\pi i \tan 1 - \frac{1}{2} 2\pi i \tan(-1)$$

$$= 2\pi i \tan 1$$

DERIVATIVE OF ANALTIC FUNCTION

If $f(z)$ is analytic in a domain D, then its derivative at any point $z = z_0$ is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

where C is an simple closed curve in D enclosing the point z_0.

Using Cauchy-Integral formula $f(z_0) = \dfrac{1}{2\pi i} \oint_C \dfrac{f(z)}{z - z_0} dz$

$$f(z_0 + \Delta z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0 - \Delta z_0} dz$$

$$\Rightarrow f(z_0 + \Delta z_0) - f(z_0) = \frac{1}{2\pi i} \oint_C f(z) \left[\frac{1}{z - z_0 - \Delta z_0} - \frac{1}{z - z_0} \right] dz = \frac{1}{2\pi i} \oint_C \frac{f(z) \Delta z_0}{(z - z_0 - \Delta z_0)(z - z_0)} dz$$

$$\Rightarrow \frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{\Delta z_0}\left[\frac{1}{z-z_0-\Delta z_0}-\frac{1}{z-z_0}\right]dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0-\Delta z_0)(z-z_0)}\,dz$$

$$\Rightarrow \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} = \lim_{\Delta z\to 0}\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0-\Delta z_0)(z-z_0)}dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

$$\Rightarrow f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

Similarly, one can prove results of higher orders.

Since, z_0 is arbitrary in D, the derivative of $f(z)$ of all orders are analytic in D if $f(z)$ is analytic in D.

Example: Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4}\, dz$, $\quad C: |z| = 3$

Cauchy Integral Formula:

$$f^3(z_0) = \frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}\, dz$$

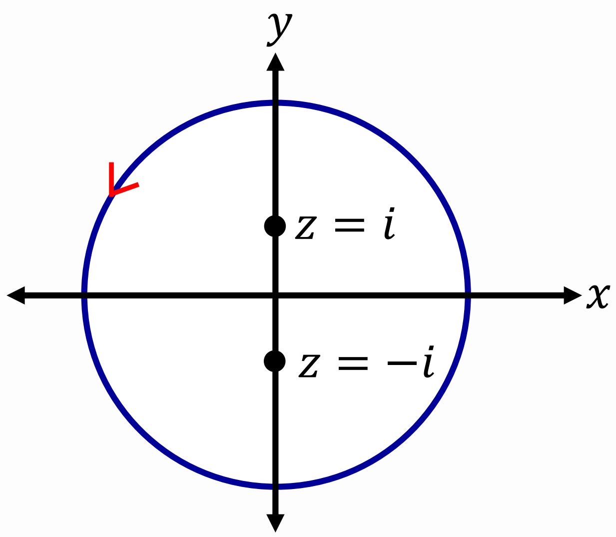
Let $f(z) = e^{2z}$, $\quad z_0 = -1$, $\quad n = 3$

$$f'(z) = 2e^{2z} \Rightarrow f'(-1) = 2e^{2.(-1)} = \frac{2}{e^2}$$

$$f''(-1) = \frac{4}{e^2} \qquad\qquad f^{(3)}(-1) = \frac{8}{e^2}$$

$$\frac{8}{e^2} = \frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}\, dz \quad \Rightarrow \oint_C \frac{e^{2z}}{(z+1)^4}\, dz = \frac{8\pi i}{3e^2}$$

Example: Evaluate $\oint_C \frac{e^{zt}}{z^2+1}\,dz\,, \qquad C : |z| = 3$



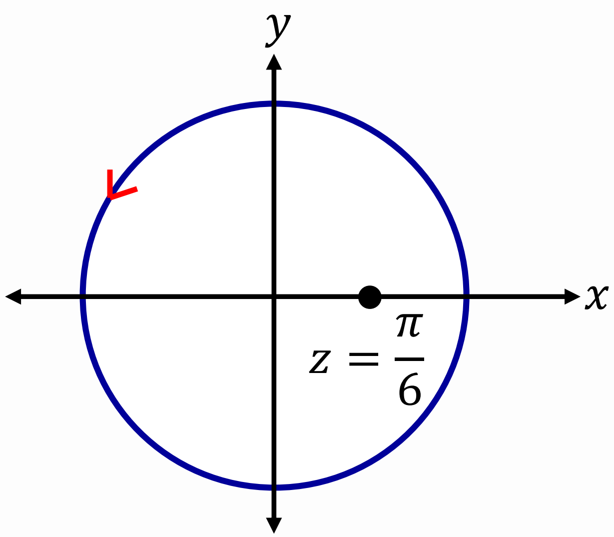
$$\oint_C \frac{e^{zt}}{z^2+1}\,dz = \oint_C \frac{e^{zt}}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dt$$

$$= \frac{1}{2i}\left[\oint_C \frac{e^{zt}}{z-i}\,dz - \oint_C \frac{e^{zt}}{z+i}\,dz\right]$$

$$= \frac{1}{2i}\,2\pi i\left[e^{it} - e^{-it}\right]$$

$$= 2\pi i \sin t$$

Example: Evaluate $\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$, $\quad C : |z| = 1$



Let $f(z) = \sin^6 z \qquad z_0 = \frac{\pi}{6}, \qquad n = 2$

Cauchy integral formula $f^{(2)}\left(\frac{\pi}{6}\right) = \frac{2!}{2\pi i} \oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^{2+1}} dz$

$$\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2} f^{(2)}\left(\frac{\pi}{6}\right)$$

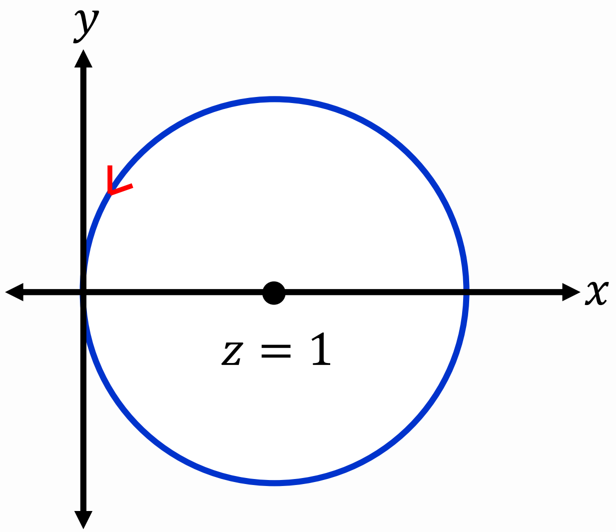
Note that $f'(z) = 6 \sin^5 z \, \cos z$

$$f''(z) = 30 \sin^4 z \, \cos^2 z + 6 \sin^5 z \cdot (-\sin z)$$

$$f''\left(\frac{\pi}{6}\right) = 30. \frac{1}{16} . \frac{3}{4} - 6. \frac{1}{32} . \frac{1}{2} = \frac{21}{16}$$

$$\Rightarrow \oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \pi i \frac{21}{16}$$

Example: Evaluate $\oint_C \frac{3z^2+z}{z^2-1}\,dz\,,\qquad C: |z-1| = 1$



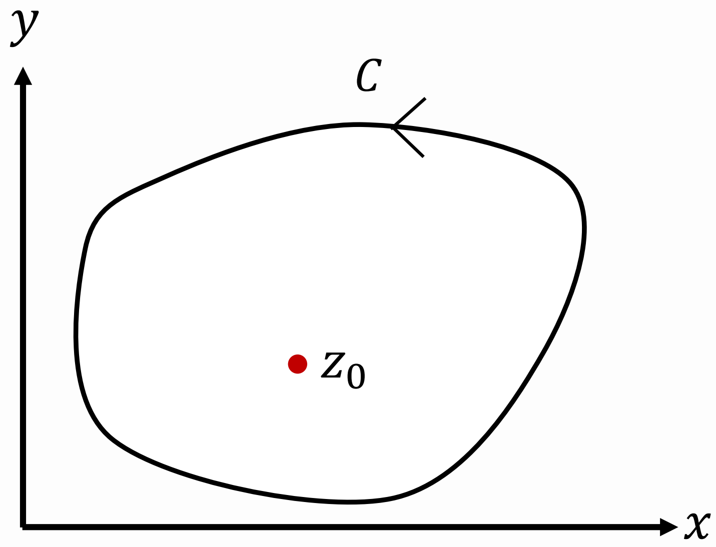
Singularities of integrand: $z^2-1=0 \qquad \Rightarrow z = \pm 1$

$$\oint_C \frac{3z^2+z}{(z-1)(z+1)}\,dz = \underbrace{\frac{1}{2}\oint_C \frac{3z^2+z}{z-1}\,dz}_{\text{Cauchy integral formula}} + \underbrace{\frac{1}{2}\oint_C \frac{3z^2+z}{z+1}\,dz}_{\text{Cauchy theorem}}$$

$$= \frac{1}{2}2\pi i\,(3+1) + 0$$

$$= 4\pi i$$

SUMMARY



Cauchy Integral Formula $\quad f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$

Derivative Formula $\quad f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$

Practice Problems

7(c) Using the Cauchy integral formula, evaluate

$$\oint_C \frac{e^z}{z(1+z)^3}\,dz$$

where C is the square ABCD with the vertices $\mathrm{A}(2,2), \mathrm{B}(-2,2), \mathrm{C}(-2,-2)$ and $\mathrm{D}(2,-2)$.

7(b). Evaluate the integral

$$\int_{\Gamma} \frac{\mathrm{e}^{\pi \mathrm{i} \mathrm{z}} \quad dz}{2z^2 - 5z + 2}$$

where Γ is the unit circle $|z| = 1$ that orients counter-clockwise.

(c) Evaluate

$$\oint_C \frac{z+1}{z^4 + 2iz^3} dz$$

by using Cauchy's integral formula for derivatives where C is given by $|z| = 1$.

[2 marks]

4. (a) Let C be the arc of the ellipse $\frac{(x-3)^2}{4}+\frac{y^2}{9}=1$ lying on the first quadrant oriented in the counter clockwise direction. Evaluate $\int\limits_C \frac{1}{z^4}dz$.

(b) Let C be the circle $|z| = 3$ oriented in counter clockwise direction. If $g(w) = \int\limits_C \frac{z^3 + 2z}{(z-w)^3} dz$,

then find $g(2)$ and $g(4i)$.

Thank You