ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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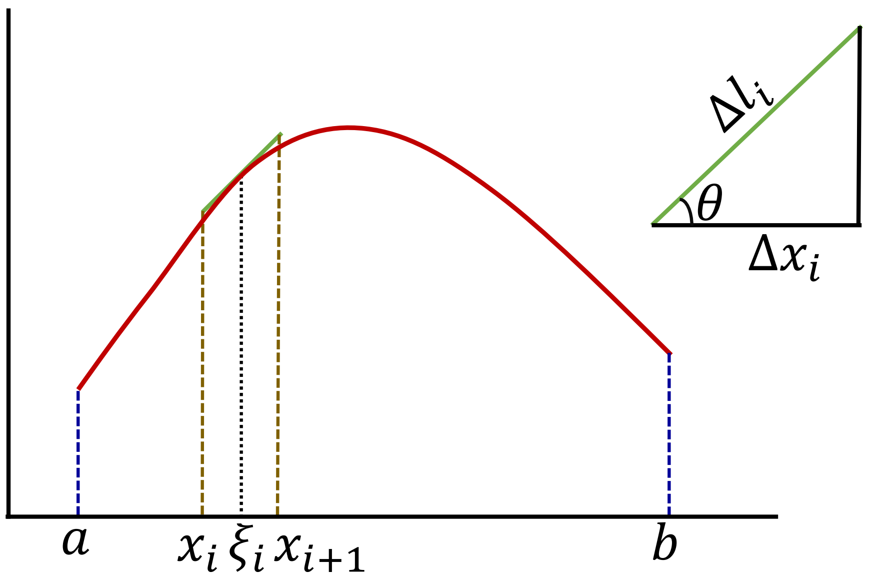
Topic

Integral Calculus – Double Integrals: Surface Area

Recall: Computation of curve length



Length of the curve $L = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta l_i$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(\xi_i))^2}\, \Delta x_i$$

$$= \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

$$\frac{\Delta x_i}{\Delta l_i} = \cos\theta \implies \Delta l_i = \frac{1}{\cos\theta}\, \Delta x_i$$

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}} \implies \frac{1}{\cos\theta} = \sqrt{1 + (f'(\xi_i))^2}$$

Computation of Surface Area $(z = f(x,y))$

Curve Length $L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dx\, dy$$

where D is the projection of the surface in the xy-plane.

Similarly, if the equation is given in the form: $x = \mu(y,z)$ or in the form $y = \psi(x,z)$ then

$$S = \iint_{\hat{D}} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\; dy\, dz \quad OR \quad \iint_{\hat{\hat{D}}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\; dx\, dz$$

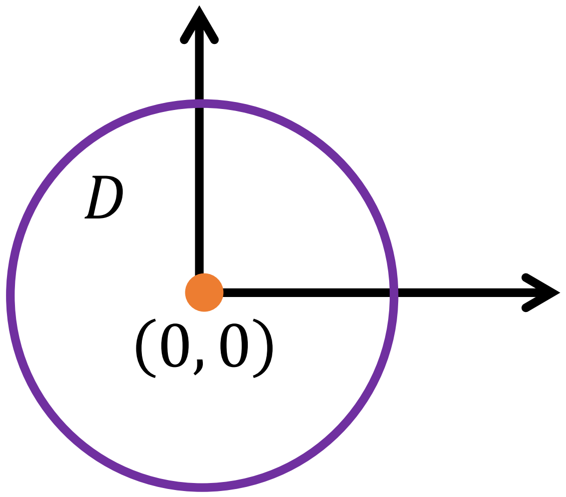
where $\hat{D}$ and $\hat{\hat{D}}$ are the domains in the yz and xz planes in which the given surface is projected.

Problem - 1 Compute the surface area of the sphere $x^2 + y^2 + z^2 = a^2$

Equation of the surface $z = \sqrt{a^2 - x^2 - y^2}$ (Upper half)

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

Domain of integration: $x^2 + y^2 \leq a^2$



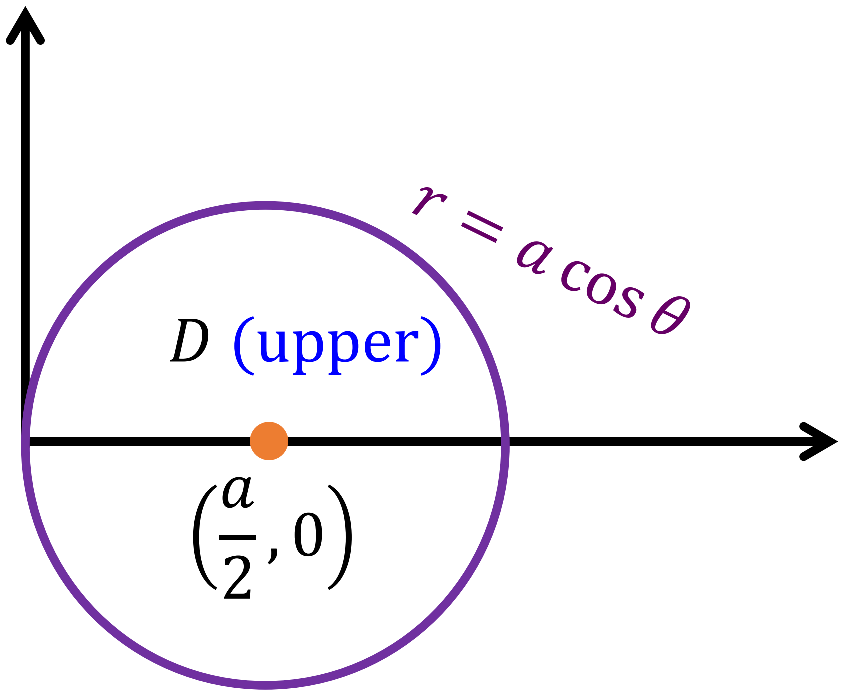
$$S = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2}\, dydx$$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

$$S = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2}\, dydx = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dydx$$

$$= 2\int_{0}^{2\pi}\int_{0}^{a} \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta = -2a\, 2\pi \sqrt{a^2 - r^2}\,\Big|_{0}^{a} = 4\pi a^2$$

Problem - 2 Find the area of that part of the sphere $x^2 + y^2 + z^2 = a^2$ that is cut off by the cylinder $x^2 + y^2 = ax$.

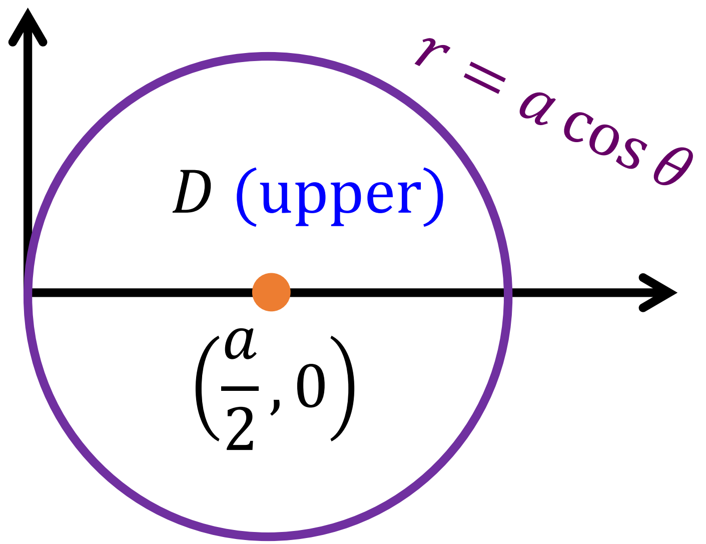


$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

$$S = 2 \cdot 2 \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2}\, dx\, dy = 4 \iint_D \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy$$

$$4\iint_D \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy = 4\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$$



$$= 4\,a\int_0^{\pi/2} \left(-\sqrt{a^2-r^2}\right)_0^{a\cos\theta} d\theta$$

$$= 4a\int_0^{\pi/2} [-a\sin\theta + a]\,d\theta$$

$$= 4a\left[\{a\cos\theta\}_0^{\frac{\pi}{2}} + a\{\theta\}_0^{\frac{\pi}{2}}\right] = 4a\left[-a + a\frac{\pi}{2}\right] = 2a^2(\pi-2)$$

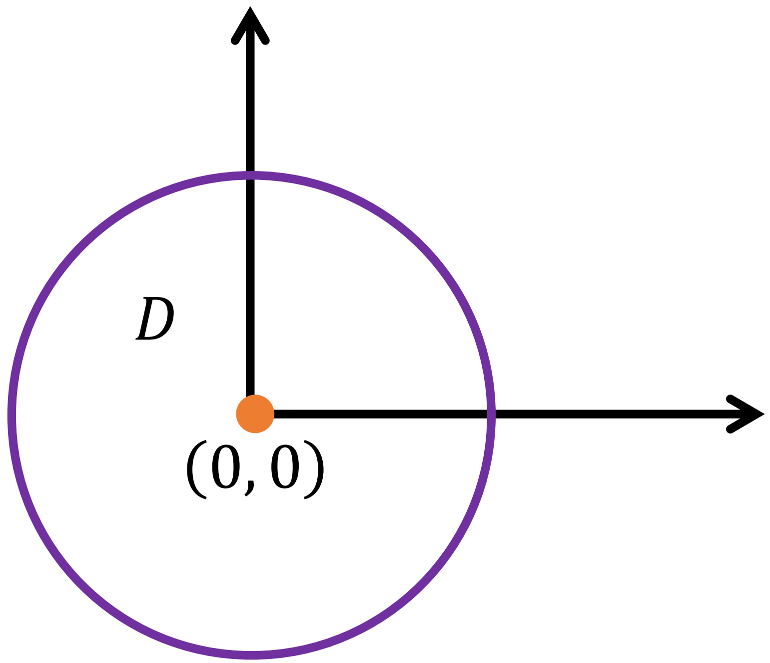
Problem - 3 Determine the surface area of the part of $z = xy$ that lies in the cylinder $x^2 + y^2 = 1.$

$z = f(x, y) = xy$ $\qquad$ $z_x = y,$ $\qquad$ $z_y = x$

$$S = \iint_D \sqrt{1 + x^2 + y^2}\; dx\, dy$$

In polar coordinate $\quad S = \int_0^{2\pi} \int_{r=0}^{1} \sqrt{1 + r^2}\; r\, dr d\theta$

$$= \int_0^{2\pi} \frac{1}{2} \frac{2}{3} \left[(1 + r^2)^{3/2}\right]_0^1 d\theta = \frac{2\pi}{3}\left(2^{3/2} - 1\right)$$



Conclusion:

Double Integrals – Application

- Surface area

QUIZ QUESTION ?

LINK FOR RESPONSES: http://www.facweb.iitkgp.ac.in/~jkumar/teach/MA11003.html

QUIZ QUESTION:

If the area of the portion of the surface $z = \sqrt{9 - x^2 - y^2}$ lying inside the cylinder $x^2 + y^2 = 3y$ is $c(\pi - d)$, then the value of $(c + d)$ is

ANS: 11

Thank You