LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

Dr. Jitendra Kumar

Professor
Department of Mathematics
Indian Institute of Technology Kharagpur
West Bengal 721302, India

Webpage: http://www.facweb.iitkgp.ac.in/~jkumar/

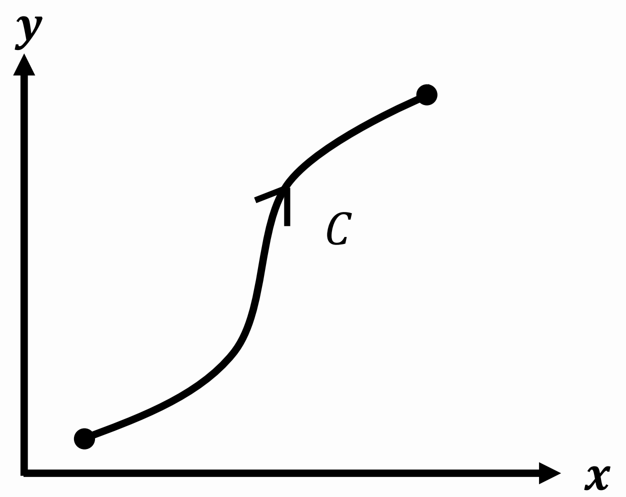
- Complex Line Integrals
- Properties of Complex line Integrals
- Evaluation of Line Integrals

COMPLEX LINE INTEGRALS

Let $f(z)$ be a continuous function of a complex variable z in some domain $D \in \mathbb{C}$.

The integral of $f(z)$ along a path C in D is denoted as

$$\int_C f(z)\,dz$$



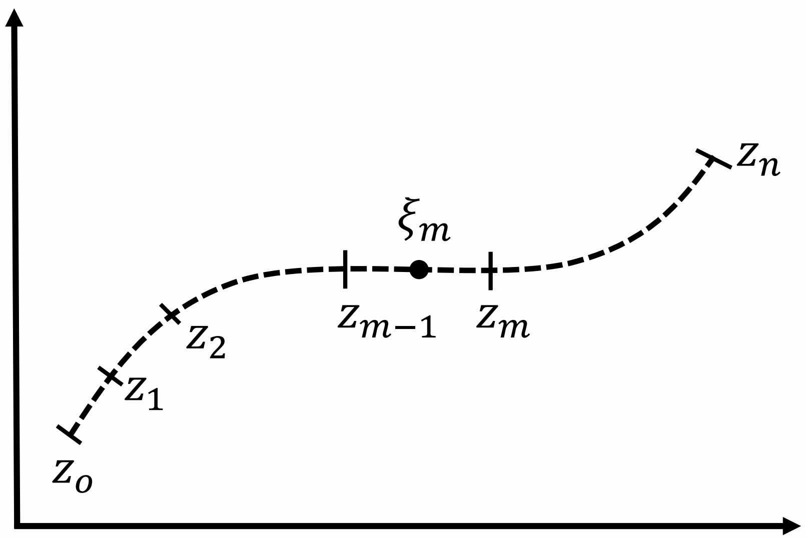
C is called the path of integration and it may be represented parametrically as

$$z(t) = x(t) + iy(t) \qquad a \leq t \leq b$$

The sense of increasing t is called the positive sense of C

LINE INTEGRALS

$$\lim_{n\to\infty}\sum_{m=1}^{n} f(\xi_m)(z_m - z_{m-1}) = \int_c f(z)\,dz$$



If C is a closed path, then the line integral is denoted by

$$\oint_C f(z)dz$$

Basic Properties of Integration

1. **Linearity:** $\int_c [k_1 f_1(z) + k_2 f_2(z)]dz = k_1 \int_c f_1(z)dz + k_2 \int_c f_2(z)dz$

Basic Properties of Integration

2. $$\int_{z_0}^{z} f(z)\, dz = -\int_{z}^{z_0} f(z)\, dz$$

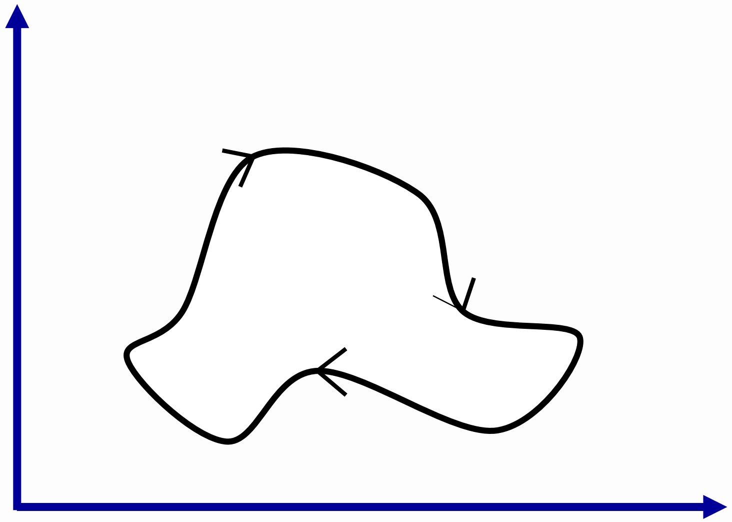
3. $$\int_{c} f(z)\, dz = \int_{c_1} f(z)\, dz + \int_{c_2} f(z)\, dz \qquad c = c_1 + c_2$$

4. Suppose $f(z)$ is integrable along a curve C having length L and suppose there exists a positive number M such that

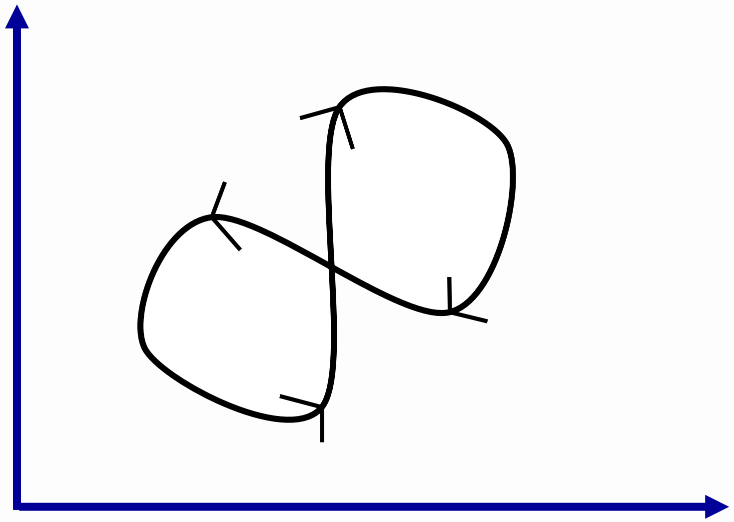
$$|f(z)| \leq M \text{ in } C, \text{ then } \left|\int_{c} f(z) dz\right| \leq ML$$

SIMPLE CLOSED CURVE

A closed curve that does not intersect (or touch) itself anywhere is called a simple closed curve.



Simple Closed Curve

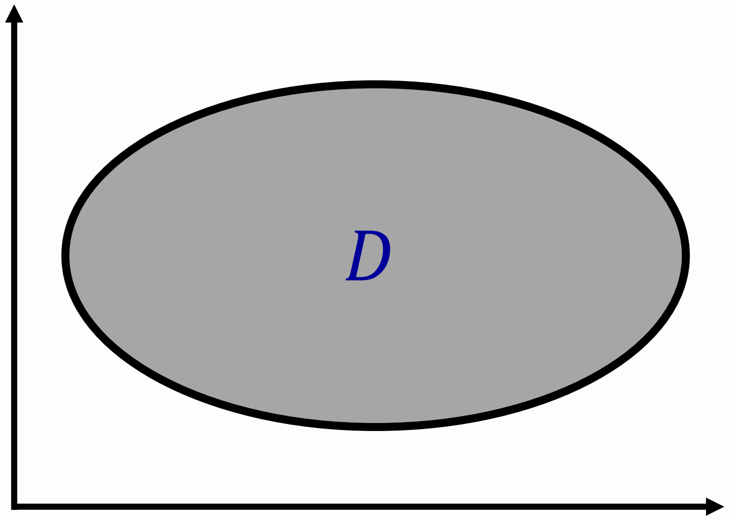


Not Simple Closed Curve

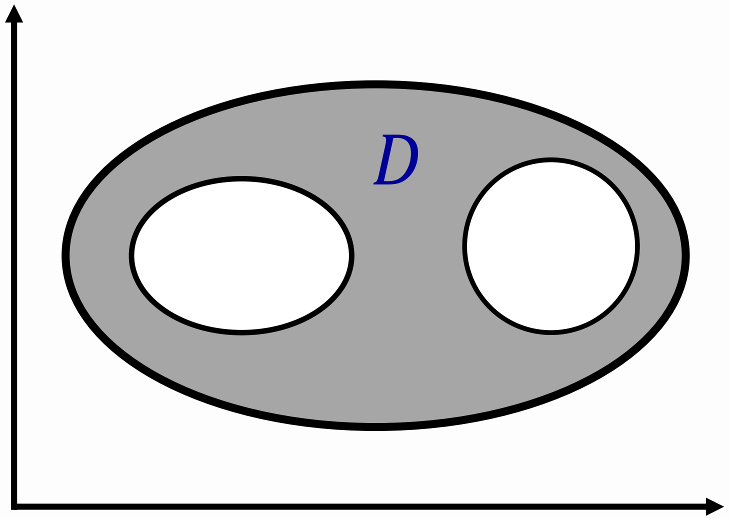
SIMPLY AND MULTIPLY CONNECTED DOMAINS

A domain D is called simply-connected if any simple closed curve which lies in D can be shrunk to a point without leaving D.

A region which is not simply connected is called multiply-connected.



Simply Connected Domain



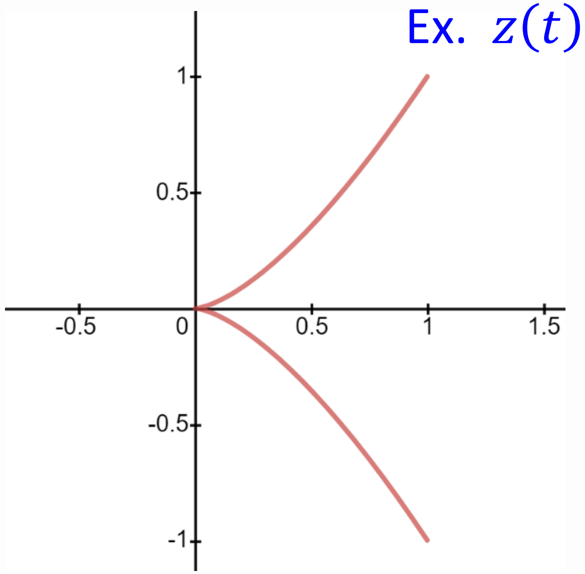
Multiply Connected Domain

SMOOTH AND PIECEWISE SMOOTH CURVE

We say that the parametrized curve $z = z(t), t \in [a, b]$ is **smooth** if $z'(t)$ exists and is continuous on $[a, b]$ and $z'(t) \neq 0$ for $t \in (a, b)$.

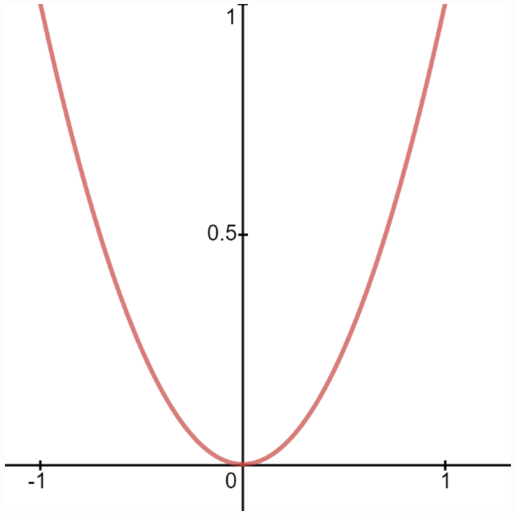
Ex. $z(t) = t^2 + it^3; \; -1 \leq t \leq 1$

non-smooth curve



Ex. $z(t) = t + it^2; \; -1 \leq t \leq 1$

smooth curve



We say that the parametrized curve is **piecewise-smooth** if z is continuous on $[a, b]$ and if there exist points $a = a_0 < a_1 < \cdots < a_n = b$, where $z(t)$ is smooth in each subinterval intervals $[a_k, b_k]$.

EVALUATION OF LINE INTEGRALS

- Let C be a smooth (piece-wise smooth) path, represented by $z = z(t)$ where $a \leq t \leq b$.

  Let $f(z)$ be continuous function on C, then

$$\int_C f(z)dz = \int_a^b f\big(z(t)\big)\,\dot{z}(t)\,dt$$

- If a continuous function f has a primitive F in D, i.e., $F'(z) = f(z)$ for all $z \in D$, then for all paths C in D joining two points z_0 and z_1 in D, we have:

$$\int_C f(z)\,dz = F(z_1) - F(z_0)$$

If a continuous function f has a primitive F in D, i.e., $F'(z) = f(z)$ for all $z \in D$,

then for all paths C in D joining two points z_0 and z_1 in D, we have: $\int_C f(z)\,dz = F(z_1) - F(z_0)$

Sketch of Proof: Let $z(t)$ be a parameterization of C (smooth curve): $z(a) = z_0$ & $z(b) = z_1$

$$\int_C f(z)\,dz = \int_a^b f\big(z(t)\big)\,\dot{z}(t)\,dt = \int_a^b F'\big(z(t)\big)\,\dot{z}(t)\,dt$$

$$= \int_a^b \frac{dF\big(z(t)\big)}{dt}\,dt = F\big(z(b)\big) - F\big(z(a)\big) = F(z_1) - F(z_0)$$

Note: Let $f(z)$ be analytic in a simply connected domain D. Then f has a primitive in D, that is, there exists $F(z)$ such that $F'(z) = f(z)$.

Example Find $\oint_c (z - z_0)^m dz$, m is an integer and C is the circle of radius ρ and center at z_0

Case I : $m \geq 0$ then $(z - z_0)^m$ is analytic

Then $\oint_c (z - z_0)^m dz = 0$

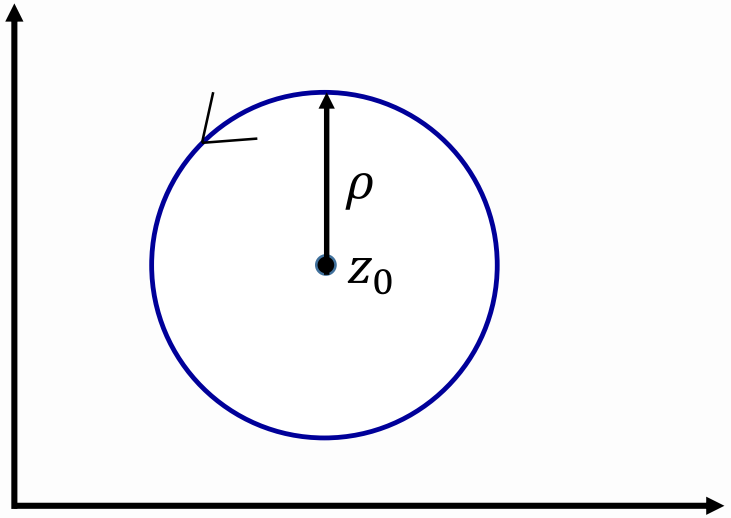
Case II : $m = -1$ i.e. $f(z) = \dfrac{1}{(z - z_0)}$

Note that the function (integrand) is not analytic inside C.

Note that C is a circle of radius ρ and center at z_0

$$\int_C f(z)dz = \int_a^b f\big(z(t)\big)\,\dot{z}(t)\,dt$$

$$z = z_0 + \rho(\cos t + i\sin t) = z_0 + \rho e^{it} \qquad 0 \le t \le 2\pi$$



$$\oint_C \frac{1}{(z - z_0)}dz = \int_0^{2\pi} \left[\rho e^{it}\right]^{-1} \rho i e^{it} dt$$

$$= \int_0^{2\pi} \rho^{-1} e^{-it} \rho i e^{it} dt$$

$$= 2\pi i$$

Case III : $m \leq -2$

$$\oint_c (z - z_0)^m dz = \int_0^{2\pi} \left[\rho e^{it}\right]^m \rho i e^{it} dt$$

$$= i\rho^{m+1} \int_0^{2\pi} e^{it(m+1)} dt = i\rho^{m+1} \left. \frac{e^{it(m+1)}}{i(m+1)} \right|_0^{2\pi} \qquad (m \neq -1)$$

$$= \rho^{m+1} \frac{1}{m+1} \left[e^{i2(m+1)\pi} - 1\right] = \rho^{m+1} \frac{1}{m+1} [1 - 1] = 0$$

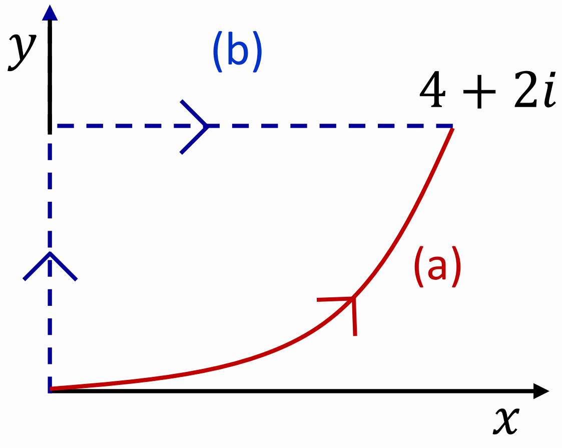
$$\Longrightarrow \oint_c (z - z_0)^m dz = \begin{cases} 2\pi i, & m = -1 \\ 0, & m \neq -1 \end{cases}$$

Remark: A complex line integral depends not only on the end points of the path but in general also on the path itself

Example: Evaluate $\int_C \bar{z}\, dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by

(a) $z = t^2 + it$ (b) The line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$

Note that $\bar{z}$ is not analytic and therefore we expect different integral values along different path.



(a) Corresponding to $z = 0$ and $z = 4 + 2i$, we have

$t = 0$ and $t = 2$ respectively.

$$\int_C \bar{z} dz = \int_{t=0}^{2} \overline{(t^2 + it)}(2t + i)dt = \int_{t=0}^{2} (t^2 - it)(2t + i)dt$$

$$= \int_{t=0}^{2} (2t^3 - it^2 + t)dt = 10 - \frac{8}{3}i$$

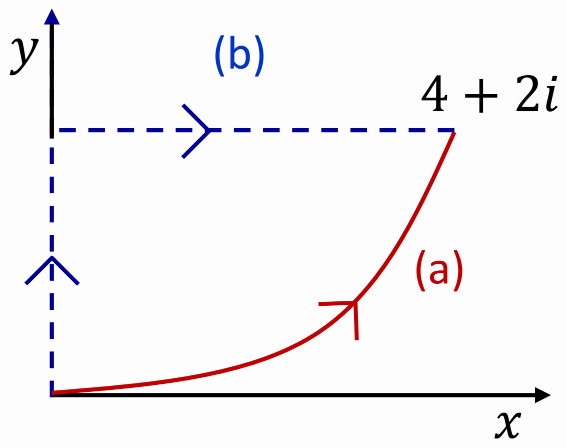
$$(b) \quad \int_C \bar{z}\, dz = \int_{C_1} \bar{z}\, dz + \int_{C_2} \bar{z}\, dz$$

$$= \int_0^2 \overline{iy}\, i\, dy + \int_0^4 \overline{(x+2i)}\, dx$$

$$= \int_0^2 y\, dy + \int_0^4 (x-2i)\, dx$$

$$= \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 16 - 2i \cdot 4$$

$$= 10 - 8i$$

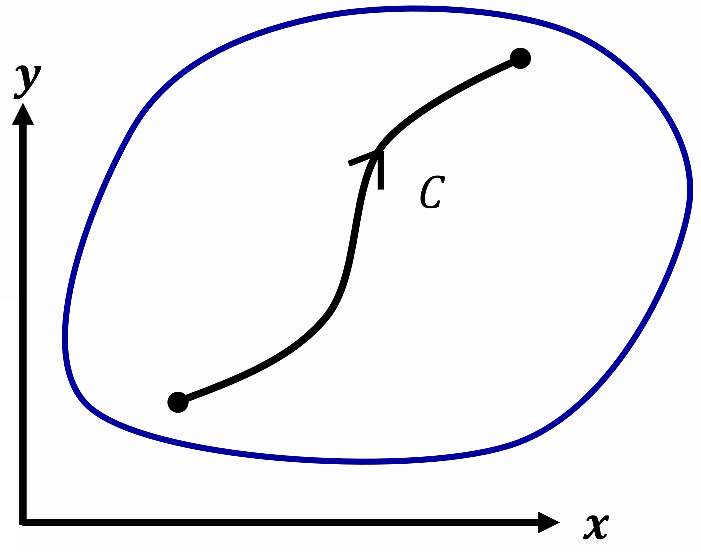


Path : $C : z = x + iy$

Along C_1: $x = 0, \; y = 0$ to 2

Along C_2: $y = 2, \; x = 0$ to 4

SUMMARY



- $C:\ z = z(t),\ a \le t \le b$

$$\int_C f(z)dz = \int_a^b f\big(z(t)\big)\,\dot{z}(t)\,dt$$

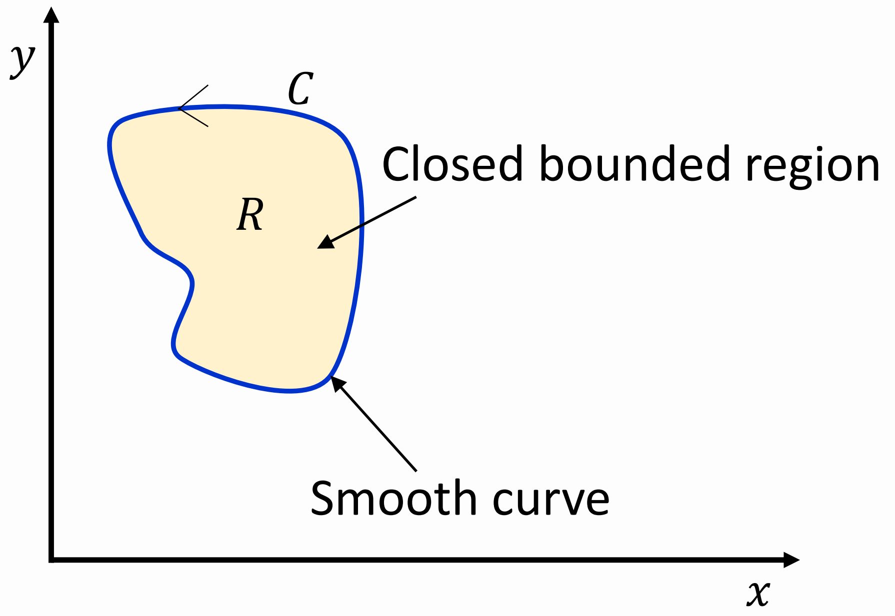
- If a continuous function f has a primitive F in D, i.e., $F'(z) = f(z)$ for all $z \in D$

$$\int_{z_0}^{z_1} f(z)\,dt = F(z_1) - F(z_0)$$

- **Cauchy Integral Theorem – Simply Connected Domain**
- **Its Generalization for Multiply Connected Domain**

Recall: GREEN'S THEOREM (transformation between double integrals and line integral)

Let R be a region in $\mathbb{R}^2$ whose boundary is a simple closed curve C which is piecewise smooth (oriented counter clockwise – when traversed on C the region R always lies left).



Let $F_1(x, y)$ and $F_2(x, y)$ be continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ everywhere in the domain R, then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C (F_1 dx + F_2 dy)$$

CAUCHY INTEGRAL THEOREM

IF $f(z)$ is analytic in a simply connected domain D, then for every simple closed C in D,

$$\oint_C f(z)\,dz = 0$$

Proof: Take an additional assumption that the derivative $f'(z)$ is continuous.

$$\oint_C f(z)dz = \oint_C (u+iv)\,(dx+idy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

We know from the C-R equations,

$$f'(z) = u_x + iv_x = v_x - iu_y$$

$$f'(z) = u_x + iv_x = v_x - iu_y$$

Since $f'(z)$ is assumed to be continuous then it implies continuity of u_x, v_x, v_y, u_y

Hence, by Green's theorem $\displaystyle\oint_C u\,dx - v\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$

R is the region bounded by C

Using C−R equations $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, we get $\displaystyle\oint_C (udx - vdy) = 0$

Similarly, we can show that $\displaystyle\oint_C (vdx + udy) = 0$

$$\boxed{\oint_C f(z)dz = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) = 0}$$

REMARK -1 Cauchy's integral theorem has been proved using Green's theorem with the added restriction that $f'(z)$ be continuous in D. However, Goursat gave a proof which removed these restrictions. Sometimes Cauchy's integral theorem is called Cauchy-Goursat Theorem.

REMARK -2

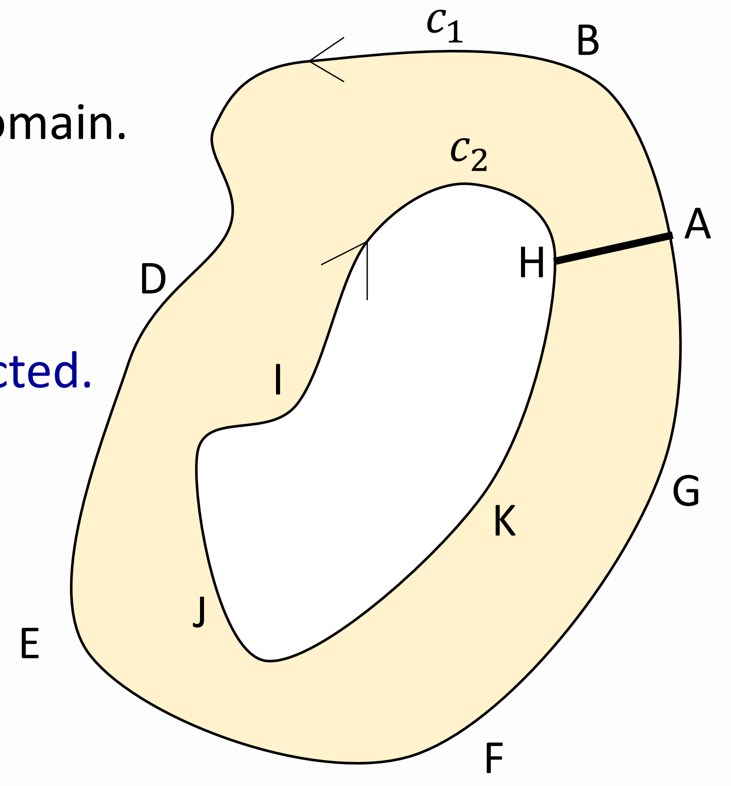
Cauchy's theorem can also be applied to multiply connected domain.

Construct cross-cut AH.

Then, the region bounded by ABDEFGAHKJIHA is simply connected.

The Cauchy's theorem implies:

$$\oint_{ABD\cdots IHA} f(z)dz = 0$$



$$\oint_{ABD\cdots IHA} f(z)dz = 0$$

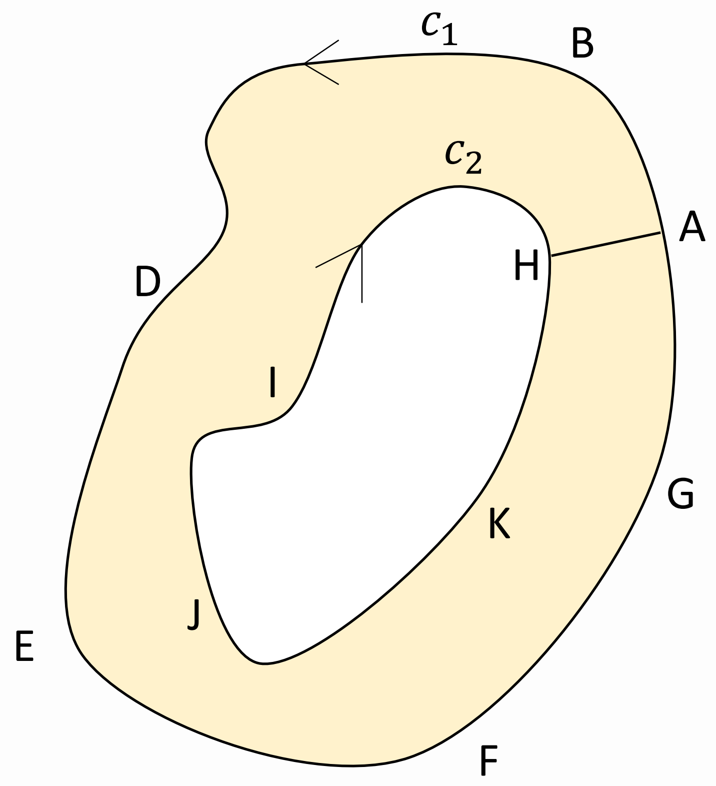
$$\Rightarrow \oint_{ABDEFGA} f(z)dz + \oint_{AH} f(z)dz + \oint_{HKJIH} f(z)dz + \oint_{HA} f(z)dz = 0$$

Using $\oint_{AH} f(z)dz = -\oint_{HA} f(z)dz$

$$\oint_{ABDEFGA} f(z)dz + \oint_{HKJIH} f(z)dz = 0$$

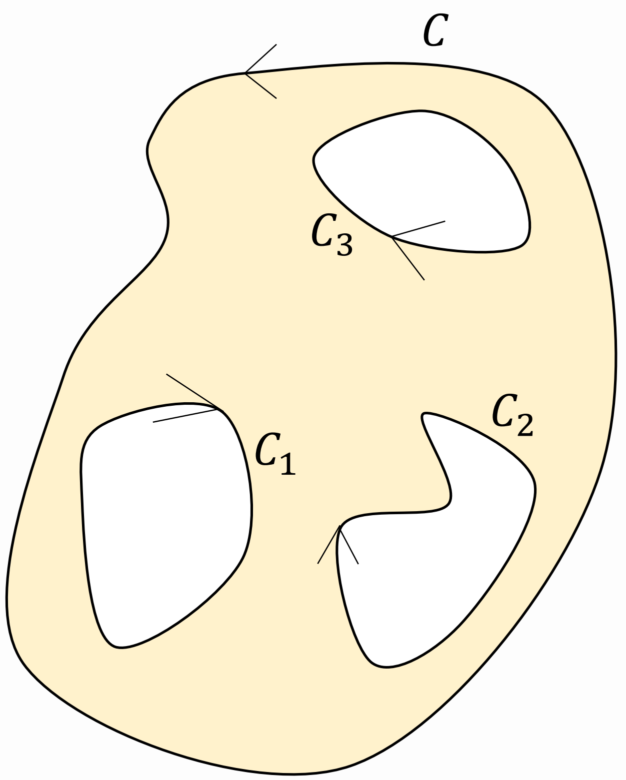
Anti-clockwise Clockwise

$$\Rightarrow \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = 0$$



More General Result:

$$\boxed{\oint_C f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz = 0}$$



REMARK - 3 As a consequence of above remark, we have following result:

Let $f(z)$ be analytic in a domain D bounded by two simple closed curve C_1 and C_2 and also on C_1 and C_2. Then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

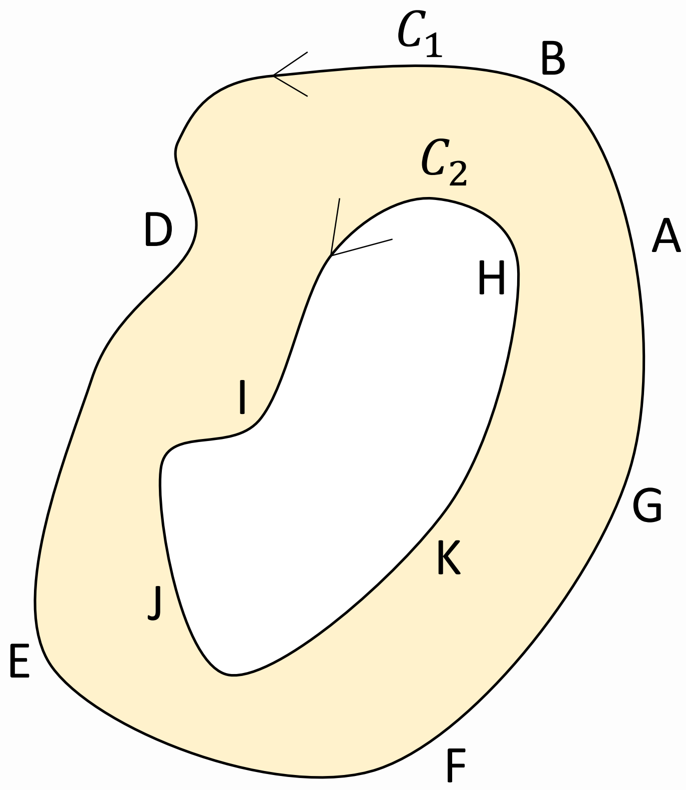
When C_1 and C_2 are both traversed counter clockwise.

From previous remark, we have

$$\oint_{ABDEFGA} f(z)dz + \oint_{HKJIH} f(z)dz = 0$$

$$\Longrightarrow \oint_{ABDEFGA} f(z)dz - \oint_{HIJKH} f(z)dz = 0$$

$$\Longrightarrow \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$



Thank You