(BB)

Slot A

Question : Let $f(x,y) = \begin{cases} \dfrac{49x^3+50y^3}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$.

1) $x = r\cos\theta$, $y = r\sin\theta$

$$f(r,\theta) = \frac{49r^3\cos^3\theta + 50r^3\sin^3\theta}{r^2}, \quad r \neq 0$$

$$= r\left[49\cos^3\theta + 50\sin^3\theta\right]$$

$$\lim_{r\to 0} f(r,\theta) = \left.\begin{matrix} 0 \\ = f(0,0) \end{matrix}\right\} \Rightarrow \quad f \text{ is continuous at } (0,0)$$

(2) It is easy to calculate $f_x(0,0) = \cancel{\ldots}\ 49$
& $f_y(0,0) = 50$

(3) Here $\displaystyle\lim_{\Delta\rho\to 0} \frac{\Delta f - df}{\Delta\rho}$, $\Delta\rho = \sqrt{h^2+k^2}$

$$= \lim_{\Delta\rho\to 0} \frac{\dfrac{49h^3+50k^3}{h^2+k^2} - (49h+50k)}{\sqrt{h^2+k^2}}$$

$$= \lim_{\Delta\rho\to 0} \frac{-49k^2h - 50h^2k}{\sqrt{h^2+k^2} \times (h^2+k^2)}$$

(Put $k = mh$)

$$= \lim_{\Delta\rho\to 0} \frac{-49m^2 - 50m}{(1+m^2)^{3/2}}, \quad \text{Thus limit}$$

does not exist & hence the function is <u>not</u> diff.
<u>This is the correct option.</u>

④ We know that if one of the partial derivative is continuous at $(0,0)$, then the function f would have been differentiable at $(0,0)$. But from ③, we have seen that the function f is NOT diff. at $(0,0)$. Thus the options involving continuity of partial derivatives at $(0,0)$ are incorrect.

Prob. B: Let $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(303)

Ans: (i) $f_x(0,0) = 0 = f_y(0,0)$

Thus f_x, f_y exists at $(0,0)$.

(ii) $$\lim_{\Delta\rho \to 0} \frac{\Delta f - df}{\Delta \rho}$$

$$= \lim_{\Delta\rho \to 0} \frac{\dfrac{hk^2}{h^2+k^4} - 0}{\sqrt{h^2+k^2}}$$

$$= \lim_{k \to 0} \frac{mk^3}{(m^2k^2+k^4)\sqrt{m^2k^2+k^2}}, \quad \text{Put } h = mk$$

$$= \lim_{k \to 0} \frac{m}{(m^2+k^2)\sqrt{1+m^2}}$$

$= \dfrac{m}{m^2\sqrt{1+m^2}}$, which is different for different values of m & thus f is not differentiable at $(0,0)$.

This is the correct option.

(iii) If one of the partial derivatives f_x or f_y is continuous at $(0,0)$ then we know that the function would have been differentiable at $(0,0)$. Thus the options involving continuity of partial derivatives are not correct.

Soln ① $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y}.$

Now, $\frac{\partial^2 z}{\partial u^2} - \frac{2\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}$

$= \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$

$= \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)(v-u)\frac{\partial z}{\partial y}$

$= \left(\sqrt{x^2-4y}\right)\frac{\partial}{\partial y}\left(\sqrt{x^2-4y}\,\frac{\partial z}{\partial y}\right)$

$= \left(\sqrt{x^2-4y}\right)^2\frac{\partial^2 z}{\partial y^2} + \left(\sqrt{x^2-4y}\right)\frac{\partial z}{\partial y}\times\frac{(-4)}{2\sqrt{x^2-4y}}$

$= (x^2-4y)\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial y}$

At the point $(2,1)$, $z_{uu} - 2z_{uv} + z_{vv} = 0.\,z_{yy} - 2z_y = -2z_y.$

$\therefore\ n = -2.$

② At the point $(3,2)$, $z_{uu} - 2z_{uv} + z_{vv} = (3^2-4.2)z_{yy} - 2z_y$

$= z_{yy} - 2z_y.$

$\therefore\ n = 1.$

③ $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = 2uv\frac{\partial f}{\partial x} + v^2\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} = u^2\frac{\partial f}{\partial x} + 2uv\frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial u\partial v} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right) = \frac{\partial}{\partial u}\left(u^2\frac{\partial f}{\partial x} + 2uv\frac{\partial f}{\partial y}\right)$$

$$= u^2\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) + 2uv\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) + 2u\frac{\partial f}{\partial x} + 2v\frac{\partial f}{\partial y}$$

$$= u^2\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial y}{\partial u}\right]$$

$$+ 2uv\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial u}\right] + 2u\frac{\partial f}{\partial x} + 2v\frac{\partial f}{\partial y}$$

$$= u^2\left[2uv\frac{\partial^2 f}{\partial x^2} + v^2\frac{\partial^2 f}{\partial y\partial x}\right] + 2uv\left[2uv\frac{\partial^2 f}{\partial x\partial y} + v^2\frac{\partial^2 f}{\partial y^2}\right]$$

$$+ 2u\frac{\partial f}{\partial x} + 2v\frac{\partial f}{\partial y}$$

$$= 2u^3v\frac{\partial^2 f}{\partial x^2} + 2uv^3\frac{\partial^2 f}{\partial y^2} + 5u^2v^2\frac{\partial^2 f}{\partial x\partial y} + 2\left(u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}\right).$$

Now, $\frac{2}{3}\left(u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}\right)$

$$= \frac{2}{3}\left[2u^2v\frac{\partial f}{\partial x} + v^2u\frac{\partial f}{\partial y} + u^2v\frac{\partial f}{\partial x} + 2uv^2\frac{\partial f}{\partial y}\right]$$

$$= 2\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)$$

$$\therefore uv\frac{\partial^2 f}{\partial u\partial v} - \frac{2}{3}\left(u\frac{\partial f}{\partial v} + v\frac{\partial f}{\partial v}\right) = 2\left(u^4v^2\frac{\partial^2 f}{\partial x^2} + u^2v^4\frac{\partial^2 f}{\partial y^2}\right) + 5u^3v^3\frac{\partial^2 f}{\partial x\partial y}$$

$$+ 2\left(u^2v\frac{\partial f}{\partial x} + uv^2\frac{\partial f}{\partial y}\right) - 2\left(x\frac{\partial f}{\partial x} + uv^2\frac{\partial f}{\partial y}\right)$$

$$= 2x^2\frac{\partial^2 f}{\partial x^2} + 2y^2\frac{\partial^2 f}{\partial y^2} + 5xy\frac{\partial^2 f}{\partial x\partial y}.$$

At $x=0$, $y=2$, we have

$$u f_{uu} - \frac{2}{3}(u f_u + v f_v) = 8 f_{yy}$$

$\therefore n = 8$.

④ At $x=3$, $y=0$, we have —

$$u f_{uu} - \frac{2}{3}(u f_u + v f_v) = 18 f_{xx}$$

$\therefore n = 18$

⑤ At $x=1$, $y=1$, we have —

$$u f_{uu} - \frac{2}{3}(u f_u + v f_v) = 2 f_{xx} + 2 f_{yy} + 5 f_{xy}$$

$\therefore a+b+c = 2+2+5 = 9$.

1.

$$\cos(u^3+u) = \frac{x^3+y^3+z^3}{3x+2y+z}.$$

Let, $w = \cos(u^3+u)$.

w is a homogeneous function of degree 2.

Therefore by Euler's theorem,

$$x w_x + y w_y + z w_z = 2w.$$

or, $-\sin(u^3+u)\cdot(3u^2+1)\,(x u_x + y u_y + z u_z) = 2\cos(u^3+u)$

or, $$x u_x + y u_y + z u_z = \frac{-2\cos(u^3+u)}{\sin(u^3+u)\,(3u^2+1)}$$

$$= -\frac{2\cot(u^3+u)}{3u^2+1}.$$

$\therefore \quad K = 2.$

2. $$e^{(u^2+6u+1)} = \frac{x^3+y^3+z^3}{\sqrt{x^4+y^4+z^4}}.$$

Let, $w = e^{(u^2+6u+1)}$

w is a homogeneous func^n of degree 1.

So, by Euler's theorem,

$$x w_x + y w_y + z w_z = w.$$

or, $e^{(u^2+6u+1)}(2u+6)\left(x u_x + y u_y + z u_z\right) = e^{(u^2+6u+1)}$

or, $$x u_x + y u_y + z u_z = \frac{e^{(u^2+6u+1)}}{e^{(u^2+6u+1)}(2u+6)}$$

$$= \frac{1}{2(u+3)}.$$

$$= \frac{3}{6(u+3)}$$

$\therefore \quad K = 3$

3. $\log(u^4+1) = \dfrac{7x^3+9y^3+z^3}{5x+6y+7z}$.

Let, $w = \log(u^4+1)$.

w is a homogeneous function of degree 2.

So, by Euler's theorem,

$$x w_x + y w_y + z w_z = 2 \log(u^4+1)$$

or, $\dfrac{1}{(u^4+1)} 4u^3 \left(x u_x + y u_y + z u_z \right) = 2 \log(u^4+1)$

or, $x u_x + y u_y + z u_z = \dfrac{2(u^4+1)\log(u^4+1)}{4u^3}$.

$$= \frac{4(u^4+1)\log(u^4+1)}{8u^3}$$

$\therefore K = 4$.

4.

$$e^{u^2} = \frac{x^2+y^2+7z^2}{2x+5y+16z}.$$

Let, $w = e^{u^2}$

w is a homogeneous function of degree 1.

By Euler's theorem we have,

$$x w_x + y w_y + z w_z = w.$$

or, $e^{u^2} . 2u \left[x u_x + y u_y + z u_z \right] = e^{u^2}$

or, $$x u_x + y u_y + z u_z = \frac{1}{2u}$$

$$= \frac{6}{12u}.$$

$\therefore$ $K = 6$.

5. $$\tan(u^5+5) = \frac{x^3+y^3+z^3}{\sqrt{x^2+y^2+z^2}}$$

Let, $w = \tan(u^5+5)$.

w is a homogeneous function of degree 2.

So by Euler's theorem,

$$x w_x + y w_y + z w_z = 2 \tan(u^5+5)$$

or, $$\sec^2(u^5+5)(5u^4)(x u_x + y u_y + z u_z) = 2 \tan(u^5+5)$$

or, $$x u_x + y u_y + z u_z = 2 \frac{\tan(u^5+5)}{\sec^2(u^5+5) \cdot 5u^4}$$

$$= \frac{2 \sin(u^5+5) \cos(u^5+5)}{5u^4}$$

$$= \frac{\sin(2u^5+10)}{5u^4}$$

$\therefore K = 1$.

Q.1 If $\int_1^\infty \left(\frac{kt}{1+t^2} - \frac{n}{t}\right) dt$ converges for $n \in \mathbb{N}$.

then the value of k is

Ans: $I = \lim_{R \to \infty} \int_1^R \left(\frac{kt}{1+t^2} - \frac{n}{t}\right) dt$

$$= \lim_{R \to \infty} \left[\frac{k}{2} \ln(1+t^2) - n \ln t\right]_1^R$$

$$= \lim_{R \to \infty} \left[\frac{k}{2} \ln(1+R^2) - n \ln R - \frac{k}{2} \ln 2\right]$$

Let $R = \frac{1}{u}$

then $I = \lim_{u \to 0} \left[\frac{k}{2} \ln\left(1+\frac{1}{u^2}\right) - n \ln\left(\frac{1}{u}\right) - \frac{k}{2} \ln 2\right]$

$$= \lim_{u \to 0} \left[\frac{k}{2}\left[\ln(u^2+1) - \ln(u^2)\right] + n \ln u - \frac{k}{2} \ln 2\right]$$

$$= \lim_{u \to 0} \left[\frac{k}{2} \ln(u^2+1) - k \ln u + n \ln u - \frac{k}{2} \ln 2\right].$$

$$= (n-k) \lim_{u \to 0} (\ln u) - \frac{k}{2} \ln 2. \quad \left[\lim_{u \to 0} \ln(u^2+1) = 0\right]$$

The above limit can be finite only if $n = k$

so $k = n$.

Q.1 Let $P_2(x,y)$ be second order Taylor polynomial approximation to $f(x,y) = x^3y + xy^2 + 2xy + 1$ about the point $(x,y)=(1,1)$ then

Solutions: $P_2(x,y) = f(1,1) + (x-1)\frac{\partial f}{\partial x}(1,1) + (y-1)\frac{\partial f}{\partial y}(1,1)$

$$+ \frac{(x-1)^2}{2!}\frac{\partial^2 f}{\partial x^2}(1,1) + (x-1)(y-1)\frac{\partial^2 f}{\partial x \partial y}(1,1) + (y-1)^2\frac{\partial^2 f}{\partial y^2}(1,1)$$

Now $P_2(x,y) = 5 + 6(x-1) + 5(y-1) + 3(x-1)^2 + 7(x-1)(y-1) + (y-1)^2$

so $P_2(2,1) = 14$

$P_2(3,1) = 29$

$P_2(1,3) = 19$

$P_2(4,1) = 50$

$P_2(1,4) = 29$.

1. Evaluate $\int\limits_0^1 45x^2(1-x^3)^2 dx$.

Ans: We first prove that $\int\limits_0^1 x^m(1-x^n)^p dx = \frac{1}{n}B(\frac{m+1}{n}, p+1)$.

For this put $x^n = z$, $x = z^{1/n}$.

$dx = \frac{1}{n}z^{\frac{1}{n-1}}dz$.

$I = \frac{1}{n}\int\limits_0^1 z^{\frac{m+1}{n}-1}(1-z)^{(p+1)-1}dz$.

$= \frac{1}{n}B(\frac{m+1}{n}, p+1)$.

For this problem, $m = 2, n = 3$ and $p = 2$. So we have $\int\limits_0^1 45x^2(1-x^3)^2 dx = 45 \times \frac{1}{3}B(1,3) = 45 \times \frac{1}{3} \times \frac{1}{3} = 5$.

2. Evaluate $\int\limits_0^1 144x^3(1-x^2)^2 dx$.

Ans: For this problem, $m = 3, n = 2$ and $p = 2$. So we have $\int\limits_0^1 144x^3(1-x^2)^2 dx = 144 \times \frac{1}{2}B(2,3) = 144 \times \frac{1}{2} \times \frac{1}{12} = 6$.

3. Evaluate $\int\limits_0^1 324x^5(1-x^3)^2 dx$.

Ans: For this problem, $m = 5, n = 3$ and $p = 2$. So we have $\int\limits_0^1 324x^5(1-x^3)^2 dx = 324 \times$

$\frac{1}{3}B(2,3) = 324 \times \frac{1}{3} \times \frac{1}{12} = 9.$

4. Evaluate $\int\limits_0^1 384x^7(1-x^4)^2dx.$

Ans: For this problem, $m = 7, n = 4$ and $p = 2$. So we have $\int\limits_0^1 384x^7(1-x^4)^2dx = 384 \times \frac{1}{4}B(2,3) = 384 \times \frac{1}{4} \times \frac{1}{12} = 8.$

5. Evaluate $\int\limits_0^1 105x^4(1-x^5)^2dx.$

Ans: For this problem, $m = 4, n = 5$ and $p = 2$. So we have $\int\limits_0^1 105x^4(1-x^5)^2dx = 105 \times \frac{1}{5}B(1,3) = 384 \times \frac{1}{5} \times \frac{1}{3} = 7.$

1. Evaluate $\int\limits_0^1 \frac{960x^3(1-x)^2}{(1+x)^7}dx$.

Ans: We know that $B(4,3) = \int\limits_0^1 y^3(1-y)^2 dy$.

Put $y = \frac{2x}{x+1}$. Then $dy = 2\frac{x+1-x}{(x+1)^2}dx = \frac{2}{(x+1)^2}dx$.

The range of x is also from 0 to 1.

$$B(4,3) = \int\limits_0^1 y^3(1-y)^2 dy = \int\limits_0^1 \left(\frac{2x}{x+1}\right)^3 \left(\frac{1-x}{1+x}\right)^2 \left(\frac{2}{(1+x)^2}\right) dx$$

$$= 2^4 \int\limits_0^1 \frac{x^3(1-x)^2}{(1+x)^7}dx.$$

Therefore $\int\limits_0^1 \frac{x^3(1-x)^2}{(1+x)^7}dx = \frac{B(4,3)}{2^4}$.

Hence $\int\limits_0^1 \frac{960x^3(1-x)^2}{(1+x)^7}dx = 960 \times \frac{B(4,3)}{2^4}$

We know that $B(4,3) = \frac{3!2!}{6!} = \frac{1}{60}$

$\int\limits_0^1 \frac{960x^3(1-x)^2}{(1+x)^7}dx = 960 \times \frac{1}{2^4} \times \frac{1}{60} = 1.$

2. Evaluate $\int\limits_0^1 \frac{960x^2(1-x)^3}{(1+x)^7}dx$.

Ans: Similarly, $\int\limits_0^1 \frac{960x^2(1-x)^3}{(1+x)^7}dx = 960 \times \frac{B(3,4)}{2^3} = 2$

3. Evaluate $\int\limits_0^1 \frac{360x(1-x)^4}{(1+x)^7} dx$.

Ans: Similarly, $\int\limits_0^1 \frac{360x(1-x)^4}{(1+x)^7} dx = 360 \times \frac{B(2,5)}{2^2}$.

We know that $B(2,5) = \frac{1!4!}{6!} = \frac{1}{30}$.

Hence $\int\limits_0^1 \frac{360x(1-x)^4}{(1+x)^7} dx = 360 \times \frac{B(2,5)}{2^2} = 360 \times \frac{1}{4} \times \frac{1}{30} = 3$.

4. Evaluate $\int\limits_0^1 \frac{672x(1-x)^5}{(1+x)^8} dx$.

Ans: 4

5. Evaluate $\int\limits_0^1 \frac{400x(1-x)^3}{(1+x)^6} dx$.

Ans: 5

Set A

1. $$\int_{10}^{\infty} (x-10)^{102} e^{-103(x-10)} dx$$

Take $(x-10) = t \Rightarrow dx = dt$

So, $$\int_{10}^{\infty} (x-10)^{102} e^{-103(x-10)} dx$$

$$= \int_{0}^{\infty} t^{102} e^{-103t} dt$$

Take $103t = u$

$$\frac{1}{\quad} \int^{\infty} \quad {}^{102} \quad {}^{-u}$$

$$\frac{1}{103^{102}} \int_0 u \quad e \cdot \frac{1}{103}\, du$$

$$= \frac{1}{103^{103}} \cdot \Gamma(103) = \frac{(102)!}{103^{103}}$$

2. $$\int_0^\infty x^{2019} e^{-2020x}\, dx$$

$$= \frac{2019!}{2020^{2020}}$$

3. $$\int_5^\infty (x-5)^{2020} e^{-2021(x-5)}\, dx$$

$$= \frac{2020!}{2021^{2021}}$$

4. $$\int_0^\infty x^{-31} e^{-32x} \, dx$$

$$= \frac{(31)!}{32^{32}}$$

5. $$\int_{32}^\infty (x-32)^{36} e^{-37(x-32)} \, dx$$

$$36!$$

$$= \frac{36!}{37^{37}}$$

Set B

1. $\displaystyle\int_0^\infty x e^{-x} \cos x \, dx.$

Say, $\displaystyle I_1 = \int_0^\infty x e^{-x} \cos x \, dx$

$\displaystyle I_2 = \int_0^\infty x e^{-x} \sin x \, dx$

$I_1 + i I_2$

$\displaystyle = \int_0^\infty x e^{-x} e^{ix} dx = \int_0^\infty x e^{-(1-i)x} dx$

$$= \int_0 x\, e\, e\, dx - \int_0$$

$$(1-i)x = t$$

or, $$dx = \frac{dt}{1-i}$$

$$\int_0^\infty x e^{-(1-i)x}\, dx = \int_0^\infty \frac{t}{(1-i)} e^{-t} \frac{1}{(1-i)}\, dt$$

$$= \int_0^\infty \frac{1}{(1-i)^2} t e^{-t}\, dt$$

$$= \frac{1}{-2i} \cdot \Gamma(2)$$

$$= \frac{i}{2}$$

$$\Rightarrow \int_0^\infty x e^{-x} \cos x = 0$$

$$\text{\&} \int_0^\infty x e^{-x} \sin x = \frac{1}{2}$$

2. $$\int_0^\infty x e^{-x} \sin x = \frac{1}{2}$$

3. $$\int_0^\infty x e^{-x} \sin 2x = \frac{4}{25}$$

4. $$\int_0^\infty x e^{-x} \cos 2x = -\frac{3}{25}$$

Say, $I_1 = \int_0^\infty x e^{-x} \cos 2x \, dx$

$I_2 = \int_0^\infty x e^{-x} \sin 2x \, dx$

$I_1 + i I_2 = \int_0^\infty x e^{-x} e^{2ix} dx$

$= \int_0^\infty x e^{-(1-2i)x} dx$

Take $(1-2i)x = t$

Or, $dx = \dfrac{dt}{(1-2i)}$

$$\to \int_0 \frac{t}{(1-2i)} e^{-t} \cdot \frac{dt}{(1-2i)}$$

$$= \frac{\Gamma(2)}{(1-2i)^2} = \frac{1}{1+4i^2-4i}$$

$$= \frac{1}{-(3+4i)}$$

$$= \frac{(3-4i)}{-(9+16)}$$

$$= -\frac{3}{25} + \frac{4}{25} i$$

5. $\int^{\infty} x^{1/3} e^{-27x} dx$

0

Take $27x = t$

$$\int_0^\infty \frac{t^{1/3}}{27^{1/3}} e^{-t} \frac{dt}{27}$$

$$= \frac{1}{3} \cdot \frac{1}{27} \int_0^\infty t^{1/3} e^{-t} \, dt$$

$$= \frac{1}{81} \Gamma\left(\frac{4}{3}\right) = \frac{1}{81} \cdot \frac{1}{3} \Gamma\left(\frac{1}{3}\right)$$

$$= \frac{1}{243} \Gamma\left(\frac{1}{3}\right)$$

JK-A

Jitendra Kumar

January 14, 2021

Question 1

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \quad \text{equals}$$

Ans: 0

Question 2

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \quad \text{equals}$$

Ans: 2

Question 3

$$\lim_{(x,y)\to(4,4)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \quad \text{equals}$$

Ans: 16

Question 4

$$\lim_{(x,y)\to(9,9)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \quad \text{equals}$$

Ans: 54

Question 5

$$\lim_{(x,y)\to(16,16)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \quad \text{equals}$$

Ans: 128

Solution:

$$\lim_{(x,y)\to(a,a)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(a,a)} \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{(x-y)} = 2a\sqrt{a}$$

JK-B

Jitendra Kumar

January 14, 2021

Question 1

Let $a, b \in \mathbb{R}$ and

$$f(x,y) = \begin{cases} \dfrac{a - e^{-x^2-y^2}}{x^2+y^2}, & (x,y) \neq (0,0) \\ b, & (x,y) = (0,0) \end{cases}$$

be a continuous function. The value of $(a+b)$ is

Ans: 2

Question 2

Let $a, b \in \mathbb{R}$ and

$$f(x,y) = \begin{cases} \dfrac{a - 2e^{-x^2-y^2}}{x^2+y^2}, & (x,y) \neq (0,0) \\ b, & (x,y) = (0,0) \end{cases}$$

be a continuous function. The value of $(a+b)$ is

Ans: 4

Question 3

Let $a, b \in \mathbb{R}$ and

$$f(x,y) = \begin{cases} \dfrac{a - 3e^{-x^2-y^2}}{x^2+y^2}, & (x,y) \neq (0,0) \\ b, & (x,y) = (0,0) \end{cases}$$

be a continuous function. The value of $(a+b)$ is

Ans: 6

Question 4

Let $a, b \in \mathbb{R}$ and

$$f(x,y) = \begin{cases} \dfrac{a - 4e^{-x^2-y^2}}{x^2+y^2}, & (x,y) \neq (0,0) \\ b, & (x,y) = (0,0) \end{cases}$$

be a continuous function. The value of $(a+b)$ is

Ans: 8

Question 5

Let $a, b \in \mathbb{R}$ and

$$f(x,y) = \begin{cases} \dfrac{a - 5e^{-x^2-y^2}}{x^2+y^2}, & (x,y) \neq (0,0) \\ b, & (x,y) = (0,0) \end{cases}$$

be a continuous function. The value of $(a+b)$ is

Ans: 10

Solution: Changing to polar coordinates

$$\lim_{(x,y)\to(0,0)} \frac{a - ce^{-x^2-y^2}}{x^2+y^2} = \lim_{r\to 0} \frac{a - ce^{-r^2}}{r^2}$$

For the existence of the limit $a = c$

$$\lim_{r\to 0} \frac{a - ce^{-r^2}}{r^2} = c \lim_{r\to 0} \frac{2re^{-r^2}}{2r} = c$$

This implies $b = c$. Hence $a + b = 2c$

Answers Test 2

Q.1) $$I = \sum_{n=1}^{k} \int_0^{2n} \frac{dx}{(x-n)^{1/5}} \qquad k \in \mathbb{N}$$

At $x = n$, functions $\frac{1}{(x-n)^{1/5}}$ becomes unbounded.

$\therefore$ its improper integral of type II.

For a particular $k \in \{1, 2, \dots\}$

$$\int_0^{2k} \frac{dx}{(x-k)^{1/5}} = \lim_{c_1 \to 0+} \int_0^{k-c_1} \frac{dx}{(x-k)^{1/5}} + \lim_{c_2 \to 0+} \int_{k+c_2}^{2k} \frac{dx}{(x-k)^{1/5}}$$

$$= \lim_{c_1 \to 0+} \left[\frac{5}{4} (x-k)^{4/5} \right]_0^{k-c_1} + \lim_{c_2 \to 0+} \left[\frac{5}{4} (x-k)^{4/5} \right]_{k+c_2}^{2k}$$

$$= \lim_{c_1 \to 0} \left[\frac{5}{4}(-c_1)^{4/5} - \frac{5}{4}(-k)^{4/5} \right]$$

$$+ \lim_{c_2 \to 0} \left[\frac{5}{4}(k)^{4/5} - \frac{5}{4}(c_2)^{4/5} \right]$$

$$= 0 - \frac{5}{4}\left[(-k)^4\right]^{1/5} + \frac{5}{4} k^{4/5} - 0$$

$$= 0$$

$$\Rightarrow I = 0$$

Question 2 :

$$I = \int_0^a \frac{1}{x} e^{-1/x} \, dx \qquad a > 0 ; \; a \in \mathbb{R}$$

function $f(x) = \frac{1}{x} e^{-1/x}$ is not bounded at $x = 0$,

$\therefore$ integral is improper integral of type II.

$$I = \lim_{c \to 0^+} \int_c^a \frac{1}{x} e^{-1/x} \, dx$$

$$= \lim_{c \to 0^+} \left[\left(\frac{1}{x} \frac{e^{-1/x}}{1/x^2} \right)_c^a - \int_c^a -\frac{1}{x^2} \frac{e^{-1/x}}{1/x^2} \, dx \right]$$

$$= \lim_{c \to 0^+} \left[\left(x e^{-1/x} \right)_c^a + \left(x^2 e^{-1/x} \right)_c^a \right]$$

$$= (a^2+a)e^{-1/a} - \lim_{c \to 0+} \left(ce^{-1/c} + c^2 e^{-1/c}\right)$$

$$I = \frac{a^2+a}{e^{1/a}}$$

Question (1) Let $f(x,y) = 2xy\, e^{-(x^2+y^2)+1}$

Then find local maximum and minimum value of $f(x,y)$.

Ans. $f(x,y) = 2xy\, e^{-(x^2+y^2)+1}$

$$f_x = 2y\, e^{-(x^2+y^2)+1} + 2xy\, e^{-(x^2+y^2)+1} \cdot (-2x)$$

$$= 2y\, e^{-(x^2+y^2)+1}\left(1-2x^2\right)$$

Thus $f_x = 0 \Rightarrow y = 0$ or $x = \pm \frac{1}{\sqrt{2}}$.

Similarly, $f_y = 2x\, e^{-(x^2+y^2)+1}\left(1-2y^2\right)$

Thus $f_y = 0 \Rightarrow x = 0$ or $y = \pm \frac{1}{\sqrt{2}}$.

Thus the critical pts are $(0,0)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ & $\left(-\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

Now $f_{xx} = 2\, e^{-(x^2+y^2)+1} \cdot 2xy \cdot (2x^2-3)$

and $f_{yy} = 2\, e^{-(x^2+y^2)+1} \cdot 2xy\,(2y^2-3)$

and $f_{xy} = 2\, e^{-(x^2+y^2)+1} (1-2x^2)(1-2y^2)$

Now $\left(f_{xx} f_{yy} - f_{xy}^2\right)\left(\frac{\pm 1}{\sqrt{2}}, \frac{\pm 1}{\sqrt{2}}\right) > 0$

and $f_{xx}\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) < 0.$

Thus $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ are pts of local maxima and the local maximum value is $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$

$= 1.$

Similarly, $\left(f_{xx} f_{yy} - f_{xy}^2\right)\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) > 0$

and $f_{xx}\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) > 0$

Thus $\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right)$ are pts of local minima and the local minimum value is

$$f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = -1.$$

Q(2) If $f(x,y) = 4xy\, e^{-(x^2+y^2)+1}$

then local maximum $= 2$

and local minimum $= -2$.

Q(3) If $f(x,y) = 6xy\, e^{-(x^2+y^2)+1}$

then local maximum $= 3$.

Q1 Let $f(x,y) = x^2 - y^2$. Then find the absolute maximum and minum value of $f(x,y)$ over the region

$$\frac{x^2}{4} + \frac{y^2}{9} \le 1.$$

Ans. Let $f(x,y) = x^2 - y^2$

and $g(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ —— (1)

Then by Lagrange's Multiplier method

we get $f_x = \lambda g_x$ & $f_y = \lambda g_y$

$\Rightarrow$ $2x = \lambda \frac{2x}{a^2}$ & $2y = -\lambda \frac{2y}{b^2}$

$\Rightarrow$ $x = 0$ and $y = 0$.

If $x = 0$ then from eqn (1) we get $y = \pm b$

and if $y = 0$ then from eqn (1) we get $x = \pm a$.

Thus the critical pts on the boundry are $(0, \pm b)$ and $(\pm a, 0)$.

For the interior $f_x = 0$ & $f_y = 0$ gives $2x = 0$ and $2y = 0$

Thus $(0,0)$ is the only critical pt in the interior.

Now $f(0, \pm b) = -b^2$

$f(\pm a, 0) = a^2$

and $f(0,0) = 0$.

Hence the absolute maximum value is a^2 and absolute minimum value is $-b^2$.

Q1. If $g(x,y) = \frac{x^2}{4} + \frac{y^2}{9} \leq 1$ then absolute maximum of $f(x,y)$ is 4 and absolute minimum of $f(x,y)$ is -9

Q2. If $g(x,y) = \frac{x^2}{9} + \frac{y^2}{4} \leq 1$

Then absolute maximum value of $f(x,y) = 9$

and absolute minimum value of $f(x,y) = -4$.

Q3 If $g(x,y) = \frac{x^2}{9} + \frac{y^2}{16} \leq 1$

Then the absolute minimum value of $f(x,y) = -16$.

Solution for set A/B Solutions: CM

Let $f(x,y) = \dfrac{my(-x^2+y^2)}{x^2+ny^2}$ for $(x,y) \neq (0,0)$

and $f(0,0)=0$. Then $\dfrac{\partial f}{\partial x}(0,0) + \dfrac{\partial f}{\partial y}(0,0) =$ —

$$\frac{\partial f(0,0)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{m \Delta y^3}{n \Delta y^3} = \frac{m}{n}$$

1. $m=4,\ n=2$ Ans: $\frac{4}{2} = 2$

(ii). $m=8,\ n=2$ Ans: $\frac{8}{2} = 4$

(iii) $m=16,\ n=2$ Ans: $\frac{16}{2} = 8$

(iv) $m=9,\ n=3$ Ans: $\frac{9}{3} = 3$

(v) $m=5,\ n=1$ Ans $\frac{5}{1} = 5$

Set- B/A

$$\frac{f(\frac{5\pi}{6}) - f(\frac{\pi}{6})}{\frac{5\pi}{6} - \frac{\pi}{6}} = \frac{m[\log \sin\frac{5\pi}{6} - \log \sin\frac{\pi}{6}]}{4\pi/6}$$

$= 0$

$f'(c) = \frac{m}{\sin c} \times \cos c = m \cot c$, $f'(c) = 0 \Rightarrow c = \frac{\pi}{2}$

(i) $\frac{6}{\pi} \times c = \frac{6}{\pi} \times \frac{\pi}{2} = 3$

(ii) $\frac{8}{\pi} \times c = \frac{8}{\pi} \times \frac{\pi}{2} = 4$

(iii) $\frac{10}{\pi} \times c = \frac{10}{\pi} \times \frac{\pi}{2} = 5$

(iv) $\frac{12}{\pi} \times c = \frac{12}{\pi} \times \frac{\pi}{2} = 6$

(v) $\frac{14}{\pi} \times c = \frac{14}{\pi} \times \frac{\pi}{2} = 7$

A - RK

$f : (0, \infty) \rightarrow \mathbb{R}$ differentiable, $\lim_{x \to \infty} f'(x) = 3$.

By LMVT
$$\frac{f(x+a) - f(x+a-b)}{(x+a) - (x+a-b)} = f'(c), \quad c \in ((x+a)-b, x+a)$$

$$\Rightarrow \lim_{x \to \infty} f(x+a) - f(x+a-b) = \lim_{x \to \infty} b \cdot f'(c)$$

As $x \to \infty$, $c \to \infty$

So, $\lim_{x \to \infty} f(x+a) - f(x+a-b) = \lim_{c \to \infty} b \cdot f'(c)$.

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① $\lim_{x \to \infty} f'(x) = 3$

$\lim_{x \to \infty} f(x+\pi) - f(x+\pi-2) = 2 \cdot 3 = 6$

② $\lim_{x \to \infty} f'(x) = 1$

$\lim_{x \to \infty} f(x+e) - f(x+e-4) = 1 \cdot 4 = 4$

③ $\lim_{x \to \infty} f'(x) = 3$

$\lim_{x \to \infty} f(x+e) - f(x+e-3) = 3 \cdot 3 = 9$

④ $\lim_{x \to \infty} f'(x) = 2$

$\lim_{x \to \infty} f(x+\pi) - f(x+\pi-5) = 2 \cdot 5 = 10$

⑤ $\lim_{x \to \infty} f'(x) = 0$

$\lim_{x \to \infty} f(x+2) - f(x+\pi-1) = 0 \cdot (\pi+1+2)$
$= 0.$

B - RK

Suppose

(6) $f : [0,1] \to \mathbb{R}$ differentiable & $f(0) = 0$.

Given $|f'(x)| \le |f(x)|$. $\forall x \in [0,1]$.

By LMVT, (for $[0,x]$)

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \quad \text{for some } c \in [0,1].$$

$\therefore \; |f(x)| \le x \cdot |f(c)|$, as $|f'(c)| \le |f(c)|$

Apply LMVT to $[0, c]$,

$|f(c)| \le c \cdot |f(c_1)|$, for some $c_1 \in [0, c]$

$\Rightarrow |f(x)| \le x^2 \, |f(c_1)|$

So, inductively $|f(x)| \le x^n \cdot |f(c_{n-1})|$, for some $c_{n-1} \in [0, c_{n-2}]$.

As f is continuous on $[0,1]$, so $|f(x)| \le k$ $\forall x \in [0,1]$.

Also, $(|f(c_{n-1})| \cdot x^n) \to 0$ as $0 < x < 1$,

$\therefore \; f(x) = 0$.

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1. $2012 + f\left(\frac{1}{2021}\right) = 2012$
2. $1 + f\left(\frac{1}{e}\right) = 1$
3. $3 + f\left(\frac{1}{2}\right) = 3$
4. $2 + 2021\, f\left(\frac{1}{4}\right) = 2$
5. $13 + f\left(\frac{1}{21}\right) = 13.$