Suppose $m < n$. Since $r \leq m$, therefore r is definitely less than n. Hence in this case the given system of equations must possess a non-zero solⁿ. The no. of solⁿ.s will be infinite.

Working rule for finding the solⁿs. of $AX = 0$

Reduce the co-eff. matrix A to echelon form by applying elementary row transformation only. This echelon form will help us to know the rank of A. Then follow the above three cases to know the solⁿ.

EX Does the following system of eqns. possess a non-zero solⁿ?

$$x + 2y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0$$

Solⁿ:
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

Rank = 3. Only zero solⁿ.

EX
$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2 \qquad \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank = 2. $3 - 2 = 1$ l.i. solⁿ. $z = c$, $y = \frac{8}{7}c$, $x = -\frac{10}{7}c$

Ex Solve completely the system of eqns.

$$x+y+z=0$$
$$2x-y-3z=0$$
$$3x-5y+4z=0$$
$$x+17y+4z=0$$

Solⁿ: $R_2 \to R_2-2R_1,\ R_3 \to R_3-3R_1,\ R_4 \to R_4-R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & -8 & 1 \\ 0 & 16 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & -24 & 3 \\ 0 & 48 & 9 \end{bmatrix} \quad R_3 \to 3R_3,\ R_4 \to 3R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 43 \\ 0 & 0 & -71 \end{bmatrix} \quad R_3 \to R_3-8R_2,\ R_4 \to R_4+16R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 43 \\ 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4+\frac{71}{43}R_3$$

$\therefore$ Rank $= 3$

Only solⁿ is the trivial solⁿ

Ex Find the solution of the following system of equations

$$3x + 4y - z - 6w = 0$$
$$2x + 3y + 2z - 3w = 0$$
$$2x + y - 14z - 9w = 0$$
$$x + 3y + 13z + 3w = 0$$

Sol^n:

$$A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix} \quad R_4 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \end{bmatrix} \quad \begin{array}{l} R_2 \to -\frac{1}{3} R_2 \\ R_3 \to -\frac{1}{5} R_3 \\ R_4 \to -\frac{1}{5} R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

The rank of A is 2. ∴ The given system of equations possesses $4-2=2$ L.I. sol^n. The given system is equivalent to the system of two eqns.

$$x+3y+13z+3w=0$$
$$y+8z+3w=0$$

Let $z=c_1$, $w=c_2$ ∴ $y=-8c_1-3c_2$; $x=11c_1+6c_2$

∴ Sol^n. is $[x,y,z,w]=[11c_1+6c_2, -8c_1-3c_2, c_1, c_2]$

where c_1 & c_2 are arbitrary constants.

$$(x,y,z,w)=(11c_1+6c_2, -8c_1-3c_2, c_1, c_2)$$
$$= c_1\underbrace{(11,-8,1,0)}_{X_1} + c_2\underbrace{(6,-3,0,1)}_{X_2}$$
$$= c_1X_1+c_2X_2$$

Dim $4-2=2$

Put	$z=1$	$z=0$
	$w=0$	$w=1$
∴	$y=-8$	$y=-3$
	$x=11$	$x=6$

∴ $(11,-8,1,0)$ and $(6,-3,0,1)$ are two L.I. sol^n. of the system

Ex Let W be the subspace of $\mathbb{R}^4$ generated by $(1,-2,5,-3)$, $(2,3,1,-4)$ and $(3,8,-3,-5)$.

Find a basis and the dimension of W.

Sol^n:

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$\dim W = 2$

Basis $= \{(1,-2,5,-3), (0,7,-9,2)\}$