Concepts Covered

Differential Equations of Higher Order

- Method of Variation of Parameters
- Worked Problem

Method of Variation of Parameters

Consider the following second order non-homogeneous linear equation:

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x) \qquad - (*)$$

Let $y = c_1y_1 + c_2y_2$, with c_1 and c_2 as arbitrary constants, be the general solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$$

We assume that $y = C_1\, y_1 + C_2\, y_2$ is the general solution of the non-homogeneous equation (*), where C_1 and C_2 are functions of x to be so chosen that (*) is satisfied.

Differentiating $y = C_1 y_1 + C_2 y_2$, we get

$$y' = C_1 y_1' + C_2 y_2' + \underbrace{C_1' y_1 + C_2' y_2}_{=0}$$

This is set for simplicity of getting a relation in C_1 and C_2. We are looking forward for one more relation so that C_1 and C_2 can be obtained. The first relation we obtain as

$$C_1' y_1 + C_2' y_2 = 0$$

Now y'' can be obtained as

$$y'' = C_1 y_1'' + C_2 y_2'' + C_1' y_1' + C_2' y_2'$$

Now we substitute y' and y'' in the given non-homogeneous equation $y'' + a_1 y' + a_2 y = f(x)$

$$C_1(y_1'' + a_1 y_1' + a_2 y_1) + C_2(y_2'' + a_1 y_2' + a_2 y_2) + C_1' y_1' + C_2' y_2' = f(x)$$

$$y'' + a_1 y' + a_2 y = f(x)$$

$$y = C_1 y_1 + C_2 y_2$$

$$y' = C_1 y_1' + C_2 y_2'$$

$$y'' = C_1 y_1'' + C_2 y_2'' + C_1' y_1' + C_2' y_2'$$

$$\Rightarrow \boxed{C_1' y_1' + C_2' y_2' = f(x)}$$

Also we have $\boxed{C_1' y_1 + C_2' y_2 = 0}$

Solving the above two equations

W: Wronskian $\neq 0$

$$C_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2 f(x)}{W} \qquad \Rightarrow C_1 = \int -\frac{y_2 f(x)}{W}\, dx + d_1$$

$$C_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f(x)}{W} \qquad \Rightarrow C_2 = \int \frac{y_1 f(x)}{W}\, dx + d_2$$

Genera Solution: $y = C_1 y_1 + C_2 y_2$

$$C_1 = \int -\frac{y_2 f(x)}{W}\, dx + d_1$$

$$C_2 = \int \frac{y_1 f(x)}{W}\, dx + d_2$$

$$\boxed{y = d_1 y_1 + d_2 y_2 + y_1 \int -\frac{y_2 f(x)}{W}\, dx + y_2 \int \frac{y_1 f(x)}{W}\, dx}$$

Example : Apply Method of Variation of Parameter to solve

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

Solution: C.F. $= c_1 e^x + c_2 e^{-x}$

Let $y = C_1 e^x + C_2 e^{-x}$ be the general solution of the given equation.

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

$$y = C_1e^x + C_2e^{-x} \Rightarrow y' = C_1e^x - C_2e^{-x} + \underbrace{C_1'e^x + C_2'e^{-x}}_{=0}$$

$$\Rightarrow y'' = C_1e^x + C_2e^{-x} + C_1'e^x - C_2'e^{-x}$$

Substituting in the given differential equation

$$C_1'e^x - C_2'e^{-x} = \frac{2}{1+e^x}$$

The Wronskian: $W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$

$$C_1 = \int -\frac{y_2 f(x)}{W}\, dx + d_1$$

$$\Rightarrow C_1 = \int \frac{e^{-x}}{1+e^x}\, dx + d_1 = \int \frac{1}{z^2(1+z)}\, dz + d_1$$

Substitute $e^x = z$

$$C_1 = \int \frac{1}{z^2(1+z)}\,dz + d_1 = \int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{z+1}\right) dz + d_1$$

$$C_1 = -\frac{1}{z} - \ln z + \ln(1+z) + d_1 = -e^{-x} - x + \ln(1+e^x) + d_1$$

$$C_2 = \int \frac{y_1 f(x)}{W}\,dx + d_2$$

$$C_2 = -\int \frac{e^x}{1+e^x}\,dx + d_2 = -\ln(1+e^x) + d_2$$

$$\boxed{y = d_1 e^x + d_2 e^{-x} - 1 - xe^x + (e^x - e^{-x})\ln(1+e^x)}$$

Concepts Covered

Differential Equations of Higher Order

- ❑ Cauchy-Euler Equations
- ❑ Solution Techniques

Cauchy-Euler Equations:

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X$$

Denoting $\frac{d}{dx} \equiv D$

$$(x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_n)\, y = X$$

Cauchy-Euler Equations: $x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X$

Choose $x = e^z$ or $z = \ln x \implies \frac{dz}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \implies x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

Denote $D \equiv \frac{d}{dx}$ $\quad D_1 \equiv \frac{d}{dz}$

$$\implies x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\implies xD \equiv D_1$$

$$\implies x^2D^2 \equiv D_1^2 - D_1 \equiv D_1(D_1 - 1)$$

$$D \equiv \frac{d}{dx} \qquad D_1 \equiv \frac{d}{dz}$$

$$x^2 D^2 \equiv D_1(D_1 - 1)$$

$$x^3 D^3 \equiv D_1(D_1 - 1)(D_1 - 2)$$

$$\vdots$$

$$x^n D^n \equiv D_1(D_1 - 1)(D_1 - 2) \cdots (D_1 - n + 1)$$

$f(D)\, y = X \implies g(D_1)\, y = Z$ where Z is a function of z only.

Example : General Solution of $(x^2D^2 - xD + 2)y = x\ln x$

General Solution

$$\boxed{y = x[c_1\cos(\ln x) + c_2\sin(\ln x)] + x\ln x}$$

Let $x = e^z$. Then $D_1 \equiv \dfrac{d}{dz}$

$$[D_1(D_1 - 1) - D_1 + 2]y = ze^z \quad \Rightarrow [D_1^2 - 2D_1 + 2]y = ze^z$$

$$\text{C.F.} = e^z(c_1\cos z + c_2\sin z) = x[c_1\cos(\ln x) + c_2\sin(\ln x)]$$

$$\text{P.I.} = \frac{1}{[D_1^2 - 2D_1 + 2]}ze^z = e^z\frac{1}{[(D_1+1)^2 - 2(D_1+1) + 2]}z$$

$$= e^z\frac{1}{D_1^2 + 1}z = e^z(1 + D_1^2)^{-1}z = ze^z = x\ln x$$

Equations Reducible to Euler-Cauchy Form:

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X$$

Take $a + bx = v$ $\quad \Rightarrow \dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = b\dfrac{dy}{dv} \quad \Rightarrow \dfrac{d^n y}{dx^n} = b^n \dfrac{d^n y}{dv^n}$

Substituting in the given differential equation, we get

$$v^n \frac{d^n y}{dv^n} + \frac{a_1}{b} v^{n-1} \frac{d^{n-1} y}{dv^{n-1}} + \cdots + \frac{a_{n-1}}{b^{n-1}} v \frac{dy}{dv} + \frac{a_n}{b^n} y = \frac{X}{b^n}$$

Example: $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=4\cos\ln(1+x)$

Take $1+x=v$ $\Rightarrow \dfrac{dy}{dx}=\dfrac{dy}{dv}\dfrac{dv}{dx}=\dfrac{dy}{dv}$ $\quad\Rightarrow \dfrac{d^2y}{dx^2}=\dfrac{d^2y}{dv^2}$

Substituting we get

$$v^2\frac{d^2y}{dv^2}+v\frac{dy}{dv}+y=4\cos\ln v$$

Consider $v=e^z$ and let $D_1\equiv\dfrac{d}{dz}$

$$[D_1(D_1-1)+D_1+1]y=4\cos z$$

$$[D_1^2 + 1]y = 4\cos z$$

C.F. $= c_1 \cos z + c_2 \sin z = c_1 \cos(\ln v) + c_2 \sin(\ln v)$

$$= c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+x))$$

P.I. $= \dfrac{1}{[D_1^2 + 1]} 4\cos z = 4\,\dfrac{z}{2} \sin z = 2z \sin z = 2\ln v \, \sin(\ln v)$

$$= 2\ln(1+x) \, \sin(\ln(1+x))$$

General Solution:

$$y = c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+x)) + 2\ln(1+x) \, \sin(\ln(1+x))$$

Conclusion

Cauchy-Euler Equations:

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X$$

Equations Reducible to Euler-Cauchy Form:

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X$$

Thank You