ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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Concepts Covered

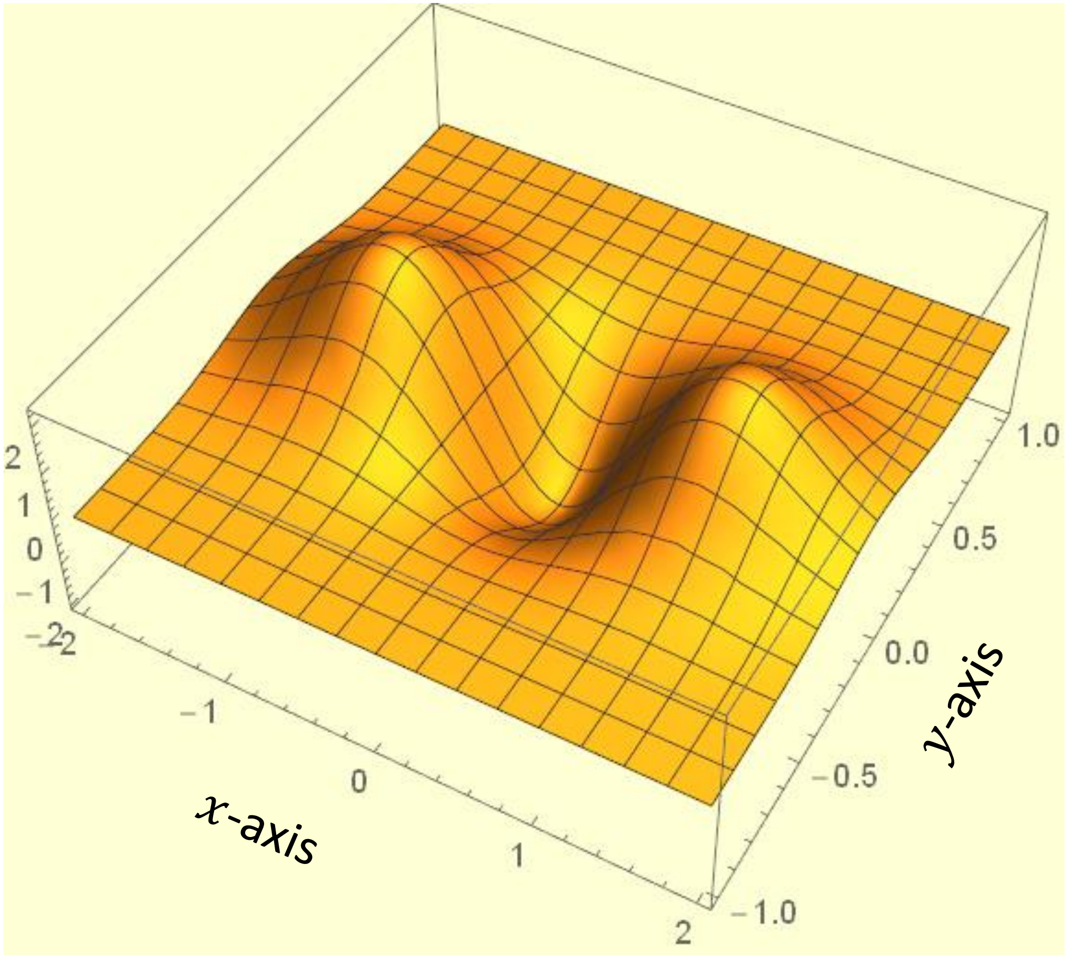
Differential Calculus

Functions of Several Variables

- ❑ Maxima and Minima (Necessary Conditions)

Local Maximum or Minimum

$$z = (4x^2 + y^2)e^{-x^2-4y^2}$$

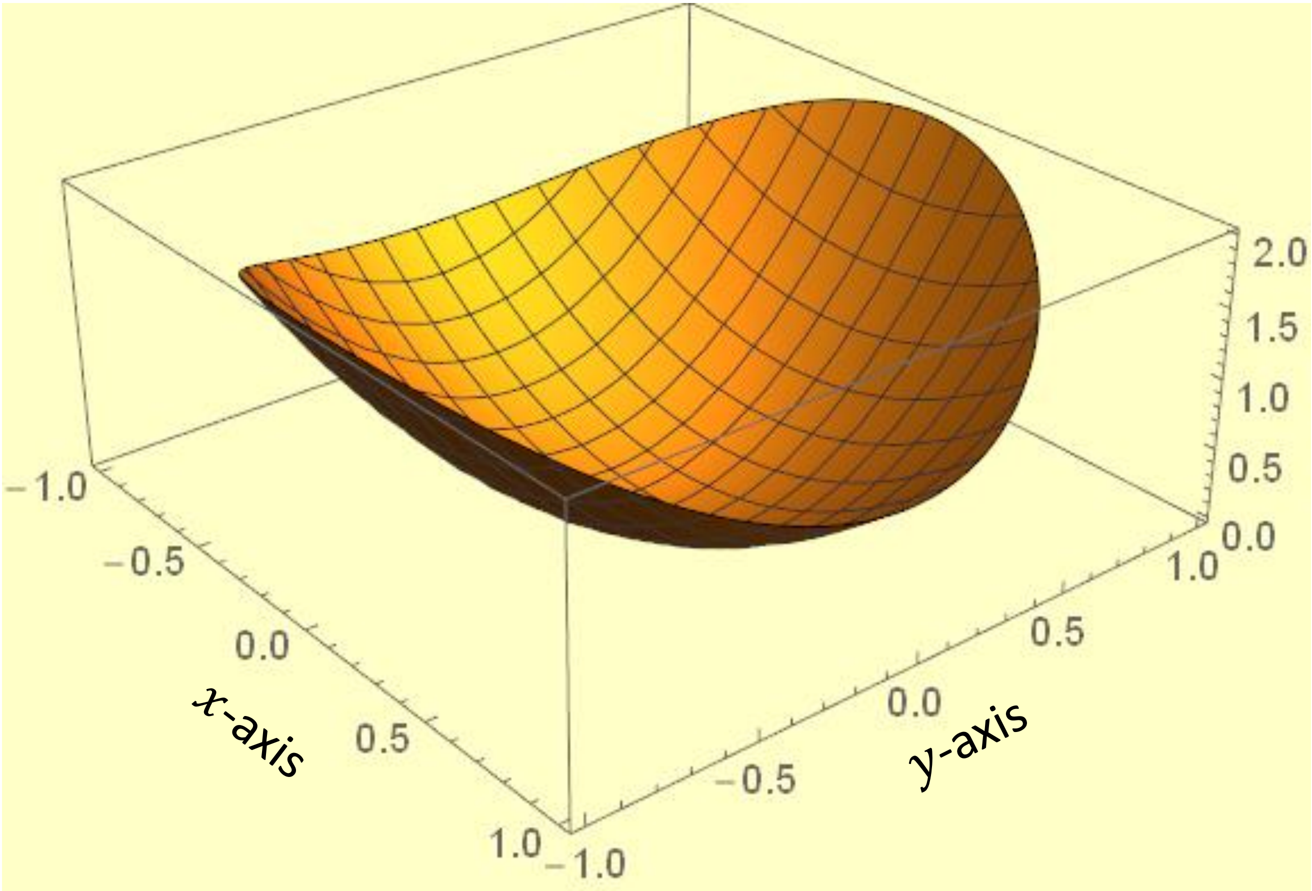


A function $z = f(x, y)$ has a maximum (or a minimum) at the point (x_0, y_0) if at every point in a neighborhood of (x_0, y_0), the function assumes a smaller value (or a larger value) than at the point itself.

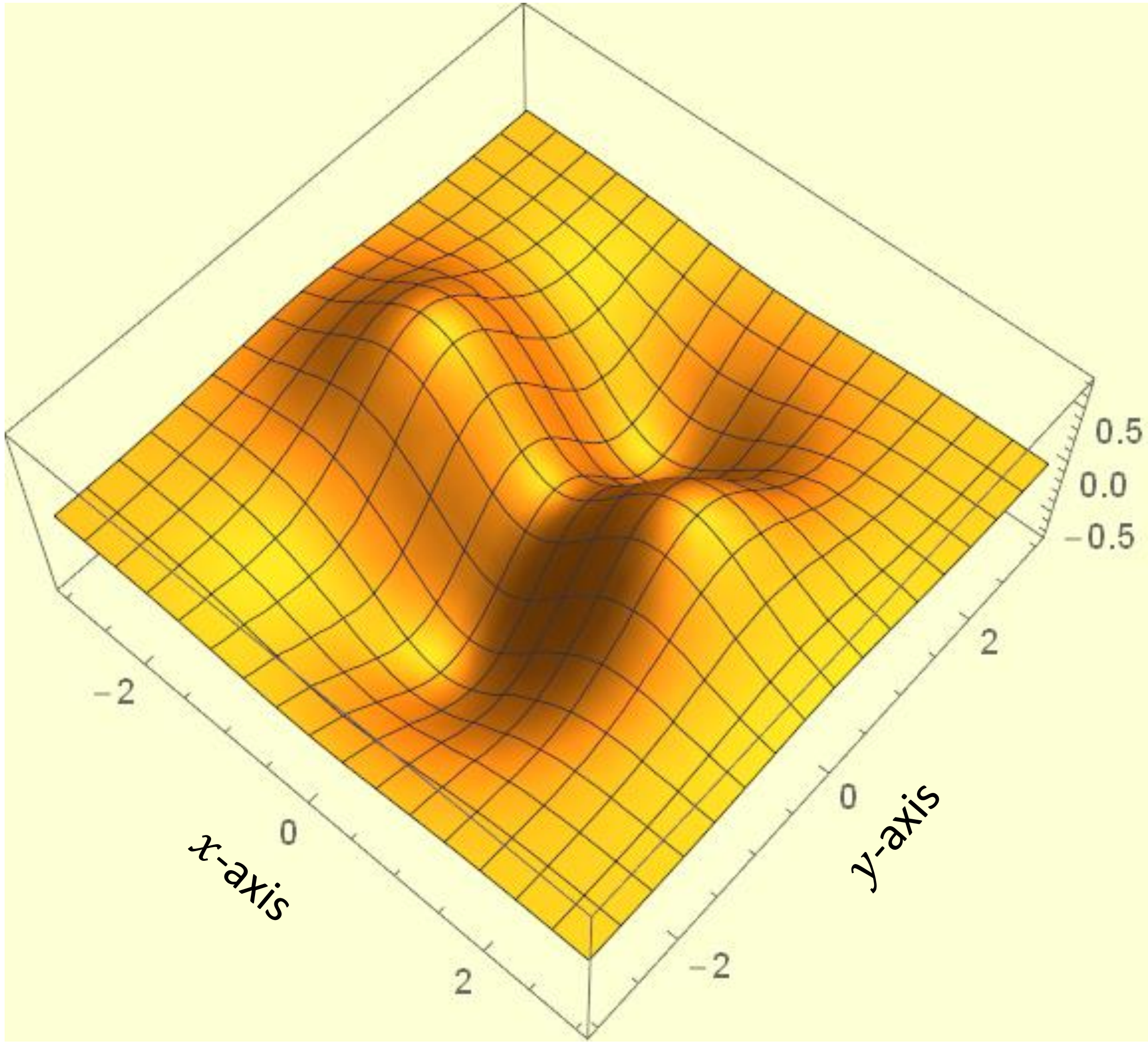
Maximum and minimum values together are called extreme values.

Absolute or Global Maximum/Minimum

$$f(x,y) = x^2 + 2y^2, \qquad x^2 + y^2 \le 1$$



The smallest and the largest values attained by a function over entire domain including the boundary of the domain are called absolute (or global) minimum and absolute (or global) maximum, respectively.



Critical point & Saddle Points

The point (x_0, y_0) is called critical point (or stationary point) of $f(x, y)$ if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

A critical point where the function has no minimum or maximum is called a saddle point.

Necessary condition for a function to have extremum

Let $f(x, y)$ be continuous and have first order partial derivatives at a point $P(a, b)$. Then necessary conditions for the existence of an extreme value of it at the point P are

$$f_x(a, b) = 0 \quad \& \quad f_y(a, b) = 0$$

(The point P is a critical point)

OR

If the point $P(a, b)$ is a relative extremum of the function $f(x, y)$ then $P(a, b)$ is also a critical point of $f(x, y)$.

Necessary Condition for a function to have extremum

Let $(a+h, b+k)$ be a point in the neighborhood of the point $P(a, b)$.

Then P will be point of maximum if

$$\Delta f = f(a+h, b+k) - f(a, b) \leq 0 \quad \text{for all sufficiently small } h \;\&\; k$$

and a point of minimum if

$$\Delta f = f(a+h, b+k) - f(a, b) \geq 0 \quad \text{for all sufficiently small } h \;\&\; k$$

Necessary Condition for a function to have extremum

Taylor series expansion about the point (a, b)

$$f(a+h, b+k) = f(a,b) + \left(h\, f_x + k\, f_y\right)_{(a,b)} + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Noting $\Delta f = f(a+h, b+k) - f(a,b)$

$$\Delta f = h\, f_x(a,b) + k\, f_y(a,b) + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

For sufficiently small h & k, the sign of Δf will depend on the sign of

$$h\, f_x(a,b) + k\, f_y(a,b)$$

$$\Delta f = h\, f_x(a,b) + k\, f_y(a,b) + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Letting $h \to 0$, we get

$$\Delta f = k\, f_y(a,b) + \frac{1}{2}k^2 f_{yy}(a,b) + \cdots$$

h	$h - 1000h^2 - 2000h^3$
0.1	-11.9
0.01	-0.092
0.001	-0.000002
0.0001	0.000089998

Note that the sign of Δf depends on the sign of $k\, f_y(a,b)$. That is

Assuming $f_y > 0$:

$\Delta f > 0 \text{ for } k > 0$

$\Delta f < 0 \text{ for } k < 0$

Assuming $f_y < 0$:

$\Delta f < 0 \text{ for } k > 0$

$\Delta f > 0 \text{ for } k < 0$

Therefore the function cannot have an extremum unless $\boxed{f_y = 0}$

$$\Delta f = h\, f_x(a,b) + k\, f_y(a,b) + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Similarly, letting $k \to 0$ we find that Δf changes sign h:

Assuming $f_x > 0$:

$\Delta f > 0 \;\text{ for }\; h > 0$

$\Delta f < 0 \;\text{ for }\; h < 0$

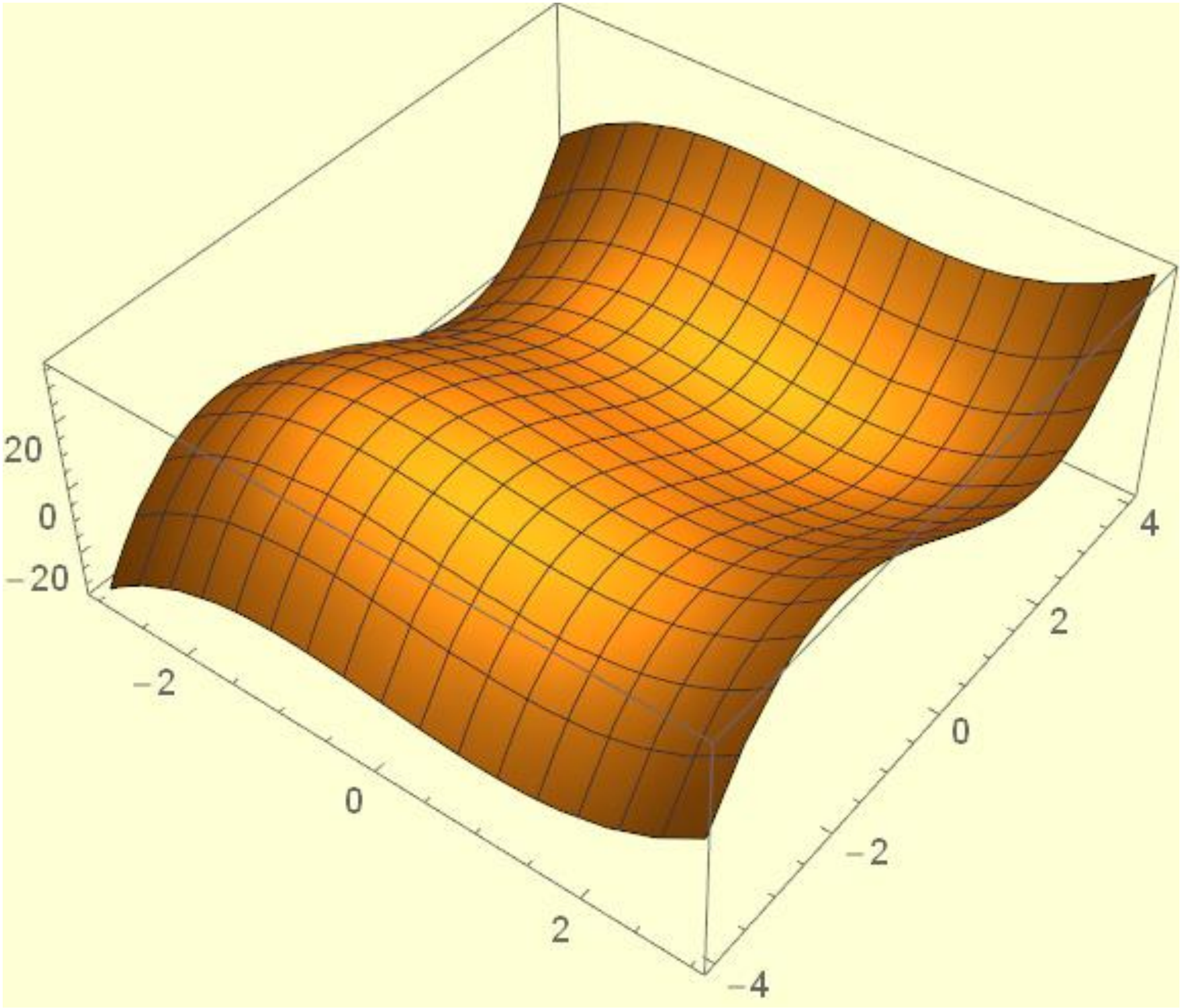
Assuming $f_x < 0$:

$\Delta f < 0 \;\text{ for }\; h > 0$

$\Delta f > 0 \;\text{ for }\; h < 0$

Therefore the function cannot have an extremum unless $f_x = 0$

Thus, the necessary conditions for the existence of an extremum at the point (a,b) is that $f_x(a,b) = 0 \;\&\; f_y(a,b) = 0$



Problem - 1

Find all critical points of the function
$f(x,y) = x^3 + y^3 - 3x - 12y + 20.$

Critical points are obtained by solving

$$f_x(x,y) = 0 \ \& \ f_y(x,y) = 0$$

$$f_x(x,y) = 0 \Longrightarrow 3x^2 - 3 = 0$$

$$f_y(x,y) = 0 \Longrightarrow 3y^2 - 12 = 0$$

Critical Points are:

$(\pm 1, \pm 2)$

QUIZ QUESTION ?

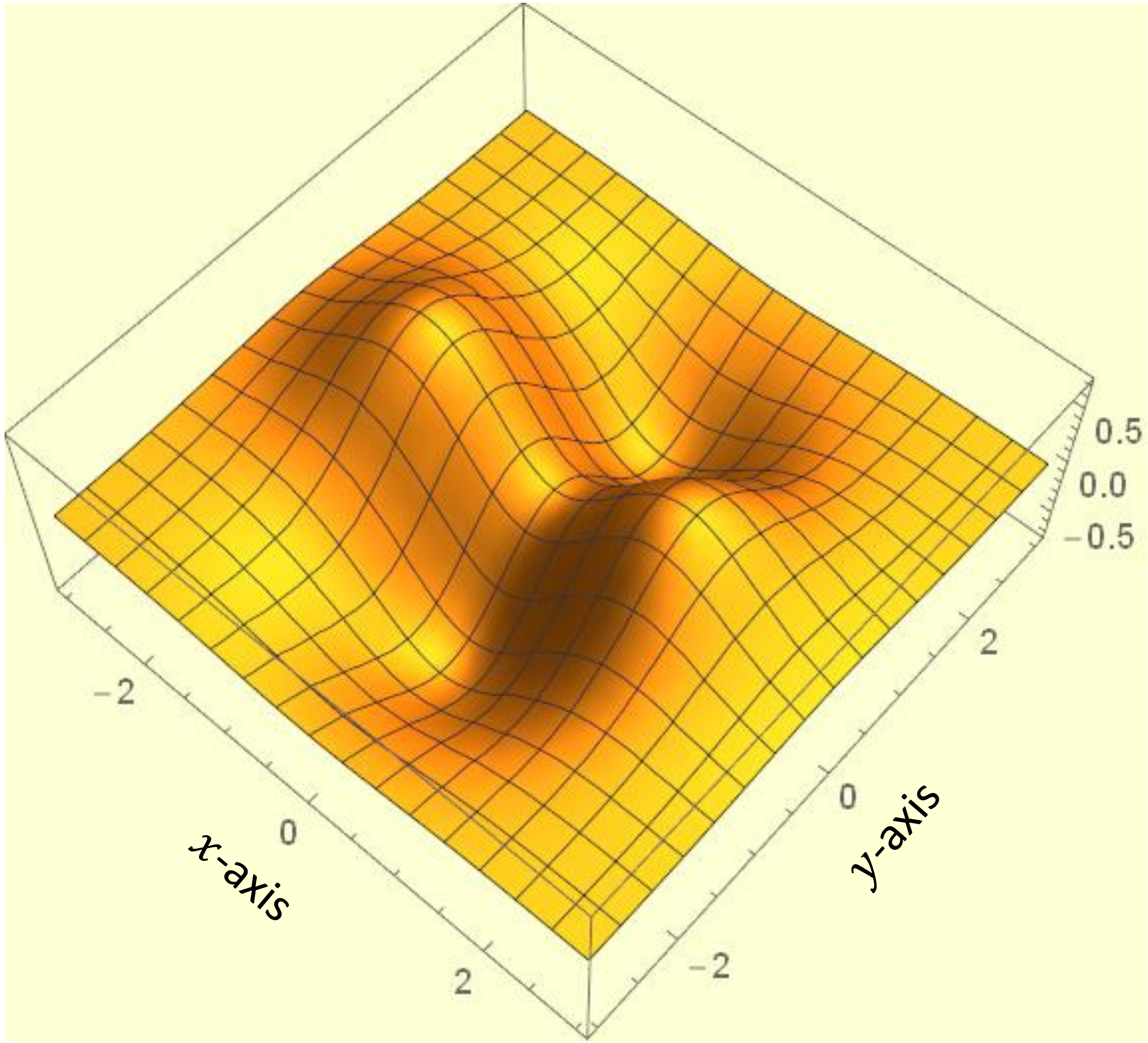
LINK FOR RESPONSES: http://www.facweb.iitkgp.ac.in/~jkumar/teach/MA11003.html

Number of critical points of the function

$$f(x,y) = (x^2 - y^2)e^{\frac{-x^2-y^2}{2}}$$

is

(A)	2	**(B)**	3
(C)	4	**(D)**	5



Problem - 2

Find all critical points of the function

$$f(x, y) = (x^2 - y^2)e^{\frac{-x^2-y^2}{2}}.$$

$$f_x = 0 \Longrightarrow \ (2 - \ (x^2 - y^2))\, x = 0$$

$$f_y = 0 \Longrightarrow \ (-2 - \ (x^2 - y^2))\, y = 0$$

Critical Points:

$$(0, 0), \left(\pm\sqrt{2}, 0\right), \left(0, \pm\sqrt{2}\right)$$

KEY TAKEAWAY

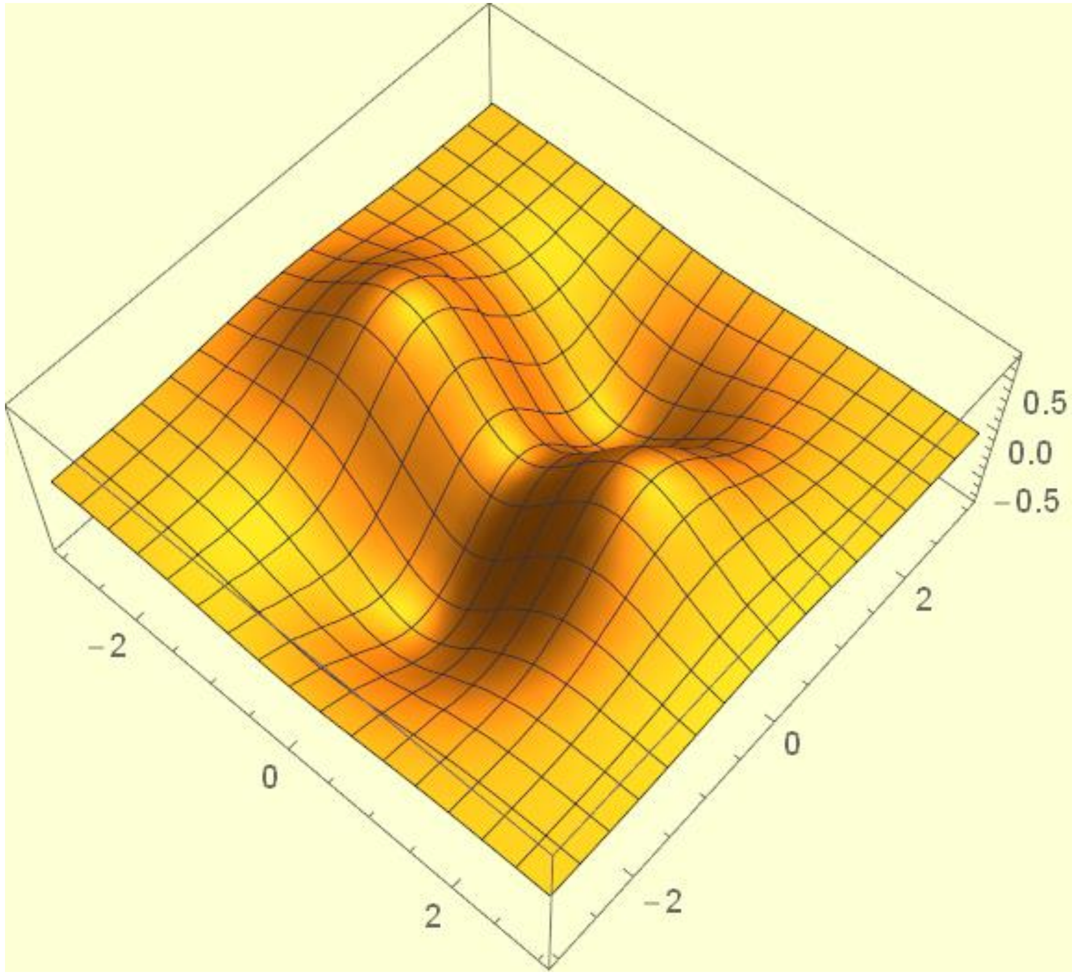
Necessary condition for extrema

$$f_x(a,b) = 0$$

$$f_y(a,b) = 0$$

Critical points are candidates for

- Local extrema
- Saddle points



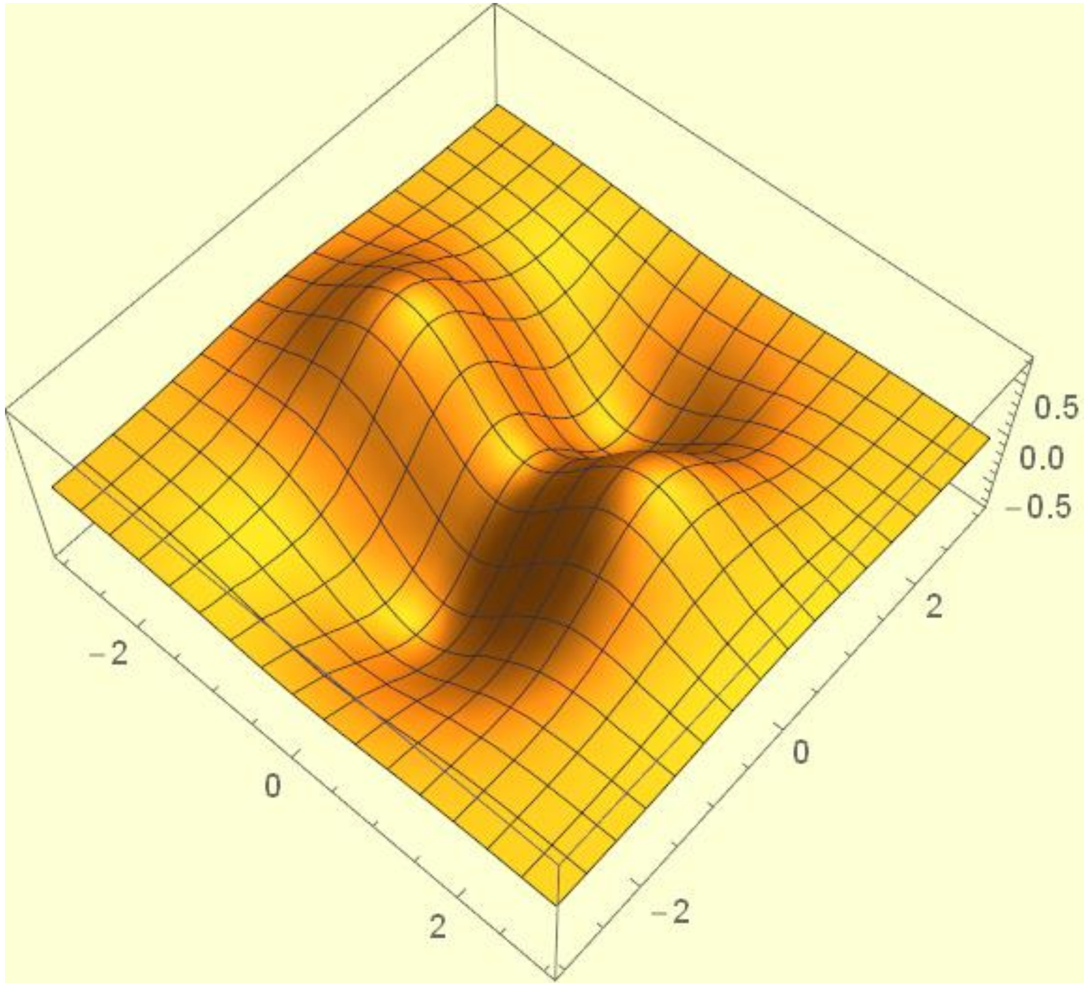
Concepts Covered

Differential Calculus

Functions of Several Variables

- ❑ Maxima and Minima (Sufficient Conditions)

Local Extrema (Previous Lecture)



A point (a, b) will be a point of local extrema if

$$\Delta f = f(a+h, b+k) - f(a, b)$$

does not change its sign for all sufficiently small h & k

Taylor's Series $\Delta f = h\, f_x(a,b) + k\, f_y(a,b)$

$$+\frac{1}{2}\left(h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Necessary Condition

$$f_x(a,b) = 0 \quad \& \quad f_y(a,b) = 0$$

Sufficient condition for a function to have extremum

Notation: $r = f_{xx}(a, b)$, $s = f_{xy}(a, b)$, $t = f_{yy}(a, b)$

Let a function $f(x, y)$ be continuous and have continuous second order partial derivatives at a point $P(a, b)$. If $P(a, b)$ is a critical point, then the point P is a point of

i. local maximum if $rt - s^2 > 0$ and $r < 0$
ii. local minimum if $rt - s^2 > 0$ and $r > 0$
iii. saddle point if $rt - s^2 < 0$
iv. test fails if $rt - s^2 = 0$ (some other way to characterize)

Sufficient condition for a function to have extremum

Consider $\Delta f = f(a+h, b+k) - f(a,b)$

$$\Delta f = h\, f_x(a,b) + k\, f_y(a,b) + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Since (a,b) is a critical point, $f_x(a,b) = 0$ & $f_y(a,b) = 0$, we have

$$\Delta f = \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{kk}\right)_{(a,b)} + \cdots$$

OR

$$\Delta f = \frac{1}{2}(h^2\, r + 2hk\, s + k^2\, t) + \cdots$$

$$\Delta f = \frac{1}{2}(h^2\, r + 2hk\, s + k^2\, t) + \cdots$$

Assuming $r \neq 0$

$$\Delta f = \frac{1}{2r}(h^2\, r^2 \; + 2hk\, rs + k^2\, rt) + \cdots$$

$$\Delta f = \frac{1}{2r}(h^2\, r^2 \; + 2hk\, rs + k^2 s^2 - k^2 s^2 + k^2\, rt) + \cdots$$

$$\Delta f = \frac{1}{2r}\left((hr + ks)^2 - k^2 s^2 + k^2 rt\right) + \cdots$$

$$\Delta f = \frac{1}{2r}\left((hr + ks)^2 + k^2(rt - s^2)\right) + \cdots$$

$$\Delta f = \frac{1}{2r}\left((hr + ks)^2 + k^2(rt - s^2)\right) + \cdots$$

Case – I: $rt - s^2 > 0$

$$\Delta f > 0 \quad \text{if } r > 0$$

$$\Delta f < 0 \quad \text{if } r < 0$$

The point (a, b) is a point of minimum if $rt - s^2 > 0,\quad r > 0$

The point (a, b) is a point of maximum if $rt - s^2 > 0,\quad r < 0$

$$\Delta f = \frac{1}{2r}\left((hr + ks)^2 + k^2(rt - s^2)\right) + \cdots$$

Case – II: $rt - s^2 < 0$

Let $k \to 0$ & $h \neq 0 \implies \Delta f > 0$ if $r > 0$

Let $k \neq 0$ & choose h such that $hr + ks = 0 \implies \Delta f < 0$ if $r > 0$

$\implies$ The sign of Δf depends on h & k

Hence no maximum/minimum of f can occur at $P(a, b)$.

$\implies$ The point $P(a, b)$ is a saddle point

$$\Delta f = \frac{1}{2r}\left((hr+ks)^2 + k^2(rt - s^2)\right) + \cdots$$

Case – III: $rt - s^2 = 0$

$$\Delta f = \frac{1}{2r}(hr+ks)^2 + \cdots$$

If we take h & k such that $hr = -ks$, then the whole second order terms of the right hand side will vanish.

Therefore, the conclusion will depend on the higher order terms.

One has to find some other way to investigate such points.

Working rules for investigating local extrema

- Find all critical points $f_x = 0 \ \& \ f_y = 0$

- For each critical point, evaluate

$$r = f_{xx}, \qquad s = f_{xy}, \qquad t = f_{yy}$$

- Identification

  - If $rt - s^2 > 0 \ \& \ r < 0$ maximum
  - If $rt - s^2 > 0 \ \& \ r > 0$ minimum
  - If $rt - s^2 < 0$ Saddle point
  - If $rt - s^2 = 0$ Test Fails

Example: Find all critical points of $f(x,y) = x^3 - 6x^2 - 8y^2$ and investigate their nature for local maximum/minimum and saddle point.

Critical points: $\left.\begin{array}{l} f_x = 0 \\ \\ f_y = 0 \end{array}\right\} \Longrightarrow (0,0) \;\; \& \;\; (4,0)$

	$(0,0)$	$(4,0)$
$r = f_{xx}$	-12	12
$s = f_{xy}$	0	0
$t = f_{yy}$	-16	-16
$rt - s^2$	192	-192

$(0,0)$ is a point of local maximum & $(4,0)$ is a saddle point.

KEY TAKEAWAY

- Necessary Conditions $f_x = 0$ & $f_y = 0$

- Sufficient Conditions

  - If $rt - s^2 > 0$ & $r < 0$ maximum
  - If $rt - s^2 > 0$ & $r > 0$ minimum
  - If $rt - s^2 < 0$ saddle point
  - If $rt - s^2 = 0$ needs further investigation

Thank You