General Method for P.I. : $\boxed{\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx}$

Concepts Covered

Differential Equations

- ❑ Particular Integral
- ❑ Special Forms of Right Hand Side Function (X)

- **Special Forms of X (e^{ax}, $\cos(ax)$, $\sin(ax)$):**

  - If a is a constant then $f(D)\, e^{ax} = f(a)\, e^{ax}$

    Note that $D\, e^{ax} = ae^{ax}$

    Similarly, $D^2 e^{ax} = a^2 e^{ax}$

    $\vdots$

    In general, $D^n e^{ax} = a^n e^{ax}$

    $\Longrightarrow f(D)\, e^{ax} = f(a) e^{ax}$

- X is of the form e^{ax}: $\quad f(D)y = e^{ax}$

We know $\quad f(D)\, e^{ax} = f(a)\, e^{ax}$

Operating $\dfrac{1}{f(D)}$ both the sides of the above equation

$$e^{ax} = \frac{1}{f(D)} f(a)\, e^{ax} = f(a)\, \frac{1}{f(D)}\, e^{ax}$$

$$\Rightarrow \boxed{\frac{1}{f(D)}\, e^{ax} = \frac{1}{f(a)} e^{ax}}, \qquad \text{provided } f(a) \neq 0$$

If $f(a) = 0$, then $(D - a)$ is a factor $f(D)$

Consider, $f(D) = (D - a)g(D)$

$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D - a)}\frac{1}{g(D)}e^{ax} = \frac{1}{(D - a)}\frac{1}{g(a)}e^{ax},$$ provided $g(a) \neq 0$

$$= \frac{1}{g(a)}\frac{1}{(D - a)}e^{ax}$$

$$\frac{1}{D - a}X = e^{ax}\int X\, e^{-ax}\, dx$$

$$= \frac{1}{g(a)}x\, e^{ax}$$

- **Short Methods for Finding P.I. :**

➢ ***X* is of the Form e^{ax}:**

I. $$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax} \qquad \text{where } f(a) \neq 0$$

II. If $f(a) = 0$, then $f(D)$ must have a factor of the type $(D - a)^r$. Then

$$\frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}$$

Example 1: General solution of the differential equation $(D^2 - 3D + 2)y = e^{3x}$

Complementary Function: $c_1 e^{x} + c_2 e^{2x}$

Particular Integral:

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} e^{3x} = \frac{1}{3^2 - 3 \times 3 + 2} e^{3x} = \frac{1}{2} e^{3x}$$

The General Solution: $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{3x}$

Example 2: Find a particular solution of $(D^3 - D^2 - D + 1)y = e^x$

$$\text{P.I.} = \frac{1}{D^3 - D^2 - D + 1} e^x \quad = \frac{1}{4} x^2 e^x$$

Example 3: Find a particular solution of $(D^2 + D + 5)y = 3$

$$\text{P.I.} = \frac{1}{D^2 + D + 5} 3 = \frac{3}{5}$$

- X is of the form $\cos \alpha x$ or $\sin \alpha x$:

  - If α, β are constants then $\phi(D^2)\sin(\alpha x+\beta)=\phi(-\alpha^2)\sin(\alpha x+\beta)$ and $\phi(D^2)\cos(\alpha x+\beta)=\phi(-\alpha^2)\cos(\alpha x+\beta)$

Easy to verify $D^2\sin(\alpha x+\beta)=-\alpha^2\sin(\alpha x+\beta)$ $\quad D^2\cos(\alpha x+\beta)=-\alpha^2\cos(\alpha x+\beta)$

We know $\phi(D^2)\sin(\alpha x+\beta)=\phi(-\alpha^2)\sin(\alpha x+\beta)$

Applying $[\phi(D^2)]^{-1}$: $\quad \sin(\alpha x+\beta)=\dfrac{1}{\phi(D^2)}\phi(-\alpha^2)\sin(\alpha x+\beta)$

IF $\phi(-\alpha^2)\neq 0$: $\quad \boxed{\dfrac{1}{\phi(D^2)}\sin(\alpha x+\beta)=\dfrac{1}{\phi(-\alpha^2)}\sin(\alpha x+\beta)}$

- X is of the form $\cos(\alpha x + \beta)$ or $\sin(\alpha x + \beta)$:

$$\frac{1}{f(D)}\cos(\alpha x + \beta): \quad \frac{1}{\phi(D^2)}\cos(\alpha x + \beta) \quad = \frac{1}{\phi(-\alpha^2)}\cos(\alpha x + \beta)$$

provided $\phi(-\alpha^2) \neq 0$

$$\frac{1}{f(D)}\sin(\alpha x + \beta): \quad \frac{1}{\phi(D^2)}\sin(\alpha x + \beta) \quad = \frac{1}{\phi(-\alpha^2)}\sin(\alpha x + \beta)$$

Example 1: Evaluate $\frac{1}{D^4 + D^2 + 1}\cos 2x = \frac{1}{(-2^2)^2+(-2^2) + 1}\cos 2x = \frac{1}{13}\cos 2x$

Example 2: Evaluate $\frac{1}{D^2 - 2D + 1}\cos 3x$

$$= \frac{1}{-9 - 2D + 1}\cos 3x = -\left(\frac{1}{2}\right)\frac{1}{D+4}\cos 3x \qquad \frac{\times (D-4)}{\times (D-4)}$$

$$= -\frac{1}{2}\frac{D-4}{D^2 - 16}\cos 3x = \frac{1}{50}(D-4)\cos 3x$$

$$= -\frac{1}{50}(4\cos 3x + 3\sin 3x)$$

If $\boldsymbol{\phi(-\alpha^2) = 0}$

Consider P.I. $= \dfrac{1}{D^2+\alpha^2}\sin\alpha x = \text{imag}\left\{\dfrac{1}{D^2+\alpha^2}\cos\alpha x + i\,\dfrac{1}{D^2+\alpha^2}\sin\alpha x\right\}$

$$= \text{imag}\left\{\frac{1}{D^2+\alpha^2}e^{i\alpha x}\right\}$$

Consider $\dfrac{1}{D^2+\alpha^2}e^{i\alpha x} = \dfrac{1}{(D-i\alpha)}\dfrac{1}{(D+i\alpha)}e^{i\alpha x} = \dfrac{1}{2i\alpha}\dfrac{1}{(D-i\alpha)}e^{i\alpha x} = \dfrac{x}{2i\alpha}e^{i\alpha x}$

$$\frac{1}{D^2+\alpha^2}\sin\alpha x = \text{imag}\left\{\frac{x}{2\alpha}\sin\alpha x - i\,\frac{x}{2\alpha}\cos\alpha x\right\} = -\frac{x}{2\alpha}\cos\alpha x$$

Rules:

$$\boxed{\frac{1}{D^2+\alpha^2}\sin\alpha x = -\frac{x}{2\alpha}\cos\alpha x}$$

$$\boxed{\frac{1}{D^2+\alpha^2}\cos\alpha x = \frac{x}{2\alpha}\sin\alpha x}$$

Example : Find the general solution of $(D^2+4)y = \sin^2 x$

C.F. $= c_1\cos 2x + c_2\sin 2x$

P.I. $= \dfrac{1}{D^2+4}\sin^2 x = \dfrac{1}{D^2+4}\left(\dfrac{1-\cos 2x}{2}\right) = \dfrac{1}{8}[1 - x\sin 2x]$

General Solution:

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{8}[1 - x\sin 2x]$$

- X is x^m or a polynomial of degree m:

Take out the lowest degree term from $f(D)$, so as to reduce it in the form

$$[1 \pm F(D)]^\alpha$$

Take it to numerator and expand it.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

Example : Evaluate $\dfrac{1}{D^3 - D^2 - 6D}(x^2+1)$

$$= \frac{1}{-6D\left(1+\frac{D}{6}-\frac{D^2}{6}\right)}(x^2+1) \quad = -\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2-\cdots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}+\cdots\right](x^2+1) = -\frac{1}{6D}\left[(x^2+1)-\frac{2x}{6}+\frac{7}{36}2\right]$$

$$= -\frac{1}{6D}\left[x^2-\frac{x}{3}+\frac{25}{18}\right] = -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

➢ X is $e^{ax}V$, where V is any function of x:

$$\boxed{\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V}$$

Example : Evaluate $\dfrac{1}{D^2+3D+2}e^{2x}\sin x$

$$= e^{2x}\frac{1}{(D+2)^2+3(D+2)+2}\sin x = e^{2x}\frac{1}{D^2+7D+12}\sin x = e^{2x}\frac{1}{7D+11}\sin x$$

$$= e^{2x}\frac{7D-11}{49D^2-121}\sin x = \frac{e^{2x}}{170}(11\sin x - 7\cos x)$$

- X is xV, where V is any function of x:

$$\boxed{\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V}$$

Example : Evaluate $\dfrac{1}{D^2 - 2D + 1}x\sin x$

$$= x\frac{1}{D^2 - 2D + 1}\sin x - \frac{2D - 2}{(D^2 - 2D + 1)^2}\sin x$$

$$= x\frac{1}{-2D}\sin x - \frac{2D - 2}{4D^2}\sin x$$

$$= \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$$

Conclusion

$$\frac{1}{\phi(D^2)}\cos ax = \frac{1}{\phi(-a^2)}\cos ax$$

$$\frac{1}{D^2+a^2}\sin ax \quad = -\frac{x}{2a}\cos ax \qquad \frac{1}{D^2+a^2}\cos ax \quad = \frac{x}{2a}\sin ax$$

- X is x^m or a polynomial of degree m: $[1 \pm F(D)]^{\alpha}$

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$$

$$\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V$$

Thank You