Complex Analysis

Definitions

Variable and function

A symbol, such as z, which can stand for any one of a set of complex numbers, is called a complex variable.

If to each value which a complex variable z can assume, there corresponds one or more values of a complex variable ω, we say that ω is a function of z and write $\omega = f(z)$. The variable z is sometimes called an independent variable, while ω is called a dependent variable. The value of a f^n. at $z = a$ is often written as $f(a)$.

Single valued and multiple valued function

If only one value of ω corresponds to each value of z, we say that ω is a single valued function of z or $f(z)$ is single valued. If more than one value of ω corresponds to each value of z, we say that ω is a multiple-valued function of z.

Ex-1 If $\omega = z^2$, then to each value of z there is only one value of ω. Hence $\omega = f(z) = z^2$ is a single-valued function of z.

Ex-2 If $\omega = z^{1/2}$, then to each value of z, there are two values of ω. Hence $\omega = f(z) = z^{1/2}$ is a multiple-valued (in this case two valued) function of z.

Whenever we speak of function, unless otherwise stated, we shall assume a single-valued function.

Limit

Let $f(z)$ be defined and single-valued in a nbd. of $z = z_0$ with the possible exception of $z = z_0$ itself (i.e. in a deleted nbd δ of z_0). We say that the number l is the limit of $f(z)$ as z approaches z_0 and write $\lim_{z \to z_0} f(z) = l$ if for any +ve no. ε (however small), we can find some positive no δ (usually depending on ε) such that $|f(z) - l| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

In such cases, we also say that $f(z)$ approaches l as z approaches z_0 and write $f(z) \to l$ as $z \to z_0$. The limit must be independent of the manner in which z approaches z_0. Geometrically, if z_0 is a point in the complex plane, then $\lim_{z \to z_0} f(z) = l$ if the difference in absolute value between $f(z)$ and l can be made as small as we wish by choosing points z sufficiently close to z_0 (excluding z_0).

Ex Let $f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$.

Then as z gets closer to i, $f(z)$ gets closer to $i^2 = -1$. We thus say that $\lim_{z \to i} f(z) = -1$. Here $\lim_{z \to i} f(z) \neq f(i)$

The limit would be in fact -1 even if $f(z)$ were not defined at $z = i$.

When the limit of a function exists, it is unique.

Theorem on limits

If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then

1. $\lim_{z \to z_0} \{f(z) + g(z)\} = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) = A + B$

2. $\lim_{z \to z_0} \{f(z) - g(z)\} = \lim_{z \to z_0} f(z) - \lim_{z \to z_0} g(z) = A - B$

3. $\lim_{z \to z_0} \{f(z) g(z)\} = \{\lim_{z \to z_0} f(z)\}\{\lim_{z \to z_0} g(z)\} = AB$

4. $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} = \dfrac{A}{B}$ if $B \neq 0$

Continuity

Let $f(z)$ be defined and single valued in a nbd. of $z = z_0$ as well as at $z = z_0$ (i.e. in a δ nbd. of z_0). The f^n. $f(z)$ is said to be continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$. This implies three conditions which must be met in order that $f(z)$ be continuous at $z = z_0$.

1. $\lim_{z \to z_0} f(z) = l$ must exist
2. $f(z_0)$ must exist i.e. $f(z)$ is defined at z_0
3. $f(z_0) = l$

Ex-1 If $f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$ then $\lim_{z \to i} f(z) = -1$. But $f(i) = 0$

Hence $\lim_{z \to i} f(z) \neq f(i)$ and the f^n. is not continuous at $z = i$

Ex-2 If $f(z) = z^2 \ \forall z$, then $\lim_{z \to i} f(z) = f(i) = -1$ and $f(z)$ is continuous at $z = i$

Ex-3 Is the f^n. $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ continuous at $z = i$?

Ans: $f(i)$ does not exist i.e. $f(z)$ is not defined at $z = i$

Thus $f(z)$ is not continuous at $z = i$.

By redefining $f(z)$ so that $f(i) = \lim_{z \to i} f(z) = 4 + 4i$ it becomes continuous at $z = i$.

Here $z = i$ is a removable discontinuity.

Ex-4 $\lim_{z \to 1+i} (z^2 - 5z + 10) = 5 - 3i$

Note: If $f(z) = u(x,y) + iv(x,y)$ is a continuous f^n. of z, then $u(x,y)$ and $v(x,y)$ are separately continuous f^n. of x and y; conversely if $u(x,y)$ and $v(x,y)$ are continuous f^ns. of x, y, then $f(z)$ is a continuous f^n. of z.

Derivatives

If $f(z)$ is single valued in some region R of the z plane, the derivatives of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such case, we say that $f(z)$ is differentiable at z.

Differentiability implies continuity, but the converse may not be true.

Analytic functions

A one valued function $f(z)$ which is defined and differentiable at each point of a domain D is said to be analytic in that domain and is referred to as an analytic function in D. A f^n. $f(z)$ is said to be analytic at a point z_0 if there exists a neighbourhood $|z-z_0|<\delta$ at all points of which $f'(z)$ exists.

The terms regular and holomorphic are sometimes used as synonymous for analytic. A f^n. $f(z)$ may be differentiable in a domain except possibly for a finite number of points. These points are called singular points or singularities of $f(z)$ in that domain.

If $f(z)$ is analytic in a domain D, then it can be proved that the derivative $f'(z)$ is itself analytic in D. It will also thus follow that a function which is differentiable once in any domain, is also infinitely differentiable in that domain.

Rules for differentiation

If $f(z)$, $g(z)$ and $h(z)$ are analytic f^ns. of z, the following differentiation rules (identical with those of elementary calculus) are valid.

1. $\frac{d}{dz}\{f(z)+g(z)\} = \frac{d}{dz}f(z) + \frac{d}{dz}g(z) = f'(z)+g'(z)$

2. $\frac{d}{dz}\{f(z)-g(z)\} = \frac{d}{dz}f(z) - \frac{d}{dz}g(z) = f'(z)-g'(z)$

3. $\frac{d}{dz}\{c\,f(z)\} = c\frac{d}{dz}f(z) = cf'(z)$ where c is a constant

4. $\frac{d}{dz}\{f(z)g(z)\} = f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)$
$= f(z)g'(z) + g(z)f'(z)$

5. $$\frac{d}{dz}\left\{\frac{f(z)}{g(z)}\right\} = \frac{g(z)\frac{d}{dz}f(z) - f(z)\frac{d}{dz}g(z)}{[g(z)]^2}$$

$$= \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2} \quad \text{if } g(z) \neq 0$$

6. Chain rule for differentiation of composite functions

If $w = f(\xi)$ where $\xi = g(z)$, then

$$\frac{dw}{dz} = \frac{dw}{d\xi}\cdot\frac{d\xi}{dz} = f'(\xi)\frac{d\xi}{dz} = f'\{g(z)\}\,g'(z)$$

Similarly, if $w = f(\xi)$ where $\xi = g(\eta)$ and $\eta = h(z)$, then

$$\frac{dw}{dz} = \frac{dw}{d\xi}\cdot\frac{d\xi}{d\eta}\cdot\frac{d\eta}{dz}$$

7. If $z = f(t)$ and $w = g(t)$ where t is a parameter, then

$$\frac{dw}{dz} = \frac{dw/dt}{dz/dt} = \frac{g'(t)}{f'(t)}$$

Necessary and sufficient conditions for $f(z)$ to be analytic

Cauchy-Riemann equations

A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region R is that, in R, u and v satisfy the relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These two relations are called Cauchy-Riemann equations (or conditions). This is not a sufficient condition. What is further required, four partial derivatives u_x, u_y, v_x, v_y are continuous in R (sufficient condition for $f(z)$ to be analytic in R).

The f^ns. $u(x,y)$ and $v(x,y)$ are sometimes called conjugate functions.

Ex Prove that the function $f(z) = u + iv$ where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \qquad z \neq 0$$

$$= 0 \qquad z = 0$$

is continuous and C-R equations are satisfied at the origin. But $f'(z)$ does not exist there.

Sol^n: Here $u = \dfrac{x^3 - y^3}{x^2+y^2}$ $\quad v = \dfrac{x^3+y^3}{x^2+y^2}$ (where $z \neq 0$)

Here we see that both u and v are finite for all values of $z \neq 0$, so u and v are rational and finite for all values of $z \neq 0$. Hence $f(z)$ is continuous where $z \neq 0$.

At the origin $u = 0$, $v = 0$ since $f(0) = 0$. Hence u and v are both continuous at the origin. So $f(z)$ is continuous at the origin. So $f(z)$ is continuous everywhere.

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \left(\frac{-y}{y}\right) = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{y}{y} = 1$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy - Riemann equations are satisfied at $z = 0$.

Again $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2} \cdot \frac{1}{x + iy} \right]$$

Let $z \to 0$ along $y = x$, then we have

$$f'(0) = \lim_{x \to 0} \frac{x^3 - x^3 + i(x^3 + x^3)}{x^2 + x^2} \cdot \frac{1}{x + ix}$$

$$= \lim_{x \to 0} \frac{2i}{2(1+i)} = \frac{1}{2}(1 - i)$$

Further, let $z \to 0$ along $y = 0$, then we have

$$f'(0) = \lim_{x \to 0} \frac{x^3(1+i)}{x^3} = 1 + i$$

Hence $f'(0)$ is not unique. Thus $f'(z)$ does not exist at the origin.

Further examples

(i) $f(z) = \bar{z}$ is not analytic at any point

(ii) $f(z) = \frac{1}{z}$, $z \neq 0$ is analytic at all pts. except $z = 0$

(iii) $$f(z) = \begin{cases} \dfrac{x^2 y^5 (x + iy)}{x^4 + y^{10}} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is not analytic at $(0,0)$ although C-R equations are satisfied there.

Harmonic functions

Any function of x, y which possesses continuous partial derivatives of first and second orders and satisfies Laplace equation, is called a harmonic function.

Theorem

If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic functions.

Proof Let $f(z) = u + iv$ be an analytic function. Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow \text{C-R eqn} \rightarrow (1)$$

Also, as u and v are the real and imaginary parts of an analytic function, therefore derivatives of u and v, of all orders, exist and are continuous functions of x and y.

So
$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \quad \text{——} \quad (2)$$

Differentiating eqn. (1), we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Adding these, we have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ by (2)

Similarly
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Such functions u and v are called conjugate harmonic functions or simply conjugate functions. v is called harmonic conjugate of u.

Determination of the conjugate function

Let $f(z) = u + iv$ be an analytic function. Being given $u(x,y)$ (say), to determine the other $v(x,y)$ or vice versa.

Since v is a f^n. of x, y, therefore

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{---(1) by C-R eqns.}$$

The RHS of Eq. (1) is of the form $M\,dx + N\,dy$ where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$ so that

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$$

and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$

Since u is a harmonic f^n., therefore it satisfies Laplace eqn. i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ or, $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}$

which makes $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence Eq. (1) satisfies the condition of exact diff. eqn. So Eq. (1) can be integrated and v can be determined.

Ex Show that the f^n. $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and find its harmonic conjugate.

Sol^n: $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$; $\frac{\partial^2 u}{\partial x^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$; $\frac{\partial^2 u}{\partial y^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\therefore u$ is a harmonic f^n.

Let v be the conjugate harmonic function.

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ by C-R eqn.

$= -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \frac{x\,dy - y\,dx}{x^2+y^2}$

$\therefore v = \tan^{-1}\frac{y}{x} + c$

Ex Prove that $u = y^3 - 3x^2y$ is a harmonic function. Determine its harmonic conjugate and find the corresponding analytic fn. $f(z)$ in terms of z.

Soln: $u = y^3 - 3x^2y$

$\therefore \frac{\partial u}{\partial x} = -6xy$ $\qquad \frac{\partial^2 u}{\partial x^2} = -6y$

$\frac{\partial u}{\partial y} = 3y^2 - 3x^2$ $\qquad \frac{\partial^2 u}{\partial y^2} = 6y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$

$\therefore$ u satisfies Laplace equation. Hence u is a harmonic fn.

Let v be the harmonic conjugate to u.

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \text{[by C-R eqn.]}$$

$$= -(3y^2 - 3x^2)\, dx - 6xy\, dy$$

$$= -(3y^2\, dx + 6xy\, dy) + 3x^2\, dx$$

Integrating, $v = -3xy^2 + x^3 + c$ which is harmonic conjugate to u.

$$f(z) = u + iv$$

$$= y^3 - 3x^2y + i(-3xy^2 + x^3 + c)$$

$$= i(x+iy)^3 + ic$$

$$= i z^3 + ic$$

Ex (a) Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.
(b) Find v such that $f(z) = u + iv$ is analytic
(c) Find $f(z)$.

Solⁿ: (a) $\frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$

$\frac{\partial^2 u}{\partial x^2} = -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$ — (1)

$\frac{\partial u}{\partial y} = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$

$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y$ — (2)

Adding (1) and (2), $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(b) $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$ — (3)

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y$ — (4)

Integrate (3) w.r.t. y keeping x constant

$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x)$

$= ye^{-x}\sin y + xe^{-x}\cos y + F(x)$ — (5)

Substituting (5) into (4),

$-ye^{-x}\sin y - xe^{-x}\cos y + e^{-x}\cos y + F'(x) = -ye^{-x}\sin y$
$- xe^{-x}\cos y - ye^{-x}\sin y$

or, $F'(x) = 0 \Rightarrow F(x) = c$

$\therefore v = e^{-x}(y\sin y + x\cos y) + c$

(c) $f(z) = u + iv = e^{-x}(x\sin y - y\cos y) + ie^{-x}(y\sin y + x\cos y)$
[not writing c]

$= e^{-x}\left\{x\left(\frac{e^{iy} - e^{-iy}}{2i}\right) - y\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\}$

$= i(x+iy)e^{-(x+iy)} = ize^{-z}$

Complex Integration

The concept of indefinite integral as the process of inverse differentiation in case of a function of a real variable also extends to a function of a complex variable if the complex function $f(z)$ is analytic. Thus in case of complex variable, if $f(z)$ is an analytic function of z and if $\int f(z)\,dz = F(z)$, then $F'(z) = f(z)$.

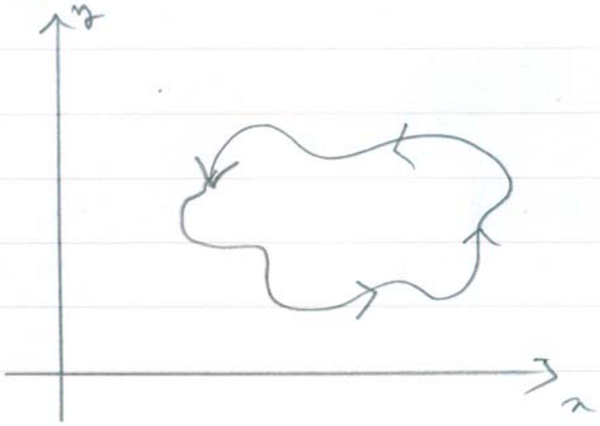
However the concept of definite integral of a function of a real variable does not out-right extend to the domain of complex variables. For example in the case of real variables for the definite integral $\int_a^b f(x)\,dx$ the path of integration is along the x axis from $x=a$ to $x=b$. But in the case of a complex function $f(z)$ the path of the definite integral $\int_a^b f(z)\,dz$ can be along any curve from $z=a$ to $z=b$ so that the value depends upon the path (curve) of integration.

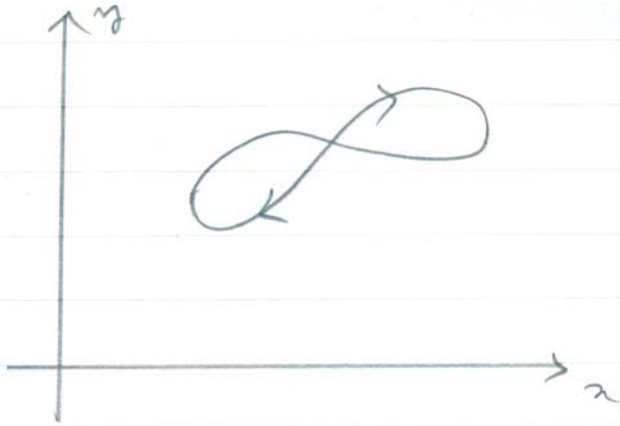
Some definitions

Continuous curve (arc) and simple closed curve

If $\varphi(t)$ and $\psi(t)$ are real functions of the real variable t assumed continuous in $t_1 \le t \le t_2$, the parametric eqn. $z = x + iy = \varphi(t) + i\psi(t) = z(t)$, $t_1 \le t \le t_2$ define a continuous curve or arc in the z plane joining the points $a = z(t_1)$ and $b = z(t_2)$. If $t_1 \neq t_2$ while $z(t_1) = z(t_2)$ i.e. $a = b$, the end points coincide and the curve is said to be closed. A closed curve which does not intersect itself anywhere is called a simple closed curve.

For example, the curve of the 1st figure is a simple closed curve while the second one is not.





Smooth curve and contour

If $\varphi(t)$ and $\psi(t)$ and thus $z(t)$ have continuous derivatives in $t_1 \le t \le t_2$, the curve is often called a smooth curve or arc. A curve which is composed of a finite number of smooth curves is called a piecewise or sectionally smooth curve or sometimes a contour. For example, the boundary of a square is a piecewise smooth curve or contour (closed contour).

Simply and multiply connected regions

A region R is called simply connected if any simple closed curve which lies in R can be shrunk to a point without leaving R. A region R which is not simply connected is called mutiply connected.

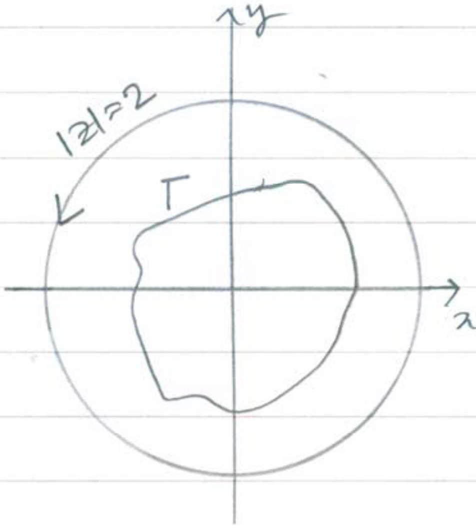


Fig (a)

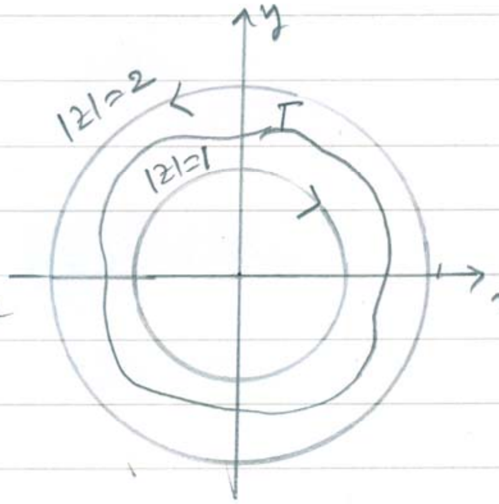


Fig (b)

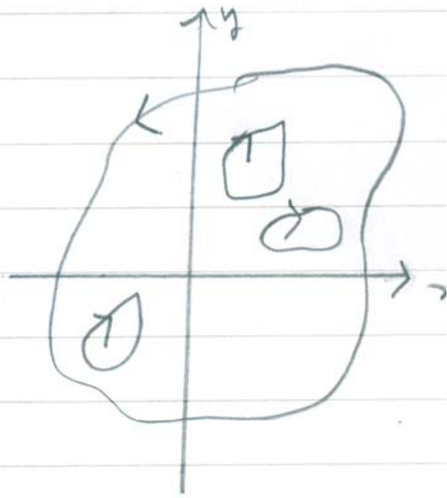


Fig (c)

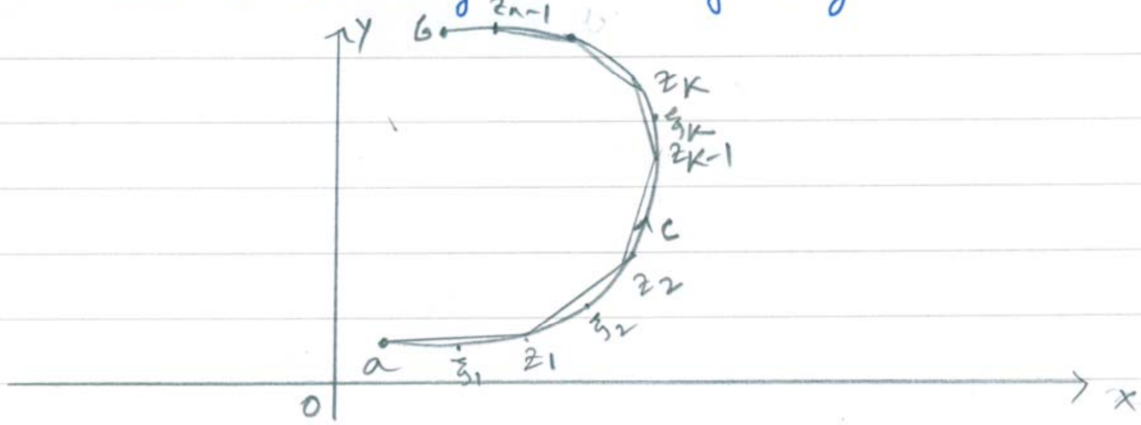
For example, suppose R is the region defined by $|z|<2$ shown in Fig (a). If T is any simple closed curve lying in R [i.e. whose points are in R], we see that it can be shrunk to a point which lies in R and thus does not leave R, so that R is simply connected. On the other hand, if R is the region defined by $1<|z|<2$, shown in Fig (b), then there is a simple closed curve T lying in R, which cannot be shrunk to a point without leaving R, so that R is multiply connected.

Intuitively, a simply connected region is one which does not have any 'holes' in it; while a multiply connected region is one which does. Fig (c) shows another multiply connected region. Thus the multiply connected region of Fig (b) and (c) have respectively one and three holes in them.

Complex line integral (Riemann definition of integration)

Let a function $f(z)$ of a complex variable z be continuou in a domain D and a, b be two points in that domain Then integral of $f(z)$ from a to b is defined as follow

Let C be any curve joining a to b and



lying entirely in the domain D so that $f(z)$ is continuous on C. Let C be subdivided into n parts by means of points $z_1, z_2,$ z_{n-1} chosen arbitrarily and call $a = z_0$, $b = z_n$. On each arc joining z_{k-1} to z_k (where k goes from 1 to n), a point ξ_k is chosen. The following sum is formed

$$S_n = f(\xi_1)(z_1 - a) + f(\xi_2)(z_2 - z_1) + \cdots + f(\xi_n)(b - z_{n-1})$$

On writing $z_k - z_{k-1} = \Delta z_k$, this becomes

$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta z_k$$

Let the number of subdivisions n increase in such a way that the largest of the chord lengths $|\Delta z_k|$ approaches zero. Then the sum S_n approaches a limit which does not depend on the mode of subdivision and we denote this limit by

$\int_a^b f(z)\,dz$ or $\int_C f(z)\,dz$ or $\oint_C f(z)\,dz$ [C closed curve]

called the complex line integral or briefly line integral of $f(z)$ along curve C or the definite integral of $f(z)$ from a to b along curve C. In such case $f(z)$ is said to be integrable along C. Note that if $f(z)$ is analytic at all points of a domain D and if C is a curve lying in D, then $f(z)$ is certainly integrable along C.

Connection between real and complex line integrals

If $f(z) = u(x,y) + iv(x,y) = u+iv$, the complex line integral $\int_C f(z)\,dz$ can be expressed in terms of real line integrals as

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

Properties of integrals

If $f(z)$ and $g(z)$ are integrable along C, then

1. $\int_C \{f(z)+g(z)\}\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

2. $\int_C A\,f(z)\,dz = A\int_C f(z)\,dz$ where A is some constant

3. $\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$

4. $\int_a^b f(z)\,dz = \int_a^m f(z)\,dz + \int_m^b f(z)\,dz$ where points a, b, m are on C

5. Suppose $f(z)$ is integrable along a curve C having finite length L and suppose $\exists$ a positive number M such that $|f(z)| \le M$ i.e. M is an upper bound of $|f(z)|$ on C, then

$$\left|\int_C f(z)\,dz\right| \le ML$$

This is known as ML inequality.