LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

Dr. Jitendra Kumar

Professor
Department of Mathematics
Indian Institute of Technology Kharagpur
West Bengal 721302, India

Webpage: http://www.facweb.iitkgp.ac.in/~jkumar/

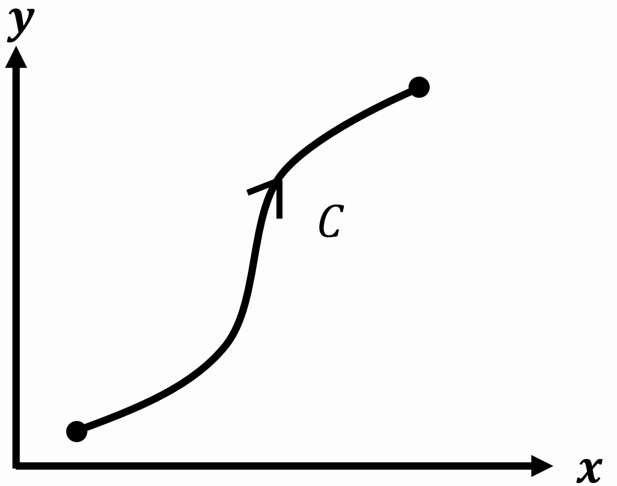
REVIEW of Complex Line Integrals

COMPLEX LINE INTEGRALS

Let $f(z)$ be a continuous function of a complex variable z in some domain $D \in \mathbb{C}$.

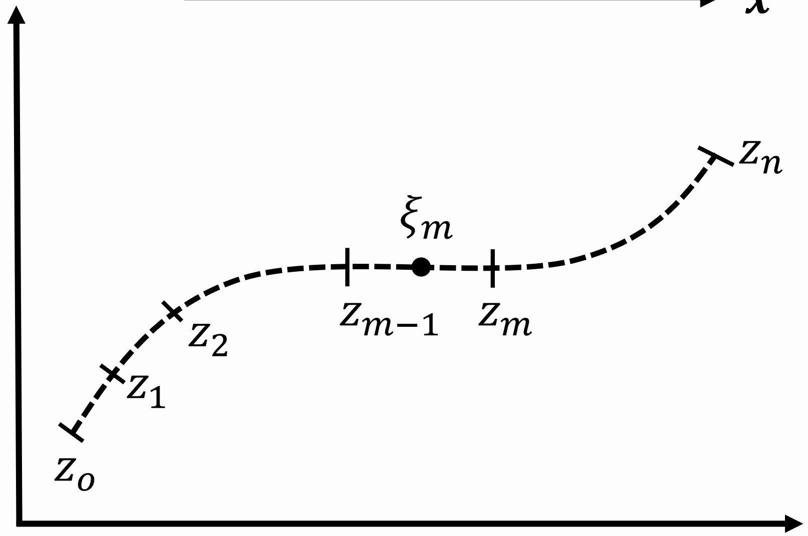
The integral of $f(z)$ along a path C in D is denoted as

$$\int_C f(z)\, dz \qquad C \text{ is called the path of integration}$$



Definition:

$$\lim_{n\to\infty} \sum_{m=1}^{n} f(\xi_m)(z_m - z_{m-1}) = \int_c f(z)\, dz$$



If C is a closed path, then the line integral is denoted by $\oint_C f(z) dz$

EVALUATION OF LINE INTEGRALS

$$\int_C f(z)\,dz = \int_C [u(x,y) + iv(x,y)][dx + idy] = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

Here the path C is piecewise smooth curve and the function $f(z)$ is continuous on C.

(A) Without Parameterize the Curve: Let the path C be given by $y = y(x);\;\; a \le x \le b$

$$\int_C f(z)\,dz = \int_a^b \{u(x, y(x)) - v(x, y(x))\, y'(x)\}\,dx + i\int_a^b \{v(x, y(x)) + u(x, y(x))\, y'(x)\}\,dx$$

EVALUATION OF LINE INTEGRALS

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Here the path C is piecewise smooth curve and the function $f(z)$ is continuous on C.

(B) Parameterize the Curve: (i) Let the path C be represented by $z = z(t)$ where $a \le t \le b$.

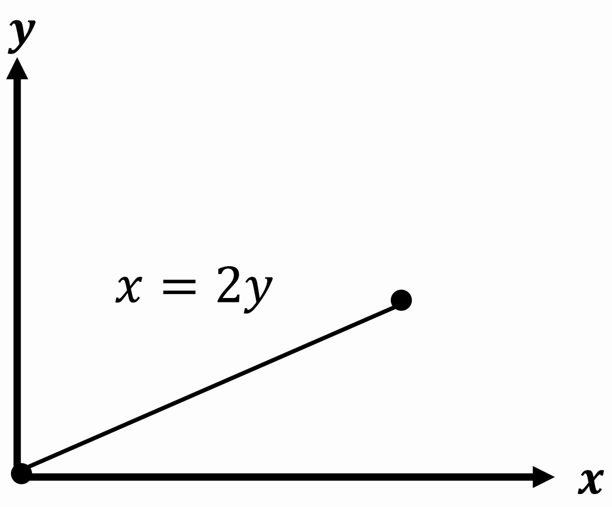
$$\int_C f(z)dz = \int_a^b f\big(z(t)\big)\,\dot{z}(t)\,dt$$

(ii) Let the path C be represented by $z = z(t) = x(t) + iy(t)$ where $a \le t \le b$.

$$\int_C f(z)\,dz = \int_a^b \{u\big(x(t),y(t)\big)x'(t) - v\big(x(t),y(t)\big)\,y'(t)\}\,dt$$

$$+i\int_a^b \{v\big(x(t),y(t)\big)x'(t) + u\big(x(t),y(t)\big)\,y'(t)\}\,dt$$

Example: Evaluate $\int_C z^2\, dz$ where C is the straight line joining $(0,0)$ to $(2,1)$



Approach - I

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy = 3y^2 + 4iy^2$$

$$dz = dx + idy = 2dy + idy = (2+i)dy$$

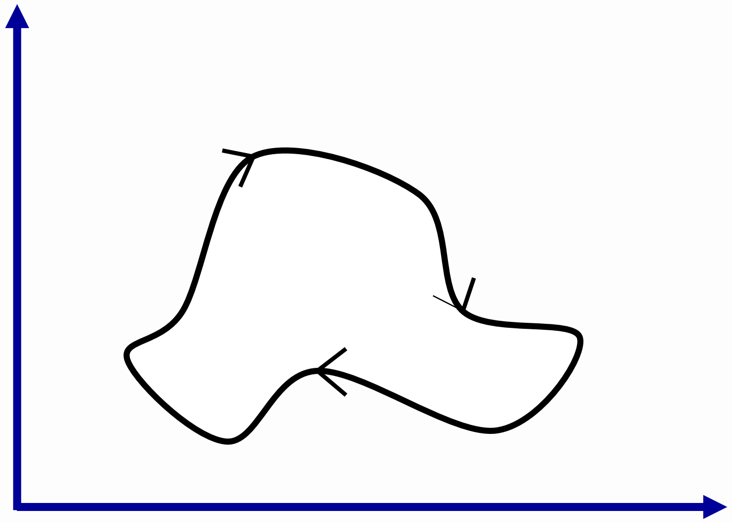
$$\int_C f(z)\, dz = \int_0^1 (3+4i)y^2(2+i)dy = \frac{2+11i}{3}$$

Approach - II Write C is parametric form $z(t) = 2t + it; 0 \le t \le 1$

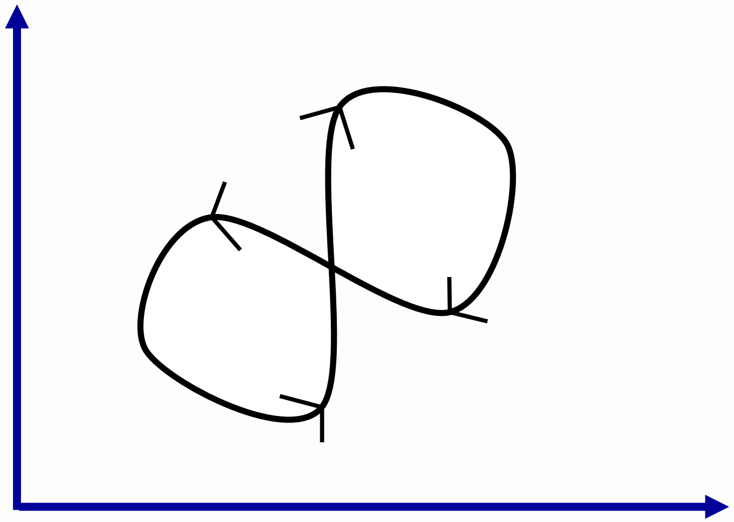
$$\int_C f(z)dz = \int_0^1 (2t+it)^2(2+i)dt = \int_0^1 (3+4i)t^2(2+i)dt = \frac{2+11i}{3}$$

SIMPLE CLOSED CURVE

A closed curve that does not intersect (or touch) itself anywhere is called a simple closed curve.



Simple Closed Curve



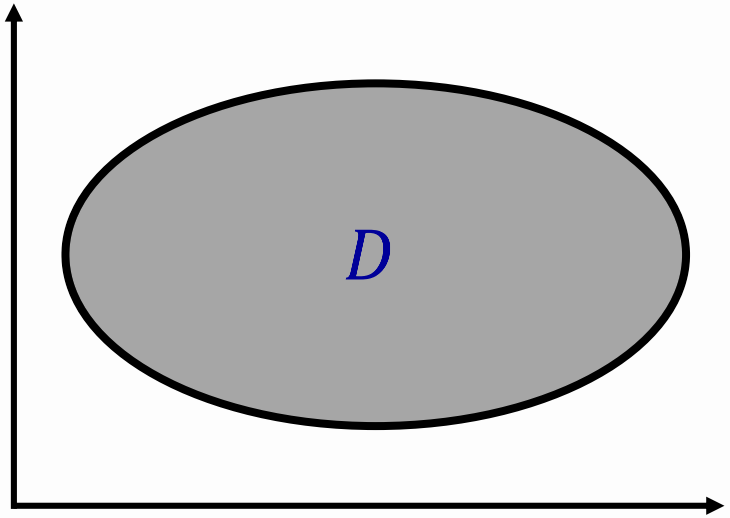
Not Simple Closed Curve

SIMPLY AND MULTIPLY CONNECTED DOMAINS

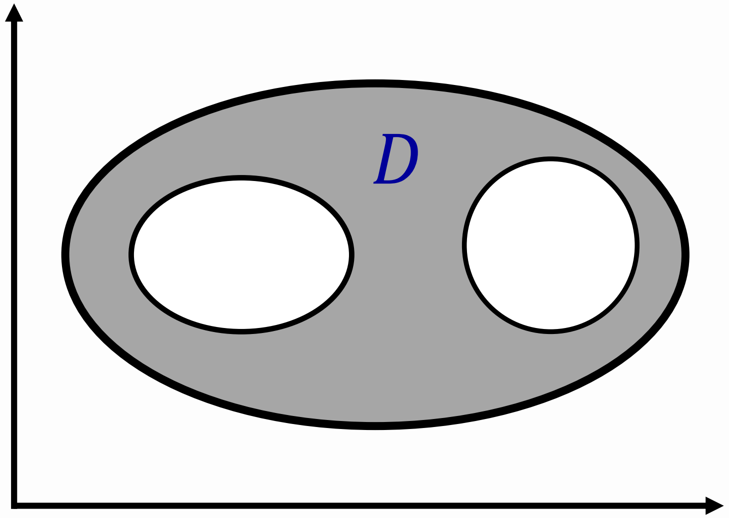
A domain D is called simply-connected if any simple closed curve which lies in D can be shrunk to a point without leaving D.

A domain D is called simply connected if every simple closed curve encloses only points of D.

A region which is not simply connected is called multiply-connected.



Simply Connected Domain



Multiply Connected Domain

Cauchy Integral Theorem

CAUCHY INTEGRAL THEOREM

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed C in D,

$$\oint_C f(z)\, dz = 0$$

Proof: Take an additional assumption that the derivative $f'(z)$ is continuous.

$$\oint_C f(z)dz = \oint_C (u + iv)\,(dx + idy)$$

$$= \oint_C (u\, dx - v\, dy) + i \oint_C (v\, dx + u\, dy)$$

We know from the C-R equations,

$$f'(z) = u_x + iv_x = v_x - iu_y$$

$$\oint_C f(z)dz = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

Since $f'(z)$ is assumed to be continuous then it implies continuity of u_x, v_x, v_y, u_y

Hence, by Green's theorem $\oint_C u\,dx - v\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$ R is the region bounded by C

Using C–R equations $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we get $\oint_C (udx - vdy) = 0$

Similarly, we can show that $\oint_C (vdx + udy) = 0$

$$\boxed{\oint_C f(z)dz = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) = 0}$$

REMARK-1

- The condition that $f(z)$ is analytic in D is sufficient for $\oint_C f(z)dz = 0$ rather than necessary.

  One can easily show that $\oint_C \frac{1}{z^2}\, dz = 0$ where C is the unit circle centred at origin

  The result does not follow due to the Cauchy theorem as $\frac{1}{z^2}$ is not analytic in $|z| < 1$

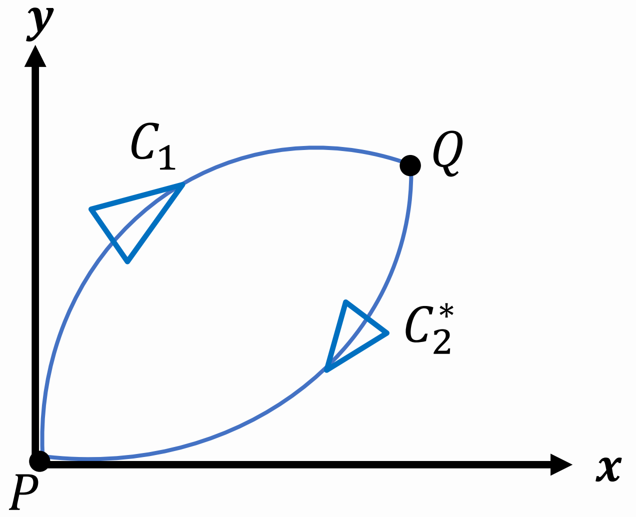
- Simply connectedness of the domain is essential one.

  One can show that $\oint_C \frac{1}{z}\, dz = 2\pi i$ where C is the unit circle lying in the annulus $\frac{1}{2} \le |z| \le \frac{3}{2}$

  Note that $\frac{1}{z}$ is analytic in the domain but the domain is not simply connected so Cauchy theorem is not applicable.

Independence of Path (a consequence of Cauchy Integral Theorem)

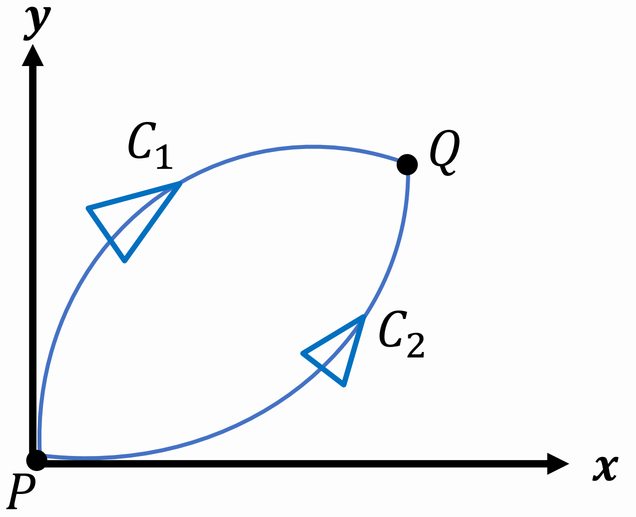
If $f(z)$ is analytic in a simply connected domain D, then $\int_C f(z)\,dz$ is independent of the path C in D.



$$\int_{C_1 \cup C_2^*} f(z)\ dz = 0 \quad \text{(using Cauchy Integral Theorem)}$$

$$\Rightarrow \int_{C_1 \cup C_2^*} f(z)\ dz = \int_{C_1} f(z)\ dz + \int_{C_2^*} f(z)\ dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\ dz - \int_{C_2} f(z)\ dz = 0 \ \Rightarrow \int_{C_1} f(z)\ dz = \int_{C_2} f(z)\ dz$$



The value of the integral between two points is independent of the path if $f(z)$ is analytic throughout a simply connected domain containing the path.

Existence of Primitive (A consequence of Independence of Path)

Let $f(z)$ be analytic in a simply connected domain D. Then f has a primitive in D, that is, there exists $F(z)$ such that $F'(z) = f(z)$.

Sketch of the Proof:

Consider a fixed point z_0 in D. Then, due to independence of path, we can define a function $F(z)$ as

$$F(z) = \int_{z_0}^{z} f(z)\ dz + A = \int_{z_0}^{z} f(\xi)\ d\xi + A$$

A depends upon z_0

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(\xi) - f(z)]d\xi < \epsilon$$

Continuity of f and independence of path

$$\Rightarrow F'(z) = f(z)$$

Another consequence of Independence of Path (Fundamental Theorem of Complex Line Integral)

Let $f(z)$ be analytic in a simply connected domain D. Then for all paths C in D joining two points z_0 and z_1 in D, we have:

$$\int_C f(z)\, dz = F(z_1) - F(z_0)$$

Here F is the primitive of f, i.e., $F' = f$.

Sketch of the Proof: Since the integral $\int_{z_0}^{z} f(z)\, dz$ is indepedent of path, we can define

$$F(z) = \int_{z_0}^{z} f(\xi)\, d\xi + A$$

A depends upon the fixed constant $z_0 \in D$

Substituting $z = z_0$, we get $F(z_0) = A$

$$\Rightarrow F(z) - F(z_0) = \int_{z_0}^{z} f(\xi)\, d\xi$$

$$\Rightarrow F(z_1) - F(z_0) = \int_{z_0}^{z_1} f(\xi)\, d\xi$$

Example: Evaluate $\int_C z^2\, dz$ where C is the straight line joining $(0,0)$ to $(2,1)$

$$\int_C z^2\, dz = \int_0^{2+i} z^2\, dz = \left[\frac{z^3}{3}\right]_0^{2+i}$$

$$= \frac{1}{3}(8 + 12i - 6 - i)$$

$$= \frac{1}{3}(2 + 11i)$$

REMARK -2

Cauchy's theorem can also be applied to multiply connected domain.

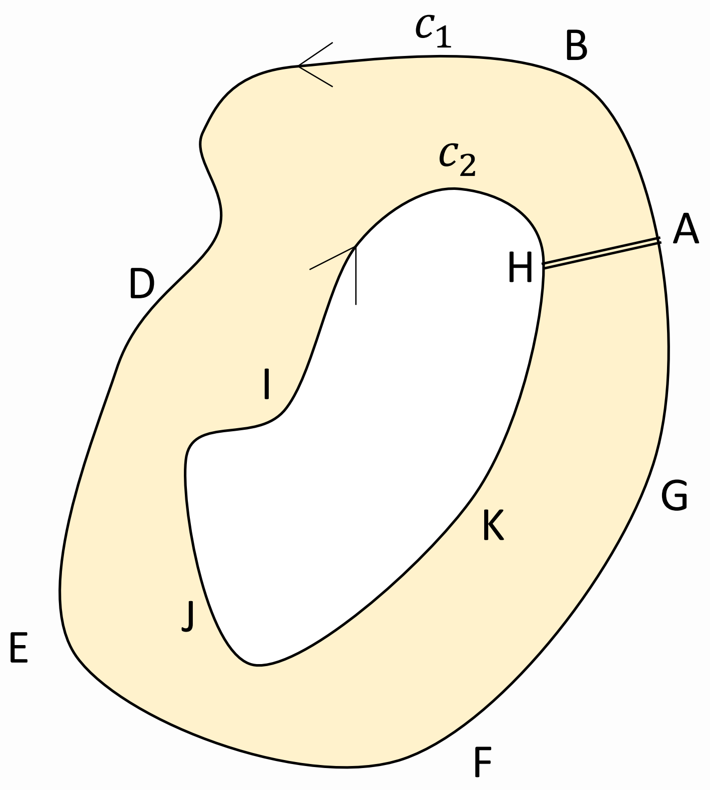
Construct cross-cut AH.

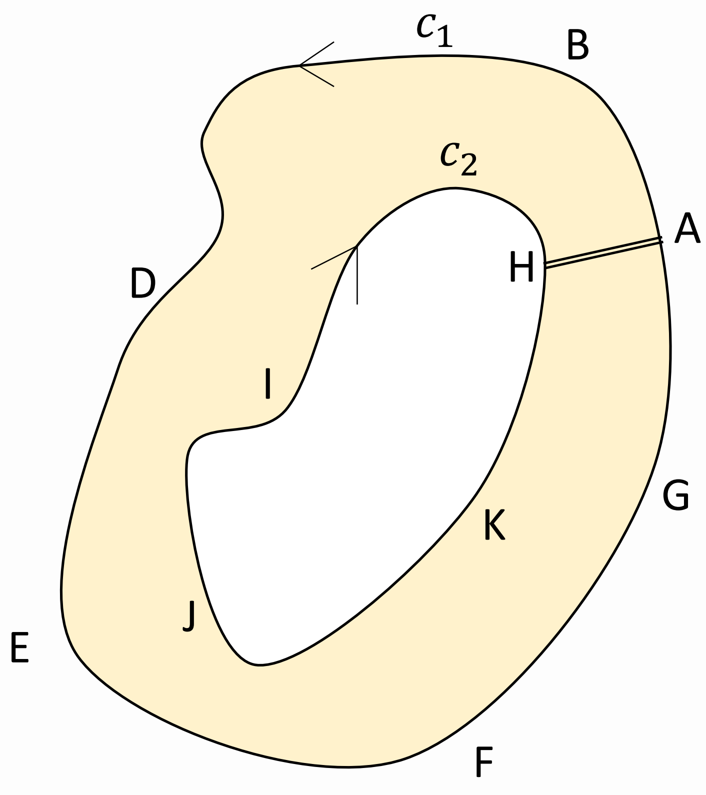
Then, the region bounded by ABDEFGAHKJIHA is simply connected.

The Cauchy's theorem implies:

$$\oint_{ABD\cdots IHA} f(z)dz = 0$$

$$\Longrightarrow \oint_{ABDEFGA} f(z)dz + \oint_{AH} f(z)dz + \oint_{HKJIH} f(z)dz + \oint_{HA} f(z)dz = 0$$





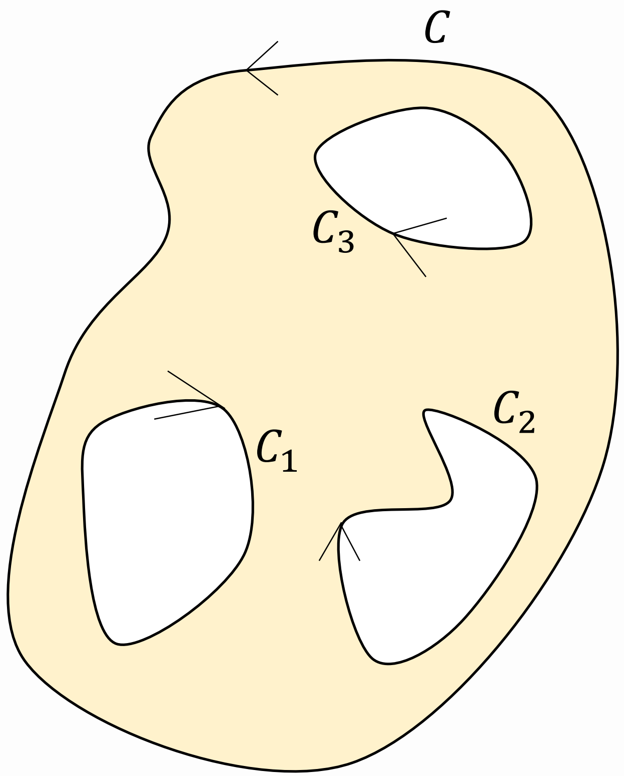
$$\Rightarrow \oint_{ABDEFGA} f(z)dz + \oint_{AH} f(z)dz + \oint_{HKJIH} f(z)dz + \oint_{HA} f(z)dz = 0$$

Using $\oint_{AH} f(z)dz = -\oint_{HA} f(z)dz$

$$\oint_{ABDEFGA} f(z)dz + \oint_{HKJIH} f(z)dz = 0$$

Anti-clockwise Clockwise

$$\Rightarrow \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = 0$$



More General Result:

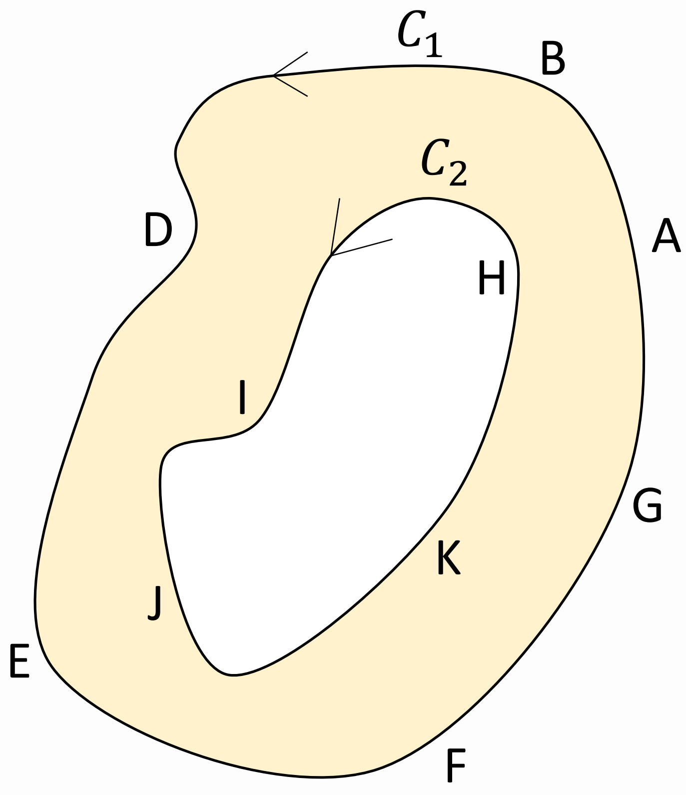
$$\boxed{\oint_{C} f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz = 0}$$

REMARK - 3 As a consequence of above remark, we have following result:

Let $f(z)$ be analytic in a domain D bounded by two simple closed curve C_1 and C_2 and also on C_1 and C_2. Then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

When C_1 and C_2 are both traversed counter clockwise.



From previous remark, we have

$$\oint_{ABDEFGA} f(z)dz + \oint_{HKJIH} f(z)dz = 0$$

$$\Longrightarrow \oint_{ABDEFGA} f(z)dz - \oint_{HIJKH} f(z)dz = 0$$

$$\Longrightarrow \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

SUMMARY (Evaluation of Line Integral)

(A) Without Parameterize the Curve: Let the path C be given by $y = y(x)$; $a \le x \le b$

$$\int_C f(z)\,dz = \int_a^b \{u(x, y(x)) - v(x, y(x))\, y'(x)\}\, dx + i \int_a^b \{v(x, y(x)) + u(x, y(x))\, y'(x)\}\, dx$$

(B) Parameterize the Curve: Let the path C be represented by $z = z(t)$ where $a \le t \le b$.

$$\int_C f(z)dz = \int_a^b f(z(t))\, \dot{z}(t)\, dt$$

(C) If $f(z)$ is Analytic in a simply connect domain D: (i) For every simple closed C in D, $\oint_C f(z)\,dz = 0$

(ii) There exists an analytic function $F(z)$ with $F'(z) = f(z)$ in D then along any path joining z_1 and z_2 in D

$$\boxed{F(z_1) - F(z_0) = \int_{z_0}^{z_1} f(z)\,dz}$$