DIFFERENTIAL EQUATIONS:

An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Examples:

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 = e^t \quad \text{— (i)}$$

$$y(y^2+1)\,dx + x(y^2-1)\,dy \quad \text{— (ii)}$$

$$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2 \quad \text{— (iii)}$$

Mathematical classifications:

ORDINARY DIFF. EQUATIONS: → Involves derivatives w.r.t single independent variables

PARTIAL DIFF. EQUATIONS: → Involves partial derivatives (more than one independent variables

ORDER OF A DIFFERENTIAL EQUATION: → The order of the highest order derivative involved

DEGREE OF A DIFFERENTIAL EQUATION: → The degree of the highest order derivative involved.

i) ODE, order - 4 degree 1

ii) ODE, order - 1 degree 1

iii) PDE, order - 3 degree 2.

LINEAR AND NONLINEAR DIFFERENTIAL EQUATION:

A differential equation is called linear if

i) every dependent variable and every derivative occur in the first degree only, and

ii) no products of dependent variables and/or derivatives occur.

If not linear than it is called nonlinear.

Note: Every linear equation is of first degree, but every first degree equation may not be linear.

$$\frac{d^2y}{dx^2} + y \cdot \frac{dy}{dx} + y = 0 \qquad \text{1st degree but non linear}$$

SOLUTION OF A DIFFERENTIAL EQUATION:

Any relation between the dependent and independent variables which satisfies the differential equation is called a solution or integral of the differential equation.

Ex. $y = \frac{A}{x} + B$ is a solution of

$$y'' + \left(\frac{2}{x}\right) y' = 0$$

Check! $y' = -\frac{A}{x^2} \Rightarrow y'' = \frac{2A}{x^3}$

Subst. in the equation: $0 = 0$.

Note: It should be noted that a solution of a differential equation does not involve the derivatives of the dep. variable w.r.t the indep. variable or variables.

Family of curves: An n-parameter family of curves is a set of relations of the form

$$\{(x,y): f(x,y,c_1,c_2,\cdots c_n)=0\}$$

Example:

i) set of concentric circles

$x^2+y^2=c$ → one parameter family if c takes non-negative real values

ii) Set of circles:

$(x-c_1)^2+(y-c_2)^2=c_3$ → three parameters family if c_1, c_2 takes all real values and c_3 takes all non-negative real values.

Note: Solution of a differential equation is a family of curves.

Formation of differential equations from a given n-parameters family of curves:

From a given family of curves containg n arbitrary constants, we can obtain an nth order differential equation whose solution is the given family:

- Differentiate the given equation n times to get n additional equations containing those arbitrary constants.
- Eliminate n arbitrary constants from the $(n+1)$ equations.
- Obtain a differential equation of the nth order.

Ex: Obtain the differential equation satisfied by

$$xy = ae^{x} + be^{-x} + x^2$$

where a & b are arbitrary constants.

Sol: Given family of curves:

$$xy = ae^{x} + be^{-x} + x^2 \quad \text{--- (1)}$$

Differentiating w.r.t x, we get

$$xy' + y = ae^{x} - be^{-x} + 2x$$

Differentiating again:

$$xy'' + 2y' = ae^{x} + be^{-x} + 2$$

Using (1) we get

$$xy'' + 2y' = xy - x^2 + 2$$

which is the desired differential equation.

Remark: Observe that the number of arbitrary constants in a solution of a differential equation depends upon the order of the differential equation. It is evident from the above example that a general solution (defined later) of an nth order differential equation will contain n arbitrary constants.

General, particular, and Singular Solution

Let $F(x, y, y', y'', \ldots y^{(n)}) = 0$ be an nth order ordinary differential equation.

i) General solution : solution containing n-independent arbitrary constants.

ii) Particular solution : Solution by giving particular values to one or more of the n-independent constants.

iii) Singular solution : cannot be obtained by any choice of independent arbitrary constant.

Example: a) $y = (x+c)^2$ is the general solution of

$$\left(\frac{dy}{dx}\right)^2 - 4y = 0 \quad — (1)$$

b) $y = x^2$ is a particular solution of (1) $(c=0)$

c) $y = 0$ is a singular solution.

Ex: Consider $yy' - x(y')^2 = 1$

General solution: $y = cx + \frac{1}{c}$

Particular solution: $y = x + 1$ $\quad (c=1)$

Singular solution: $y^2 = 4x$.

Explicit & Implicit solutions:

Explicit : $y = y(x)$

Implicit : $F(x, y) = 0$

Example: $y'' + k^2 y = 0$

solution: $y = C_1 \cos kx + C_2 \sin kx$

↳ explicit solution

Example: $x + 3yy' = 0$

Solution: $x^2 + 3y^2 = C$

↳ Implicit solution

Equation of first order and first degree :

We shall consider two standard forms of differential equation:

i) $\dfrac{dy}{dx} = f(x, y)$

ii) $M(x, y)\, dx + N(x, y)\, dy = 0.$

Solution methods:

• **Separation of variables:** If a differential equation can be written in the form

$$f_1(y)\,\frac{dy}{dx} = f_2(x) \qquad \text{— } ①$$

then we say variables are separable in the given differential equation.

Solution of ①:

$$\int f_1(y)\,dy = \int f_2(x)\,dx + C \quad (\text{how?})$$

<u>Example:</u>

$$\frac{dy}{dx} = e^{x-2y} + x^2 e^{-2y}$$

$$\Rightarrow \quad e^{2y}\frac{dy}{dx} = e^{x} + x^2$$

Integrating both side:

$$\frac{e^{2y}}{2} = e^{x} + \frac{x^3}{3} + C_1$$

or $$\boxed{e^{2y} = 2e^{x} + \frac{2}{3}x^3 + C}$$

Equation reducible to separation of variables:

Consider $\frac{dy}{dx} = f(ax+by+c)$ —(1)

OR $\frac{dy}{dx} = f(ax+by)$

Subst. $ax+by+c = v$ OR $ax+by = v$

$\Rightarrow a + b\cdot\frac{dy}{dx} = \frac{dv}{dx}$

Then (1) reduces to

$$\frac{1}{b}\left[\frac{dv}{dx} - a\right] = f(v)$$

$$\Rightarrow \frac{dv}{dx} = bf(v) + a$$

$$\Rightarrow \int \frac{dv}{bf(v)+a} = \int dx$$

Example: $\frac{dy}{dx} = \sec(x+y)$

Sol: Let $x+y = v \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Then the diff. eq. becomes:

$$\frac{dv}{dx} = \sec v + 1 \quad \text{(separable form)}$$

$$= \frac{1+\cos v}{\cos v} = \frac{2\cos^2\frac{v}{2}}{2\cos^2\frac{v}{2} - 1}$$

$$\Rightarrow \int \left[1 - \frac{1}{2}\sec^2\left(\frac{v}{2}\right)\right] dv = \int dx$$

$$\Rightarrow v - \tan\left(\frac{v}{2}\right) = x + c$$

Subst. $v = x+y$: $\boxed{y - \tan\left(\frac{x+y}{2}\right) = c}$

Homogeneous equations: A differential equation of first order and first degree is said to be homog. if it is of the form or can be put in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{— (1)}$$

<u>Solution:</u> $\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$(1) \Rightarrow v + x\frac{dv}{dx} = f(v)$$

$$\Rightarrow x\frac{dv}{dx} = f(v) - v \quad \text{(separable form)}$$

$$\Rightarrow \int \frac{dv}{f(v)-v} = \int \frac{dx}{x} + C.$$

Example: $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$

<u>Sol:</u> $$\frac{dy}{dx} = -\frac{x^3+3xy^2}{y^3+3x^2y} = -\frac{1+3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3+3\left(\frac{y}{x}\right)}$$

Subst. $\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{1+3v^2}{v^3+3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^4-6v^2-1}{v^3+3v}$$

$$\Rightarrow -\int \frac{4(v^3+3v)}{v^4+6v^2+1}\,dv = \int 4\frac{dx}{x}$$

$$\Rightarrow -\ln(v^4+6v^2+1) = 4\ln x + \ln C \quad (x>0)$$

$$\Rightarrow x^4 C(v^4+6v^2+1) = 1$$

$$\Rightarrow Cx^4\left(\frac{y^4}{x^4}+6\frac{y^2}{x^2}+1\right) = 1$$

$$\Rightarrow \boxed{C(y^4+6y^2x^2+x^4) = 1}$$

Equation reducible to homogeneous form:

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad , \text{ where } \frac{a}{a'} \neq \frac{b}{b'} \ (*) \qquad \text{—} ①$$

Take $\left.\begin{array}{l} x = X+h \\ y = Y+k \end{array}\right\} \quad \text{—} ②$

where X & Y are new variables and h & k are constants to be so chosen that the resulting equation in X and Y becomes homogeneous

② $\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dY}{dX}$

$$\left[\begin{array}{l} y(x) = Y(X)+k \\ \text{or } Y(X) = y(x)+k \\ \dfrac{dY}{dX} = \dfrac{d}{dX}\left[y(x)+k\right] = \dfrac{dy}{dx}\cdot\underbrace{\dfrac{dx}{dX}}_{=1} \end{array}\right]$$

① $\Rightarrow$

$$\frac{dY}{dX} = \frac{aX+bY+ah+bk+c}{a'X+b'Y+a'h+b'k+c'} \qquad \text{—} ③$$

In order to make ③ homog. choose h and k such that

$$\left.\begin{array}{l} ah+bk+c = 0 \quad \text{and} \\ a'h+b'k+c' = 0 \end{array}\right\} \text{(alway possible because } ab'-a'b \neq 0\text{)}$$

Getting h & k we have $X = x-h$ & $Y = y-k$

$$\Rightarrow \frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y} = \frac{a+b\left(\frac{Y}{X}\right)}{a'+b'\left(\frac{Y}{X}\right)} \qquad \text{homogeneous in } X \text{ \& } Y$$

(*) In case $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{1}{\lambda} \Rightarrow a' = \lambda a$ & $b' = \lambda b$

Subst. $\dfrac{dy}{dx} = \dfrac{ax+by+c}{\lambda(ax+by)+c'} = f(ax+by)$ (Can be solved by subst $ax+by = u$).

Ex: $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ — ①

Sol. Take $x = X+h$ & $y = Y+K$ so that $\frac{dy}{dx} = \frac{dY}{dX}$.

$$\frac{dY}{dX} = \frac{X+2Y+(h+2K-3)}{2X+Y+(2h+K-3)} \quad — ②$$

Choose h, K so that $\left.\begin{array}{l} h+2K-3=0 \\ 2h+K-3=0 \end{array}\right\} \Rightarrow h=1 \;\&\; K=1.$

So from ① $X = x-1$ $Y = y-1$

② $\Rightarrow \frac{dY}{dX} = \frac{X+2Y}{2X+Y}$

Take $Y = vX \Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$

$$X\frac{dv}{dX} = \left(\frac{1+2v}{2+v}\right) - v = \frac{1-v^2}{2+v}$$

$$\Rightarrow \frac{dX}{X} = \left[\frac{1}{2}\left(\frac{1}{1+v}\right) + \frac{3}{2}\left(\frac{1}{1-v}\right)\right] dv$$

Integrating:

$$\Rightarrow \ln X + \ln C = \frac{1}{2}\left[\ln(1+v) - 3\ln(1-v)\right]$$

$$\Rightarrow 2\ln(XC) = \ln\left(\frac{1+v}{(1-v)^3}\right) \Rightarrow X^2C^2 = \frac{1+v}{(1-v)^3}$$

sub: $v = \frac{y-1}{x-1}$

$$\boxed{C^2(x-y)^3 = x+y-2}$$

Exact Differential Equations

If M and N are functions of x and y, the equation $M\,dx + N\,dy = 0$ is called exact when there exists a function $f(x,y)$ such that

$$d(f(x,y)) = M\,dx + N\,dy$$

or

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M\,dx + N\,dy$$

Theorem: The necessary and sufficient condition for the differential equation

$$M\,dx + N\,dy = 0$$

to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{— (1)}$$

<u>Proof</u>: The condition is necessary '$\Rightarrow$'

Let the equation be exact, then

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M\,dx + N\,dy$$

Equating coefficients of dx & dy, we get:

$$M = \frac{\partial f}{\partial x} \qquad N = \frac{\partial f}{\partial y}$$

Assuming f to be continuous upto 2nd order partial derivatives, we obtain

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the equation is exact then M & N satisfy (1).

Now we show that the condition ① is sufficient.

We assume the $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and show that the equation $M\,dx + N\,dy$ is exact.

That means we find a function $f(x,y)$ such that

$$df = M\,dx + N\,dy.$$

Let $g(x,y) = \int M\,dx$ be the partial integral of M such that

$$\frac{\partial g}{\partial x} = M.$$

We first prove that $\left(N - \frac{\partial g}{\partial y}\right)$ is a function of y only.

Consider $\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x \partial y}$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y \partial x} \qquad \left(\text{assuming } \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Now consider:

$$f = g(x,y) + \int \left(N - \frac{\partial g}{\partial y}\right) dy. \text{ and then}$$

$$df = dg + d\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right) = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy + \overbrace{\frac{\partial}{\partial x}\Big(\underbrace{\int \left(N - \frac{\partial g}{\partial y}\right) dy}_{\text{function of } y}\Big)}^{=0} dx$$

$$+ \frac{\partial}{\partial y}\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right) dy$$

$$= \frac{\partial g}{\partial x}\,dx + \cancel{\frac{\partial g}{\partial y}\,dy} + N\,dy - \cancel{\frac{\partial g}{\partial y}\,dy}$$

$$= M\,dx + N\,dy.$$

$\Rightarrow$ The given differential equation is exact.

Remark: The solution of an exact differential equation $Mdx + Ndy = 0$ can be written as

$$f = c$$

i.e.,

$$\int_{(y\ \text{const.})} M\,dx + \int \underbrace{\left(N - \frac{\partial g}{\partial y}\right)}_{\text{function of } y \text{ alone}} dy = c$$

OR

$$\int_{(y\ \text{const})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c.$$

Example: Solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

Sol: $M = x^2 - 4xy - 2y^2 \qquad N = y^2 - 4xy - 2x^2$

$$\frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x} \Rightarrow \text{the equation is exact.}$$

Hence, there exists a function $f(x,y)$ such that

$$d(f(x,y)) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M\,dx + N\,dy$$

$$\Rightarrow \frac{\partial f}{\partial x} = x^2 - 4xy - 2y^2 \quad \& \quad \frac{\partial f}{\partial y} = y^2 - 4xy - 2x^2$$

Int. of 1st w.r.t. x

$$\Rightarrow f = \frac{x^3}{3} - 2x^2y - 2xy^2 + c_1(y)$$

On differentiation w.r.t y:

from above.

$$\frac{\partial f}{\partial y} = -2x^2 - 4xy + c_1'(y) = y^2 - 4xy - 2x^2$$

$$\Rightarrow c_1'(y) = y^2 \Rightarrow c_1(y) = \frac{y^3}{3} + c_2$$

Hence: $f = c_3 \Rightarrow \dfrac{x^3}{3} - 2x^2y - 2xy^2 + \dfrac{y^3}{3} + c_2 = c_3$

$$\Rightarrow \boxed{x^3 - 6xy(x+y) + y^3 = c}$$

Example: Show that the differential equation

$$(3xy+y^2)\,dx + (x^2+xy)\,dy = 0$$

is not exact and hence it cannot be solved by the method discussed above.

Sol: Chek: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$3x+2y \neq 2x+y$$

So the given equation is not exact.

However, if we proceed with the method given above, we get

$$\frac{\partial f}{\partial x} = 3xy + y^2 \qquad \frac{\partial f}{\partial y} = x^2+xy$$

$$\Rightarrow \quad f = \frac{3}{2}x^2y + y^2x + f_1(y)$$

$$\frac{\partial f}{\partial y} = \frac{3}{2}x^2 + 2y\,x + f_1'(y) = x^2 + xy$$

$$\Rightarrow \quad f_1'(y) = \underbrace{-\frac{x^2}{2} - xy}_{\text{depends on } x \,\&\, y}$$

(Not possible to solve)

Thus, there is no $f(x,y)$ exists and hence it can not be solved in this way.

Exact Differential Equations: Integrating Factors

If an equation of the form $M\,dx + N\,dy = 0$ is not exact, it is sometimes possible to choose a function of x & y such that after multiplying all terms of the equation, it becomes exact. Such a multiplier is called an integrating factor. That is, if $I(x,y)$ is an integrating factor then the differential equation

$$I(x,y)\,M(x,y)\,dx + I(x,y)\,N(x,y)\,dy = 0$$

becomes exact.

Note: Although an equation of the form $M\,dx + N\,dy = 0$ always has integrating factor(s), there is not general rule of finding them. We now discuss some methods of finding integrating factors.

Rule I: By inspection

This method is based on recognition of some standard exact differentials that occur frequently in practice.

i) $d(xy) = y\,dx + x\,dy$

ii) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$ or $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

iii) $d\left(\ln \frac{y}{x}\right) = \frac{x\,dy - y\,dx}{xy}$ or $d\left(\ln \frac{x}{y}\right) = \frac{y\,dx - x\,dy}{xy}$

iv) $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$ or $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2+x^2}$

v) $d(\ln xy) = \frac{y\,dx + x\,dy}{xy}$

Ex. Solve the differential equation

$$y(y^2+1)\,dx + x(y^2-1)\,dy = 0$$

(Check! it is not exact D.E.)

Sol: Rewriting:

$$y^2(y\,dx + x\,dy) + y\,dx - x\,dy = 0$$

Divinding it by y^2: (I.F.)

$$y\,dx + x\,dy + \frac{y\,dx - x\,dy}{y^2} = 0$$

$$\Rightarrow d(xy) + d\left(\frac{x}{y}\right) = 0$$

$$\Rightarrow xy + \frac{x}{y} = c \quad \Rightarrow \boxed{xy^2 + x = cy}$$

Ex. Solve $(y^2e^x + 2xy)\,dx - x^2\,dy = 0$

(Check! it is not exact D.E.)

We know that

$$d\left(\frac{x^2}{y}\right) = \frac{2x}{y}\,dx - \frac{x^2}{y^2}\,dy$$

Diving the given equation by y^2, we get:

$$\left(e^x + \frac{2x}{y}\right)dx - \frac{x^2}{y^2}\,dy = 0$$

$$\Rightarrow d(e^x) + d\left(\frac{x^2}{y}\right) = 0$$

$$\Rightarrow \boxed{e^x + \frac{x^2}{y} = c}$$

Rule II: $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$.

In this case $I(x,y) = \dfrac{1}{Mx+Ny}$ is an integrating factor.

Example: $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ — ①

Sol.

$$M = x^2y - 2xy^2 \qquad N = -(x^3 - 3x^2y)$$

$$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2$$

$$= x^2y^2 \neq 0.$$

$$I.F. = \frac{1}{x^2y^2}$$

Multiplying ① by I.F.

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad \text{— ②}$$

Now equation ② is an exact differential equation (check!)

If u is the exact differential of ② then:

$$\frac{\partial u}{\partial x} = \frac{1}{y} - \frac{2}{x} \Rightarrow u = \frac{x}{y} - 2\ln x + \varphi(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{x}{y^2} + \varphi'(y) = -\frac{x}{y^2} + \frac{3}{y} \Rightarrow \varphi'(y) = \frac{3}{y}$$

$$\Rightarrow \varphi(y) = 3\ln y + C_1$$

$$\Rightarrow \boxed{\frac{x}{y} - 2\ln x + 3\ln y + C_1 = C_2}$$

or $\boxed{\frac{x}{y} - 2\ln x + 3\ln y = C}$

Rule III: $M\,dx + N\,dy = 0$ is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$ then $\frac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$

Ex: Solve

$$(xy \sin xy + \cos xy)\, y\, dx + (xy \sin xy - \cos xy)\, x\, dy = 0$$

$$M = (xy \sin xy + \cos xy)\, y \qquad N = (xy \sin xy - \cos xy)\, x$$

$$Mx - Ny = (xy \sin xy + \cos xy)\, xy - (xy \sin xy - \cos xy)\, xy$$

$$= 2xy \cos xy \neq 0$$

$$IF = \frac{1}{2xy \cos xy}$$

Multiplying the given equation by I.F.

$$\frac{1}{2}\left(y \tan xy + \frac{1}{x}\right) dx + \frac{1}{2}\left(x \tan xy - \frac{1}{y}\right) dy = 0$$

it must be exact (check!)

Solution: $\boxed{\frac{x}{y} \sec xy = c}$

Rule IV: An integrating factor for an equation of the form

$$x^a y^b (my\,dx + nx\,dy) + x^r y^s (py\,dx + qx\,dy) = 0$$

is $x^h y^k$ where h & k can be obtained by applying the condition that after multiplication by $x^h y^k$ the equation must become exact. Here a, b, m, n, r, s, p, q are constants.

Example: Solve $(3x+2y^2)y\,dx + 2x(2x+3y^2)\,dy = 0$

This equation can be rewritten as

$$x(3y\,dx + 4x\,dy) + y^2(2y\,dx + 6x\,dy) = 0$$

multiplying the integrating factor $x^h y^k$, we get

$$(3x^{h+1}y^{k+1} + 2x^h y^{k+3})\,dx + (4x^{h+2}y^k + 6x^{h+1}y^{k+2})\,dy = 0$$

If it is exact we must have

$$3(k+1)x^{h+1}y^k + 2(k+3)x^h y^{k+2} = 4(h+2)x^{h+1}y^k + 6(h+1)x^h y^{k+2}$$

This is satisfied if

$$3(k+1) = 4(h+2) \quad \&$$
$$2(k+3) = 6(h+1)$$

Solving these we get $h=1,\ k=3$.

Integrating factor is xy^3.

Solution: $\boxed{x^3y^4 + x^2y^6 = c}$

Rule V : Most general approach :

The idea is to multiply the given differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

by a function $I(x,y)$ and then try to choose $I(x,y)$ so that the resulting equation

$$I(x,y)\,M(x,y)\,dx + I(x,y)\,N(x,y)\,dy = 0 \quad \text{—— ①}$$

becomes exact.

The above equation is exact if and only if

$$\frac{\partial(IM)}{\partial y} = \frac{\partial(IN)}{\partial x} \quad \text{——} \quad (*)$$

If a function I satisfying $(*)$ can be found then the given equation ① will be exact. However solving $(*)$ is very difficult so we consider some special cases.

i) An integrating factor I that is either as function of x alone or
ii) a function of y alone.

In the case i), the equation $(*)$ reduces to

$$IM_y = IN_x + NI_x \Rightarrow I_x = \frac{IM_y - IN_x}{N}$$

If $\dfrac{M_y - N_x}{N}$ is a function of x only, say $f(x)$ then
$I(x) = e^{\int f(x)\,dx}$ is an integrating factor. (by solving $\frac{dI}{I} = f(x)\,dx$)

In the case ii) If $\frac{1}{M}(N_x - M_y)$ is a function of y alone, say $f(y)$
then $I(y) = e^{\int f(y)\,dy}$ is an I.F.

Example: Solve $(x^2+y^2+x)\,dx + xy\,dy = 0$ — (1)

$$M = x^2+y^2+x \qquad N = xy$$

$$\frac{\partial M}{\partial y} = 2y \quad \& \quad \frac{\partial N}{\partial x} = y$$

$$\therefore \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y-y) = \frac{1}{x}$$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = x.$$

Multiplying (1) by x:

$(x^3+xy^2+x^2)\,dx + x^2y\,dy = 0$ This must be an exact D.E.

Solution: $(3x^4 + 6x^2y^2 + 4x^3) = c$

Ex: Solve $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

$$M = 2xy^4e^y + 2xy^3 + y \qquad N = x^2y^4e^y - x^2y^2 - 3x.$$

$$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -8xy^3e^y - 8xy^2 - 4$$

$$= -4(2xy^3e^y + 2xy^2 + 1)$$

$$= -\frac{4}{y}\cdot(2xy^4e^y + 2xy^3 + y) = -\frac{4}{y}\cdot M$$

$$\Rightarrow \text{I.F.} = e^{\int -\frac{4}{y}dy} = y^{-4}$$

Solution: $\boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c}$

Exact Differential Equations (Summary)

Necessary and sufficient condition of $M(x,y)dx + N(x,y)dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Integrating Factors

Rule I: By Inspection

Example:

$$d(xy) = ydx + xdy, \quad d\left(\ln xy\right) = \frac{ydx + xdy}{xy} \quad etc.$$

Rule II: $Mdx + Ndy = 0$ **is homogeneous and** $Mx + Ny \neq 0$ **then** $I(x,y) = \dfrac{1}{(Mx + Ny)}$ **is an integrating factor**

Rule III: $Mdx + Ndy = 0$ **is of the form** $f_1(xy)ydx + f_2(xy)xdy = 0$ **and** $Mx - Ny \neq 0$ **then** $\dfrac{1}{(Mx - Ny)}$ **is an integrating factor**

Rule IV: $Mdx + Ndy = 0$ **is of the form** $x^a y^b(mydx + nxdy) + x^r y^s(pydx + qxdy) = 0$ **then** $I(x,y) = x^h y^k$ **may be taken as an integrating factor, where** h, k **are obtained so that the differential equation after multiplication by** $I(x,y)$ **becomes exact**

Rule V: Most general approach

If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is function of x alone say $f(x)$, then $I(x) = e^{\int f(x)dx}$ is an I.F.

If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is function of y alone say $f(y)$, then $I(y) = e^{\int f(y)dy}$ is an I.F.

Linear Differential Equation:

A first order differential equation is called linear if it can be written in the form:

$$\frac{dy}{dx} + P(x)\,y = Q(x) \qquad (\text{linear in } y)$$

Rewritten as

$$\underbrace{dy + Py\,dx}_{\text{Compare with } M dx + N dy \text{ to get}} = Q(x)\,dx \qquad \text{— (1)}$$

$$M = Py \qquad N = 1$$

Observe that $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{1}(P - 0) = P$ (function of x alone)

Hence $\text{I.F.} = e^{\int P dx}$

multiplying (1) by $e^{\int P dx}$

$$e^{\int P dx}\,dy + Py\,e^{\int P dx}\,dx = Q(x)\,e^{\int P dx}\cdot dx$$

$$\Rightarrow \quad d\left(e^{\int P dx}\cdot y\right) = Q\,e^{\int P dx}\,dx$$

Integrating:

$$e^{\int P dx}\cdot y = \int Q\,e^{\int P dx}\,dx + C$$

OR

$$\boxed{y \cdot \text{IF} = \int Q \cdot \text{I.F.}\,dx + C}$$

Note: Sometimes a differential equation cannot be put in the form $\frac{dy}{dx} + P(x)\,y = Q(x)$ which is linear in y, but in the form

$$\frac{dx}{dy} + P_1(y)\,x = Q_1(y)$$

which is linear in x, then

$$I.F. = e^{\int P_1 dy}$$

and the solution

$$x \cdot I.F. = \int Q_1 \, I.F. \, dy + c$$

Ex. Solve $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\,y = \frac{4x^2}{1+x^2} \quad (\text{linear in } y)$$

$$I.F. = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2$$

Solution: $y \cdot I.F = \int Q \cdot I.F\, dx + C \Rightarrow y\cdot(1+x^2) = \int 4x^2\,dx + c$

$$\Rightarrow \boxed{y(1+x^2) = \frac{4}{3}x^3 + c}$$

Ex. Solve $(x+2y^3)\frac{dy}{dx} = y \Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y^2$

$$I.F. = e^{\int -\frac{1}{y}dy} = e^{-\ln y} = \frac{1}{y}$$

$$\Rightarrow x\cdot\frac{1}{y} = \int 2y^2 \cdot \frac{1}{y}\,dy + c$$

$$= \int 2y\,dy + c$$

$$\Rightarrow \boxed{\frac{x}{y} = y^2 + c}$$

Equation reducible to linear form:

An equation of the form

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \qquad \text{— (1)}$$

Putting $f(y) = v \Rightarrow f'(y)\frac{dy}{dx} = \frac{dv}{dx}$

Equation (1) reduces to:

$$\frac{dv}{dx} + Pv = Q \qquad (\text{linear in } v)$$

A special case: Bernoulli's Equation

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \qquad \text{— (2)}$$

where P & Q are constants or function of x and n is a constant except 0 & 1 is called Bernoulli's differential equation.

Note that equation (2) can be written as

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$$

Subst: $\frac{1}{y^{n-1}} = v \Rightarrow (1-n)\,y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{1}{(1-n)} \cdot \frac{dv}{dx} + Pv = Q$$

$$\Rightarrow \frac{dv}{dx} + P(1-n)v = Q(1-n) \qquad (\text{linear in } v)$$

Ex. $(x^2-2x+2y^2)\,dx+2xy\,dy=0$

Sol: Rewriting:

$$2xy\frac{dy}{dx}+x^2-2x+2y^2=0$$

or $$2y\frac{dy}{dx}+\frac{2y^2}{x}=\frac{2x-x^2}{x}$$

Subst. $y^2=v \Rightarrow 2y\frac{dy}{dx}=\frac{dv}{dx}$

$$\Rightarrow \frac{dv}{dx}+\frac{2}{x}\cdot v=(2-x) \qquad \text{(linear in } v\text{)}$$

$$\text{I.F.}=e^{\int \frac{2}{x}\cdot dx}=x^2$$

$$\Rightarrow v\cdot x^2=\int(2-x)x^2\,dx+C$$

$$\Rightarrow \boxed{y^2x^2=\frac{2}{3}x^3-\frac{x^4}{4}+C}$$

Ex: $\frac{dy}{dx}-y\tan x=-y^2\sec x$

Sol: Dividing by y^2:

$$\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}\cdot\tan x=-\sec x$$

putting $\frac{1}{y}=v \Rightarrow -\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$

$$\Rightarrow -\frac{dv}{dx}-v\tan x=-\sec x \Rightarrow \frac{dv}{dx}+v\tan x=\sec x. \qquad \text{(linear in } v\text{)}$$

$$\text{I.F.}=e^{\int\tan x\,dx}=e^{\ln\sec x}=\sec x$$

Solution: $$v\cdot\sec x=\int\sec^2x\,dx+C$$

$$\Rightarrow v\cdot\sec x=\tan x+C$$

$$\Rightarrow \boxed{y^{-1}\sec x=\tan x+C}$$

Ex. Solve $\frac{dz}{dx} + \frac{z}{x}\ln z = \frac{z}{x}(\ln z)^2$, $x>0, z>0$

Sol: Dividing by z:

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\ln z = \frac{1}{x}(\ln z)^2$$

Subst. $\ln z = t \Rightarrow \frac{1}{z}\cdot\frac{dz}{dt} = \frac{dt}{dx}$

$$\Rightarrow \frac{dt}{dx} + \frac{1}{x}t = \frac{1}{x}t^2$$

$$\Rightarrow \frac{1}{t^2}\frac{dt}{dx} + \frac{1}{t}\cdot\frac{1}{x} = \frac{1}{x}$$ Bernoulli's equation

Subst. $\frac{1}{t} = v \Rightarrow -\frac{1}{t^2}\frac{dt}{dx} = \frac{dv}{dx}$

$$\Rightarrow -\frac{dv}{dx} + \frac{1}{x}\cdot v = \frac{1}{x}$$

$$\text{I.F.} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

Solution: $v\cdot\frac{1}{x} = -\int \frac{1}{x}\cdot\frac{1}{x}dx + C$

$$\frac{v}{x} = \frac{1}{x} + C$$

$$\Rightarrow v = 1 + Cx$$

$$\Rightarrow \boxed{(\ln z)^{-1} = 1 + Cx}$$

Linear Differential Equations of Higher order with constant Coefficients

The general form of linear equation with constant coeff.

$$\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n y = X \qquad \text{—①}$$

where $a_1, a_2, \ldots a_n$, are constants and X is a function of x.

General solution = Complementary function (C.F.) + Particular Integral (P.I.)

C.F. ↓ Contain n arbitrary constants

P.I. ↓ free from arbitrary constants

P.I.: If v is any particular solution, then

$$\frac{d^nv}{dx^n} + a_1 \frac{d^{n-1}v}{dx^{n-1}} + \cdots + a_n v = X$$

C.F.: It is the general solution of the homog. equation

$$\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots a_n y = 0 \qquad \text{—②}$$

Remarks:

1. Let y_1, y_2 be any two solutions of ②, then $c_1y_1 + c_2y_2$ is also a solution of ②, where c_1, c_2 are arbitrary constants:

$$\frac{d^n}{dt^n}(c_1y_1+c_2y_2) + a_1\frac{d^{n-1}}{dt^{n-1}}(c_1y_1+c_2y_2) + \cdots + a_n(c_1y_1+c_2y_2)$$
$$= c_1\left(\frac{d^n}{dt^n}y_1 + a_1\frac{d^{n-1}}{dt^{n-1}}y_1 + \cdots + a_ny_1\right) + c_2\left(\frac{d^n}{dt^n}y_2 + a_1\frac{d^{n-1}}{dt^{n-1}}y_2 + \cdots + a_ny_2\right)$$
$$= 0$$

Generalization: If $y_1, y_2 \cdots y_n$ be any n linearly independent solutions of (2) then $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is the general solution of (2), where $c_1, c_2 \cdots c_n$ are arbitrary constants.

* discussed later!

2. If u be the general solution of (2) and v be any particular solution of (1), then $(u+v)$ is the general solution of (1).

$(C.F. + P.I.)$

as it will contain n arbitrary constants.

$$\frac{d^n}{dt^n}(u+v) + a_1 \frac{d^{n-1}}{dt^{n-1}}(u+v) + \cdots + a_n(u+v)$$

$$= \left(\frac{d^n}{dt^n}u + a_1 \frac{d^{n-1}}{dt^{n-1}}u + \cdots + a_n u\right) + \left(\frac{d^n}{dt^n}v + a_1 \frac{d^{n-1}}{dt^{n-1}}v + \cdots + a_n v\right)$$

$$= 0 + X = X$$

OPERATOR: $\frac{d}{dx}, \frac{d^2}{dx^2} \cdots$

For the sake of convenience, the operators

$\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ are denoted by $D, D^2, \cdots, D^3, \cdots$

PRODUCT OF OPERATORS:

$$(D-\alpha)(D-\beta)y = (D-\beta)(D-\alpha)y$$, α, β being any constant.

L.H.S: $(D-\alpha)(D-\beta)y = (D-\alpha)\left(\frac{dy}{dx} - \beta y\right)$

$$= \frac{d}{dx}\left(\frac{dy}{dx} - \beta y\right) - \alpha\left(\frac{dy}{dx} - \beta y\right)$$

$$= \frac{d^2y}{dx^2} - \beta\frac{dy}{dx} - \alpha\frac{dy}{dx} + \alpha\beta y$$

$$= \frac{d^2y}{dx^2} - (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y$$

$$= \left[D^2 - (\alpha+\beta)D + \alpha\beta\right] y$$

Similarly, one can show that

$$(D-\beta)(D-\alpha)y = [D^2 - (\alpha+\beta)D + \alpha\beta]y$$

$$\therefore \quad (D-\alpha)(D-\beta) \equiv (D-\beta)(D-\alpha)$$

So the order of operational factors is immaterial.

Also note that

$$\underbrace{(D-\alpha)(D-\beta)}\, y = \underbrace{[D^2 - (\alpha+\beta)D + \alpha\beta]}\, y$$

same

Note: Treating D as a number, the ordinary laws of multiplication works.

In general:

$$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]\, y = X$$

$$\Rightarrow \left[(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)\right] y = X$$

Solution of $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$

Write the equation in operator form

$$(D^2 + a_1 D + a_2) y = 0$$

- Case of non-repeated roots:

Suppose α_1 and α_2 are two non-repeated roots

$$(D-\alpha_1)(D-\alpha_2) y = 0$$

A solution of the equation:

$$(D-\alpha_2) y = 0$$

$$\Rightarrow \frac{dy}{dx} = \alpha_2 y \quad \Rightarrow \quad y = e^{\alpha_2 x}$$

Similarily the other solution:

$$y = e^{\alpha_1 x}$$

Thus the general solution:

$$c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$$

GENERALIZATION:

If $\alpha_1, \alpha_2, \dots \alpha_n$ are distinct roots of

$(D^n + a_1 D^{n-1} + \dots + a_n) = 0$ then

$e^{\alpha_1 x}, e^{\alpha_2 x}, \dots, e^{\alpha_n x}$ will be n different independent solutions of the given equation and

$$c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \dots + c_n e^{\alpha_n x}$$

Is the general solution of the homogeneous equation

CASE OF REPEATED ROOTS:

$$(D-\alpha)(D-\alpha)y=0$$

Let $(D-\alpha)y = Z$

then $(D-\alpha)Z = 0 \Rightarrow Z = c_1 e^{\alpha x}$

Now solving

$$(D-\alpha)y = c_1 e^{\alpha x}$$

$$\Rightarrow \frac{dy}{dx} - \alpha y = c_1 e^{\alpha x} \qquad (\text{linear in } y)$$

$$I.F. = e^{\int -\alpha\, dx} = e^{-\alpha x}$$

$$\Rightarrow y \cdot e^{-\alpha x} = \int c_1 e^{\alpha x} \cdot e^{-\alpha x}\, dx + c_2$$

$$\Rightarrow y = (c_1 x + c_2)\, e^{\alpha x}$$

GENERALIZATION: If a root α is repeated r times, the solution is

$$y = \left[c_1 x^{r-1} + c_2 x^{r-2} + \cdots + c_r\right] e^{\alpha x}$$

CASE OF IMAGINARY ROOTS:

Let $\alpha + i\beta$ and $\alpha - i\beta$ be two conjugate roots, the solution will be

$$y = \bar{c}_1 e^{(\alpha+i\beta)x} + \bar{c}_2 e^{(\alpha - i\beta)x}$$

$$\Rightarrow \quad y = \bar{c}_1 e^{\alpha x} e^{i\beta x} + \bar{c}_2 e^{\alpha x} e^{-i\beta x}$$

$$= e^{\alpha x}\left[\bar{c}_1 e^{i\beta x} + \bar{c}_2 e^{-i\beta x}\right]$$

$$= e^{\alpha x}\cdot\left[\bar{c}_1\{\cos\beta x + i\sin\beta x\} + \bar{c}_2\{\cos\beta x - i\sin\beta x\}\right]$$

$$= e^{\alpha x}\left[(\bar{c}_1 + \bar{c}_2)\cos\beta x + i(\bar{c}_1 - \bar{c}_2)\sin\beta x\right]$$

$$= e^{\alpha x}\left[c_1\cos\beta x + c_2\sin\beta x\right]$$

GENERALIZATION:

It can similarly be shown that if $(\alpha+i\beta)$ & $(\alpha-i\beta)$ are conjugate imaginary roots, each repeated r times, then the solution is

$$e^{\alpha x}\left[(p_1 + p_2 x + \cdots + p_r x^{r-1})\cos\beta x + (q_1 + q_2 x + \cdots + q_r x^{r-1})\sin\beta x\right]$$

Linear Dependence and Independence:

Two functions f and g are called linearly dependent on an open interval I if

$$f(x) = c\, g(x) \quad \forall x \text{ in } I \quad \text{for some constant } c.$$

$$\left(\text{OR } g(x) = c\, f(x)\right)$$

Otherwise they are called *linearly independent*.

Example: $\cos x$, $\sin x$

WRONSKIAN Test: To test whether two solutions of

$$y'' + p(x)\, y' + q(x)\, y = 0$$

are linearly independent.

Define the wronskian of solutions y_1 and y_2 to be

$$W(x) = y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x)$$

$$= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Theorem: Let y_1 & y_2 be solutions of $y'' + p(x)y' + q(x)y = 0$ on an open interval I. Then

1. Either $W(x) = 0 \ \forall x$ in I, or $W(x) \neq 0 \ \forall x \in I$

2. y_1 and y_2 are linearly indep. on I iff $W(x) \neq 0$ on I.

Example: $y_1(x) = \cos x$ & $y_2(x) = \sin x$, solution of $y'' + y = 0$

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x$$

$$= 1 \neq 0$$

So y_1 & y_2 are linearly independent.

Example: Consider $y'' + xy = 0$ and its two solutions

$$y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12960}x^9 + \cdots$$

$$y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \frac{1}{45{,}360}x^{10} + \cdots$$

$\forall x \in \mathbb{R}$.

Sol: Note that calculating Wronskian at any nonzero x will be difficult, so we consider $x=0$ for Wronskian

$$W(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Nonvanishing of the wronskian at this point alone is enough to conclude linear independence of these solution.

Finding the general solution of a second order homogeneous linear equation if we know one solution.

Let y_1 be some known particular solution of the equation

$$y'' + p\,y' + q\,y = 0 \quad \text{—} (1)$$

Aim is to find another solution y_2 of the given equation so that y_1 & y_2 are linearly independent.

Since y_1 & y_2 are two solution, they will satisfy (1), i.e.,

$$y_2'' + p_1\,y_2' + q_2\,y_2 = 0 \quad \times y_1$$

$$y_1'' + p\,y_1' + q\,y_1 = 0 \quad \times y_2 \qquad \text{Substract}$$

$$(y_1 y_2'' - y_1'' y_2) + p\,(y_1 y_2' - y_1' y_2) + q\,(y_1 y_2 - y_1 y_2) = 0 \quad \text{—} (2)$$

Recall: $\quad W = y_1 y_2' - y_1' y_2$

$$W' = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_1' y_2'} - y_1'' y_2$$

$$\Rightarrow \quad W' + pW = 0 \quad \text{—} (3)$$

Solving (3): $\quad W = c\,e^{-\int p\,dx}$

$$\Rightarrow \quad \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{1}{y_1^2}\,c \cdot e^{-\int p\,dx}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{1}{y_1^2}\,c \cdot e^{-\int p\,dx}$$

$$\Rightarrow \quad \frac{y_2}{y_1} = \int \frac{c\, e^{-\int p\, dx}}{y_1^2}\, dx + c'$$

Since we are looking for a (particular) solution, we can set

$c' = 0$ & $c = 1$;

$$y_2 = y_1 \int \frac{e^{-\int p\, dx}}{y_1^2}\, dx$$

Example: Given that $y = x$ is a solution of the differential equation

$$(1-x^2)\, y'' - 2x y' + 2y = 0 \qquad 0 < x < 1.$$

Find the other linearly indep. solution.

Sol: $\quad p(x) = -\dfrac{2x}{1-x^2} \qquad y_1 = x.$

$$\Rightarrow \quad y_2 = x \int \frac{e^{-\int \frac{-2x}{1-x^2} \cdot dx}}{x^2}\, dx$$

$$= x \cdot \int \frac{e^{-\ln(1-x^2)}}{x^2}\, dx \quad = x \int \frac{1}{x^2(1-x^2)}\, dx$$

$$= x \int \left[\frac{1}{x^2} + \frac{1}{1-x^2} \right] dx$$

$$= x \int \left[\frac{1}{x^2} + \frac{1}{2} \frac{1}{1+x} + \frac{1}{2(1-x)} \right] dx$$

$$= x \left[-\frac{1}{x} + \frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(1-x) \right]$$

$$\Rightarrow \quad \boxed{y_2 = -1 + \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right)}$$

C.F.

$$f(D)y = 0$$

Auxiliary equation $f(m) = 0$

roots $\alpha_1, \alpha_2, \dots \alpha_n$.

Case I: Roots are real and non-repeated

$$C.F. = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + \dots + C_n e^{\alpha_n x}$$

Case II: Roots are real but repeated, say.

$\alpha_1 = \alpha_2 = \alpha$; $\alpha_3, \alpha_4 \dots \alpha_n$

$$C.F. = (C_1 + C_2 x) e^{\alpha x} + C_3 e^{\alpha_3 x} + \dots + C_n e^{\alpha_n x}$$

Case III: Roots are complex and non-repeated

$\alpha \pm i\beta, \alpha_3, \alpha_4 \dots \alpha_n$

$$C.F. = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + C_3 e^{\alpha_3 x} + \dots + C_n e^{\alpha_n x}$$

Case IV: Roots are complex and repeated

$\alpha \pm i\beta, \alpha \pm i\beta, \alpha_5, \alpha_6, \dots \alpha_n$

$$C.F. = e^{\alpha x}\left((C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x\right) + C_5 e^{\alpha_5 x} + \dots$$
$$\dots + C_n e^{\alpha_n x}.$$

Evaluation of C.F. :

Ex. Solve the differential equation

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$$

In operator form:

$$(D^4 - 2D^3 + 5D^2 - 8D + 4)\,y = 0$$

Auxiliary equation: $m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$

Its roots: $m = 1, 1, 2i, -2i$

The general solution:

$$y = (C_1 + C_2 x)e^x + C_3 \cos 2x + C_4 \sin 2x.$$

Ex. Suppose roots of the auxiliary eq. are

$$1, 2, 2, 1 \pm 2i, 1 \pm 2i$$

GENERAL SOLUTION:

$$y = c_1 e^x + (c_2 + c_3 x)e^{2x} + e^x\left[(c_4 + c_5 x)\cos 2x + (c_6 + c_7 x)\sin 2x\right]$$

Determination of particular integral:

Diff. Eq. $f(D)\,y = X$

$$\boxed{P.I. = \frac{1}{f(D)}.X}$$

1. General method of getting P.I.

$$\boxed{\frac{1}{(D-\alpha)}X = e^{\alpha x}\int X e^{-\alpha x}\,dx}$$

Proof: Let $y = \frac{1}{D-\alpha}X$

On operating $D-\alpha$ both sides, we get

$$(D-\alpha)y = X$$

$$\Rightarrow \frac{dy}{dx} - \alpha y = X \qquad \text{(linear equation in } y\text{)}$$

$$I.F. = e^{\int -\alpha\,dx} = e^{-\alpha x}$$

$$\Rightarrow y\cdot e^{-\alpha x} = \int X e^{-\alpha x}\,dx + c$$

$$\Rightarrow \boxed{y = e^{\alpha x}\int X e^{-\alpha x}\,dx} + c\,e^{\alpha x}$$

Ex. Solve $(D^2+a^2)y = \sec ax$

Auxiliary equation: $m^2+a^2=0 \Rightarrow m = \pm ai$

$$C.F. = C_1\cos ax + C_2 \sin ax$$

$$P.I. = \frac{1}{D^2+a^2}\sec ax = \frac{1}{(D-ia)(D+ia)}\cdot \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax$$

Consider $\frac{1}{D-ia}\sec ax = e^{iax}\int \sec ax\, e^{-iax}\,dx$

$$= e^{iax}\int \sec ax\,[\cos ax - i\sin ax]\,dx$$

$$= e^{iax}\int\left(1 - i\,\frac{\sin ax}{\cos ax}\right)dx$$

$$= e^{iax}\left[x + \frac{i}{a}\ln|(\cos ax)|\right]$$

Similarly $\frac{1}{D+ia}\sec ax = e^{-iax}\left[x - \frac{i}{a}\ln|(\cos ax)|\right]$

Hence,

$$P.I. = \frac{1}{2ia}\left[e^{iax}\left\{x+\frac{i}{a}\ln(\cos ax)\right\} - e^{-iax}\left\{x - \frac{i}{a}\ln(\cos ax\right.\right.$$

$$= \frac{1}{2ia}\left[x\left(e^{iax}-e^{-iax}\right) + \frac{i}{a}\ln|(\cos ax)|\left\{e^{iax}+e^{-iax}\right\}\right]$$

$$= \frac{x}{a}\cdot \sin ax + \frac{1}{a^2}\ln|(\cos ax)|\cdot \cos ax.$$

GENERAL SOLUTION:

$$y = C_1\cos ax + C_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\ln|(\cos ax)|\cos ax$$

2. Short Methods for finding P.I. (Proofs: Shanti Narayan)

- X is of the form $\boxed{e^{ax}}$

i) $$\boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}$$ where $f(a) \neq 0$

ii) If $f(a) = 0$, then $f(D)$ must have a factor of the type $(D-a)^r$.

Then, $$\boxed{\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{\underline{|r}} e^{ax}}$$

Ex. $$\text{P.I.} = \frac{1}{D^3 - D^2 - D + 1} \cdot e^x$$

$$= \frac{1}{(D-1)^2 (D+1)} e^x$$

$$= \frac{1}{(D-1)^2} \frac{1}{2} e^x$$

$$= \frac{1}{2} \cdot \frac{x^2}{\underline{|2}} \cdot e^x = \frac{1}{4} x^2 e^x.$$

Ex. $$\text{P.I.} = \frac{1}{D^2+D+5} \cdot 7$$

$$= \frac{1}{D^2+D+5} \cdot 7 e^{0x}$$

$$= 7 \cdot \frac{1}{D^2+D+5} \cdot e^{0x} = \frac{7}{5}.$$

Proof of:

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad \text{where } f(a) \neq 0$$

let $f(D) = D^n + C_1 D^{n-1} + \cdots + C_{n+1} D + C_n$

Consider

$$f(D)\, e^{ax} = \left[D^n + C_1 D^{n-1} + \cdots + C_{n-1} D + C_n\right] e^{ax}$$

$$= \left[a^n + C_1 a^{n-1} + \cdots + C_{n-1} a + C_n\right] e^{ax}$$

$$f(D)\, e^{ax} = f(a)\, e^{ax}$$

Operating both side by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)} f(D)\, e^{ax} = \frac{1}{f(D)} \left(f(a)\, e^{ax}\right)$$

$$\Rightarrow e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$$

$$\Rightarrow \boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}$$

• X is $\cos ax$ or $\sin ax$

$$\boxed{P.I. = \frac{1}{f(D)}\cos ax = \frac{1}{\varphi(D^2)}\cos ax = \frac{1}{\varphi(-a^2)}\cos ax}$$

provided $\varphi(-a^2) \neq 0$

$\boxed{\text{Replace } D^2 \text{ by } -a^2}$

Ex. $$P.I. = \frac{1}{D^4+D^2+1}\cos 2x = \frac{1}{(D^2)^2+D^2+1}\cos 2x$$

$$= \frac{1}{16-4+1}\cos 2x = \frac{1}{13}\cos 2x$$

Ex. $$P.I. = \frac{1}{D^2-2D+1}\cos 3x$$

$$= \frac{1}{-9-2D+1}\cos 3x = \frac{1}{-2D-8}\cos 3x$$

$$= -\frac{1}{2}\cdot\frac{1}{D+4}\cdot\cos 3x$$

$$= -\frac{1}{2}\frac{D-4}{D^2-16}\cos 3x$$

$$= \frac{1}{50}(D-4)\cos 3x$$

$$= \frac{1}{50}\cdot(-3\cdot\sin 3x - 4\cos 3x)$$

$$= -\frac{1}{50}\left(3\sin 3x + 4\cos 3x\right).$$

If $\varphi(-a^2) = 0$:

Ex. $$\frac{1}{D^2+a^2} \sin ax$$

$$= \text{imag}\left\{ \frac{1}{D^2+a^2} \cos ax + i \frac{1}{D^2+a^2} \sin ax \right\}$$

$$= \text{imag}\left\{ \frac{1}{D^2+a^2} \cdot e^{iax} \right\}$$

Consider. $$\frac{1}{D^2+a^2} e^{iax} = \frac{1}{(D-ai)(D+ia)} \cdot e^{iax}$$

$$= \frac{1}{D-ai} \cdot \frac{1}{2ia} e^{iax}$$

$$= \frac{1}{2ia} \cdot \frac{x}{1} \cdot e^{iax}$$

$$= \frac{x}{2ia} \{\cos ax + i \sin ax\}$$

$$= \frac{x}{2a} \sin ax - i \frac{x}{2a} \cos ax$$

$$\text{P.I.} = -\frac{x}{2a} \cos ax .$$

Rules:

$$\boxed{\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax}$$

$$\boxed{\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax}$$

Ex. Solve $(D^2+4)y = \sin^2 x$

$$C.F. = C_1 \cos 2x + C_2 \sin 2x.$$

$$P.I. = \frac{1}{D^2+4} \sin^2 x$$

$$= \frac{1}{D^2+4} \cdot \frac{1}{2}(1-\cos 2x)$$

$$= \frac{1}{2} \cdot \left[\frac{1}{4} - \frac{1}{D^2+4} \cdot \cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} - \frac{x}{2 \cdot 2} \sin 2x\right]$$

$$= \frac{1}{8}\left[1 - x \sin 2x\right]$$

General solution:

$$y = c_1 \cos 2x + C_2 \sin 2x + \frac{1}{8}\left[1 - x \sin 2x\right].$$

- X is x^m or a polynomial of degree m.

Take out the lowest degree term from $f(D)$, so as to reduce it in the form

$$[1 \pm F(D)]^{\alpha}.$$

Take it to numerator and expand it.

Ex. Find $\frac{1}{D^3-D^2-6D}\cdot(x^2+1)$

$$= \frac{1}{-6D\left(1+\frac{D}{6}-\frac{D^2}{6}\right)}(x^2+1)$$

$$= -\frac{1}{6D}\left[1+\left(\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}(x^2+1)$$

$$= -\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2-\cdots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}\cdots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[(x^2+1)-\frac{1}{6}(2x)+\frac{7}{36}\cdot 2\right]$$

$$= -\frac{1}{6D}\left[x^2-\frac{x}{3}+\frac{25}{18}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

- X is $e^{ax}V$, where V is any function of x.

Rule: $$\boxed{\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V.}$$

Ex.

$$\text{P.I.} = \frac{1}{D^2+3D+2} e^{2x} \sin x$$

$$= e^{2x} \cdot \frac{1}{(D+2)^2+3(D+2)+2} \cdot \sin x$$

$$= e^{2x} \cdot \frac{1}{D^2+7D+12} \sin x$$

$$= e^{2x} \cdot \frac{1}{7D+11} \sin x$$

$$= e^{2x} \frac{7D-11}{49D^2-121} \sin x$$

$$= e^{2x} \cdot \frac{7D-11}{-170} \sin x$$

$$= -\frac{e^{2x}}{170}\left(7 \cdot \cos x - 11 \sin x\right)$$

$$= \frac{e^{2x}}{170}\left(11 \sin x - 7 \cos x\right).$$

- X is xV

Rule $$\boxed{\frac{1}{f(D)}(x\cdot V) = x\cdot\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V}$$

Ex.

$$P.I. = \frac{1}{D^2-2D+1}\cdot x\sin x$$

$$= x\frac{1}{D^2-2D+1}\sin x - \frac{(2D-2)}{(D^2-2D+1)^2}\sin x$$

$$= x\frac{1}{-2D}\sin x - \frac{(2D-2)}{(-2D)^2}\sin x$$

$$= +\frac{x}{2}\cos x - \frac{(2D-2)}{4(-1)}\sin x$$

$$= \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$$

$$= \frac{1}{2}(x\cos x + \cos x - \sin x).$$

$$f(D)y = X$$

$$P.I. = \frac{1}{f(D)} X$$

1. General rule $\quad \frac{1}{D-\alpha} X = e^{\alpha x} \int X e^{-\alpha x} \, dx$

2. Short Methods:

a) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad ; \quad f(a) \neq 0$

a′) If $f(a) = 0$; $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{\lfloor r} e^{ax}$;

$$f(D) = (D-a)^r \, \Phi(D)$$

b) $\frac{1}{\Phi(D^2)} \cos ax = \frac{1}{\Phi(-a^2)} \cos ax \quad ; \quad \Phi(a^2) \neq 0$

b′) $\frac{1}{\Phi(D^2)} \sin ax = \frac{1}{\Phi(-a^2)} \sin ax \; ; \quad \Phi(-a^2) \neq 0$

b″) $\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$

b‴) $\frac{1}{D^2+a^2} \sin ax = \frac{x}{2a} \sin ax$

c) $\frac{1}{f(D)} \left(e^{ax} V \right) = e^{ax} \frac{1}{f(D+a)} V$

d) $\frac{1}{f(D)} (xV) = x \cdot \frac{1}{f(D)} V - \frac{f'(D)}{\{f(D)\}^2} V$

Method of variation of parameters

Consider the following second order non-homogeneous linear equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x) \tag{1}$$

Let $y = c_1y_1 + c_2y_2$, with c_1 and c_2 as arbitrary constants, be the general solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$$

We assume that

$$y = C_1y_1 + C_2y_2 \tag{2}$$

is the general solution of the non-homogeneous equation (1), where C_1 and C_2 are functions of x to be so chosen that (1) is satisfied.

Differentiating (2) we get

$$y^{'} = C_1y_1^{'} + C_2y_2^{'} + \underbrace{C_1^{'}y_1 + C_2^{'}y_2}_{=0} \tag{3}$$

For simplicity, in order to find C_1 and C_2 we assume that

$$C_1^{'}y_1 + C_2^{'}y_2 = 0 \tag{4}$$

Differentiating (3) again,

$$y^{''} = C_1y_1^{''} + C_2y_2^{''} + C_1^{'}y_1^{'} + C_2^{'}y_2^{'} \tag{5}$$

Substituting y, y' and y'' in (1) we get

$$C_1\left(y_1^{''} + a_1y_1^{'} + a_2y_1\right) + C_2\left(y_2^{''} + a_1y_2^{'} + a_2y_2\right) + C_1^{'}y_1^{'} + C_2^{'}y_2^{'} = f(x)$$

$$\implies \quad C_1^{'}y_1^{'} + C_2^{'}y_2^{'} = f(x) \tag{6}$$

Solving the equations (4) and (6):

$$C_1^{'} = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2^{'} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix}} = -\frac{y_2f(x)}{W}$$

Here W is called Wronskian. It is non-zero because y_1 and y_2 are linearly independent. Similarly

$$C_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f(x)}{W}$$

After integrating:

$$C_1 = \int -\frac{y_2 f(x)}{W} dx + d_1 \quad \text{and} \quad C_2 = \int \frac{y_1 f(x)}{W} dx + d_2$$

Hence the general solution of the non-homogeneous equation

$$\boxed{y = d_1 y_1 + d_2 y_2 + y_1 \int -\frac{y_2 f(x)}{W} dx + y_2 \int \frac{y_1 f(x)}{W} dx}$$

Example: Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x} \tag{7}$$

Solution:

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

Let $y = C_1 e^x + C_2 e^{-x}$ be the general solution of the given equation.

$$y' = C_1 e^x - C_2 e^{-x} + \underbrace{C_1' e^x + C_2' e^{-x}}_{=0}$$

$$y'' = C_1 e^x + C_2 e^{-x} + C_1' e^x - C_2' e^{-x}$$

Substituting in (7)

$$C_1' e^x - C_2' e^{-x} = \frac{2}{1+e^x}$$

The Wronskian

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Hence

$$C_1 = -\frac{1}{2} \int -e^{-x} \frac{2}{1+e^x} dx + d_1 = \int \frac{e^{-x}}{1+e^x} dx + d_1$$

Substitute $e^x = z \Rightarrow e^x dx = dz$

$$C_1 = \int \frac{1}{z^2(1+z)} dz + d_1 = \int \frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z} dz + d_1$$

$$C_1 = -\frac{1}{z} - \ln z + \ln(1+z) + d_1 = -e^{-x} - x + \ln(1+e^x) + d_1$$

Similarly

$$C_2 = -\frac{1}{2}\int e^x \frac{2}{1+e^x} dx + d_1 = -\ln(1+e^x) + d_2$$

The general solution of the differential equation

$$\boxed{y = d_1 e^x + d_2 e^{-x} - 1 - xe^x + (e^x - e^{-x})\ln(1+e^x)}$$

Cauchy-Euler Equations

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X \tag{8}$$

or

$$(x^n D^n + a_1 D^{n-1} + \cdots + a_n) y = X \tag{9}$$

is called Euler-Cauchy equation.

Working Rule: To solve equation (8) we change the variable from x to z by putting $x = e^z$ i.e. $z = \ln(x)$.

$$z = \ln(x) \Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dz}}$$

We define a new operator

$$x\frac{d}{dx} \equiv \frac{d}{dz} \equiv D_1$$

Again

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D_1(D_1 - 1)y$$

Thus we have the following formulas for $D \equiv \frac{d}{dx}$ and $D_1 \equiv \frac{d}{dz}$

$$\begin{aligned}
&xD = D_1 \\
&x^2D^2 = D_1(D_1 - 1) \\
&x^3D^3 = D_1(D_1 - 1)(D_1 - 2) \\
&\vdots \\
&x^nD^n = D_1(D_1 - 1)(D_1 - 2)\ldots(D_1 - n + 1)
\end{aligned}$$

Substituting these operator relations in the equation (9), we obtain a linear differential equation with constant coefficient

$$f(D_1)y = Z, \quad \text{where Z becomes a function of z only}$$

Example 1.

$$(x^2D^2 - xD + 2)y = x\ln x \tag{10}$$

Let $x = e^z$ so that $z = \ln x$ and $D_1 \equiv \frac{d}{dz}$ then the equation (10) becomes

$$[D_1(D_1 - 1) - D_1 + 2]\, y = ze^z$$

Auxiliary equation $m^2 - 2m + 2 = 0$ and its roots are $m = 1 \pm i$
Hence

C.F. $= e^z\,[c_1 \cos(z) + c_2 \sin(z)] = x\,[c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))]$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D_1^2 - 2D_1 + 2} ze^z = e^z \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} z \\
&= e^z \frac{1}{D_1^2 + 1} z = e^z (1 + D_1^2)^{-1} z = e^z z = x\ln(x)
\end{aligned}$$

General solution

$$\boxed{y = x\,[c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))] + x\ln(x)}$$

Equations reducible to Euler-Cauchy form There can be several forms of equation which can be reduced to Euler-Cauchy form

Example 1: Solve

$$\frac{d^3y}{dx^3} - \frac{4}{x}\frac{d^2y}{dx^2} + \frac{5}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 1$$

Solution: $y = c_1x^2 + c_2x^{(5+\sqrt{21})/2} + c_3x^{(5-\sqrt{21})/2} - x^3/5$

Example 2:

$$2x^2y\frac{d^2y}{dx^2} + 4y^2 = x^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

Hint: $y = z^2$

Solution:

$$y = z^2 \Rightarrow \frac{dy}{dx} = 2z\frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\left(\frac{dz}{dx}\right)^2 + 2z\frac{d^2z}{dx^2}$$

Substituting these values in the differential equation we get

$$x^2\frac{d^2z}{dx^2} - x\frac{dz}{dx} + z = 0$$

or

$$[x^2D^2 - xD + 1]z = 0$$

Substitute $x = e^t \Leftrightarrow \ln x = t$

$$\Rightarrow x\frac{dz}{dx} = \frac{dz}{dt} \Rightarrow xD \equiv D_1$$

Similarly

$$x^2D^2 = D_1(D_1 - 1)$$

Then the equation becomes

$$[D_1^2 - 2D_1 + 1]z = 0 \Rightarrow z = [c_1 + c_2t]e^t$$

$$\Rightarrow z = [c_1 + c_2\ln(x)]x$$

$$\Rightarrow \boxed{y = (c_1 + c_2\ln(x))^2x^2}$$

Example 3: A differential equation of the form

$$(a+bx)^n\frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_ny = X$$

can be reduced to Euler-Cauchy equation by putting

$$a + bx = v \Rightarrow \frac{dv}{dx} = b$$

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = b\frac{dy}{dv}$$

Again

$$\frac{d^2y}{dx^2} = b^2\frac{d^2y}{dv^2} \text{ or in general } \frac{d^ny}{dx^n} = b^n\frac{d^ny}{dv^n}$$

Substituting these derivatives in the equation, we get

$$v^n \frac{d^n y}{dv^n} + \frac{a_1}{b} v^{n-1} \frac{d^{n-1} y}{dv^{n-1}} + \cdots + \frac{a_{n-1}}{b^{n-1}} v \frac{dy}{dv} + \frac{a_n}{b^n} y = \frac{X}{b^n}$$

which is an standard Euler-Cauchy equation.

Example 4: solve

$$(1+x)^2 \frac{d^2 y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\ln(1+x)$$

Solution: Let $(1+x) = v \Rightarrow \frac{dv}{dx} = 1$.
Hence $\frac{dy}{dx} = \frac{dy}{dv}$ and $\frac{d^2y}{dx^2} = \frac{d^2y}{dv^2}$ and the differential equation becomes

$$v^2 \frac{d^2 y}{dv^2} + v\frac{dy}{dv} + y = 4\cos\ln v$$

Put $v = e^z \Rightarrow \ln(v) = z$ and let $D_1 \equiv \frac{d}{dz}$

$$\left[D_1(D_1 - 1) + D_1 + 1\right] y = 4\cos z$$

$$\left(D_1^2 + 1\right) y = 4\cos z$$

$$\begin{aligned} \text{C.F.} = c_1\cos(z) + c_2\sin(z) &= c_1\cos(\ln v) + c_2\sin(\ln v) \\ &= c_1\cos(\ln(1+x)) + c_2\sin(\ln(1+x)) \end{aligned}$$

$$\text{P.I.} = 2z\sin z = 2\ln(v)\sin(\ln(v)) = 2\ln(1+x)\sin(\ln(1+x)).$$

The general solution

$$\boxed{y = c_1\cos(\ln(1+x)) + c_2\sin(\ln(1+x)) + 2\ln(1+x)\sin(\ln(1+x)).}$$