Indian Institute of Technology Kharagpur
Department of Mathematics
MA11003 - Advanced Calculus
Problem Sheet - 5
Autumn 2020

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1. Find $\dfrac{df}{dt}$ at $t = 0$ for the following functions,

   (a) $f(x, y) = x\cos y + e^x \sin y$, where $x(t) = t^2 + 1$, $y(t) = t^3 + t$

   (b) $f(x, y, z) = x^3 + xz^2 + y^3 + xyz$, where $x(t) = e^t$, $y(t) = \cos t$, $z(t) = t^3$

   (c) $f(x_1, x_2, x_3) = 2x_1^2 - x_2x_3 + x_1x_3^2$ where $x_1(t) = 2\sin(t)$, $x_2(t) = t^2 - t + 1$, $x_3(t) = 3^{-t}$

2. (a) Using implicit differentiation, find $\dfrac{dy}{dx}$ from the followings:

   i. $x^y + y^x = c$,
   ii. $xy^2 + \exp(x)\sin(y^2) + \tan^{-1}(x + y) = c$
   iii. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + 1 = 0$
   iv. $\ln(x^2 + y^2) + \tan^{-1}(y/x) = 0$

   (b) Using implicit differentiation, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ from the followings:

   i. $xy^2z^2 + \sin(yz) - \exp(xz^2) = 0$,
   ii. $x\tan^{-1}(\frac{y}{z}) + y\tan^{-1}(\frac{z}{x}) + z\tan^{-1}(\frac{x}{y}) = 0$
   iii. $xy^2 + z^3 + \sin(xyz) = 0$
   iv. $x - yz + \cos(xyz) - x^2z^2 = 1$

3. If $u = f(r, s, t)$, where $r = \dfrac{x}{y}$, $s = \dfrac{y}{z}$ and $t = \dfrac{z}{x}$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

4. If $v = f(u)$ where u is a homogeneous function of x and y of degree n, then prove that

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nu\frac{dv}{du}$$

5. Check whether the following functions are homogeneous or not, if so, determine the degree of the function:

   (a) $\tan^{-1}\dfrac{y}{x} + \sin^{-1}\dfrac{x}{y}$
   (b) $\cos^{-1}\left(\dfrac{y}{\sqrt{x^2 + y^2}}\right)$
   (c) $\dfrac{x^2}{y} + \dfrac{y^2}{x}$
   (d) $\dfrac{x}{y}\sin\left(\dfrac{y}{x}\right)$
   (e) $x^{2/3}y^{4/3}\tan\dfrac{y}{x}$
   (f) $\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$
   (g) $x^2y^2 + xy^3 + x^2y + x^3y$
   (h) $\dfrac{x^2 + y^2}{x^3 + y^3}$

6. If $f(x,y) = \frac{y}{x} + \frac{x}{y}$, then show that $xf_x + yf_y = 0$.

7. If $u = \frac{x^2+y^2}{\sqrt{x+y}}, (x,y) \neq (0,0)$, what should be the value of k so that $xu_x + yu_y = ku$?

8. If $y = f(x+ct) + \phi(x-ct)$, then show that $y_{tt} = c^2 y_{xx}$.

9. If $u = e^{-mx}\sin(nt-mx)$, prove that $u_t = \frac{n}{2m^2}u_{xx}$.

10. If $x^x.y^y.z^z = k$(constant), then show that at the point (x,y,z) where $x=y=z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{xlog_e(ex)}.$$

11. If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$.

12. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, then prove that $xu_x + yu_y = \sin 2u$.

13. If $u = \sin^{-1}\sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$, then show that

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right).$$

14. If $u(x,y) = x\log(\frac{y}{x})$, for $xy \neq 0$, then show that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$.

15. If $u = \frac{(ax^3+by^3)^n}{3n(3n-1)} + xf(\frac{y}{x})$, then prove that

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = (ax^3+by^3)^n$$

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16. If $z = x^m f(\frac{y}{x}) + y^n g(\frac{x}{y})$, then show that
$x^2z_{xx} + 2xyz_{xy} + y^2z_{yy} + mnz = (m+n-1)(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y})$.

17. Let $f(x,y)$ and $g(x,y)$ be two homogeneous functions of degree m and n respectively, where $m \neq 0$ and $h = f+g$. If $(x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y}) = 0$, then show that $f = \alpha g$, for some scalar α.

18. By the transformation $\xi = a + \alpha x + \beta y, \eta = b - \beta x + \alpha y$ where α, β, a, b are all constant and $\alpha^2 + \beta^2 = 1$, the function $u(x,y)$ is transfered into $U(\xi,\eta)$. Prove that $U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2 = u_{xx}u_{yy} - u_{xy}^2$.

19. If z be a differentiable function of x and y (rectangular cartesian co-ordinates) and let $x = r\cos\theta, y = r\sin\theta (r, \theta$ are polar co-ordinates), then show that

(a) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$

(b) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \theta^2}$.

20. Given $w = (x_1^2 + x_2^2 + \cdots + x_n^2)^k$, for $n \geq 2$. Then for what values of k, the following relation holds:

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \cdots + \frac{\partial^2 w}{\partial x_n^2} = 0.$$

21. Let $u(x, y)$ be such that all its second order partial derivatives exists. If $x = r\cos\theta, y = r\sin\theta$, then show that

$$r^2\frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial \theta^2} - r\frac{\partial u}{\partial r} = (x^2 - y^2)(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}) + 4xy\frac{\partial^2 u}{\partial x \partial y}.$$

22. Let $u(x, y)$ be such that all its second order partial derivatives exists. If $x = \xi\cos\alpha - \eta\sin\alpha, \quad y = \xi\sin\alpha + \eta\cos\alpha$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}.$$