# ADVANCED CALCULUS
# MA11003

## SECTION 11, 12, & 15CD

Dr. Jitendra Kumar

Professor
Department of Mathematics
Indian Institute of Technology Kharagpur
West Bengal 721302, India

**Webpage:** http://www.facweb.iitkgp.ac.in/~jkumar/

# Mean Value Theorems

- Rolle's Theorem
- Proof of Rolle's Theorem
- Worked Problems

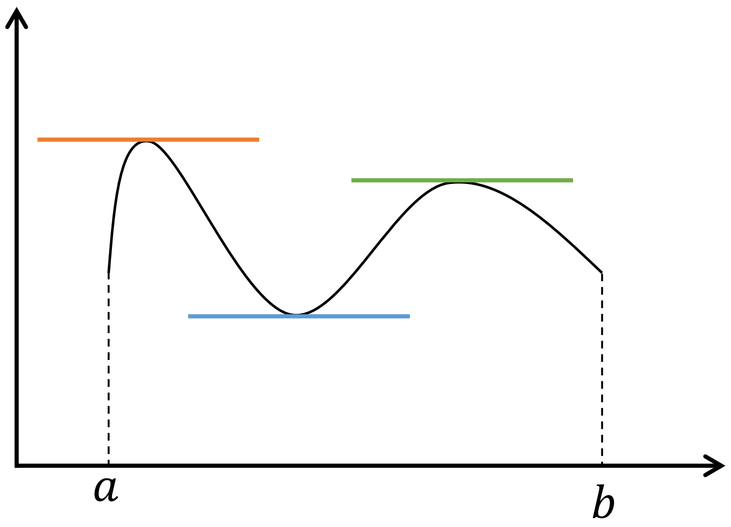
# Rolle's Theorem

If a function $f$ is

a) Continuous in $[a, b]$

b) Differentiable in $(a, b)$

c) $f(a) = f(b)$



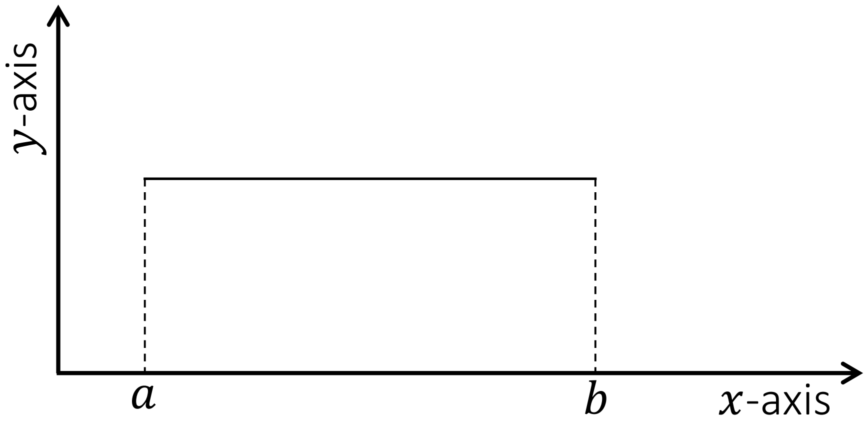

Then there exists a number $c \in (a, b)$ such that $f'(c) = 0$

# Proof of Rolle's Theorem

Suppose $M$ & $m$ are maximum and minimum of $f$ in $[a, b]$

(Extreme value theorem: A continuous function on $[a,\ b]$ reaches its maximum and minimum)

## Case –I $(M = m)$



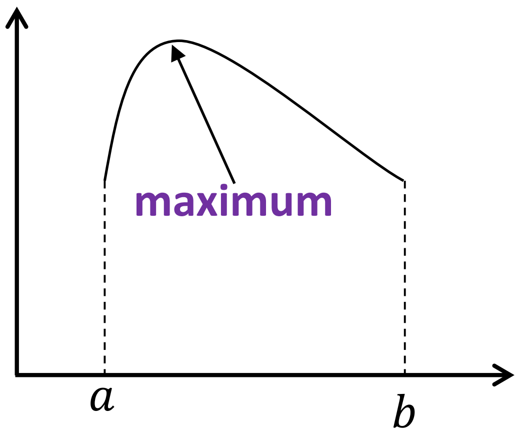

In this case:
$f(x) = M = m =$ constant

This implies $f'(x) = 0, \quad \forall\, x \in (a, b)$

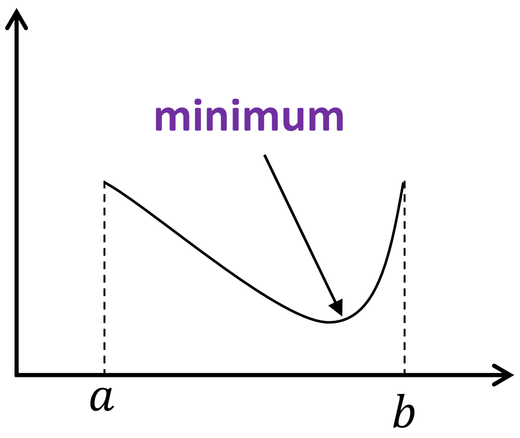
# Proof of Rolle's Theorem

Suppose $M$ & $m$ are maximum(local) and minimum(local) of $f$ in $[a, b]$

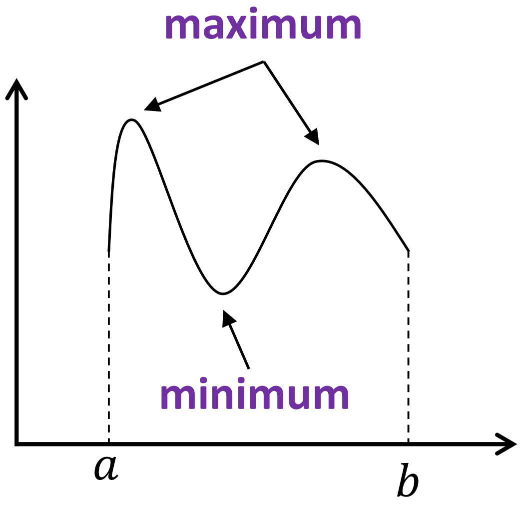
## Case –II $(M \neq m)$




**maximum is different**




**minimum is different**




**both are different**

# **Proof of Rolle's Theorem: Case –II ($M \neq m$)**

Suppose $M$ is different from the equal values $f(a)$ & $f(b)$ and let $f(c) = M$

Since $f(c)$ is the maximum value, we have

$$f(c+\Delta x) - f(c) \leq 0, \quad \text{for} \quad \Delta x > 0 \ \text{ or } \ \Delta x < 0$$

$$\frac{f(c+\Delta x) - f(c)}{\Delta x} \leq 0 \quad \text{for} \quad \Delta x > 0 \Longrightarrow \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c) \leq 0$$

$$\frac{f(c+\Delta x) - f(c)}{\Delta x} \geq 0 \quad \text{for} \quad \Delta x < 0 \quad \Longrightarrow f'(c) \geq 0$$

$f'(c) = 0$

# Remark - 1

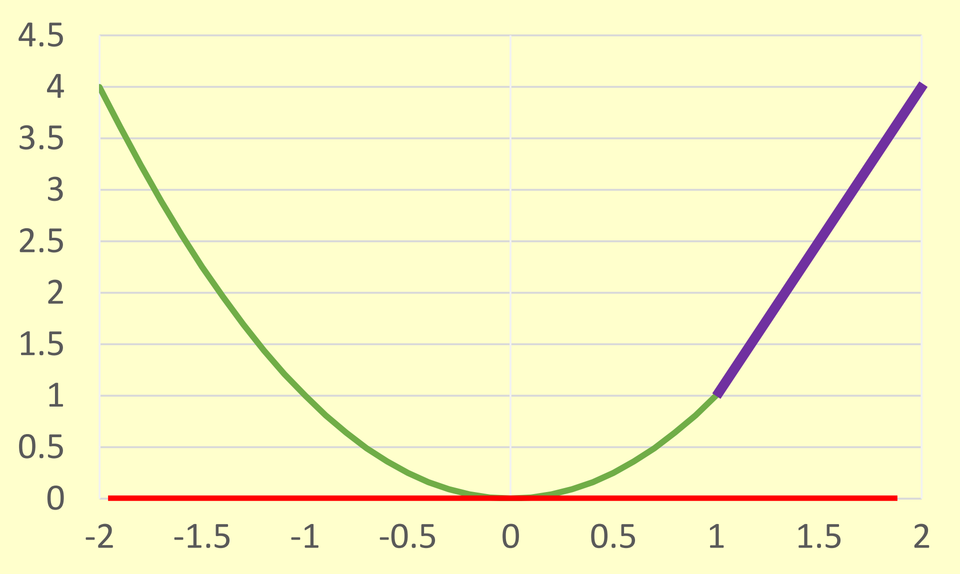
*The hypotheses of Rolle's theorem are sufficient but not necessary for the conclusion.*

$$\text{Continuity in } [a\ b], \text{differentiability in } (a,b), \big(f(a) = f(b)\big) \Longrightarrow f'(c) = 0$$

*Meaning, if all three hypotheses are met then conclusion is guaranteed. However, if the hypotheses are not met then you may or may not reach the conclusion.*

Consider:

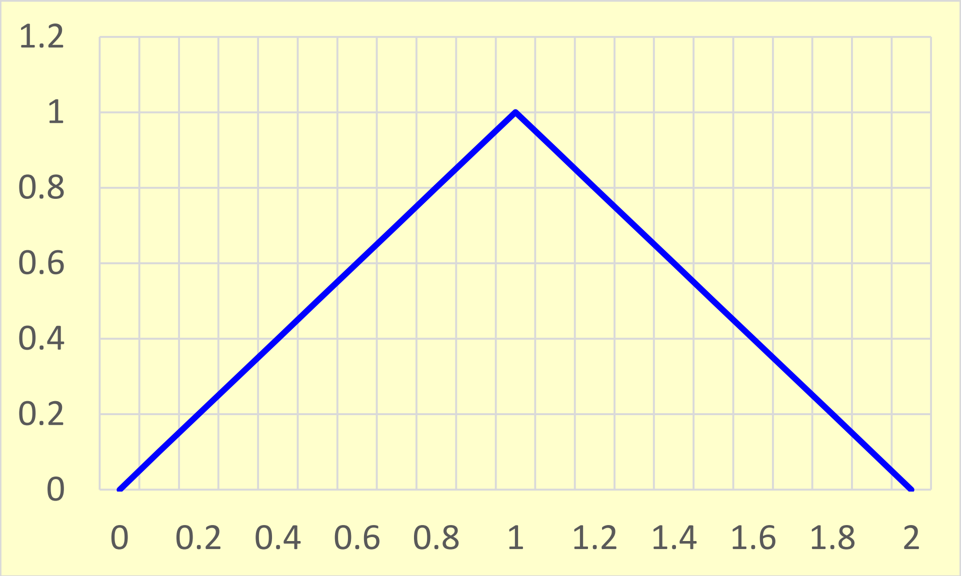
$$f(x) = \begin{cases} x^2; & -2 \le x \le 1 \\ 3x - 2; & 1 < x \le 2 \end{cases}$$




$f'(0) = 0$

Further, consider:

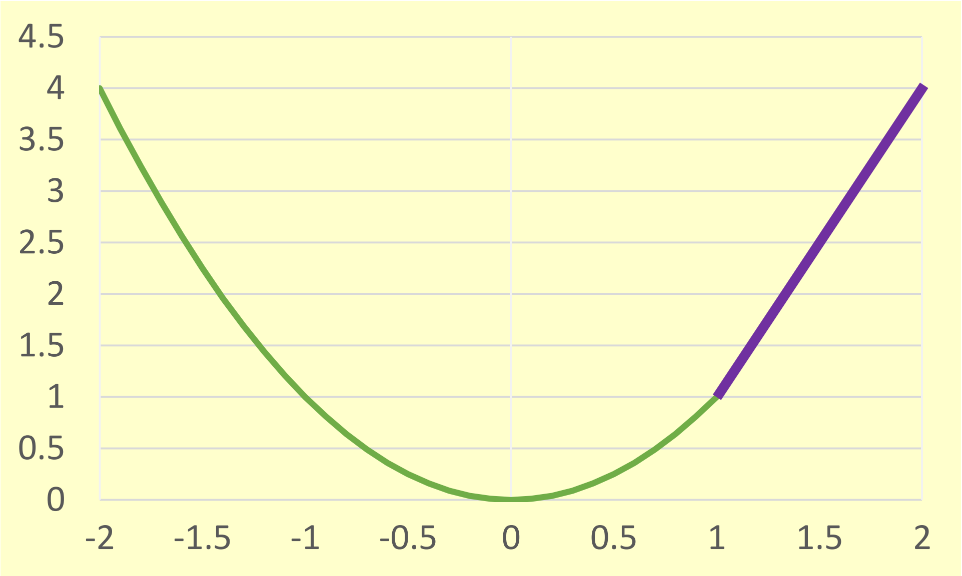
$$f(x) = \begin{cases} x; & 0 \le x \le 1 \\ 2 - x; & 1 < x \le 2 \end{cases}$$




$f'(x) \neq 0$, for any $x \in (0, 2)$

Previous Example:

$$f(x) = \begin{cases} x^2; & -2 \le x \le 1 \\ 3x - 2; & 1 < x \le 2 \end{cases}$$



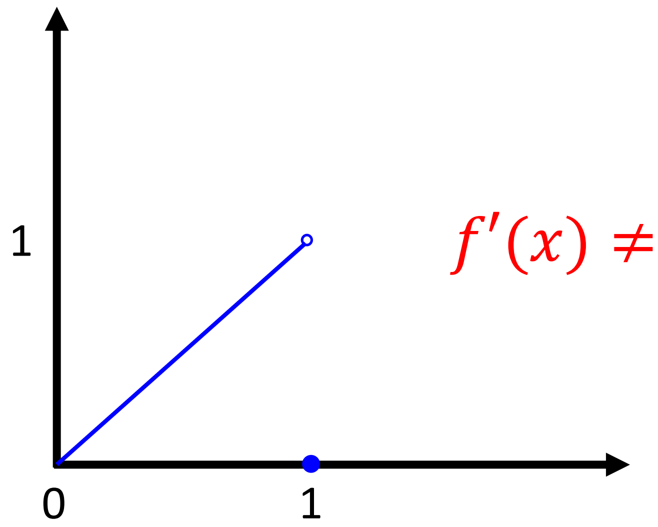

$f'(0) = 0$

# Remark - 2

*The continuity condition for the function on the closed interval* $[a, b]$ *is essential.*

Consider:

$$f(x) = \begin{cases} x; & 0 \le x < 1 \\ 0; & x = 1 \end{cases}$$



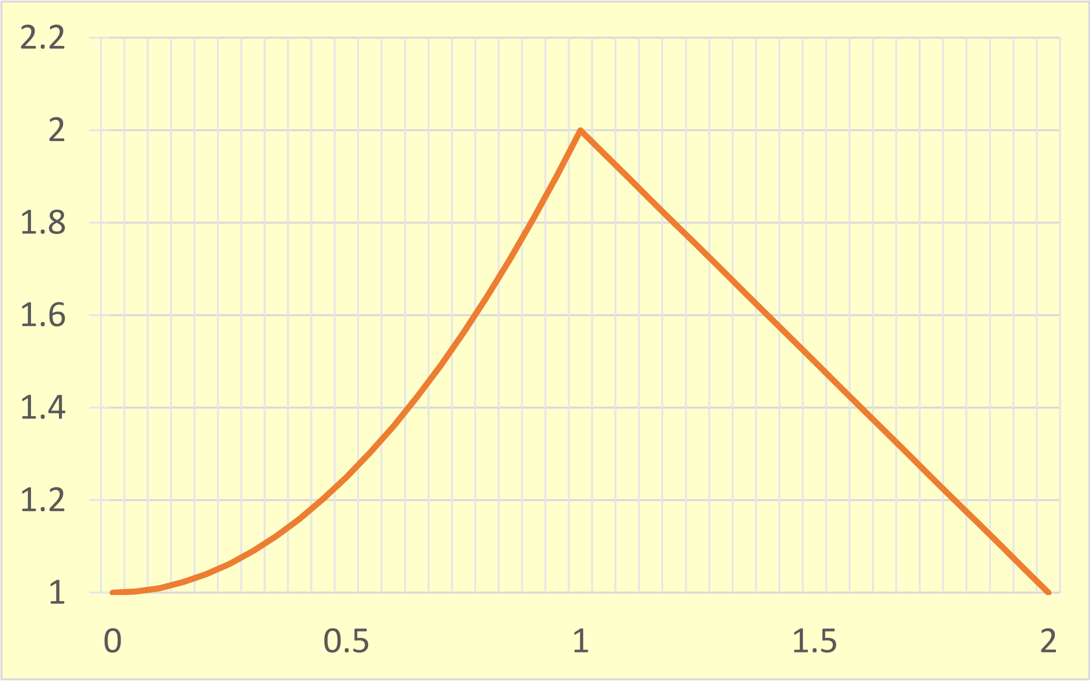

$f'(x) \neq 0$ for any $x \in (0, 1)$.

Note that $f$ is continuous and differentiable on $(0, 1)$, and also $f(0) = f(1)$.

## Example - 1

Discuss the applicability of Rolle's theorem to the function

$$f(x) = \begin{cases} x^2 + 1\,, & x \in [0, 1] \\ 3 - x, & x \in (1, 2] \end{cases}$$



**Example - 2** Using Rolle 's Theorem, show that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one real root in $[0, 1]$.

Suppose $f(x) = x^{13} + 7x^3 - 5$ has more than one real root in $[0, 1]$.

Take any two roots, say $\alpha$ and $\beta$, that is, we have $f(\alpha) = 0 = f(\beta)$, $0 < \alpha < \beta < 1$

Rolle 's Theorem implies $f'(c) = 0$ for some $c \in (\alpha, \beta)$

This implies $13\mathrm{c}^{12} + 21\mathrm{c}^2 = 0$ for some $c \in (\alpha, \beta)$

Note that $c > 0$ and therefore $13\mathrm{c}^{12} + 21\mathrm{c}^2 \neq 0$.

It contradicts our assumption of more than one real root.

On the other hand $f(0) = -5$ and $f(1) = 3$.

It confirms the existence of at least one root.

## Key Takeaway

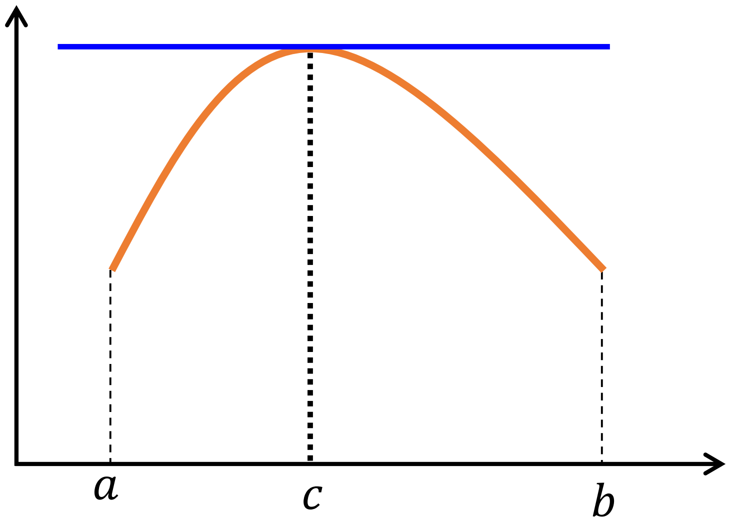
# Rolle's Theorem

If a function $f$ is

$a)$ Continuous in $[a, b]$

$b)$ Differentiable in $(a, b)$

$c)$ $f(a) = f(b)$




Then there exists a number $c \in (a, b)$ such that $f'(c) = 0$

# Mean Value Theorems

- Lagrange's Mean Value Theorem
- Cauchy's Mean Value Theorem

# Lagrange’s Mean Value Theorem

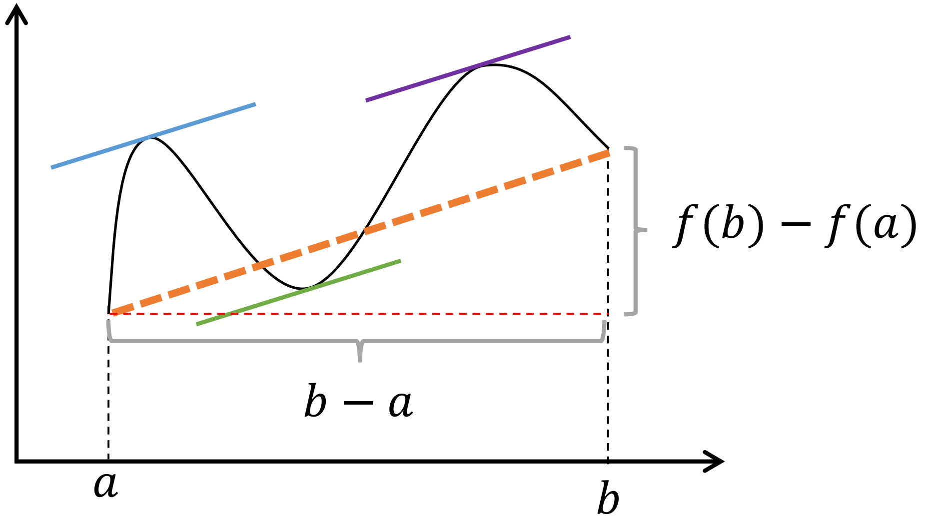
If a function $f$ is

a) Continuous in $[a, b]$

b) Differentiable in $(a, b)$

Then there exists a number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$




In other words, there is at least one tangent line in the interval that is parallel to the line segment that goes through the endpoints of the curve.

# Proof of Lagrange's Mean Value Theorem

Define a function

$$\phi(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right] x$$

Note that the function $\phi(x)$ satisfies all the conditions of Rolle's theorem.

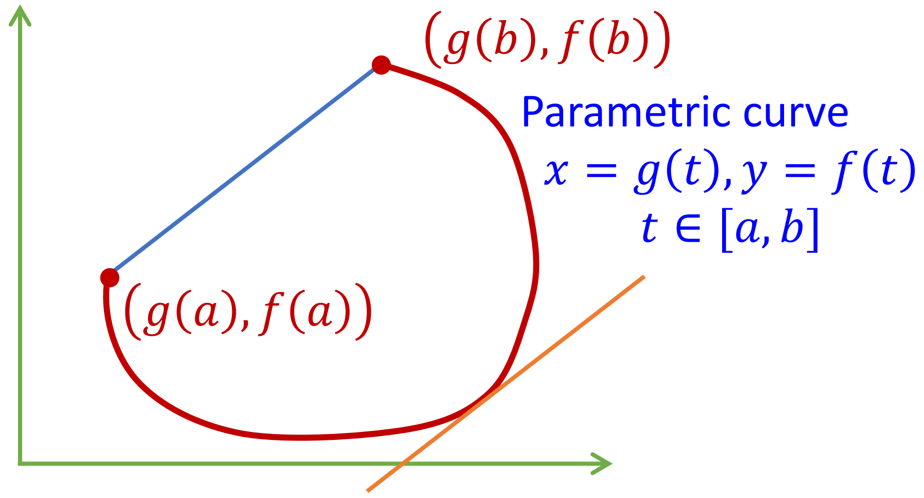
Therefore, Rolle's Theorem gives

$$\phi'(c) = 0 \text{ for some } c \in (a, b) \quad \Rightarrow f'(c) - \left[\frac{f(b) - f(a)}{b - a}\right] = 0$$

# Generalized Mean Value Theorem (Cauchy's MVT)

If $f(x)$ and $g(x)$ are two functions continuous in $[a, b]$ and differentiable in $(a, b)$, and $g'(x)$ does not vanish anywhere inside the interval then $\exists$ a point $c$ in $(a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$




**Proof:**

Define a function

$$\phi(x) = \big(f(x) - f(a)\big) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right]\big(g(x) - g(a)\big)$$

Note that $g(b) \neq g(a)$ **?**

Application of Rolle's theorem follows the result.

## Generalized MVT

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

What if $g(x) = x$ ?

## Lagrange's MVT

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

What if $f(b) = f(a)$ ?

## Rolle's MVT

$$0 = f'(c)$$

**Example - 1** Using Mean Value Theorem, show that

$$|\cos e^{x} - \cos e^{y}| \leq |x - y| \qquad x, y \leq 0$$

Consider $f(t) = \cos e^{t}$ in the interval $[x, y]$. Also assume $x \neq y$ otherwise equality holds.

Apply Lagrange's mean value theorem

$$\frac{\cos e^{x} - \cos e^{y}}{x - y} = f'(c), \qquad c \in (x, y)$$

$$|\cos e^{x} - \cos e^{y}| = |x - y|\,|e^{c}\sin e^{c}|$$

This further implies

$$|\cos e^{x} - \cos e^{y}| \leq |x - y| \max_{c \in (x,y)} |e^{c}\sin e^{c}| < |x - y|$$

# Example - 2

Let $f$ be a differentiable function on $[-2, 2]$ such that $f(-2) = 1,\ f(2) = 5$ and $|f'(x)| \leq 1$ for all $x \in [-2,2]$. Using mean value theorem, find the value of $f(0)$.

Using Lagrange's Mean Value Theorem on $[-2, 0]$ , we get

$$\frac{f(0) - f(-2)}{0 - (-2)} = f'(c_1), \text{for some } \ c_1 \in (-2,0) \Rightarrow -1 \leq \frac{f(0) - 1}{2} \leq 1, \text{since } |f'(x)| \leq 1$$

$$\Rightarrow -1 \leq f(0) \leq 3$$

Using Lagrange's Mean Value Theorem on $[0, 2]$ , we get

$$\frac{f(2) - f(0)}{2 - 0} = f'(c_2) \ \ \Rightarrow -7 \leq -f(0) \leq -3 \Rightarrow 7 \geq f(0) \geq 3$$

This implies: $f(0) = 3$

## Example - 3

The function $f : [0, 2] \to \mathbb{R}$ satisfies $f'(x) = \dfrac{1}{5 - x^2}$ and $f(0) = 2$. Use Lagrange's Mean Value Theorem to estimate the bounds of $f(1)$.

Using Lagrange's Mean Value Theorem on $[0, 1]$ , we get

$$\frac{f(1) - f(0)}{1} = f'(c), \text{for some } c \in (0, 1) \quad \Rightarrow f(1) - 2 = f'(c)$$

The derivative $f'(c)$ can be estimated as $\dfrac{1}{5} < \dfrac{1}{5 - c^2} < \dfrac{1}{4}$

$$\Rightarrow \frac{1}{5} < f(1) - 2 < \frac{1}{4} \qquad \Rightarrow \frac{11}{5} < f(1) < \frac{9}{4}$$

*Thank You*