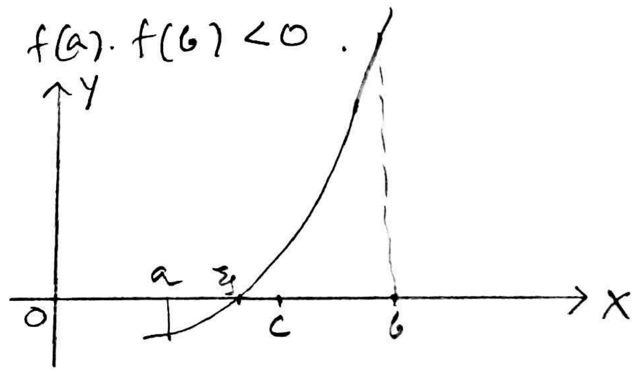
Solution of algebraic and transcendental equations

Order / rate of convergence

An iterative method is said to be of order $p$ or has rate of convergence $p$, if $p$ is the +ve real no. for which $\exists$ a finite constant $c \neq 0$ such that $|\varepsilon_{n+1}| \leq c|\varepsilon_n|^p$ where $\varepsilon_n = x_n - \xi$ is the error in nth iterate. $c$ depends on derivatives of $f(x)$ at $x = \xi$. $\xi$ is the exact root.

Bisection method

Let $\xi$ be a root of the eqn. $f(x) = 0$ lying in the interval $[a, b]$ i.e. $f(a) \cdot f(b) < 0$.




The interval $[a, b]$ is divided into two equal intervals $[a, c]$ and $[c, b]$ each of length $\frac{b-a}{2}$ and $c = \frac{a+b}{2}$ [figure].

If $f(c) = 0$, then $c$ is an exact root.

Now if $f(c) \neq 0$, then the root lies either in the interval $[a, c]$ or in the interval $[c, b]$. If $f(a) \cdot f(c) < 0$, then the interval $[a, c]$ is taken as new interval otherwise $[c, b]$ is taken as the new interval.

Let the new interval be $[a_1, b_1]$ and use the same process to select the next new interval. In the next step, let the new interval be $[a_2, b_2]$. The process of bisection is continued until either the mid point of the interval is a root or the length $(b_n - a_n)$ of the interval $[a_n, b_n]$ at nth step is sufficiently small. The number $a_n$ and $b_n$ are the approximate roots of the eqn. $f(x)=0$. Finally, $x_n = \frac{a_n + b_n}{2}$ is taken as the approximate value of the root $\xi$.

It may be noted that when the reduced interval be $[a_1, b_1]$, then the length of the interval is $\frac{b-a}{2}$, when the interval be $[a_2, b_2]$, then the length is $\frac{b-a}{2^2}$. At the nth step, the length of the interval being $\frac{b-a}{2^n}$. In the final step, when $\xi = \frac{a_n + b_n}{2}$ is chosen as a root, then the length of the interval being $\frac{b-a}{2^{n+1}}$ and hence the error does not exceed $\frac{b-a}{2^{n+1}}$.

Thus, if $\varepsilon$ be the error at the nth step then the lower bound of $n$ is obtained from the following relation

$$\frac{|b-a|}{2^n} \leq \varepsilon$$

The lower bound of $n$ is obtained by rewriting this inequation as

$$n \geq \frac{\log(|b-a|) - \log \varepsilon}{\log 2}$$

Hence the minimum no. of iterations reqd. to achieve the accuracy $\varepsilon$ is $\frac{\log\left(\frac{|b-a|}{\varepsilon}\right)}{\log 2}$. For ex, if the length of the interval is $|b-a| = 1$ and $\varepsilon = \cdot 0001$, then $n$ is given by $n \geq 14$.