Indian Institute of Technology Kharagpur
Department of Mathematics
MA11004 - Linear Algebra, Numeircal and Complex Analysis
Problem Sheet - 6
Spring 2021

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1. Find the following limits (if exist):

   (a) $\lim_{z \to 0} \frac{\bar{z}}{z}$.

   (b) $\lim_{z \to 0} \frac{\operatorname{Im}(z)}{|z|}$.

   (c) $\lim_{z \to 1+i} (z^2 - 5z + 10)$.

   (d) $\lim_{z \to 0}. \left[\frac{1}{1 - e^{\frac{1}{y}}} + iz^2 \right]$.

2. Test the continuity of the following functions at $z = 0$:

   $(a)\ f(z) = \begin{cases} \frac{\operatorname{Re}(z)\operatorname{Im}(z)}{|z|^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$ $(b)\ f(z) = \begin{cases} \frac{\operatorname{Re}(z)}{|z|} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$ $(c)\ f(z) = \begin{cases} \frac{\operatorname{Im}(z^2)}{z^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$

3. Test the differentiability of the following functions at $z = 0$:

   (a) $f(z) = |z|$.

   (b) $f(z) = \operatorname{Re}(z)$.

   (c) $f(z) = |z|^2$.

4. Let

$$f(z) = \begin{cases} \frac{\bar{z}^3}{z^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

   Show that

   (a) $f(z)$ is continuous everywhere on $\mathbb{C}$.

   (b) The complex derivative $f'(0)$ does not exist.

5. Show that the function $f(z) = |\operatorname{Re}(z)\operatorname{Im}(z)|^{1/2}$ satisfies the Cauchy-Riemann equations at $z = 0$, but $f'(0)$ does not exist.

6. Show that following functions are harmonic and find their harmonic conjugates:

   (a) $u(x, y) = 4xy - x^3 + 3xy^2$.

   (b) $u(x, y) = e^{-x}(x \sin y - y \cos y)$.

   (c) $u(x, y) = x^3 - 3xy^2$.

   (d) $u(r, \theta) = r^2 \sin 2\theta$.

7. Using Cauchy Riemann-equations, show that $f(z) = (1 + 2i)x^2y^2$ is nowhere analytic.

8. Let

$$f(z) = \begin{cases} \frac{\bar{z}^2}{|z|} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Show that $f(z)$ is continuous everywhere, but nowhere analytic on $\mathbb{C}$.

9. Let $f(z) = u + iv$ be analytic in a domain D. Prove that f is constant in D if any one of the followings hold:

   (a) $f'(z)$ vanishes in D.

   (b) $\operatorname{Re} f(z) = u =$ constant.

   (c) $\operatorname{Im} f(z) = v =$ constant.

   (d) $|f(z)| =$ constant (non zero).

10. Let $f(z) = u + iv$ be an analytic function in a domain D such that $v = u^2$. Show that $f(z)$ is constant.

11. Show that there exist no analytic function f such that $\operatorname{Re} f(z) = y^2 - 2x$.

12. Prove the following statements:

   (a) A real-valued function of a complex variable either has derivative zero or the derivative does not exist.

   (b) If $f(z)$ is continuous at $z = z_0$, then $|f(z)|$ is also continuous at $z = z_0$.

   (c) If $f(z)$ satisfies the Cauchy-Riemann equations at $z = z_0$, then $(f(z))^n$ also satisfies the Cauchy Riemann equations at $z = z_0$.

   (d) If u and v are harmonic conjugates to each other in some domain D, then u and v must be constant there.

   (e) A necessary condition for a complex valued function $f(z) = u+iv$ to be differentiable at $z = z_0$ is that $\left(\frac{\partial f}{\partial \bar{z}}\right)_{\text{at} z=z_0} = 0$.