

ADVANCED CALCULUS

MA11003

SECTION 11, 12, & 15CD

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Concepts Covered

Differential Calculus

Functions of Several Variables

□ Continuity

Continuity

A function $z = f(x, y)$ is said to be continuous at a point (x_0, y_0) if

I. $f(x, y)$ is defined at (x_0, y_0)

II. $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ exists

III. $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$

If a function $f(x, y)$ is continuous at every point in a domain D , then it is said to be continuous in D .

Continuity ($\epsilon - \delta$ Definition)

A function $z = f(x, y)$ is said to be continuous at a point (x_0, y_0) , if for a given $\epsilon > 0$, there exist a real number $\delta > 0$ such that

$$|f(x, y) - f(x_0, y_0)| < \epsilon \quad \text{whenever} \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

Removable Discontinuity

- I.* $f(x, y)$ is defined at (x_0, y_0)
- II.* $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ exists
- III.* $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) \neq f(x_0, y_0)$

Problem – 1:

Discuss the continuity of $f(x, y) = \begin{cases} \frac{2x^4 + 3y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ at origin.

We need to check if $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^4 + 3y^4}{x^2 + y^2} = 0$

Changing to polar coordinate:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^4 + 3y^4}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{2r^4 \cos^4 \theta + 3r^4 \sin^4 \theta}{r^2} = 0$$

Hence the function is continuous

Problem – 2:

Discuss the continuity of $f(x, y) = \begin{cases} \frac{(x - y)^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ at origin.

Choosing the path $y = mx$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x - y)^2}{x^2 + y^2} = \frac{(1 - m)^2}{(1 + m^2)} \quad \text{Limit depends on the path}$$

The limit does not exist. Hence the function is **not continuous at (0,0)**

Problem – 3:

Discuss the continuity of $f(x, y) = \begin{cases} \frac{\sin \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ at origin.

Changing to polar coordinate:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} = \lim_{r \rightarrow 0} \frac{\sin \sqrt{r^2}}{\sqrt{r^2}} = \lim_{r \rightarrow 0} \frac{\sin r}{r} = 1$$

The limit exists. But the function is **not continuous at (0,0)** as

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \neq 0 \quad (\text{Removable Discontinuity})$$

Problem – 4:

Discuss the continuity of $f(x, y) = \begin{cases} \frac{x^4 y^4}{(x^2 + y^4)^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ at origin.

Choosing the path $y^2 = mx$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 y^4}{(x^2 + y^4)^3} = \frac{m^2}{(1 + m^2)^3} \quad \text{Limit depends on the path}$$

The limit does not exist. Hence the function is **not continuous** at $(0,0)$

Problem – 5:

Discuss the continuity of $f(x, y) = \begin{cases} \frac{x^2 + y^2}{xy}, & xy \neq 0 \\ 0, & \text{elsewhere} \end{cases}$ at origin.

Choosing the path $y = mx$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{xy} = \frac{1 + m^2}{m} \quad \text{Limit depends on the path}$$

The limit does not exist. Hence the function is not continuous at (0,0)

Remark:

Changing to polar coordinate (subst. $x = r \cos \theta$, $y = r \sin \theta$) and investigating the limit of the resulting expression as $r \rightarrow 0$ is often very useful.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta}{r^2} = 0 \quad \text{Limit is 0}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^2 \cos^2 \theta}{r^2} = \cos^2 \theta \quad \text{Limit does not exist}$$

Remark:

Changing to polar coordinate (subst. $x = r \cos \theta$, $y = r \sin \theta$) *does not always help* and the transformation may tempt us to *false conclusion*.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^2 \theta \sin \theta}{r^4 \cos^4 \theta + r^2 \sin^2 \theta} = \lim_{r \rightarrow 0} \frac{r \cos^2 \theta \sin \theta}{r^2 \cos^4 \theta + \sin^2 \theta}$$

If we fix θ , then $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2} = 0$

Taking the path $r \sin \theta = r^2 \cos^2 \theta$ ($y = x^2$)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{r \rightarrow 0} \frac{r^2 \cos^2 \theta r^2 \cos^2 \theta}{r^4 \cos^4 \theta + r^4 \cos^4 \theta} = \frac{1}{2}$$

Limit does not exist

Problem – 6:

Discuss the continuity of $f(x, y) = \begin{cases} \frac{x^2 y^2}{x^3 + y^3}, & (x^3 + y^3) \neq 0 \\ 0, & \text{elsewhere} \end{cases}$ at origin.

Changing to polar coordinate ($x = r \cos \theta$, $y = r \sin \theta$):

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^3 + y^3} = \lim_{r \rightarrow 0} r \underbrace{\left(\frac{\cos^2 \theta \sin^2 \theta}{\cos^3 \theta + \sin^3 \theta} \right)}_{\text{Unbounded}}$$

Note that If we fix θ the limit is **ZERO**

Thus we cannot conclude that the **limit is ZERO**


Continuity of $f(x, y) = \begin{cases} \frac{x^2 y^2}{x^3 + y^3}, & (x^3 + y^3) \neq 0 \\ 0, & \text{elsewhere} \end{cases}$

Along the path $y = mx$ $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^3 + y^3} = 0$

Along the path $y = -xe^x$ $\lim_{x \rightarrow 0} \frac{x^2 x^2 e^{2x}}{x^3 - x^3 e^{3x}} = \lim_{x \rightarrow 0} \frac{x e^{2x}}{1 - e^{3x}} = -\frac{1}{3}$

The limit does not exist. Hence the function is **not continuous at (0,0)**

Problem – 7:

 Discuss the continuity of $f(x, y) = \begin{cases} \frac{e^{-\frac{1}{x^2+y^2}}}{x^4 + y^4}, & (x, y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$ at origin.

Changing to polar coordinate ($x = r \cos \theta$, $y = r \sin \theta$):

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{e^{-\frac{1}{x^2+y^2}}}{x^4 + y^4} &= \lim_{r \rightarrow 0} \frac{e^{-\frac{1}{r^2}}}{r^4(\cos^4 \theta + \sin^4 \theta)} = \lim_{r \rightarrow 0} \frac{e^{-\frac{1}{r^2}}}{r^4(1 - 2\cos^2 \theta \sin^2 \theta)} \\ &= \lim_{r \rightarrow 0} \frac{e^{-\frac{1}{r^2}}}{r^4 \left(1 - \frac{1}{2} \sin^2 2\theta\right)} \end{aligned}$$

$$\lim_{r \rightarrow 0} \frac{e^{-\frac{1}{r^2}}}{r^4 \left(1 - \frac{1}{2} \sin^2 2\theta\right)} = 0$$

Noting: $\frac{1}{2} \leq \left(1 - \frac{1}{2} \sin^2 2\theta\right) \leq 1$

For $r \neq 0$

$$0 < \frac{e^{-\frac{1}{r^2}}}{r^4 \left(1 - \frac{1}{2} \sin^2 2\theta\right)} < \frac{2e^{-\frac{1}{r^2}}}{r^4}$$

Using Sandwich (Squeeze) Theorem, we get the limit.

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{2e^{-\frac{1}{r^2}}}{r^4} &= \lim_{t \rightarrow \infty} 2e^{-t^2} t^4 \\ &= \lim_{t \rightarrow \infty} \frac{2t^4}{e^{t^2}} = \lim_{t \rightarrow \infty} \frac{4t^2}{e^{t^2}} \\ &= \lim_{t \rightarrow \infty} \frac{4}{e^{t^2}} = 0 \end{aligned}$$

KEY TAKEAWAY

CONTINUITY

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Needs to be careful while changing to polar coordinate!

Thank You