ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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Concepts Covered

Differential Calculus

Functions of Several Variables

- Differentiation of Composite Functions
- Differentiation of a Function Defined Implicitly

Differentiation of Composite Functions

Recall:

For the case $z = f(x, y)$

$$x = \phi(u, v), \qquad y = \psi(u, v)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

For the case $z = f(x)$

$$x = \phi(u, v)$$

$$\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v}$$

For the case $z = f(x)$

$$x = \phi(u, v, w)$$

$$\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v}$$

$$\frac{\partial z}{\partial w} = \frac{dz}{dx}\frac{\partial x}{\partial w}$$

Example: Find $\partial z/\partial u$ and $\partial z/\partial v$ if $z = \tan^{-1} x$ and $x = e^u + \ln v$

$$\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u} = \frac{1}{1+x^2}e^u = \frac{1}{1+(e^u+\ln v)^2}e^u$$

$$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v} = \frac{1}{1+x^2}\frac{1}{v} = \frac{1}{1+(e^u+\ln v)^2}\frac{1}{v}$$

Derivative of a function defined implicitly

Case – I : Functions of single variable

Let the function y of x be defined as $F(x,y) = 0$

Let $z = F(x,y) = 0$

$$\frac{dz}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \qquad \Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial y}; \qquad \frac{\partial F}{\partial y} \neq 0$$

Case – II : Functions of two Variables

Let the function z of x & y be defined as $F(x, y, z) = 0$

Differentiating F with respect to x

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \qquad \text{OR} \qquad \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial z}; \qquad \frac{\partial F}{\partial z} \neq 0$$

Differentiating F with respect to y

$$\Rightarrow \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \qquad \text{OR} \qquad \frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\bigg/\frac{\partial F}{\partial z}; \qquad \frac{\partial F}{\partial z} \neq 0$$

Example: Find $\partial z/\partial x$ and $\partial z/\partial y$ of $x^2 + y^2 + z^2 - c = 0$

Differentiating with respect to x

$$2x + 2z\frac{\partial z}{\partial x} = 0 \quad \Longrightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}$$

Differentiating with respect to y

$$2y + 2z\frac{\partial z}{\partial y} = 0 \quad \Longrightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$$

Thank You