Vector space (Def^n)

Let V be a non-empty set and R be the field of real nos. Let there be two compositions, one is '+' between two members of V and another is '.', between a member of V and a member of R. V is said to be a vector space or linear space and the elements of V are called vectors if the following axioms hold:

I (i) $\alpha+\beta \in V \ \forall \alpha \ \& \ \beta$ in V [closure property under '+']

(ii) $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma \ \forall \alpha, \beta$ and γ in V [Associative property under '+']

(iii) There is a unique vector in V, called the zero vector and denoted by 0 such that for every α in V, $\alpha+0=\alpha$.

(iv) Corresponding to each element α in V, $\exists$ an element '$-\alpha$' in V, such that $\alpha+(-\alpha)=0$

(v) $\alpha+\beta = \beta+\alpha \ \forall \ \alpha, \beta \in V$ [commutative property under '+']

II (i) $a.\alpha \in V \ \forall a \in R$ and $\forall \alpha \in V$

(ii) $1.\alpha = \alpha \ \forall \alpha \in V$

(iii) $(ab).\alpha = a.(b\alpha) \ \forall a, b \in R$ and $\forall \alpha \in V$

(iv) $a.(\alpha+\beta) = a.\alpha + a.\beta \ \forall a \in R$ and $\forall \alpha, \beta \in V$

(v) $(a+b).\alpha = a.\alpha + b.\alpha \ \forall a, b \in R$ and $\forall \alpha \in V$

A complex vector space is obtained, if instead of real nos. we take complex nos. as scalars.

Ex Let V be the set of all $m \times n$ matrices with entries from an arbitrary field K. Then V is a vector space over K w.r.t. the operations of matrix addition and scalar multiplication.

§ The set of all ordered n-tuples over F is a V.S. denoted by $V_n(F)$.

$V_2(F) = \{(a_1, a_2) ; a_1, a_2 \in F\}$ is the V.S. of all ordered pairs over F

$V_3(F) = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in F\}$ is the V.S. of all ordered triads over F.

Subspaces

Let W be a subset of a vector space over a field K. W is called a subspace of V if W is itself a vector space over K w.r.t. the operations of vector addition and scalar multiplication on V.

Theorem

W is a subspace of V if and only if

(i) W is nonempty

(ii) W is closed under vector addition: $v, w \in W \Rightarrow v+w \in W$

(iii) W is closed under scalar multiplication: $v \in W$ implies $kv \in W$ for every $k \in K$.

Corollary

W is a subspace of V if and only if

(i) $0 \in W$

(ii) $v, w \in W \Rightarrow av + bw \in W$ for every $a, b \in K$.

Ex Let V be any vector space. Then the set $\{0\}$ consisting of the zero vector alone and also the entire space V are subspaces of V.

Ex Consider any homogeneous system of linear equations in n unknowns

$$a_{11}x_1 + \cdots + a_{1n}x_n = 0$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n = 0$$

The set W of all solutions of the homogeneous system is a subspace of R^n. The solution set of a nonhomogeneous system of linear eqns. in n knowns is not a subspace of R^n.

Vector subspace spanned by a given set of vectors

A vector space which arises as a set of all linear combinations of any given set of vectors, is said to be spanned by the given set of vectors.

Ex The vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ generate the vector space R^3. For any vector $(a,b,c) \in R^3$ is a linear combination of e_i; $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1) = ae_1 + be_2 + ce_3$

The n-vector space

The set of all n-vectors of a field F is called the n-vector space over F and it is usually denoted by $V_n(F)$ or V_n.

Basis and dimension

A vector space V is said to be of finite dimension n or to be n dimensional, written $\dim V = n$, if $\exists$ L.I. vectors $e_1, e_2, \cdots, e_n$ which span V. The sequence $\{e_1, e_2, \cdots, e_n\}$ is then called a basis of V.

Theorem

Let V be a finite dimensional vector space. Then every basis V has the same number of elements.

Ex In R^3, $\{(1,0,0), (0,1,0), (0,0,1)\}$ and $\{(2,-1,0), (3,5,1), (1,1,2)\}$ are two bases.

Ex Let U be the V.S. of all 2×3 matrices over R. Then the matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ form a basis of U. Thus $\dim U = 6$

Theorem

Let V be of finite dimension n. Then

(i) Any set of $n+1$ or more vectors is L.D.

(ii) Any L.I. set is part of a basis

(iii) A L.I. set with n elements is a basis.

Theorem

Let W be a subspace of a n dimensional V.S. V. Then $\dim W \leq n$. In particular if $\dim W = n$, then $W = V$.

Ex Let W be a subspace of R^3. $\dim R^3 = 3$

$\dim W$ can be $0, 1, 2, 3$.

(i) $\dim W = 0$, then $W = \{0\}$ is a point

(ii) $\dim W = 1$, then W is a line through the origin

(iii) $\dim W = 2$, " W " " plane " " "

(iv) $\dim W = 3$, " W " the entire space R^3.

Row rank and column rank of a matrix

Let $A = [a_{ij}]$ be any $m \times n$ matrix. Each of the m rows of A consists of n elements. Therefore the row vectors of A are n vectors. These row vectors of A will span a subspace R of V_n. This subspace R is called the row space of A. The dimension r of R is called the row rank of A. In other words, the row rank of a matrix A is equal to the maximum no. of linearly independent rows of A.

Similarly the defn. for column rank of a matrix

Theorem

Row equivalent matrices have the same row rank

Column " " " " " column "

Row " " " " " " "

Column " " " " " row "

Theorem

The rank, row rank and column rank of a matrix are all equal.

14.1.19

Solution of a system of linear equations using rank concept:

Non-homogeneous system

Theorem (a) Existence A linear system of m equations in n unknowns $x_1, x_2, \ldots x_n$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \ldots \quad \ldots \quad \ldots \quad &\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \text{—— (1)}$$

has solutions if and only if the coefficient matrix A and the augmented matrix $\tilde{A}$ have the same rank.

(b) Uniqueness

The system (1) has exactly one solution if and only if this common rank r of A and $\tilde{A}$ is equal to n.

(c) Infinitely many solutions

If this rank r is less than n, the system (1) has infinitely many solutions. All of these are obtained by determining r suitable unknowns in terms of $n-r$ unknowns, to which arbitrary values can be assigned.

(d) Gauss elimination

If solutions exist, they all can be obtained by the Gauss elimination.

The homogeneous linear system

Theorem

A homogeneous linear system

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\} \quad \text{—(A)}$$

always has a trivial solution $x_1 = 0, \ldots, x_n = 0$. Nontrivial solⁿ. exist if and only if rank $A < n$. If rank $A = r < n$, these solutions together with $x = 0$ form a vector space of dimension $n - r$, called the solution space of (A).

In particular if X_1 and X_2 are solution vectors of (A), then $X = c_1X_1 + c_2X_2$, c_1, c_2 are any scalars is a solution of (A).

$$A(c_1X_1 + c_2X_2) = c_1(AX_1) + c_2(AX_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

The solution space of (A) is also called the null space of A because $Ax = 0$ for every x in the solution space. Its dimension is called the nullity of A.

$$\therefore \text{rank } A + \text{nullity } A = n$$

where n is the no. of unknowns.