ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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Concepts Covered

Differential Calculus

Functions of Several Variables

- Maxima and Minima

Local Extrema (Previous Lecture)

A point (a, b) will be a point of local extrema if

$$\Delta f = f(a+h, b+k) - f(a, b)$$

does not change its sign for all sufficiently small h & k

Taylor's Series $\Delta f = h\, f_x(a, b) + k\, f_y(a, b) + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$

Necessary Condition

$f_x(a, b) = 0$ & $f_y(a, b) = 0$

Sufficient condition for a function to have extremum

Notation: $r = f_{xx}(a,b)$, $s = f_{xy}(a,b)$, $t = f_{yy}(a,b)$

Consider $\Delta f = f(a+h, b+k) - f(a,b)$

$$\Delta f = h\, f_x(a,b) + k\, f_y(a,b) + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Since (a,b) is a critical point, $f_x(a,b) = 0$ & $f_y(a,b) = 0$, we have

$$\Delta f = \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

$$\Delta f = \frac{1}{2r}\left((hr+ks)^2 + k^2(rt - s^2)\right) + \cdots \qquad r \neq 0$$

$$\Delta f = \frac{1}{2r}\left((hr+ks)^2 + k^2(rt-s^2)\right) + \cdots$$

Case – I: $rt - s^2 > 0$

$$\Delta f > 0 \quad \text{if } r > 0$$

$$\Delta f < 0 \quad \text{if } r < 0$$

The point (a, b) is a point of minimum if $rt - s^2 > 0,\quad r > 0$

The point (a, b) is a point of maximum if $rt - s^2 > 0,\quad r < 0$

$$\Delta f = \frac{1}{2r}\left((hr + ks)^2 + k^2(rt - s^2)\right) + \cdots$$

Case – II: $rt - s^2 < 0$

Let $k \to 0$ & $h \neq 0 \quad \Longrightarrow \quad \Delta f > 0 \quad \text{if } r > 0$

Let $k \neq 0$ & choose h such that $hr + ks = 0 \quad \Longrightarrow \quad \Delta f < 0 \quad \text{if } r > 0$

$\Longrightarrow$ The sign of Δf depends on h & k

Hence no maximum/minimum of f can occur at $P(a, b)$.

$\Longrightarrow$ The point $P(a, b)$ is a saddle point

$$\Delta f = \frac{1}{2r}\left((hr + ks)^2 + k^2(rt - s^2)\right) + \cdots$$

Case – III: $rt - s^2 = 0$

$$\Delta f = \frac{1}{2r}(hr + ks)^2 + \cdots$$

If we take h & k such that $hr = -ks$, then the whole second order terms of the right hand side will vanish.

Therefore, the conclusion will depend on the higher order terms.

One has to find some other way to investigate such points.

Working rules for investigating local extrema

- Find all critical points $f_x = 0 \;\; \& \;\; f_y = 0$
- For each critical point, evaluate

$$r = f_{xx}, \qquad s = f_{xy}, \qquad t = f_{yy}$$

- Identification
  - If $rt - s^2 > 0 \;\&\; r < 0$ maximum
  - If $rt - s^2 > 0 \;\&\; r > 0$ minimum
  - If $rt - s^2 < 0$ Saddle point
  - If $rt - s^2 = 0$ Test Fails needs further investigation

Example: Find all critical points of $f(x, y) = x^3 - 6x^2 - 8y^2$ and investigate their nature for local maximum/minimum and saddle point.

Critical points: $\left.\begin{array}{l} f_x = 0 \\ \\ f_y = 0 \end{array}\right\} \Longrightarrow (0,0) \ \ \& \ \ (4,0)$

	$(0,0)$	$(4,0)$
$r = f_{xx}$	-12	12
$s = f_{xy}$	0	0
$t = f_{yy}$	-16	-16
$rt - s^2$	192	-192

$(0,0)$ is a point of local maximum & $(4, 0)$ is a saddle point.

Problem - 1 Discuss local extrema of the function $f(x,y) = (4x^2 + y^2)e^{-x^2-4y^2}$

$$f_x(x,y) = 2x\, e^{-x^2-4y^2}(4 - 4x^2 - y^2)$$

$$f_y(x,y) = 2y\, e^{-x^2-4y^2}(1 - 16x^2 - 4y^2)$$

Critical Points:

$$(0,0), \left(0, \frac{1}{2}\right), \left(0, -\frac{1}{2}\right), (1,0), (-1,0)$$

$$f_x(x,y) = 2\,e^{-x^2-4y^2}(4x - 4x^3 - xy^2)$$

$$r = f_{xx}(x,y) = 2\,e^{-x^2-4y^2}(4 - 20x^2 + 8x^4 - y^2 + 2\,x^2y^2)$$

$$f_y(x,y) = 2\,e^{-x^2-4y^2}(y - 16y\,x^2 - 4y^3)$$

$$s = f_{xy}(x,y) = 4\,xy\,e^{-x^2-4y^2}(-17 + 16x^2 + 4y^2)$$

$$f_y(x,y) = 2\,e^{-x^2-4y^2}(y - 16y\,x^2 - 4y^3)$$

$$t = f_{yy}(x,y) = 2\,e^{-x^2-4y^2}(1 - 20y^2 - 16x^2 - 128\,x^2y^2 + 32y^4)$$

$$r = 2\, e^{-x^2-4y^2}(4 - 20x^2 + 8x^4 - y^2 + 2\, x^2y^2)$$

$$s = 4\, xy\, e^{-x^2-4y^2}(-17 + 16x^2 + 4y^2)$$

$$t = 2\, e^{-x^2-4y^2}(1 - 20y^2 - 16x^2 - 128\, x^2y^2 + 32y^4)$$

Identification

$P_1(0,0)$: $r = 8$ $s = 0$ $t = 2$

$$rt - s^2 = 16 > 0$$

$\Longrightarrow$ The point $P_1(0,0)$ is a local minimum.

$$r = 2\, e^{-x^2-4y^2}(4 - 20x^2 + 8x^4 - y^2 + 2\, x^2y^2) \qquad s = 4\, xy\, e^{-x^2-4y^2}(-17 + 16x^2 + 4y^2)$$

$$t = 2\, e^{-x^2-4y^2}(1 - 20y^2 \; - 16x^2 - 128\, x^2y^2 + 32y^4)$$

$P_{2/3}(0, \pm 1/2)$: $\quad r = \dfrac{15}{2e} \qquad s = 0 \qquad t = -\dfrac{4}{\mathrm{e}} \qquad rt - s^2 = \; -\dfrac{30}{e^2} \; < 0$

$\Longrightarrow$ The point $P_{2/3}$ are saddle points

$P_{4/5}(\pm 1{,}0)$: $\quad r = -\dfrac{16}{e} \qquad s = 0 \qquad t = -\dfrac{30}{\mathrm{e}} \qquad rt - s^2 = \dfrac{480}{e^2} > 0$

$\Longrightarrow$ The point $P_{4/5}$ are local maxima.

Problem - 2 Discuss local extrema of the function $f(x,y) = y^2 + x^2y + x^4$

$f_x = 2xy + 4x^3$ $\quad f_y = 2y + x^2$ $\quad$ Stationary points: $(0,0)$

$r = f_{xx}(0,0) = 0$ $\quad s = f_{xy}(0,0) = 0$ $\quad t = f_{yy}(0,0) = 2$

$\Longrightarrow rt - s^2 = 0$ $\quad$ Test fails!

Consider $\quad \Delta f = f(0+h, 0+k) - f(0,0) = k^2 + h^2k + h^4$

$$= \left(\frac{k}{2} + h^2\right)^2 + \frac{3}{2}k^2 > 0, \qquad \forall\, h \neq 0, k \neq 0$$

$\Longrightarrow$ $(0,0)$ is a point of local minimum.

Problem - 3 Discuss local extrema of the function $f(x,y) = 2x^4 - 3x^2y + y^2$

$f_x = 8x^3 - 6xy$ $\quad$ $f_y = -3x^2 + 2y$ $\quad$ Stationary points: $(0,0)$

$r = f_{xx}(0,0) = 0$ $\quad$ $s = f_{xy}(0,0) = 0$ $\quad$ $t = f_{yy}(0,0) = 2$

$\Longrightarrow rt - s^2 = 0$ $\quad$ Test fails!

Consider $\quad \Delta f = f(0+h, 0+k) - f(0,0) = 2h^4 - 3h^2k + k^2$

$$= 2h^4 - 2h^2k - h^2k + k^2 = 2h^2(h^2 - k) - k(h^2 - k)$$

$$= (h^2 - k)(2h^2 - k)$$

For $k < 0,\ \Delta f > 0$ $\quad$ For $h^2 < k < 2h^2,\ \Delta f < 0$

$\Longrightarrow (0,0)$ is a saddle point

Problem - 4 Find the absolute maximum and minimum values of

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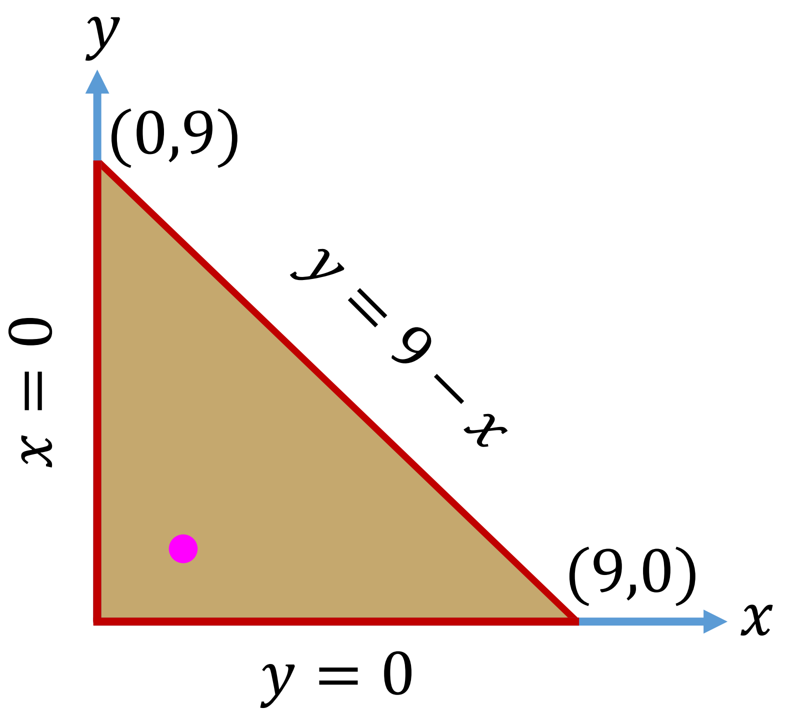
Calculus and Analytical Geometry

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular plate in the first quadrant bounded by the lines $x = 0, y = 0, y = 9 - x$



Interior Points: Stationary points

$$\left.\begin{aligned} f_x &= 2 - 2x = 0 \\ f_y &= 2 - 2y = 0 \end{aligned}\right\} \quad (x,y) = (1,1)$$

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

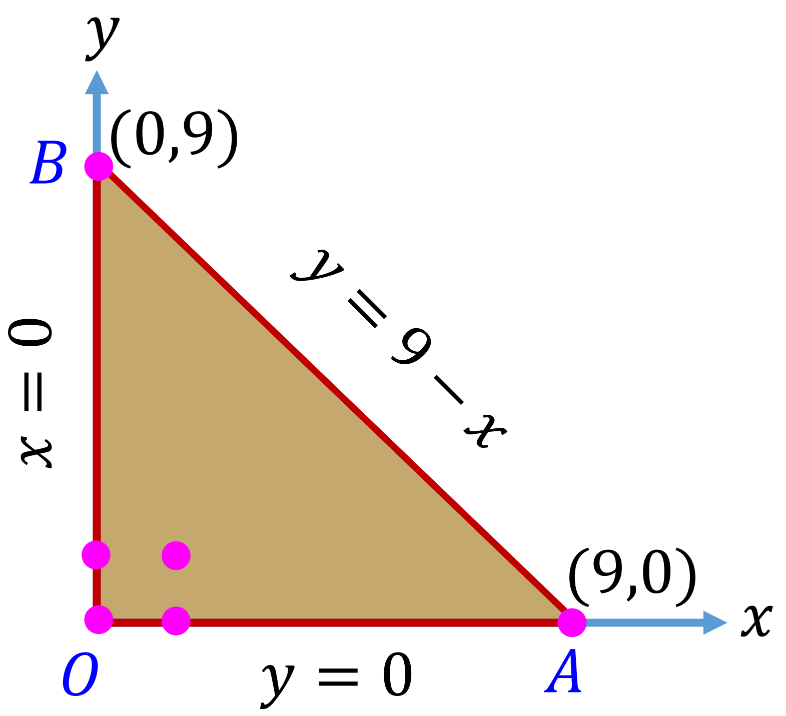
Boundary Points:

Along OA $\quad f = 2 + 2x - x^2, \qquad x \in [0,9]$

Stationary points $\quad f_x = 0 \Longrightarrow x = 1$

Possible candidates (points) for extrema along this boundary:

$(0,0) \qquad (9,0) \qquad (1,0)$



Along OB $\quad f = 2 + 2y - y^2, \qquad y \in [0,9]$

Possible candidates (points) for extrema along this boundary:

$(0,0) \qquad (0,9) \qquad (0,1)$

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

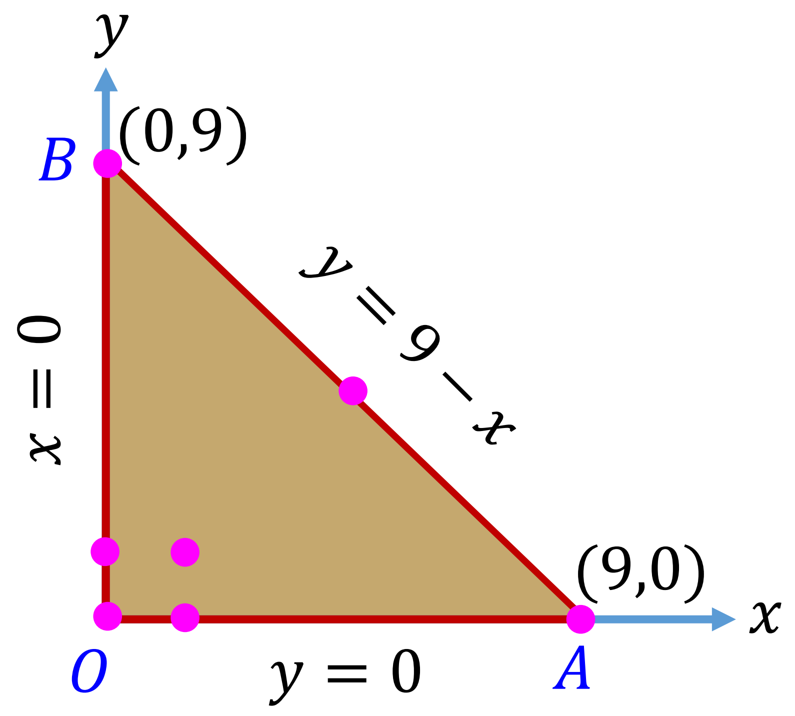
Boundary Points:

Along AB: $\quad y = 9 - x$

$$f = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2, \qquad x \in [0,9]$$

$$f = -61 + 18x, -2x^2, \qquad x \in [0,9]$$

$$f_x = 0 \implies (x, y) = \left(\frac{9}{2}, \frac{9}{2}\right)$$



(x, y)	(1,1)	(0,0)	(1,0)	(9,0)	(0,1)	(0,9)	(9/2,9/2)
f	**4**	2	3	$\mathbf{-61}$	3	$\mathbf{-61}$	$-41/2$

The Maximum is **4** and the minimum value is $\mathbf{-61}$

KEY TAKEAWAY

Maxima/minima can occur only at

- Boundary points of the domain (closed and bounded domain)
- Critical points $(f_x = 0 = f_y)$

Thank You