The Cayley-Hamilton Theorem

Every square matrix satisfies its characteristic eqn.

Ex Find the characteristic eqn. of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and verify that it is satisfied by A and hence obtain A^{-1}

Sol^n: $|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$

$$= (2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{-1(2-\lambda)+1\}$$
$$+ 1\{1-(2-\lambda)\}$$

$$= (2-\lambda)(3-4\lambda+\lambda^2) + (\lambda-1) + (\lambda-1)$$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

$\therefore$ the characteristic eqn of the matrix A is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$\therefore$ To verify $A^3 - 6A^2 + 9A - 4I = 0$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = A \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+ 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = 0$$

Multiplying by A^{-1},

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I\right]$$

$$A^2 - 6A + 9I = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Ex Use C-H theorem to find A^{-1} where $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$

Solⁿ: C. eqn. $\lambda^2 - 7\lambda + 7 = 0$

i.e. $A^2 - 7A + 7I_2 = 0$

$A - 7I_2 + 7A^{-1} = 0$

$$\therefore A^{-1} = -\frac{1}{7}(A - 7I_2)$$

$$= \frac{1}{7}\begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$$

Ex Use Cayley-Hamilton theorem to express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A, when

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

Solⁿ: $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)+1$

$= \lambda^2 - 5\lambda + 7$

$\therefore$ characteristic eqn. is $\lambda^2 - 5\lambda + 7 = 0$

By C-H theorem $A^2 - 5A + 7I = 0$

$\therefore A^2 = 5A - 7I$

$\therefore A^3 = 5A^2 - 7A$

$A^4 = 5A^3 - 7A^2$

$A^5 = 5A^4 - 7A^3$

$$2A^5 - 3A^4 + A^2 - 4I = 2(5A^4 - 7A^3) - 3A^4 + A^2 - 4I$$

$$= 7A^4 - 14A^3 + A^2 - 4I$$

$$= 7(5A^3 - 7A^2) - 14A^3 + A^2 - 4I$$

$$= 21A^3 - 48A^2 - 4I$$

$$= 21(5A^2 - 7A) - 48A^2 - 4I$$

$$= 57A^2 - 147A - 4I$$

$$= 57(5A - 7I) - 147A - 4I$$

$$= 138A - 403I$$

Ex Use C-H theorem to find A^{50} where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Solⁿ:

$$A^2 - 2A + I_2 = 0$$

$$A^2 - A = A - I_2$$

$$A^3 - A^2 = A^2 - A$$

$$= A - I_2$$

$$\cdots\cdots\cdots$$

$$A^{50} - A^{49} = A - I_2$$

$\therefore$ $A^{50} = 50A - 49I_2$ by adding

$$= \begin{pmatrix} 50 & 50 \\ 0 & 50 \end{pmatrix} - \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} = \begin{pmatrix} 1 & 50 \\ 0 & 1 \end{pmatrix}$$