ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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DIFFERENTIATION UNDER INTEGRAL SIGN

- **Leibnitz Rule**
- **Derivation**
- **Worked Problems**

Mean Value Theorems (Recall)

- Lagrange Mean Value Theorem

$$\frac{f(b)-f(a)}{b-a}=f'(\xi\,), \qquad \xi\in(a,b)$$

- Mean Value Theorem of Integral Calculus

$$\int_a^b g(x)\,dx=(b-a)\,g(\xi\,), \qquad \xi\in(a,b)$$

Note

$$\frac{1}{b-a}\int_a^b f'(x)\,dx=f'(\xi)$$

Let $f'(x)=g(x)$

Leibnitz Rule

Let $\Phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)dx$

where $u_1(\alpha)$ & $u_2(\alpha)$ posses continuous first order derivatives with respect to α.

$$\frac{\mathrm{d}\phi}{\mathrm{d}\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\alpha)\,dx + f(u_2(\alpha),\alpha)\frac{\mathrm{d}u_2}{\mathrm{d}\alpha} - f(u_1(\alpha),\alpha)\frac{\mathrm{d}u_1}{d\alpha}$$

Leibnitz Rule

Let $$\Phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\, dx$$

$$\Delta\Phi = \Phi(\alpha + \Delta\alpha) - \Phi(\alpha)$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)dx - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)dx$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_1(\alpha)} f(x,\alpha+\Delta\alpha)dx + \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha+\Delta\alpha)dx$$

$$+ \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)dx - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)dx$$

$$\Delta\Phi = \int_{u_1(\alpha+\Delta\alpha)}^{u_1(\alpha)} f(x,\alpha+\Delta\alpha)dx + \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha+\Delta\alpha)dx$$

$$+ \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)dx - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)dx$$

$$\Delta\Phi = \int_{u_1(\alpha)}^{u_2(\alpha)} [\, f(x,\alpha+\Delta\alpha) - f(x,\alpha)]\, dx + \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)dx$$

$$- \int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)dx$$

Using Mean Value theorem

$$\int_{u_1(\alpha)}^{u_2(\alpha)} [f(x,\alpha+\Delta\alpha) - f(x,\alpha)]\ dx = \Delta\alpha \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\xi_1)\,dx \qquad \xi_1 \in (\alpha,\alpha+\Delta\alpha)$$

$$\int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)dx = f(\xi_2,\alpha+\Delta\alpha)[u_2(\alpha+\Delta\alpha) - u_2(\alpha)]$$

$$\xi_2 \in \left(u_2(\alpha), u_2(\alpha+\Delta\alpha)\right)$$

$$\int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)dx = f(\xi_3,\alpha+\Delta\alpha)[u_1(\alpha+\Delta\alpha) - u_1(\alpha)]$$

$$\xi_3 \in \left(u_1(\alpha), u_1(\alpha+\Delta\alpha)\right)$$

$$\Delta\Phi = \int_{u_1(\alpha)}^{u_2(\alpha)} [\, f(x, \alpha + \Delta\alpha) - f(x, \alpha)]\, dx \; + \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x, \alpha + \Delta\alpha) dx$$

$$- \int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x, \alpha + \Delta\alpha) dx$$

$$\frac{\Delta\Phi}{\Delta\alpha} = \frac{\Delta\alpha}{\Delta\alpha} \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x, \xi_1)\, dx + \; f(\xi_2, \alpha + \Delta\alpha) \frac{[u_2(\alpha + \Delta\alpha) - u_2(\alpha)]}{\Delta\alpha}$$

$$+ f(\xi_3, \alpha + \Delta\alpha) \frac{[u_1(\alpha + \Delta\alpha) - u_1(\alpha)]}{\Delta\alpha}$$

$$\frac{\Delta\phi}{\Delta\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x, \xi_1)\, dx + f(\xi_2, \alpha + \Delta\alpha)\frac{\Delta u_2}{\Delta\alpha} - f(\xi_3, \alpha + \Delta\alpha)\frac{\Delta u_1}{\Delta\alpha}$$

$\xi_1 \in (\alpha, \alpha + \Delta\alpha)$

$\xi_2 \in \left(u_2(\alpha), u_2(\alpha + \Delta\alpha)\right)$

$\xi_3 \in \left(u_1(\alpha), u_1(\alpha + \Delta\alpha)\right)$

Taking the limit as $\Delta\alpha \to 0$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x, \alpha)\, dx + f(u_2(\alpha), \alpha)\frac{\mathrm{d}u_2}{\mathrm{d}\alpha} - f(u_1(\alpha), \alpha)\frac{\mathrm{d}u_1}{d\alpha}$$

Particular Case: Assume that $u_1(\alpha)$ and $u_2(\alpha)$ are some constants. Then

$$\frac{d\Phi(\alpha)}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha} dx \quad \text{OR} \quad \frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha} dx$$

Example - 1 Show that $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}dx = \frac{\pi}{2}\ln(1+a), \qquad if\ a \geq 0$

Let $\Phi(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}dx$

$$\Phi'(a) = \int_0^\infty \frac{1}{(1+a^2x^2)(1+x^2)}dx = \int_0^\infty \frac{1}{1-a^2}\left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right]dx$$

Solve

$$= \frac{1}{1-a^2}\left[\tan^{-1} x - a\tan^{-1} ax\right]\Big|_0^\infty = \frac{1}{1-a^2}\frac{\pi}{2}(1-a)$$

$$\Phi'(a) = \frac{1}{1+a}\frac{\pi}{2}$$

$$\Phi(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}dx$$

Integrating $\phi(a) = \frac{\pi}{2}\ln(1+a) + c$

Note that $\phi(0) = 0 \implies 0 = \frac{\pi}{2}\ln 1 + c \implies c = 0$

$$\implies \phi(a) = \frac{\pi}{2}\ln(1+a)$$

Example - 2 $$\int_0^\infty e^{-x^2}\cos\alpha x\,dx = \frac{\sqrt{\pi}}{2}e^{-\frac{\alpha^2}{4}}$$

$$\phi(\alpha) = \int_0^\infty e^{-x^2}\cos\alpha x\,dx \qquad \Longrightarrow \phi'(\alpha) = -\int_0^\infty e^{-x^2}\,x\;\sin\alpha x\;\,dx$$

Integrating right hand side by parts

$$\phi'(\alpha) = \frac{e^{-x^2}}{2}\sin\alpha x\,\Big|_0^\infty + \int_0^\infty\left(-\frac{e^{-x^2}}{2}\right)\cos\alpha x\;\,\alpha\,dx = -\frac{\alpha}{2}\phi(\alpha)$$

$$\phi'(\alpha) = -\frac{\alpha}{2}\phi(\alpha) \quad \Longrightarrow \quad \frac{\phi'(\alpha)}{\phi(\alpha)} = -\frac{\alpha}{2}$$

$$\phi(\alpha) = \int_0^\infty e^{-x^2}\cos\alpha x\,dx$$

$$\ln\phi(\alpha) = -\frac{\alpha^2}{4} + c \quad \Longrightarrow \quad \phi(\alpha) = c_1 e^{-\frac{\alpha^2}{4}}$$

Note that $\phi(0) = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \quad \Longrightarrow \quad \frac{\sqrt{\pi}}{2} = c_1$

$$\Longrightarrow \int_0^\infty e^{-x^2}\cos\alpha x\,dx = \frac{\sqrt{\pi}}{2}e^{-\frac{\alpha^2}{4}}$$

Example - 3 Let $\phi(\alpha) = \int_{\alpha}^{\alpha^2} \frac{\sin \alpha x}{x}\, dx$. Find $\phi'(\alpha)$ where $\alpha \neq 0$.

$$\phi'(\alpha) = \int_{\alpha}^{\alpha^2} \frac{\cos \alpha x}{x}\, x\, dx + 2\alpha \frac{\sin \alpha^3}{\alpha^2} - \frac{\sin \alpha^2}{\alpha}$$

$$= \frac{\sin \alpha x}{\alpha}\Big|_{\alpha}^{\alpha^2} + \frac{2 \sin \alpha^3}{\alpha} - \frac{\sin \alpha^2}{\alpha}$$

$$= \frac{3 \sin \alpha^3 - 2 \sin \alpha^2}{\alpha}$$

Conclusion:

Leibnitz Rule

$$\Phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha) dx$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\alpha)\, dx + f(u_2(\alpha),\alpha)\frac{\mathrm{d}u_2}{\mathrm{d}\alpha} - f(u_1(\alpha),\alpha)\frac{\mathrm{d}u_1}{d\alpha}$$

Thank You