All orders forward differences can be written in a tabular form. This difference table is called forward difference table or diagonal difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Backward differences

The first order backward difference of $f(x)$ is defined as

$$\nabla f(x) = f(x) - f(x-h) \qquad \nabla \equiv \text{backward diff. operator}$$

Thus $\nabla f(x_1) = f(x_1) - f(x_0)$

i.e. $\nabla y_1 = y_1 - y_0$

Similarly $\nabla y_2 = y_2 - y_1$

$\nabla y_n = y_n - y_{n-1}$

The 2nd order differences are

$$\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1) = y_2 - 2y_1 + y_0$$

$$\nabla^2 y_3 = y_3 - 2y_2 + y_1$$

In general, $\nabla^k y_i = y_i - {}^kC_1 y_{i-1} + {}^kC_2 y_{i-2} - \cdots (-1)^k y_{i-k}$

The table shows how the backward differences of all orders can be formed. The backward difference table is sometimes called horizontal difference table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
x_1	y_1	∇y_1			
x_2	y_2	∇y_2	$\nabla^2 y_2$		
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$	
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$

It is clear that $\nabla[f(x+h)] = \Delta f(x)$, $\nabla^2[f(x+2h)] = \Delta^2 f(x)$

and $\nabla^n[f(x+nh)] = \Delta^n f(x)$

Also $\Delta^r f(x) =$ constant if $r=n$
$= 0$ if $r>n$

Newton's forward difference interpolation formula

Let $y=f(x)$ be a f^n. whose explicit form is unknown. But the values of y at the equispaced points $x_0, x_1, \cdots x_n$ i.e. $y_i = f(x_i)$, $i = 0, 1, \cdots n$ are known. Since $x_0, x_1, \cdots x_n$ are equispaced, hence $x_i = x_0 + ih$, $i = 0, 1, \cdots, n$ where h is the spacing. It is required to construct a polynomial $\varphi(x)$ of degree less than or equal to n satisfying the conditions $y_i = \varphi(x_i)$, $i=0,1,\cdots n$ — (1)

Since $\varphi(x)$ is a polynomial of degree at most n, so $\varphi(x)$ can be taken in the following form

$$\varphi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad —(2)$$

where $a_0, a_1, \ldots a_n$ are constants whose values are to be determined, using (1).

To determine the values of a_i's, substituting $x = x_i$, $i = 0, 1, 2, \ldots n$

When $x = x_0$

$$\phi(x_0) = a_0 \quad \text{or} \quad a_0 = y_0$$

For $x = x_1$, $\phi(x_1) = a_0 + a_1(x_1 - x_0)$

or, $y_1 = y_0 + a_1 h$

$$\Rightarrow a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

For $x = x_2$, $\phi(x_2) = a_0 + a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1)$

$$\Rightarrow y_2 = y_0 + \frac{y_1 - y_0}{h} \cdot 2h + a_2(2h)h$$

$$\Rightarrow a_2 = \frac{y_2 - 2y_1 + y_0}{2!\,h^2} = \frac{\Delta^2 y_0}{2!\,h^2}$$

In this way, $a_3 = \dfrac{\Delta^3 y_0}{3!\,h^3} \quad \ldots \quad a_n = \dfrac{\Delta^n y_0}{n!\,h^n}$

Using these values,

$$\phi(x) = y_0 + (x - x_0)\frac{\Delta y_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 y_0}{2!\,h^2} + \cdots$$

$$+ (x - x_0)(x - x_1)\cdots(x - x_{n-1})\frac{\Delta^n y_0}{n!\,h^n} \quad \text{—(3)}$$

Introducing the condⁿ. $x_i = x_0 + ih$, $i = 0, 1, \ldots n$ for equispaced pts. and a new variable u as $x = x_0 + uh$

$$\therefore x - x_i = (u - i)h$$

So (3) becomes

$$\phi(x) = y_0 + (uh)\frac{\Delta y_0}{h} + uh(u-1)h\frac{\Delta^2 y_0}{2!\,h^2} + \cdots$$

$$+ uh(u-1)h(u-2)h \cdots (u-\overline{n-1})h\frac{\Delta^n y_0}{n!\,h^n}$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots + \frac{u(u-1)\cdots(u-\overline{n-1})}{n!}\Delta^n y_0$$

where $u = \frac{x - x_0}{h}$

This is known as Newton's forward difference interpolation formula.

Ex The following table gives the value of e^x for certain equidistant values of x. Find the value of e^x when $x = 0{\cdot}612$ using Newton's forward difference interpolation formula.

x:	0·61	0·62	0·63	0·64	0·65
y:	1·840431	1·858928	1·877610	1·896481	1·915541

Soln: The forward difference table is

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	0·61	1·840431	0·018497	0·000185	0·000004	−·000004
x_1	0·62	1·858928	0·018682	0·000189	0·0	
x_2	0·63	1·877610	0·018871	0·000189		
x_3	0·64	1·896481	0·019060			
x_4	0·65	1·915541				

Here $x_0 = 0.61$, $x = 0.612$, $h = 0.01$, $u = \frac{x - x_0}{h} = \frac{0.612 - 0.61}{0.01} = 0.2$

Then

$$y(0.612) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 1.840431 + 0.2 \times 0.018497 + \frac{0.2(0.2-1)}{2} \times 0.000185$$

$$+ \frac{0.2(0.2-1)(0.2-2)}{6} \times 0.000004$$

$$= 1.840431 + 0.003699 - 0.000015 + 0.00000019$$

$$= 1.844115$$

Error in Newton's forward formula

The error in any polynomial interpolation formula is

$$E(x) = (x - x_0)(x - x_1) \cdots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$= u(u-1)(u-2) \cdots (u-n) h^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad [\text{by } x = x_0 + uh]$$

where ξ lies between $\min\{x_0, x_1, \ldots x_n, x\}$ and $\max\{x_0, x_1, \ldots x_n, x\}$

Newton's forward formula is used to compute the approximate value of $f(x)$ when the argument x is near the beginning of the table. But this formula is not appropriate to compute $f(x)$ when x is at the end of the table. In this situation Newton's backward formula is appropriate.

Newton's backward difference interpolation formula

Suppose, a set of values $y_0, y_1, \cdots y_n$ of the f^n. $y=f(x)$ is given at $x_0, x_1, \cdots, x_n$ i.e. $y_i = f(x_i)$ $i=0,1,\cdots n$. Let us consider the polynomial $\phi(x)$ in the following form

$$f(x) \approx \phi(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + \cdots$$
$$+ a_n(x-x_n)(x-x_{n-1})\cdots(x-x_1) \quad -(1)$$

The constants a_i's are to be determined using the conditions

$y_i = \phi(x_i)$ $\quad i=0,1,\cdots n \quad -(2)$

Substituting $x = x_n, x_{n-1}, \cdots, x_1$ in (1), we obtain

$$\phi(x_n) = a_0 \quad \text{or} \quad a_0 = y_n$$

$$\phi(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n) \quad \text{or}, \quad y_{n-1} = y_n + a_1(-h) \Rightarrow a_1 = \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_n}{h}$$

$$\phi(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$= y_n + \frac{y_n - y_{n-1}}{h}(-2h) + a_2(-2h)(-h)$$

$$\Rightarrow y_{n-2} = 2y_{n-1} - y_n + a_2\, 2!\, h^2 \quad \Rightarrow a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2!\, h^2} = \frac{\nabla^2 y_n}{2!\, h^2}$$

$$a_3 = \frac{\nabla^3 y_n}{3!\, h^3} \quad \cdots \quad a_n = \frac{\nabla^n y_n}{n!\, h^n}$$

When the values of a_i's are substituted in (1), then the polynomial $\varphi(x)$ becomes

$$\varphi(x) = y_n + (x-x_n)\frac{\nabla y_n}{h} + (x-x_n)(x-x_{n-1})\frac{\nabla^2 y_n}{2!h^2}$$
$$+ \cdots + (x-x_n)(x-x_{n-1})\cdots(x-x_1)\frac{\nabla^n y_n}{n!\,h^n} \quad -(3)$$

We introduce a new variable as $v = \frac{x-x_n}{h}$. Also for equispaced points $x_i = x_0 + ih$. Then $x - x_{n-i} = (x_n + vh) - (x_0 + \overline{n-i}h)$

$$= (x_n - x_0) + (v - \overline{n-i})h$$
$$= (v+i)h \qquad i = 0, 1, \cdots n$$

$= nh + vh - \overline{n-i}\,h$
$= nh - nh + ih + vh$
$= (v+i)h$

Using the above result (3) becomes

$$\varphi(x) = y_n + vh\frac{\nabla y_n}{h} + vh(v+1)h\frac{\nabla^2 y_n}{2!h^2} + \cdots$$
$$+ vh(v+1)h(v+2)h \cdots (v+n-1)h\frac{\nabla^n y_n}{n!\,h^n}$$

$$= y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$
$$+ \frac{v(v+1)(v+2)\cdots(v+n-1)}{n!}\nabla^n y_n$$

This formula is known as Newton's backward difference interpolation formula.

Ex From the following table of values of x and $f(x)$ determine the value of $f(0.29)$ using Newton's backward interpolation formula.

x :	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$:	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139