Linear transformation

Let U and V be two vector spaces over the same field F.
A mapping $T : U \to V$ is said to be a linear mapping or linear transformation if it satisfies the following conditions

1. $T(\alpha+\beta) = T(\alpha) + T(\beta) \;\forall\; \alpha, \beta$ in U
2. $T(c\alpha) = cT(\alpha) \;\forall\; c$ in F and all α in U

These two conditions can be replaced by a single condition

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad \forall\; a, b \text{ in } F \;\&\; \forall\; \alpha, \beta \text{ in } U.$$

Ex The function $T : V_3(R) \to V_2(R)$ defined by $T(a,b,c) = (a,b)$ $\forall\; a,b,c \in R$ is a L.T. from $V_3(R)$ to $V_2(R)$.

Sol^n: Let $\alpha = (a_1, b_1, c_1)$, $\beta = (a_2, b_2, c_2) \in V_3(R)$

If $a, b \in R$, then

$$T(a\alpha + b\beta) = T\left[a(a_1, b_1, c_1) + b(a_2, b_2, c_2)\right]$$
$$= T(aa_1 + ba_2,\, ab_1 + bb_2,\, ac_1 + bc_2)$$
$$= (aa_1, ab_1) + \cancel{b}$$
$$= (aa_1 + ba_2,\, ab_1 + bb_2)$$
$$= (aa_1, ab_1) + (ba_2, bb_2)$$
$$= a(a_1, b_1) + b(a_2, b_2)$$
$$= aT(a_1, b_1, c_1) + bT(a_2, b_2, c_2)$$
$$= aT(\alpha) + bT(\beta)$$

$\therefore$ T is a L.T.

Ex Let $F : R^2 \to R^2$ be a mapping defined by $F(x,y) = (x+1, y+2)$.
Here $F(0,0) = (1,2) \neq 0$. i.e. the zero vector is not mapped into the zero vector. Hence F is not linear.

Image or range of a linear mapping

Let $U(F)$ and $V(F)$ be two vector spaces and let T be a L.T. from U to V. Then the range of T written as $R(T)$ (or image of T written as $Im(T)$) is the set of all vectors β in V such that $\beta = T(\alpha)$ for some α in U.

$$Im\, T = \{ T(\alpha) \in V, \alpha \in U \}$$

Kernel of a linear transformation

Let $U(F)$ and $V(F)$ be two vector spaces and let T be a L.T. from U to V. Then the kernel of T (or null space of T) written as $Ker\, T$ (or $N(T)$) is the set of all vectors α in U such that $T(\alpha) = 0$ (zero vector of V). Thus

$$Ker\, T = \{ \alpha \in U ;\ T(\alpha) = 0 \in V \}$$

Theorem

If $U(F)$ and $V(F)$ are two vector spaces and T is a L.T. from U to V, then $Im\, T$ is a subspace of V.

Proof Obviously $Im(T)$ is a non-empty subset of V. Let $\beta_1, \beta_2 \in Im(T)$. Then $\exists$ vectors α_1, α_2 in U such that $T(\alpha_1) = \beta_1$, $T(\alpha_2) = \beta_2$.

Let a, b be any elements of the field F. We have

$$a\beta_1 + b\beta_2 = aT(\alpha_1) + bT(\alpha_2) = T(a\alpha_1 + b\alpha_2) \quad \because T \text{ is a L.T.}$$

Now U is a V.S. Therefore $\alpha_1, \alpha_2 \in U$ and $a, b \in F \Rightarrow a\alpha_1 + b\alpha_2 \in U$

Consequently, $T(a\alpha_1 + b\alpha_2) = a\beta_1 + b\beta_2 \in Im(T)$

Thus $a, b \in F$ and $\beta_1, \beta_2 \in Im\, T \Rightarrow a\beta_1 + b\beta_2 \in Im(T)$

$\therefore Im(T)$ is a subspace of V.

Theorem

Let $U(F)$ and $V(F)$ are two vector spaces and T is a L.T. from U to V, then the kernel of T or the null space of T is a subspace of U.

Proof Let $\text{Ker } T = \{\alpha \in U : T(\alpha) = 0 \in V\}$

$\because T(0) = 0 \in V$, therefore at least $0 \in \text{Ker } T$

Thus $\text{Ker } T$ is a non-empty subset of U.

Let $\alpha_1, \alpha_2 \in \text{Ker } T$. Then $T(\alpha_1) = 0$ and $T(\alpha_2) = 0$

Let $a, b \in F$. Then $a\alpha_1 + b\alpha_2 \in U$ and

$$T(a\alpha_1 + b\alpha_2) = a\,T(\alpha_1) + b\,T(\alpha_2) \quad [\because T \text{ is a L.T.}]$$
$$= a.0 + b.0$$
$$= 0 + 0 = 0 \in V$$

$\therefore a\alpha_1 + b\alpha_2 \in \text{Ker } T$

Thus $a, b \in F$ and $\alpha_1, \alpha_2 \in \text{Ker } T \Rightarrow a\alpha_1 + b\alpha_2 \in \text{Ker } T$.

Thus $\text{Ker } T$ is a subspace of U.

Rank and nullity of a L.T.

$$\rho(T) = \dim \text{Im}(T)$$
$$\nu(T) = \dim N(T)$$

Theorem $\quad T : U \to V$

$$\text{rank } T + \text{nullity } T = \dim U$$

Theorem

$T: U \to V$. $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$ be a basis of U. Then $T(\alpha_1), T(\alpha_2) \cdots T(\alpha_n)$ generate $\operatorname{Im} T$.

Ex A mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3,\ 2x_1 + x_2 + 2x_3,\ x_1 + 2x_2 + x_3),$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$. Show that T is a L.T. Find $\operatorname{Ker} T$, $\dim \operatorname{Ker} T$, $\operatorname{Im} T$ and $\dim \operatorname{Im} T$.

Sol^n: First to show $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

$$\operatorname{Ker} T = \{(x_1, x_2, x_3) : T(x_1, x_2, x_3) = (0,0,0)\}$$

Let $(x_1, x_2, x_3) \in \operatorname{Ker} T$

$\therefore x_1 + x_2 + x_3 = 0$

$2x_1 + x_2 + 2x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

Solving $(x_1, x_2, x_3) = k(1, 0, -1)$, $k \in \mathbb{R}$

$\operatorname{Ker} T = L\{(1, 0, -1)\}$

$\therefore \dim \operatorname{Ker} T = 1$

Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a basis of $\mathbb{R}^3$. $\operatorname{Im} T$ is the linear span of vectors $T(\alpha_1), T(\alpha_2), T(\alpha_3)$. $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$.

$$T(1,0,0) = (1,2,1)$$
$$T(0,1,0) = (1,1,2)$$
$$T(0,0,1) = (1,2,1)$$

$$\therefore \operatorname{Im} T = L\{(1,2,1), (1,1,2)\}$$

$\dim$ of $\operatorname{Im} T = 2$

Ex Determine the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vector $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ of $\mathbb{R}^3$ to the vectors $(1,1)$, $(2,3)$, $(3,2)$ respectively. Find $\operatorname{Ker} T$ & $\operatorname{Im} T$.

Solⁿ: $(x,y,z) \in \mathbb{R}^3$

$$(x,y,z) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$$

$$\therefore c_1 = x \quad c_2 = y \quad c_3 = z$$

$$\begin{aligned} T(x,y,z) &= c_1 T(1,0,0) + c_2 T(0,1,0) + c_3 T(0,0,1) \\ &= c_1(1,1) + c_2(2,3) + c_3(3,2) \\ &= x(1,1) + y(2,3) + z(3,2) \\ &= (x+2y+3z,\ x+3y+2z) \end{aligned}$$

Let $(x,y,z) \in \operatorname{Ker} T$

$$\therefore T(x,y,z) = (0,0)$$

$$x+2y+3z=0$$

$$x+3y+2z=0$$

$$(x,y,z) = k(-5,1,1)$$

$$\therefore \operatorname{Ker} T = L\{(-5,1,1)\}$$

$\operatorname{Im} T$ is the linear span of $T(\alpha_1), T(\alpha_2), T(\alpha_3)$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ is any basis of $\mathbb{R}^3$.

$\because \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$,

$$\operatorname{Im} T = L\{(1,1), (2,3), (3,2)\}$$

Linear mappings and system of linear equations

Consider a system of m linear equations in n unknowns over a field R:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$- \quad - \quad - \quad - \quad -$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

which is equivalent to the matrix eqn. $Ax = b$. Now the matrix A may also be viewed as the linear mapping

$$A : R^n \to R^m$$

Thus the soln. of eqn. $Ax = b$ may be viewed as the preimage of $b \in R^m$ under the linear mapping $A : R^n \to R^m$. Furthermore, the solution of the associated homogeneous eqn. $Ax = 0$ may be viewed as the kernel of the linear mapping $A : R^n \to R^m$.

$$\dim(\operatorname{Ker} A) + \dim(\operatorname{Im} A) = \dim R^n$$
$$\Rightarrow \dim(\operatorname{Ker} A) = \dim R^n - \dim(\operatorname{Im} A)$$
$$= n - \operatorname{rank} A$$

n is the no. of unknowns in the homogeneous system $Ax = 0$

Dim of soln. space is $n - r$.

(i) Injective mapping (one-one)

If $a \neq a' \Rightarrow f(a) \neq f(a')$

or equivalently if $f(a) = f(a') \Rightarrow a = a'$

(ii) Surjective mapping (onto)

If every $b \in B$ is the image of at least one $a \in A$

(iii) Bijective mapping

Both one-one and onto.

Theorem $T : U \to V$ be a L.T. Then T is 1-1 iff $\operatorname{Ker} T = \{0\}$

Proof: Let T is 1-1. Since $T(\theta) = \theta'$ in V, θ is a preimage of θ' and since T is 1-1, θ is the only pre-image of θ'.
So $\operatorname{Ker} T = \{\theta\}$.

Conversely, let $\operatorname{Ker} T = \{\theta\}$. Let α, β be 2 elements of U such that $T(\alpha) = T(\beta)$ in V. Now $\theta' = T(\alpha) - T(\beta) = T(\alpha - \beta)$

$\therefore \alpha - \beta \in \operatorname{Ker} T$ and $\because \operatorname{Ker} T = \{\theta\}$, $\alpha = \beta$.

$\therefore T(\alpha) = T(\beta) \Rightarrow \alpha = \beta$ $\therefore T$ is 1-1.

Theorem Let U & V be two V.S. of same dimension. $T : U \to V$ is a L.T. Then T is 1-1 $\Leftrightarrow$ T is onto.

Proof: Let T be 1-1. $\therefore \operatorname{Ker} T = \{0\}$ and $\dim \operatorname{Ker} T = 0$

$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim U \Rightarrow \dim \operatorname{Im} T = \dim U$

$\therefore \dim \operatorname{Im} T = \dim V$ $\therefore \operatorname{Im} T = V$ $\therefore T$ is onto.

Ex $T : F^4 \to F^2$ s.t. $N(T) = \{(x, y, z, w) \in F^4 : x = 5y, z = 7w\}$
Prove that T is onto.

Sol^n: $x - 5y = 0$, $z - 7w = 0$ $\quad \dim N(T) = 2$ $\quad \dim N(T) + \dim R(T) = 4$

$\therefore \dim R(T) = 2 = \dim F^2$

$\therefore R(T) = F^2$ $\therefore T$ is onto.

Matrix representation of a linear transformation

Let U and V be finite dimensional vector space over F with $\dim U = n$ and $\dim V = m$. Let $T: U \to V$ be a L.T. T is completely determined by its action on a given basis of U. Let $(\alpha_1, \alpha_2, \ldots \alpha_n)$ be an ordered basis of U and $(\beta_1, \beta_2, \ldots \beta_m)$ be an ordered basis of V. T is completely determined by the images $T(\alpha_1), T(\alpha_2) \ldots, T(\alpha_n)$. Each $T(\alpha_i)$ in V is a linear combination of the vectors $\beta_1, \ldots, \beta_m$.

$$\text{Let } T(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \ldots + a_{m1}\beta_m$$
$$T(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \ldots + a_{m2}\beta_m$$
$$\ldots$$
$$T(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + \ldots + a_{mn}\beta_m$$

where a_{ij} are unique scalars in F determined by the ordered basis $(\beta_1, \ldots \beta_m)$. Let $\xi = x_1\alpha_1 + \ldots + x_n\alpha_n$ be an arb. vector of U and let $T(\xi) = y_1\beta_1 + \ldots + y_m\beta_m$ $\quad x_i, y_i \in F$

$$T(\xi) = T(x_1\alpha_1 + \ldots x_n\alpha_n)$$
$$= x_1 T(\alpha_1) + \ldots + x_n T(\alpha_n)$$
$$= x_1(a_{11}\beta_1 + a_{21}\beta_2 + \ldots + a_{m1}\beta_m) + \ldots + x_n(a_{1n}\beta_1 + \ldots + a_{mn}\beta_m)$$

$\because \{\beta_1, \ldots \beta_m\}$ is L.I.

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ &\ldots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{aligned}\right\} \text{ or } \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & & a_{2n} \\ & & \\ a_{m1} & & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

or $Y = AX$ where $A = (a_{ij})_{m \times n}$, X is the co-ordinate vector of an arb. element ξ in U relative to the ordered basis $(\alpha_1, \ldots \alpha_n)$ and Y is the co-ord. vector of $T(\xi)$ in V relative to the o.b $(\beta_1, \ldots \beta_m)$.

$Y = Ax$ is the matrix representation of the linear mapping relative to the chosen ordered bases of U and V.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$ is said to be the matrix associated with the linear mapping T relative to the chosen ordered bases of U and V.

Ex A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$

Find the matrix of T relative to the ordered bases

(i) $((1,0,0), (0,1,0), (0,0,1))$ of $\mathbb{R}^3$ and $((1,0), (0,1))$ of $\mathbb{R}^2$

(ii) $((0,1,0), (1,0,0), (0,0,1))$ of $\mathbb{R}^3$ and $((0,1), (1,0))$ of $\mathbb{R}^2$.

Solⁿ: (i) $T(1,0,0) = (3,1) = 3(1,0) + 1(0,1)$

$T(0,1,0) = (-2,-3) = -2(1,0) - 3(0,1)$

$T(0,0,1) = (1,-2) = 1(1,0) - 2(0,1)$

$$\therefore T = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$$

(ii) $T(0,1,0) = (-2,-3) = -3(0,1) - 2(1,0)$

$T(1,0,0) = (3,1) = 1(0,1) + 3(1,0)$

$T(0,0,1) = (1,-2) = -2(0,1) + 1(1,0)$

$$\therefore T = \begin{pmatrix} -3 & 1 & -2 \\ -2 & 3 & 1 \end{pmatrix}$$

Ex The matrix of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases $((0,1,1),(1,0,1),(1,1,0))$ of $\mathbb{R}^3$ and $((1,0),(1,1))$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find T. Also find the matrix of T relative to the ordered bases $((1,1,0),(1,0,1),(0,1,1))$ of $\mathbb{R}^3$ and $((1,1),(0,1))$ of $\mathbb{R}^2$.

Sol^n: $T(0,1,1) = 1(1,0) + 2(1,1) = (3,2)$

$T(1,0,1) = 2(1,0) + 1(1,1) = (3,1)$

$T(1,1,0) = 4(1,0) + 0(1,1) = (4,0)$

Let $(x,y,z) \in \mathbb{R}^3$ and let $(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$

$c_2 + c_3 = x, \quad c_1 + c_3 = y, \quad c_1 + c_2 = z$

$\therefore c_1 = \frac{1}{2}(y+z-x), \quad c_2 = \frac{1}{2}(z+x-y), \quad c_3 = \frac{1}{2}(x+y-z)$

$T(x,y,z) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$

$= c_1(3,2) + c_2(3,1) + c_3(4,0)$

$= (3c_1 + 3c_2 + 4c_3, \ 2c_1 + c_2)$

$= (2x+2y+z, \ \frac{1}{2}(-x+y+3z)), \quad (x,y,z) \in \mathbb{R}^3$

$T(1,1,0) = (4,0); \ T(1,0,1) = (3,1); \ T(0,1,1) = (3,2)$

Let $(4,0) = c_1(1,1) + c_2(0,1)$; $c_1 = 4, \ c_1 + c_2 = 0 \ \therefore c_2 = -4$

$(3,1) = c_1(1,1) + c_2(0,1)$; $c_1 = 3, \ c_1 + c_2 = 1 \Rightarrow c_2 = -2$

$(3,2) = c_1(1,1) + c_2(0,1)$; $c_1 = 3, \ c_1 + c_2 = 2 \Rightarrow c_1 = 3, c_2 = -1$

$\therefore$ The matrix of $T = \begin{pmatrix} 4 & 3 & 3 \\ -4 & -2 & -1 \end{pmatrix}$