ADVANCED CALCULUS
MA11003

SECTION 11, 12

Dr. Jitendra Kumar

Professor
Department of Mathematics
Indian Institute of Technology Kharagpur
West Bengal 721302, India

Webpage: http://www.facweb.iitkgp.ac.in/~jkumar/

QUIZ QUESTION ?

LINK FOR RESPONSES: http://www.facweb.iitkgp.ac.in/~jkumar/teach/MA11003.html

QUIZ QUESTION:

If $y(x)$ is the solution of

$$x\frac{dy}{dx} + y = y^2; \quad y(1) = 2$$

Then the value of $y(-3)$ is ________

Concepts Covered

Differential Equations

- Exact Differential Equations
- Solution

Exact Differential Equations

If M and N are functions of x and y, the equation $Mdx + Ndy = 0$ is called exact there exists a function $f(x, y)$ such that

$$d\big(f(x,y)\big) = Mdx + Ndy$$

or

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

Theorem: The necessary and sufficient condition for the differential equation

$$Mdx + Ndy = 0 \quad \text{to be exact is} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof: The condition is necessary. Let the equation be exact, then

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M\,dx + N\,dy$$

Equating coefficients of dx & dy , we get $\quad M = \frac{\partial f}{\partial x} \qquad N = \frac{\partial f}{\partial y}$

Assuming f to be continuous up to 2nd order partial derivatives, we obtain

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad \Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Now we show that the given condition is sufficient.

We assume $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and show that the equation $Mdx + Ndy$ is exact

That means we find a function $f(x, y)$ such that $df = Mdx + Ndy$

Let $g(x, y) = \int M\, dx$ be the partial integral of M such that $\dfrac{\partial g}{\partial x} = M$

We first show that $\left(N - \dfrac{\partial g}{\partial y}\right)$ is a function of y only

Given $g(x,y) = \int M dx$ and $\dfrac{\partial g}{\partial x} = M$

Consider $\dfrac{\partial}{\partial x}\left(N - \dfrac{\partial g}{\partial y}\right) = \dfrac{\partial N}{\partial x} - \dfrac{\partial^2 g}{\partial x \partial y} = \dfrac{\partial N}{\partial x} - \dfrac{\partial^2 g}{\partial y \partial x}$ Assuming $\dfrac{\partial^2 g}{\partial x \partial y} = \dfrac{\partial^2 g}{\partial y \partial x}$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

Now consider $f = g(x,y) + \int \left(N - \dfrac{\partial g}{\partial y}\right) dy$

We will show that $df = M dx + N dy$

$$f = g(x,y) + \int \left(N - \frac{\partial g}{\partial y}\right) dy$$

$$\frac{\partial g}{\partial x} = M$$

$\left(N - \frac{\partial g}{\partial y}\right)$ is a function of y only

$$df = dg + d\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right)$$

$$= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial}{\partial x}\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right) dx + \frac{\partial}{\partial y}\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right) dy$$

$$= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + N dy - \frac{\partial g}{\partial y} dy = M dx + N dy$$

$\implies$ The given differential equation is exact

Remark: The solution of an exact differential equation

$$Mdx + Ndy = 0 \qquad (df = 0)$$

can be written as $f = c$

$$\int M\,dx + \int \left(N - \frac{\partial g}{\partial y}\right) dy = c \qquad \left(N - \frac{\partial g}{\partial y}\right) \text{ is a function of } y \text{ only}$$

$$\boxed{\int M\,dx + \int (\text{term of } N \text{ not containing } x)\, dy = c}$$

Example: Solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

$$M = x^2 - 4xy - 2y^2 \qquad N = y^2 - 4xy - 2x^2$$

$$\frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x} \qquad \Longrightarrow \quad \text{the equation is exact}$$

Hence, there exists a function $f(x, y)$ such that

$$d\big(f(x,y)\big) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

$$\frac{\partial f}{\partial x} = x^2 - 4xy - 2y^2 \qquad \& \qquad \frac{\partial f}{\partial y} = y^2 - 4xy - 2x^2$$

$$\frac{\partial f}{\partial x} = x^2 - 4xy - 2y^2 \qquad \frac{\partial f}{\partial y} = y^2 - 4xy - 2x^2$$

Integration w.r.t. x

$$f = \frac{x^3}{3} - 2x^2y - 2xy^2 + c_1(y)$$

On differentiation w.r.t. y

$$\frac{\partial f}{\partial y} = -2x^2 - 4xy + c_1'(y) = y^2 - 4xy - 2x^2 \implies c_1'(y) = y^2 \implies c_1(y) = \frac{y^3}{3} + c_2$$

Solution $f = c_3 \implies \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} + c_2 = c_3$

$$\boxed{x^3 - 6xy(x+y) + y^3 = c}$$

DIRECT APPROACH

Given Differential Equation $(x^2 - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dy = 0$

Solution: $\int M\, dx + \int (\text{term of } N \text{ not containing } x)\, dy = c$

$$\int (x^2 - 4xy - 2y^2)\, dx + \int y^2\, dy = c$$

$$\Longrightarrow \boxed{\frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c}$$

Example: Show that the differential equation $(3xy + y^2)dx + (x^2 + xy)dy = 0$ is not exact and hence it cannot be solved by the method discussed above

Check: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \quad \Longrightarrow 3x + 2y \neq 2x + y$

So the given equation is not exact

However, if we proceed with the method given above, we get

$$\frac{\partial f}{\partial x} = 3xy + y^2 \qquad \frac{\partial f}{\partial y} = x^2 + xy$$

$$\frac{\partial f}{\partial x} = 3xy + y^2 \qquad \frac{\partial f}{\partial y} = x^2 + xy$$

$$f = \frac{3}{2}x^2y + y^2x + c_1(y)$$

$$\frac{\partial f}{\partial y} = \frac{3}{2}x^2 + 2yx + c_1'(y) = x^2 + xy \qquad c_1'(y) = -\frac{x^2}{2} - xy$$

Thus, there is no $f(x, y)$ exists and hence it can not be solved in this way.

Conclusion

The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$d\big(f(x, y)\big) = M\,dx + N\,dy$$

SOLUTION: $f = c$

Concepts Covered

Differential Equations

- ❑ Exact Differential Equations
- ❑ Integrating Factors

Exact Differential Equations (RECALL)

The necessary and sufficient condition for the differential equation

$$Mdx + Ndy = 0$$

to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Differential Equations: Integrating Factors

If an equation of the form $Mdx + Ndy = 0$ is not exact.

It is sometimes possible to choose a function of x & y such that after multiplying all terms of the equation, it becomes exact. Such a multiplier is called an **integrating factor**.

That is, if $I(x,y)$ is an **integrating factor** then the differential equation

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$$

becomes exact.

Note: Although an equation of the form $Mdx + Ndy = 0$

has always integrating factor(s), there is not general rule of finding them.

We will discuss some methods of finding integrating factors.

Rule 1 : By inspection

This method is based on recognition of some standard exact differentials that occur frequently in practice.

Rule 1 : By inspection

$i)\quad d(xy) = y\,dx + x\,dy$

$ii)\quad d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$ or $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

$iii)\quad d\left(\ln\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{xy}$ or $d\left(\ln\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{xy}$

$iv)\quad d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$ or $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2 + x^2}$

$v)\quad d(\ln xy) = \frac{y\,dx + x\,dy}{xy}$

Example: Solve the differential equation $y(y^2+1)\,dx + x(y^2-1)\,dy = 0$

(Check! It is not exact D.E.) $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$d(xy) = y\,dx + x\,dy$$

$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$$

Rewrite $y^2(ydx + xdy) + ydx - xdy = 0$

Dividing it by y^2: (I.F.)

$$\Longrightarrow ydx + xdy + \frac{ydx - xdy}{y^2} = 0 \quad \Longrightarrow d(xy) + d\left(\frac{x}{y}\right) = 0$$

$$\Longrightarrow xy + \frac{x}{y} = c \quad \Longrightarrow \boxed{xy^2 + x = cy}$$

More General Approach

The idea is to multiply the given differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

by a function $I(x, y)$ and then try to choose $I(x, y)$ so that the resulting equation $I(x, y)M(x, y)dx + I(x, y)N(x, y)dy = 0$ becomes exact

The above equation is exact if and only if $\dfrac{\partial(IM)}{\partial y} = \dfrac{\partial(IN)}{\partial x}$

If a function I satisfying $\frac{\partial(IM)}{\partial y} = \frac{\partial(IN)}{\partial x}$ can be found then the given equation will be **exact**.

However, solving the above (PDE) is very difficult so we consider some special cases.

i) An **integrating factor** I that is either as function of x alone or

ii) A function of y alone.

In the case i), the above PDE reduces to $IM_y = IN_x + NI_x$

$$I_x = \frac{IM_y - IN_x}{N}$$

Given: $I_x = \dfrac{IM_y - IN_x}{N}$

If $\dfrac{M_y - N_x}{N}$ is a **function of x only**, say $f(x)$ then by solving $\dfrac{dI}{I} = f(x)\,dx$

we get an integrating factor $I(x) = e^{\int f(x)dx}$

In the case ii) If $\dfrac{1}{M}(N_x - M_y)$ is a function of y alone, say $g(y)$

Then $I(y) = e^{\int g(y)dy}$ is an integrating factor

Example 1: Consider $(x^2+y^2+x)dx + xy\,dy = 0$ $\quad M = x^2+y^2+x \quad N = xy$

$$\frac{\partial M}{\partial y} = 2y \quad \& \quad \frac{\partial N}{\partial x} = y \qquad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y-y) = \frac{1}{x}$$

Integrating factor: $e^{\int \frac{1}{x}dx} = x$

Multiplying the given differential equation by x: $(x^3+xy^2+x^2)dx + x^2y\,dy = 0$

This must be an exact differential equation.

Solution: $\boxed{(3x^4+6x^2y^2+4x^3) = c}$

Example 2: Consider $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

$M = 2xy^4e^y + 2xy^3 + y \qquad N = x^2y^4e^y - x^2y^2 - 3x$

$$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1 \qquad \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -8xy^3e^y - 8xy^2 - 4 = -4(2xy^3e^y + 2xy^2 + 1)$$

$$= -\frac{4}{y}(2xy^4e^y + 2xy^3 + y) = -\frac{4}{y}M$$

Integrating factor : $e^{\int \frac{-4}{y}dy} = y^{-4}$ **Solution:** $\boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c}$

Some more rules for finding Integrating Factors:

- $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0.$

  In this case $I(x, y) = \dfrac{1}{Mx + Ny}$ is an integrating factor

- $Mdx + Ndy = 0$ is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$

  then $\dfrac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$

Conclusion

Integrating Factors

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$$

If $\dfrac{M_y - N_x}{N}$ is a **function of x only**, say $f(x)$

$$\text{IF: } I(x) = e^{\int f(x)\,dx}$$

If $\dfrac{1}{M}(N_x - M_y)$ is a **function of y only**, say $g(y)$

$$\text{IF: } I(y) = e^{\int g(y)dy}$$

Concepts Covered

Differential Equations

- First Order Linear Differential Equations
- Equations Reducible to Linear DEs
- Solution Techniques

Integrating Factors (RECALL)

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$$

If $\dfrac{M_y - N_x}{N}$ is a **function of x only**, say $f(x)$

$$\text{IF: } I(x) = e^{\int f(x)\,dx}$$

If $\dfrac{1}{M}(N_x - M_y)$ is a **function of y only**, say $g(y)$

$$\text{IF: } I(y) = e^{\int g(y)dy}$$

Linear Differential Equation

A first order differential equation is called linear if it can be written in the form

$$\frac{dy}{dx} + P(x)\, y = Q(x) \qquad \text{(linear in } y\text{)}$$

Rewritten as $dy + P\, y\, dx = Q(x)\, dx \qquad M = Py \qquad N = 1$

Observe that $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{1}(P - 0) = P(x)$

Hence I.F.: $e^{\int P\, dx}$

Given Differetial Equation: $dy + P\,y\,dx = Q(x)\,dx$

I.F. : $e^{\int Pdx}$

$$e^{\int Pdx}dy + Pye^{\int Pdx}dx = Q(x)e^{\int Pdx}dx$$

$$d(e^{\int Pdx}y) = Q\,e^{\int Pdx}\,dx$$

Integrating $e^{\int P\,dx}\,y = \int Qe^{\int Pdx}dx + c$

$$\boxed{y \times I.F. = \int (Q \times\ I.F.)dx + c}$$

Note: Sometimes a differential equation cannot be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

which is linear in y, but in the form

$$\frac{dx}{dy} + P_1(y)x = Q_1(y) \qquad \text{(linear in } x\text{)}$$

$$I.F. = e^{\int P_1(y)\,dy}$$

$$\boxed{x \times I.F. = \int (Q_1 \times I.F.)\,dy + c}$$

Example 1: Consider $(1+x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2} \qquad \text{(linear in } y\text{)}$$

$$I.F. = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1 + x^2$$

Solution: $y \times I.F. = \int Q \times I.F.\ dx + c$

$$\Rightarrow y(1+x^2) = \int 4x^2\,dx + c$$

$$\Rightarrow y(1+x^2) = \frac{4}{3}x^3 + c$$

Example 2: Consider $(x+2y^3)\dfrac{dy}{dx} = y$

Rewrite $\dfrac{dx}{dy} - \dfrac{1}{y}x = 2y^2$

I.F. $= e^{\int -\frac{1}{y}dy} = e^{-\ln y} = \dfrac{1}{y}$

Solution: $x\dfrac{1}{y} = \displaystyle\int 2y^2\frac{1}{y}dy + c \quad \Longrightarrow \frac{x}{y} = y^2 + c$

Equation Reducible to Linear Form:

An equation of the form $f'(y)\dfrac{dy}{dx} + Pf(y) = Q(x)$

Substituting $f(y) = v$ $\quad \Longrightarrow f'(y)\dfrac{dy}{dx} = \dfrac{dv}{dx}$

Equation reduces to: $\dfrac{dv}{dx} + Pv = Q$

A Special Case: Bernoulli's Equation

An equation of the form $$\frac{dy}{dx} + Py = Qy^n$$

where P & Q are constants or function of x and n is a constant except 0 & 1 is called **Bernoulli's Differential Equation**

The above equation can be written as $$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$$

Substitute: $$\frac{1}{y^{n-1}} = v \quad \Longrightarrow \quad (1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{1}{(1-n)}\frac{dv}{dx} + Pv = Q \qquad \Longrightarrow \quad \frac{dv}{dx} + P(1-n)v = Q(1-n)$$

Example 1: Consider $(x^2 - 2x + 2y^2)\, dx + 2xy\, dy = 0$

Rewrite: $2xy\dfrac{dy}{dx} + x^2 - 2x + 2y^2 = 0$ OR $2y\dfrac{dy}{dx} + \dfrac{2y^2}{x} = \dfrac{2x - x^2}{x}$

Substitution $y^2 = v$ $\Longrightarrow 2y\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\dfrac{dv}{dx} + \dfrac{2}{x}v = (2 - x)$ I.F. $= e^{\int \frac{2}{x} dx} = x^2$

$v\, x^2 = \int (2 - x)x^2\, dx + c$ $\Longrightarrow y^2x^2 = \dfrac{2}{3}x^3 - \dfrac{x^4}{4} + c$

Example 2: $\dfrac{dy}{dx} - y\tan x = -y^2 \sec x$

Dividing by y^2: $\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{y}\tan x = -\sec x$ Subst. $\dfrac{1}{y} = v$ $\Rightarrow -\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\dfrac{dv}{dx} + v\tan x = \sec x$ I.F. $= e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$

Solution: $v \sec x = \int \sec^2 x \, dx + c \Rightarrow v\sec x = \tan x + c$

$y^{-1}\sec x = \tan x + c$

Conclusion

Linear Differential Equations of Order - 1

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Equations Reducible to Linear DEs of Order - 1

$$f'(y)\frac{dy}{dx} + Pf(y) = Q(x)$$

Thank You