LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

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Numerical Integration

- Trapezoidal Rule
- Simpson's Rule

Numerical Integration

Applications: To find complicated integrals like: $\int_0^1 e^{-x^2}\, dx$ $\int_0^\pi x^\pi \sin(\sqrt{x})\, dx$

Newton's Cotes Integration formulas :

These formulas are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate.

$$I = \int_a^b f(x)\, dx \approx \int_a^b P_n(x)\, dx$$ where $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$

The Trapezoidal Rule: (Single Application)

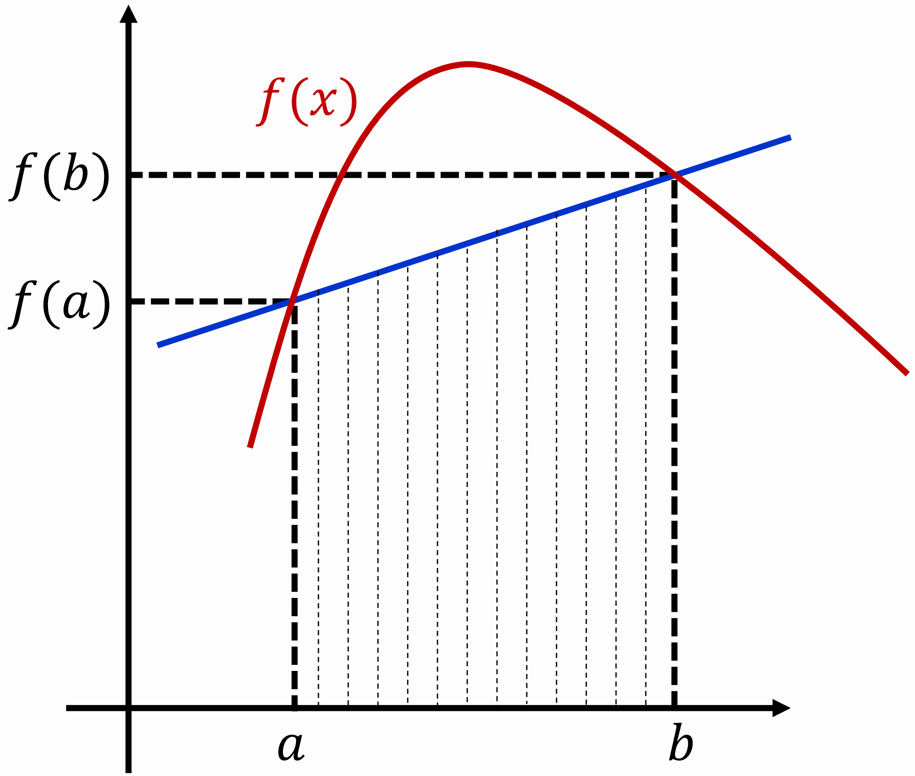
$$I = \int_a^b f(x)\,dx \approx \int_a^b P_1(x)\,dx$$

$$= \int_a^b \left\{ f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right\} dx$$

$$= f(a)(b - a) + \frac{f(b) - f(a)}{b - a}\,\frac{1}{2}\,(b - a)^2$$

$$= f(a)(b - a) + \frac{1}{2}(b - a)\big(f(b) - f(a)\big)$$

$$\Rightarrow \int_a^b f(x)dx \approx (b - a)\frac{[f(b) + f(a)]}{2}$$



Example : Using trapezoidal rule integral numerically the function

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Compare result with exact value of integral 1.640433.

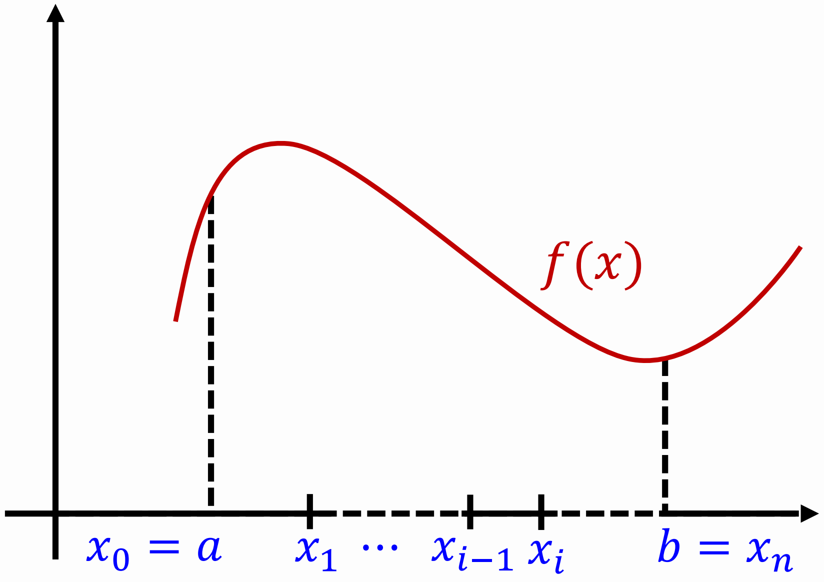
Solution: The function values $f(0) = 0.2$ $f(0.8) = 0.232$

$$\int_0^{0.8} f(x)dx \approx \frac{0.2 + 0.232}{2}(0.8 - 0)$$

$$= 0.1728$$

The Multiple Application of Trapezoidal Rule

To improve accuracy of the trapezoidal rule we divide the integration interval from a to b into a number of segments and apply the method to each segment.



Consider there are $n+1$ equally spaced base points $x_0, x_1, \ldots, x_n$.

Denote $h = \dfrac{(b-a)}{n}$

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

The Multiple Application of Trapezoidal Rule

$$I = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$\approx h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

$$= \frac{h}{2}\left[f(x_0) + 2\big(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\big) + f(x_n)\right]$$

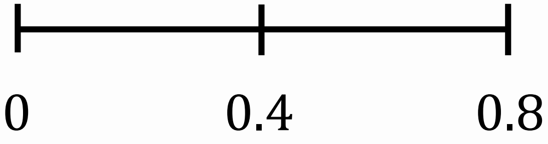
$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

Example : Use the two-segment trapezoidal rule to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$.

Solution:



Exact Value: 1.640433

$$h = \frac{0.8 - 0}{2} = 0.4$$

$$f(0) = 0.2, \quad f(0.4) = 2.456, \quad f(0.8) = 0.232.$$

$$I = \int_0^{0.8} f(x)dx \approx \frac{h}{2}\{0.2 + 2(2.456) + 0.232\} = 1.0688$$

Weighted Mean Value Theorem

Assume f and g are continuous in $[a, b]$. If g never changes sign in $[a, b]$, then

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x) dx \quad \text{where } c \in (a, b) \ \&\ g \text{ is integrable.}$$

Discrete Mean Value Theorem

Let $f \in C^0[a,\ b]$ and let x_j be $(n+1)$ points in $[a,\ b]$ and C_j be $(n+1)$ constants,

all having the same sign. Then there exists $\xi \in [a, b]$ such that

$$\sum_{j=0}^{n} C_j f(x_j) = f(\xi) \sum_{j=0}^{n} C_j$$

In particular, if $C_j = 1 \ \ \forall\, j$, then $\dfrac{1}{n+1} \displaystyle\sum_{j=0}^{n} f(x_j) = f(\xi)$

Error bounds for the Trapezoidal rule

Single application: We know, $f(x) - P_1(x) = (x - x_0)(x - x_1)\dfrac{f''(t)}{2}$

Integrating above equation from x_0 to $x_1 = x_0 + h$ gives

t depends on x and lies between x_0 & x_1.

$$E = \int_{x_0}^{x_0+h} f(x)dx - \frac{h}{2}[f(x_0) + f(x_1)] = \int_{x_0}^{x_0+h} (x - x_0)(x - x_1)\frac{f''(t)}{2}dx$$

Note that $(x - x_0)(x - x_1)$ does not change the sign in $[x_0, x_0 + h]$

Applying weighted mean value theorem, we get

$$E = \frac{f''(\tilde{t})}{2}\int_{x_0}^{x_0+h} (x - x_0)(x - x_0 - h)\, dx$$

Substitute $x = x_0 = v \quad \Longrightarrow dx = dv.$

Error bounds for the Trapezoidal rule

$$E = \frac{f''(\tilde{t})}{2} \int_{x_0}^{x_0+h} (x - x_0)(x - x_0 - h)dx \qquad \text{where } \tilde{t} \in (x_0,\ x_1)$$

Substitute $x - x_0 = v \quad \Longrightarrow dx = dv.$

$$= \frac{f''(\tilde{t})}{2} \int_0^h v(v-h)\, dx$$

$$= \frac{f''(\tilde{t})}{2} \left[\frac{1}{3} h^3 - \frac{h}{2} h^2 \right]$$

$$= -\frac{h^3}{12} f''(\tilde{t})$$

2. **Error in multiple application :**

$$E = \sum_{i=0}^{n-1} \left\{ -\frac{h^3}{12} f''(\tilde{t_i}) \right\} = -\frac{h^3}{12} \sum_{i=0}^{n-1} f''(\tilde{t_i})$$

Using discrete mean value theorem

$$= -\frac{h^3}{12}\, n\, f''(\hat{t})$$

where $\hat{t}$ lies between a and b

$$\boxed{E = -\frac{(b-a)}{12} h^2 f''(\hat{t})}$$

Error bounds: Let $M_2 = \max\limits_{[x_0,\ x_n]} |f''(x)|$. Then, $\boxed{|E| \le \frac{(b-a)h^2}{12} M_2}$

Example: Evaluate the following integral using trapezoidal rule with $n = 2, 4$

$$\int_0^1 \frac{dx}{3+2x}$$

Compare numerical values with the exact solution. Find the bound on the error.

Also find the number of sub-intervals required if the error is to be less than 5×10^{-4}

Solution: Case 1: Number of sub-intervals $= 2$

$$\Rightarrow h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$$

Hence $I_1 = \frac{0.5}{2}\big(f(0) + 2f(0.5) + f(1)\big) = \frac{0.5}{2}\left(\frac{1}{3} + 2 \times \frac{1}{4} + \frac{1}{5}\right) = 0.25833$

Case 2: Number of sub-intervals $= 4$

$$\Rightarrow h = \frac{1-0}{4} = \frac{1}{4}$$

Hence,

$$I_2 = \frac{1}{4}\frac{1}{2}\left[f(0) + 2\left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right) + f(1)\right]$$

$$= \frac{1}{8}\left[\frac{1}{3} + 2\left(\frac{2}{7} + \frac{1}{4} + \frac{2}{9}\right) + \frac{1}{5}\right]$$

$$= 0.25615$$

Exact solution: $\frac{1}{2}\ln\frac{5}{3} = 0.25541$

$E_1 = |0.25541 - 0.258331| = 0.00292$

$E_2 = |0.25541 - 0.25615| = 0.00074$

$I_1 = 0.25833$

$I_2 = 0.25615$

Error bounds :

$$|E| \leq \frac{(b-a)h^2}{12} M_2$$

$$M_2 = \max_{[x_0,\ x_n]} |f''(x)|$$

$$f(x) = \frac{1}{3+2x} \Longrightarrow f'(x) = -\frac{2}{(3+2x)^2} \Longrightarrow f''(x) = \frac{8}{(3+2x)^3}$$

$$M_2 = \max_{[0,1]} \frac{8}{(3+2x)^3} = \frac{8}{27}$$

Hence, $|\text{Error}| \leq \frac{(b-a)h^2}{12} M_2 = \frac{1}{12} h^2 \frac{8}{27} = \frac{2h^2}{81}$

For $h = 0.5$, $|\text{Error}| \leq 0.00617$

For $h = 0.25$, $|\text{Error}| \leq 0.00154$

Given, $E = 5 \times 10^{-4}$

$$\Rightarrow \frac{(b-a)h^2}{12} M_2 \leq 5 \times 10^{-4}$$

$$\Rightarrow \frac{(b-a)(b-a)^2}{12n^2} \frac{8}{27} \leq 5 \times 10^{-4}$$

$$\Rightarrow \frac{1 \times 8}{12 \times 27 \times 5 \times 10^{-4}} \leq n^2$$

$$\Rightarrow n \geq 7.03$$

Since, n is an integer, we require $n = 8$.

Simpson's 1/3rd Rule

$$I = \int_a^b f(x)dx \approx \int_a^b P_2(x)dx$$

Let $x_0 = a,\; x_1,\; x_2 = b$

$$I \approx \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

$$= \frac{1}{2h^2} f(x_0) \int_{x_0}^{x_2} (x-x_1)(x-x_1+x_1-x_2)dx - \frac{1}{h^2} f(x_1) \int_{x_0}^{x_2} (x-x_0)(x-x_0+x_0-x_2)dx$$

$$+ \frac{1}{2h^2} f(x_2) \int_{x_0}^{x_2} (x-x_0)(x-x_0+x_0-x_1)dx$$

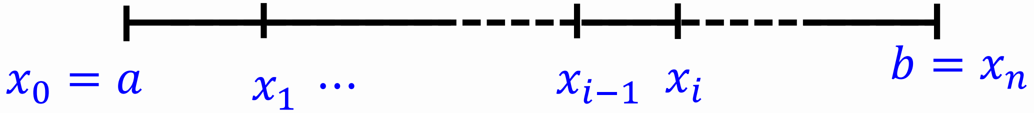
$$I \approx \frac{1}{2h^2} f(x_0) \int_{x_0}^{x_2} (x - x_1)(x - x_1 + x_1 - x_2)dx - \frac{1}{h^2} f(x_1) \int_{x_0}^{x_2} (x - x_0)(x - x_0 + x_0 - x_2)dx$$

$$+ \frac{1}{2h^2} f(x_2) \int_{x_0}^{x_2} (x - x_0)(x - x_0 + x_0 - x_1)dx$$

$$I \approx \frac{f(x_0)}{2h^2}\left[\frac{1}{3}(h^3 + h^3) - h.0\right] - \frac{f(x_1)}{h^2}\left[\frac{1}{3}(2h)^3 - \frac{2h}{3}(2h)^2\right] + \frac{f(x_2)}{2h^2}\left[\frac{1}{3}(2h)^3 + \left(\frac{-h}{2}\right)(2h)^2\right]$$

$$I \approx \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$ Simpson’s 1/3rd Rule

Multiple Application of Simpson's Rule



$$b - a = nh$$

$$I = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$\approx \frac{h}{3}\{f(x_0) + 4f(x_1) + f(x_2)\} + \frac{h}{3}\{f(x_2) + 4f(x_3) + f(x_4)\} + \cdots + \frac{h}{3}\{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\}$$

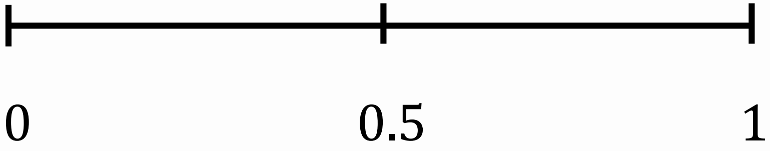
$$= \frac{h}{3}\left\{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)\right\}$$

Error: Single application: $E = -\frac{h^5}{90} f^{(4)}(\xi);\ \xi \in (a, b)$

Multiple application: $E = -\frac{b-a}{180} h^4 f^{(4)}(\xi);\ \tilde{\xi} \in (a, b)$

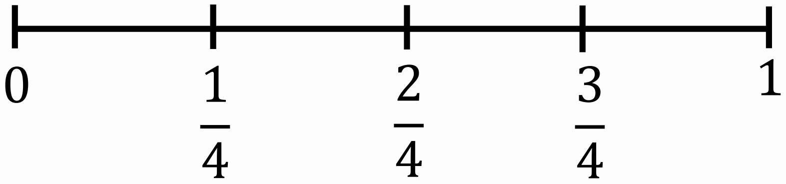
Example: Evaluate $\int_0^1 \frac{dx}{3+2x}$ using Simpson's rule with $n = 2, 4$. Compare with the exact solution.

Solution: For $n = 2$



$$I \approx \frac{h}{3}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

$$= \frac{0.5}{3}\left[\frac{1}{3} + 4 \times \frac{1}{4} + \frac{1}{5}\right] = 0.25556$$



For $n = 4$

Exact solution: $\frac{1}{2}\ln\frac{5}{3} = 0.25541$

$$I \approx \frac{h}{3}\left[f(0) + 4\left\{f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right\} + 2f\left(\frac{1}{2}\right) + f(1)\right] = 0.25542$$

Numerical Integration

Trapezoidal Rule

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$$E = -\frac{(x_n - x_0)}{12} h^2 f''(\xi)$$

$$\xi \in (x_0, x_n)$$

Simpson's 1/3 Rule

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{h}{3}\left\{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)\right\}$$

$$E = -\frac{x_n - x_0}{180} h^4 f^{(4)}(\xi); \;\; \xi \in (x_0, x_n)$$

Thank You