Vandermonde matrix

Definition A Vandermonde matrix is a square matrix of the form

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{n-1} \end{bmatrix}$$

Theorem If A is a Vandermonde matrix, then

$$\det A = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

Proof (by induction) We proceed by induction on the order n of the matrix. If $n=1$, there is nothing to show. If $n=2$

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \quad \therefore \det A = x_2 - x_1$$

If $n=3$
$$A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_2^2 - x_1^2 \\ x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & x_2 + x_1 \\ 1 & x_3 + x_1 \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1)(x_3 + \cancel{x_1} - x_2 - \cancel{x_1})$$

$$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$$

Now suppose the claim holds for $n-1$. By row operations

$$\det A = \det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 0 & x_2-x_1 & x_2^2-x_1^2 & \cdots & & x_2^{n-1}-x_1^{n-1} \\ - & - & - & & - & - \\ 0 & x_n-x_1 & x_n^2-x_1^2 & & & x_n^{n-1}-x_1^{n-1} \end{bmatrix}_{n\times n}$$

$$= \det \begin{bmatrix} x_2-x_1 & x_2^2-x_1^2 & & x_2^{n-1}-x_1^{n-1} \\ & & & \\ x_n-x_1 & x_n^2-x_1^2 & & x_n^{n-1}-x_1^{n-1} \end{bmatrix}_{(n-1)\times(n-1)}$$

$$= \det\left(\begin{bmatrix} x_2-x_1 & 0 & \cdots & 0 \\ 0 & x_3-x_1 & & 0 \\ 0 & 0 & & x_n-x_1 \end{bmatrix} \begin{bmatrix} 1 & x_2+x_1 & \cdots \\ 1 & x_3+x_1 & \cdots \\ 1 & x_n+x_1 & \cdots \end{bmatrix} \right)$$

$$= \prod_{j=2}^{n} (x_j - x_1) \det \begin{bmatrix} 1 & x_2+x_1 & \cdots \\ 1 & x_3+x_1 & \cdots \\ 1 & x_n+x_1 & \cdots \end{bmatrix}$$

$$= \prod_{j=2}^{n} (x_j - x_1) \det\left(\begin{bmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} \\ 0 & 1 & x_1 & \cdots & x_1^{n-3} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \right)$$

$$= \prod_{j=2}^{n} (x_j - x_1) \det \begin{bmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ 1 & x_n & \cdots & x_n^{n-2} \end{bmatrix}$$

$$= \prod_{j=2}^{n} (x_j - x_1) \prod_{2\le i<j\le n} (x_j - x_i) = \prod_{1\le i<j\le n} (x_j - x_i)$$

Uniqueness of polynomial interpolation

Given $(x_i, y_i)_{i=0}^{n}$ with x_i's distinct. There exists one and only one polynomial $P_n(x)$ of degree $\leq n$ such that $P_n(x_i) = y_i$ for $i = 0, 1, 2, \cdots n$.

Proof: Suppose P_n and Q_n are two different polynomials of degree $\leq n$ of which both interpolate the same data. Then the polynomial $P_n - Q_n$ is of degree $\leq n$ and the value of this polynomial is zero at $n+1$ data points. But a polynomial of degree n has at most n zeros unless it is a zero polynomial. Therefore $P_n - Q_n = 0$ i.e. $P_n = Q_n$.

[Fundamental theorem of Algebra — Every polynomial of degree n that is not identically 0, has maximum n roots (including multiplicities). These roots may be real or complex. In particular, this implies that if a polynomial of degree n has more than n roots, then it must be identically zero.]