ADVANCED CALCULUS
MA11003

SECTION 11 & 12

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Concepts Covered

Differential Equations of Higher Order

- ❑ Complementary Function
- ❑ Solution Techniques

RECALL

Linear Differential Equations of Higher Order with Constant Coefficients

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X$$

General solution $=$ Complementary Function (C.F.) + Particular Integral (P.I.)

If $y_1, y_2 \ldots y_n$ be any n linearly independent solutions of homogeneous differential equation, then

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$ ($c_1, c_2, \cdots, c_n$ are arbitrary constants)

is the general solution of the homogeneous differential equation.

Solution of Homogeneous Linear Equations (Complementary Function)

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$$

$$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]y = 0$$

$$\implies [(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)]y = 0$$

Treating the operator D as a number, the ordinary laws of multiplication works.

Consider $\dfrac{d^2y}{dx^2} + a_1 \dfrac{dy}{dx} + a_2 y = 0$

Write the equation in operator form: $(D^2 + a_1 D + a_2)\, y = 0$

Write the auxiliary equation: $(m^2 + a_1 m + a_2) = 0$

Case of Non-Repeated Roots:

Suppose α_1 and α_2 are two **non-repeated roots** of the auxiliary equations

$$(D^2 + a_1 D + a_2)\, y = 0 \implies (D - \alpha_1)(D - \alpha_2)\, y = 0$$

$$\implies (D - \alpha_2)(D - \alpha_1)\, y = 0$$

Consider $(D-\alpha_1)(D-\alpha_2)y=0$

A solution of the above equation: $(D-\alpha_2)y=0 \quad \Longrightarrow \dfrac{dy}{dx}=\alpha_2 y \quad \Longrightarrow y=e^{\alpha_2 x}$

Similarly, consider $(D-\alpha_2)(D-\alpha_1)\,y=0$

A solution of the above equation: $(D-\alpha_1)y=0 \quad \Longrightarrow \dfrac{dy}{dx}=\alpha_1 y \quad \Longrightarrow y=e^{\alpha_1 x}$

Thus the general solution: $y=c_1 e^{\alpha_1 x}+c_2 e^{\alpha_2 x}$

Generalization

Consider $\dfrac{d^n y}{dx^n} + a_1 \dfrac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$

If $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are distinct roots of $(m^n + a_1 m^{n-1} + \cdots + a_n) = 0$ then

$$e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}$$

will be n different independent solution of the given equation and

$$y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \cdots + c_n e^{\alpha_n x}$$

is the general solution of the homogeneous equation.

Case of Repeated Roots

$(D-\alpha)(D-\alpha)\,y = 0$

Let $(D-\alpha)y = z$ then $(D-\alpha)z = 0 \implies z = c_1 e^{\alpha x}$

Now solving $(D-\alpha)y = c_1 e^{\alpha x} \implies \dfrac{dy}{dx} - \alpha y = c_1 e^{\alpha x}$ (linear in y) I.F. $= e^{-\alpha x}$

Solution: $y\, e^{-\alpha x} = \int c_1 e^{\alpha x} e^{-\alpha x} dx + c_2 \implies y = (c_1 x + c_2) e^{\alpha x}$

Generalization: If a root α is repeated r times

Then, the solution is: $y = (c_1 x^{r-1} + c_2 x^{r-2} + \cdots + c_r) e^{\alpha x}$

Case of Imaginary Roots: Let $\alpha + i\beta$ and $\alpha - i\beta$ be two conjugate roots

Solution: $y = \bar{c}_1 e^{(\alpha+i\beta)x} + \bar{c}_2\, e^{(\alpha-i\beta)x}$

$$y = \bar{c}_1 e^{\alpha x} e^{i\beta x} + \bar{c}_2 e^{\alpha x} e^{-i\beta x} \implies y = e^{\alpha x}(\bar{c}_1 e^{i\beta x} + \bar{c}_2 e^{-i\beta x})$$

$$y = e^{\alpha x}[\bar{c}_1\{\cos\beta x + i\sin\beta x\} + \bar{c}_2\{\cos\beta x - i\sin\beta x\}]$$

$$y = e^{\alpha x}[(\bar{c}_1 + \bar{c}_2)\cos\beta x\} + i(\bar{c}_1 - \bar{c}_2)\sin\beta x]$$

$$y = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]$$

Generalization:

It can similarly be shown that if $(\alpha + i\beta)$ & $(\alpha - i\beta)$

are conjugate imaginary roots, **each repeated r times**, then the solution is

$$y = e^{\alpha x}\left[(p_1 + p_2 x + \cdots + p_r x^{r-1})\cos\beta x + (q_1 + q_2 x + \cdots + q_r x^{r-1})\sin\beta x\right]$$

$p_i, q_i, i = 1,2, \dots, r$ are arbitrary constants

Complementary Function (Summary): $f(D)\,y = 0$ Aux. Eq.: $f(m) = 0$ Roots: $\alpha_1, \alpha_2, \ldots, \alpha_n$

Case I: Roots are real and non-repeated $y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \cdots + c_n e^{\alpha_n x}$

Case II: Roots are real but repeated, say $\alpha_1 = \alpha_2 = \alpha; \alpha_3, \alpha_4, \ldots, \alpha_n$

$$y = (c_1 + c_2 x)e^{\alpha x} + c_3 e^{\alpha_3 x} + \cdots + c_n e^{\alpha_n x}$$

Case III: Roots are complex and non-repeated, say $\alpha \pm i\beta, \alpha_3, \alpha_4, \ldots, \alpha_n$

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{\alpha_3 x} + \cdots + c_n e^{\alpha_n x}$$

Case IV: Roots are complex and repeated, say $\alpha \pm i\beta, \alpha \pm i\beta, \alpha_5, \alpha_6, \ldots, \alpha_n$

$$y = e^{\alpha x}\big((c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\big) + c_5 e^{\alpha_5 x} + \cdots + c_n e^{\alpha_n x}$$

Example 1: Solution of $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$

In operator form: $(D^2 - 5D + 6)y = 0$

Auxiliary equation: $(m^2 - 5m + 6) = 0$

Roots: $m = 2, 3$

The general solution: $y = c_1 e^{2x} + c_2\, e^{3x}$

Example 2: Solution of $\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} + 5\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 4y = 0$

In operator form: $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$

Auxiliary equation: $(m^4 - 2m^3 + 5m^2 - 8m + 4) = 0$

Roots: $m = 1, 1, 2i, -2i$

The general solution: $y = (c_1 + c_2 x)e^x + c_3 \cos 2x + c_4 \sin 2x$

Conclusion

Solution of $f(D)y = 0$ Complementary Function

Auxiliary Equation: $f(m) = 0$

Nature of the roots of the auxiliary equation is important for writing the solution.

Concepts Covered

Differential Equations

- Particular Integral
- Solution Techniques

Determination of Particular Integral :

$f(D)\, y = X$ Particular Integral (P.I.) $= \dfrac{1}{f(D)} X$ $\dfrac{1}{f(D)}$ is called the inverse operator

Note that the operator $f(D)$ can be expressed as $(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)$

Particular Integral (P.I.) $= \dfrac{1}{(D-\alpha_1)} \dfrac{1}{(D-\alpha_2)} \cdots \dfrac{1}{(D-\alpha_n)} X$

- **General Method for P.I. :** $\boxed{\dfrac{1}{D-a} X = e^{ax} \int X e^{-ax} dx}$

- **General Method for P.I. :** $\dfrac{1}{D-a}X = e^{ax}\displaystyle\int Xe^{-ax}dx$

Proof: Let $y = \dfrac{1}{D-a}X \quad \Rightarrow (D-a)y = X \quad \Rightarrow \dfrac{dy}{dx} - ay = X$

$$\Rightarrow ye^{-ax} = \int Xe^{-ax}\,dx + C$$

C may be taken as 0 for P.I.

$$\Rightarrow \boxed{y = e^{ax}\int X\,e^{-ax}\,dx} + C\,e^{ax}$$

Example : Solve $(D^2 + a^2)\, y = \sec ax$

C.F. = $c_1 \cos ax + c_2 \sin ax$

P.I.$= \dfrac{1}{(D^2 + a^2)} \sec ax = \dfrac{1}{2ia}\left[\dfrac{1}{D - ia} - \dfrac{1}{D + ia}\right] \sec ax$

Consider $\dfrac{1}{D - ia} \sec ax = e^{iax} \displaystyle\int \sec ax\, e^{-iax} dx = e^{iax}\left[x + \dfrac{i}{a} \ln|\cos ax|\right]$

Similarly $\dfrac{1}{D + ia} \sec ax = e^{-iax}\left[x - \dfrac{i}{a} \ln|\cos ax|\right]$

$$\text{P.I.} = \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\sec ax$$

$$= \frac{1}{2ia}\left[e^{iax}\left\{x + \frac{i}{a}\ln|\cos ax|\right\} - e^{-iax}\left\{x - \frac{i}{a}\ln|\cos ax|\right\}\right]$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\ln|\cos ax|\cos ax$$

General Solution:

$$y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\ln|\cos ax|\cos ax$$

Thank You