

ADVANCED CALCULUS

MA11003

SECTION 11, 12, & 15CD

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TRIPLE INTEGRALS

Triple Integral

Divide the region V into n sub regions of respective volumes $\delta V_1, \delta V_2, \dots, \delta V_n$

Let (x_j, y_j, z_j) be an arbitrary point in the j th sub-region.

Consider the sum
$$\sum_{j=1}^n f(x_j, y_j, z_j) \delta V_j$$

Represent Volume if $f = 1$

If the limit exists as $n \rightarrow \infty$ and $\delta V_j \rightarrow 0$ then

$$\iiint_V f(x, y, z) dV = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j, y_j, z_j) \delta V_j$$

Triple Integral - Evaluation

$$\int \int \int_V f(x, y, z) dV = \int_{z=a}^b \left\{ \int_{y=\psi_1(z)}^{\psi_1(z)} \left\{ \int_{x=f_1(y,z)}^{f_2(y,z)} f(x, y, z) dx \right\} dy \right\} dz$$

Note: Similar to double integrals, the order of integration is immaterial if the limits of integration are constants.

$$\begin{aligned} \int_a^b \int_c^d \int_e^f f(x, y, z) dx dy dz &= \int_e^f \int_c^d \int_a^b f(x, y, z) dz dy dx \\ &= \int_c^d \int_e^f \int_a^b f(x, y, z) dz dx dy \end{aligned}$$

Problem -1: Evaluate $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$

$$I = \int_0^a \int_0^x e^{x+y+z} \Big|_0^{x+y} dy dx = \int_0^a \int_0^x (e^{2(x+y)} - e^{x+y}) dy dx$$

$$= \int_0^a \frac{e^{2(x+y)}}{2} \Big|_0^x dx - \int_0^a e^{x+y} \Big|_0^x dx = \frac{1}{2} \int_0^a \{(e^{4x} - e^{2x}) - 2(e^{2x} - e^x)\} dx$$

$$= \frac{1}{2} \int_0^a (e^{4x} - 3e^{2x} + 2e^x) dx = \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$$

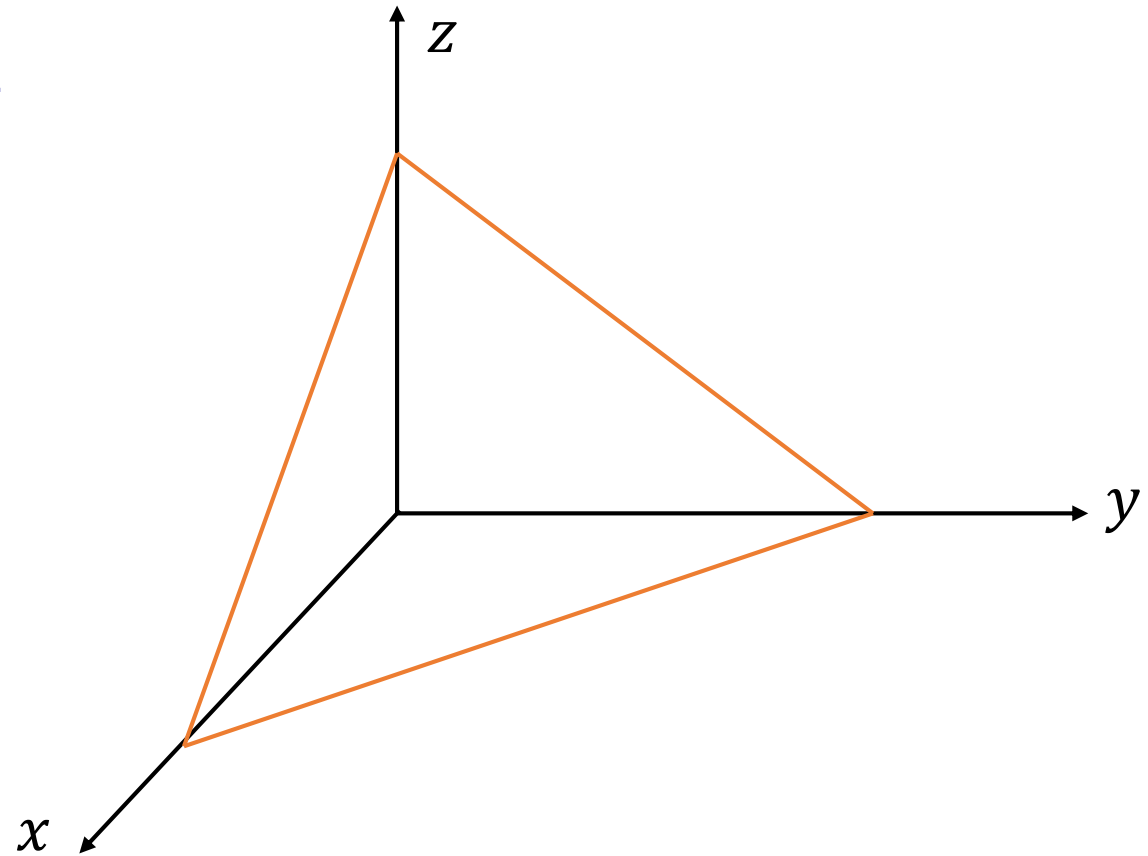
Problem -2: Evaluate $I = \iiint_R \frac{dx dy dz}{(x + y + z + 1)^3}$

R is the region bounded by
 $x = 0, y = 0, z = 0$ & $x + y + z = 1$

$$I = \int \int \int \frac{1}{(x + y + z + 1)^3} dz dy dx$$

$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2} (x + y + z + 1)^{-2} \right]_0^{1-x-y} dy dx$$

$$I = \int_0^1 \int_0^{1-x} \left[-\frac{1}{2} (x + y + z + 1)^{-2} \right]_0^{1-x-y} dy dx = \frac{1}{2} \left[\log 2 - \frac{5}{8} \right]$$

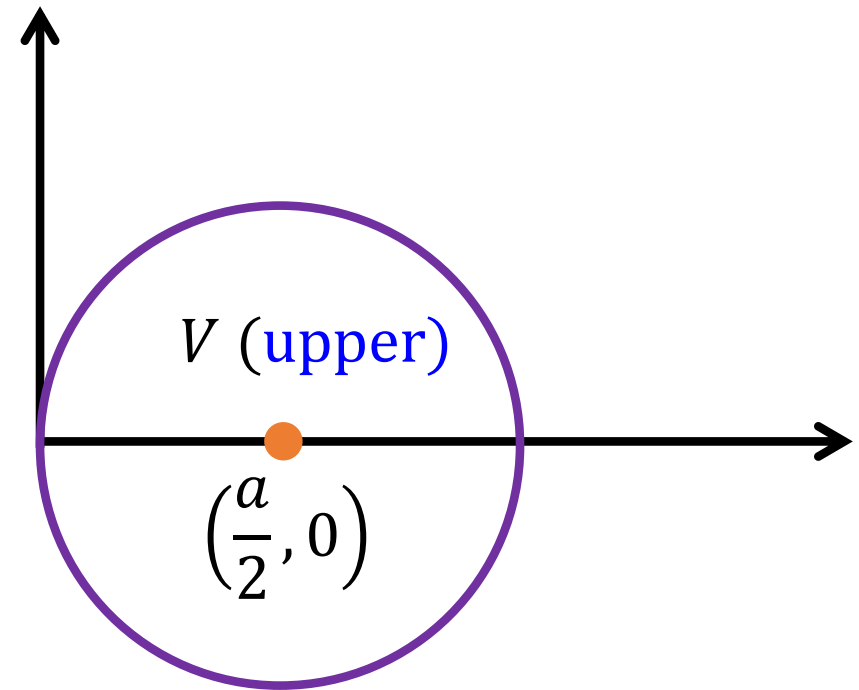


Problem -3: Using triple integral find the volume common to a sphere $x^2 + y^2 + z^2 = a^2$ and a circular cylinder $x^2 + y^2 = ax$.

$$V = 4 \int \int \int_V dx \, dy \, dz = 4 \int \int \int_V dz \, dy \, dx$$

$$= 4 \int \int \int dz \, dy \, dx$$

$$= 4 \int_0^a \int_{y=0}^{\sqrt{ax-x^2}} \sqrt{a^2 - x^2 - y^2} \, dy \, dx$$



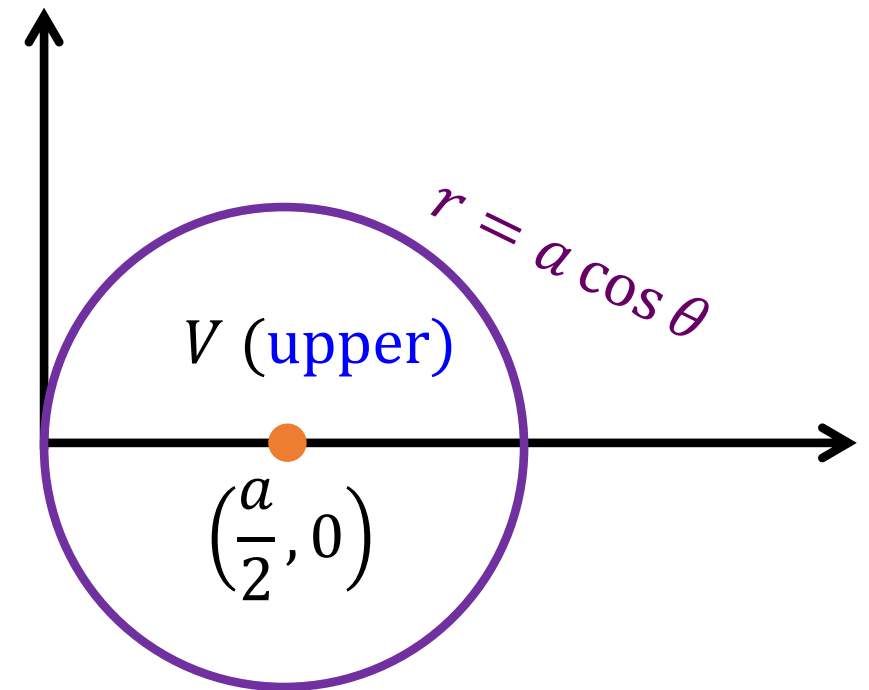
$$= 4 \int_0^a \int_{y=0}^{\sqrt{ax-x^2}} \sqrt{a^2 - x^2 - y^2} \, dy \, dx$$

$$= 4 \int_0^{\frac{\pi}{2}} \int_{r=0}^{a \cos \theta} \sqrt{a^2 - r^2} \, r \, dr \, d\theta$$

$$= 4 \left(-\frac{1}{2} \cdot \frac{2}{3} \right) \int_0^{\frac{\pi}{2}} \left[(a^2 - r^2)^{\frac{3}{2}} \right]_0^{a \cos \theta} d\theta$$

$$= -\frac{4}{3} a^3 \int_0^{\frac{\pi}{2}} (\sin^3 \theta - 1) \, d\theta$$

$$= \frac{2}{3} a^3 \left(\pi - \frac{4}{3} \right)$$



$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

Triple Integrals – Change of Variables

Change of Variables in Triple Integrals:

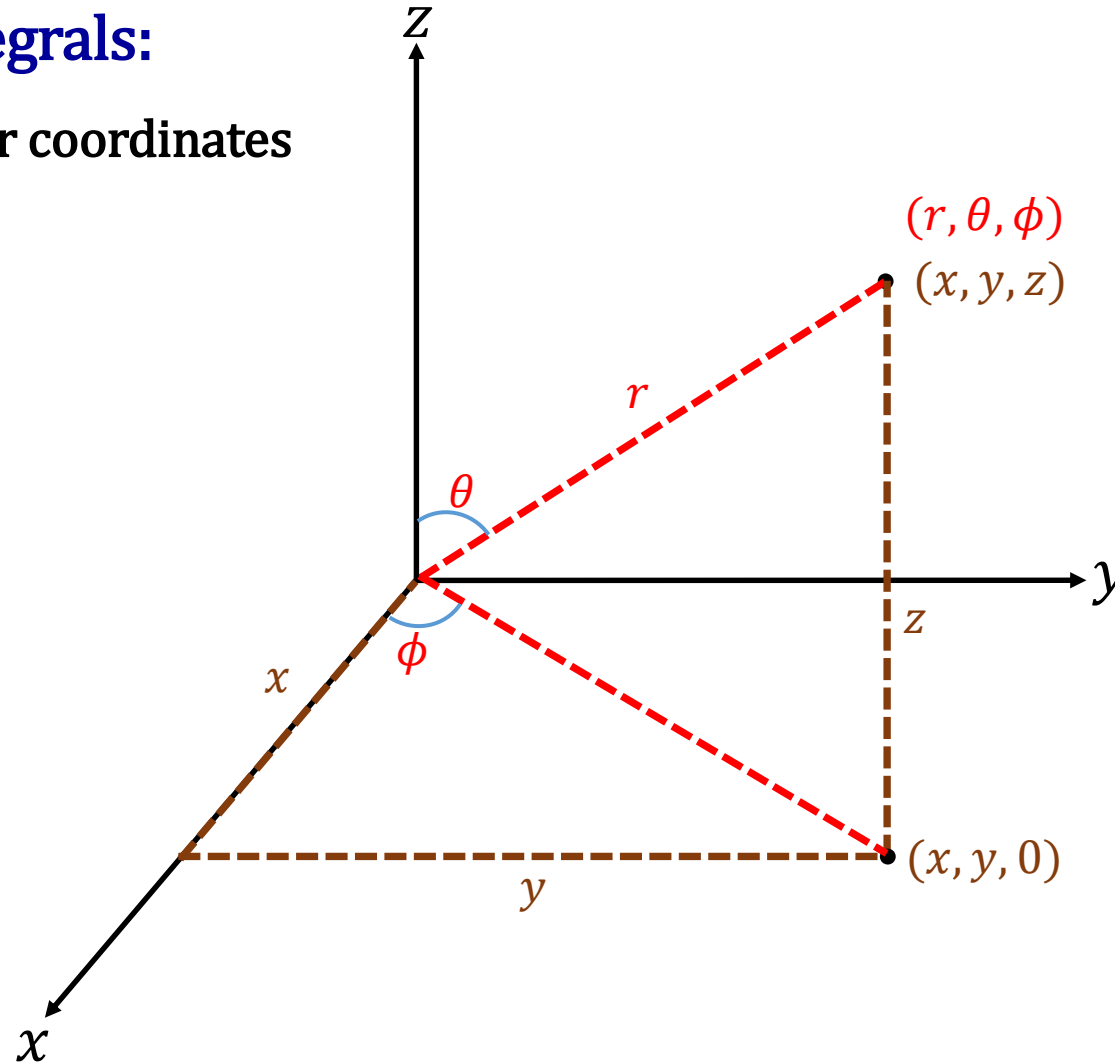
Cartesian coordinates to spherical polar coordinates

$$x = r \sin \theta \cos \phi,$$

$$y = r \sin \theta \sin \phi,$$

$$z = r \cos \theta$$

Note that $x^2 + y^2 + z^2 = r^2$



Change of Variables in Triple Integrals:

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi$$

Cartesian coordinates to spherical polar coordinates

$$z = r \cos \theta$$

$$\int \int \int_D f(x, y, z) \, dx dy dz = \int \int \int_{\hat{D}} f(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) |J| \, dr \, d\theta \, d\phi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix} = r^2 \sin \theta$$

Change of Variables in Triple Integrals:

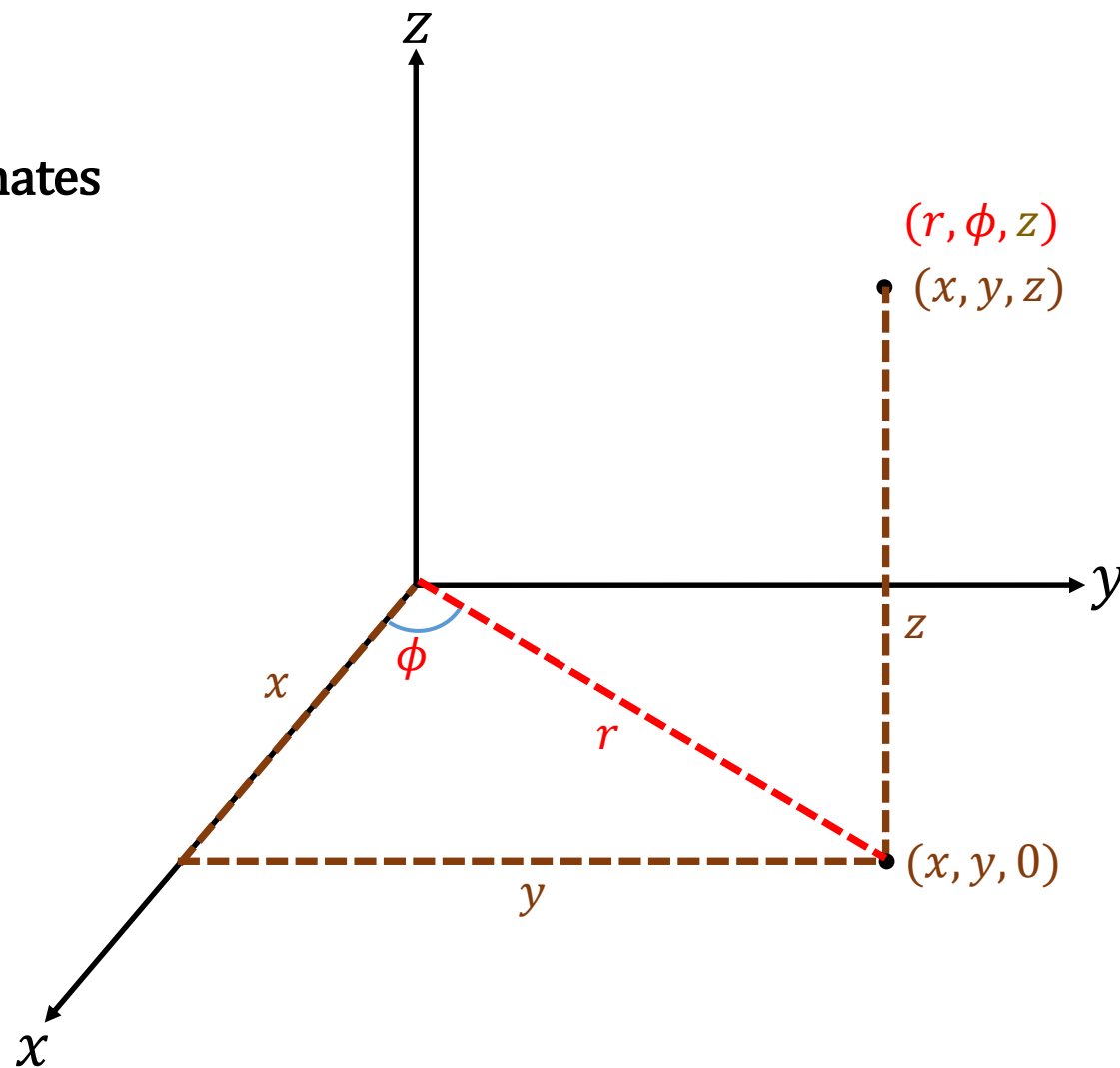
Cartesian coordinates to cylindrical coordinates

$$x = r \cos \phi,$$

$$y = r \sin \phi,$$

$$z = z$$

Note that $x^2 + y^2 = r^2$



Change of Variables in Triple Integrals:

$$x = r \cos \phi \quad y = r \sin \phi \quad z = z$$

Cartesian coordinates to cylindrical coordinates

$$\int \int \int_D f(x, y, z) \, dx dy dz = \int \int \int_{\hat{D}} f(r \cos \phi, r \sin \phi, z) |J| \, dr d\phi dz$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \phi & -r \sin \phi & 0 \\ \sin \phi & r \cos \phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

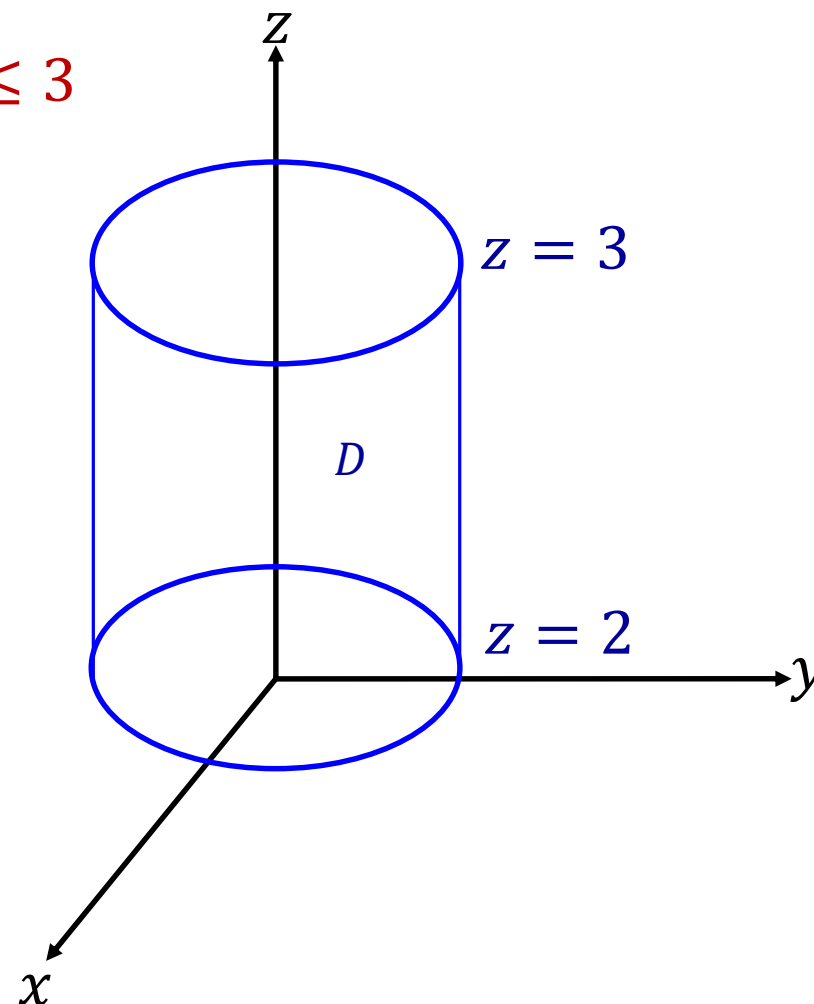
Problem -1: Changing to cylindrical coordinate, evaluate

$$\iiint_D z(x^2 + y^2) dx dy dz \quad D: x^2 + y^2 \leq 1, 2 \leq z \leq 3$$

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

Note that $x^2 + y^2 = r^2$ and $J = r$

$$I = \int_{z=2}^3 \int_{\phi=0}^{2\pi} \int_{r=0}^1 z r^2 r dr d\phi dz$$



$$I = \int_{z=2}^3 \int_{\phi=0}^{2\pi} \int_{r=0}^1 z r^2 r dr d\phi dz$$

$$= \int_{z=2}^3 \int_{\phi=0}^{2\pi} \frac{1}{4} z d\phi dz$$

$$= \frac{2\pi}{4} \int_{z=2}^3 z dz$$

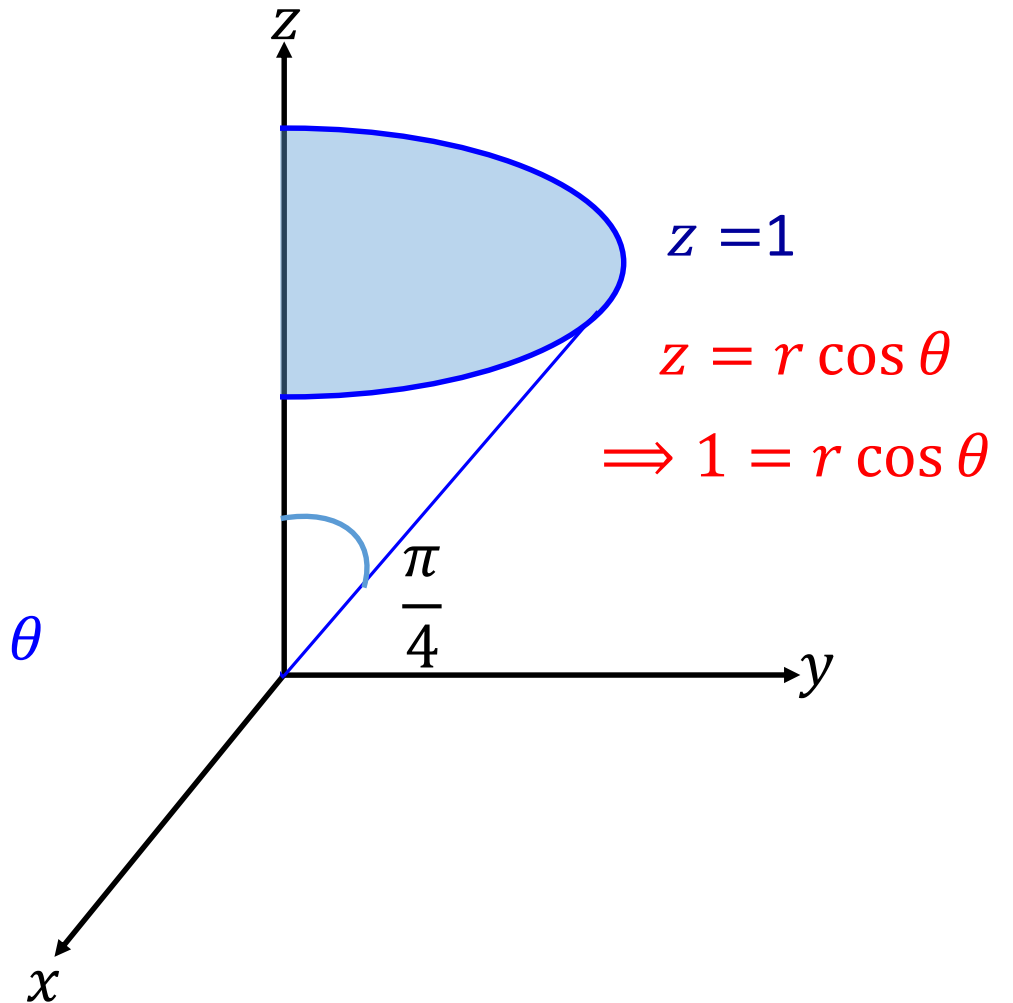
$$= \frac{5\pi}{4}$$

Problem -2: Changing to spherical coordinate, evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{1}{\sqrt{x^2+y^2+z^2}} dz dy dx$$

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta$$

$$J = r^2 \sin \theta, \quad x^2 + y^2 + z^2 = r^2$$

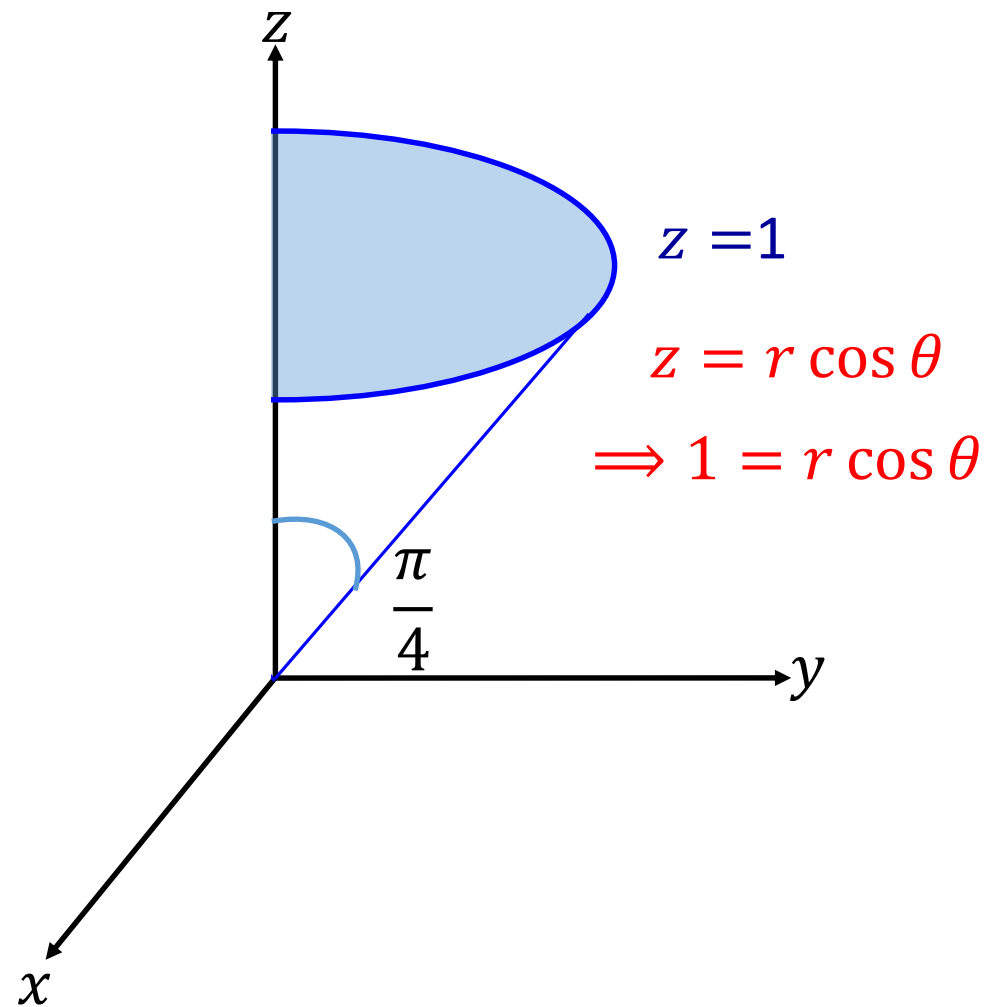


$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{1}{\sqrt{x^2+y^2+z^2}} dz dy dx$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_{r=0}^{\sec \theta} \frac{1}{r} r^2 \sin \theta dr d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \frac{1}{2} \sec^2 \theta \sin \theta d\theta d\phi$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sec \theta \Big|_0^{\frac{\pi}{4}} d\phi = \frac{(\sqrt{2} - 1)\pi}{4}$$



Problem -3: Changing to spherical coordinate, evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2+y^2+z^2}} dz dy dx$$

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta$$

$$J = r^2 \sin \theta, \quad x^2 + y^2 + z^2 = r^2$$

$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{\frac{\pi}{2}} \int_{r=0}^1 \frac{r^2 \sin \theta}{\sqrt{1-r^2}} dr d\phi d\theta$$

$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{\frac{\pi}{2}} \int_{r=0}^1 \frac{r^2 \sin \theta}{\sqrt{1-r^2}} dr d\phi d\theta$$

$$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin \theta d\phi d\theta$$

$$= \frac{\pi}{4} \frac{\pi}{2} [-\cos \theta]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{8}$$

First evaluate $\int_{r=0}^1 \frac{r^2}{\sqrt{1-r^2}} dr$

$$= \int_0^{\frac{\pi}{2}} \sin^2 t dt \quad (\text{sub. } r = \sin t)$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2t) dt$$

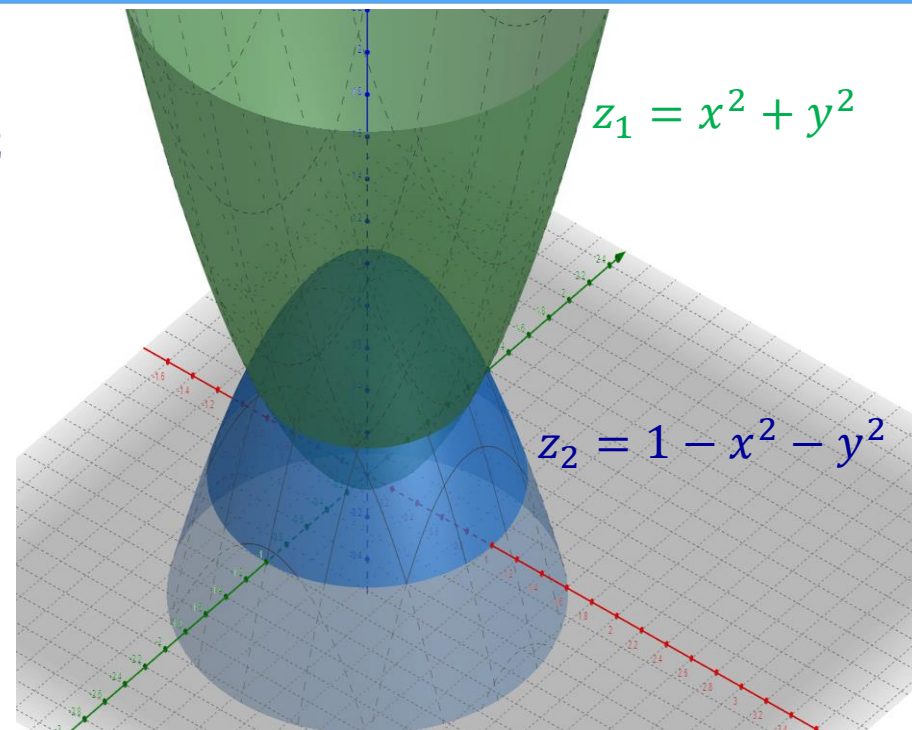
$$= \frac{\pi}{4}$$

Problem -4: Find the volume of the solid formed by two
Paraboloids: $z_1 = x^2 + y^2$ & $z_2 = 1 - x^2 - y^2$

Intersecting curve:

$$x^2 + y^2 = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 = \frac{1}{2}$$

(Projection on xy plane)



$$V = \iiint_V dx \, dy \, dz = \int \int \int dz \, dy \, dx$$

Changing to Cylindrical coordinates

$$V = 4 \int \int \int r \, dz \, dr \, d\theta = \frac{\pi}{4}$$

Conclusion:

Triple Integrals – Change of Variables

- Spherical coordinates
- Cylindrical coordinates

Thank You