Vector space

Vectors : Def^n. Any ordered n-tuple of numbers is called an n-vector. By an ordered n-tuple we mean a set consisting of n numbers in which the place of each number is fixed. If $x_1, x_2, \ldots x_n$ be any n numbers, then the ordered n-tuple $X = (x_1, x_2, \ldots, x_n)$ is called an n vector. The n numbers $x_1, x_2, \ldots x_n$ are called components of the n-vector $X = (x_1, x_2, \ldots, x_n)$. A vector may be written either as a row vector or as a column vector. If A be a matrix of the type $m \times n$, then each row of A will be an n-vector and each column of A will be an m vector. A vector whose components are all zero is called a zero vector and will be denoted by O. If k be any number and X be any vector, then relative to the vector X, k is called a scalar.

Algebra of vectors

Since an n-vector is nothing but a row matrix or a column matrix, therefore we can develop an algebra of vectors in the same manner as the algebra of matrices.

Equality of two vectors, addition of two vectors, multiplication of a vector by a scalar (number).

Properties of addition and scalar multiplication of vectors : If X, Y, Z be any 3 n vectors and p, q be any two numbers, then

(i) $X + Y = Y + X$ (ii) $X + (Y + Z) = (X + Y) + Z$

(iii) $p(X + Y) = pX + pY$ (iv) $(p + q)X = pX + qX$ (v) $p(qX) = (pq)X$

Linearly dependent set of vectors

A set of r n-vectors $X_1, X_2, \cdots X_r$ is said to be linearly dependent if there exist r scalars (numbers) $k_1, k_2, \cdots k_r$, not all zero, such that $k_1X_1 + k_2X_2 + \cdots + k_rX_r = 0$, where 0, denotes the n vector whose components are all zero.

Linearly independent set of vectors

A set of r n vectors $X_1, X_2, \cdots, X_r$ is said to be linearly independent if every relation of the type $k_1X_1 + k_2X_2 + \cdots + k_rX_r = 0$ implies $k_1 = k_2 = \cdots = k_r = 0$.

A vector as a linear combination of vectors

A vector X which can be expressed in the form

$$X = k_1X_1 + k_2X_2 + \cdots + k_rX_r$$

is said to be a linear combination of the set of vectors $X_1, X_2, \cdots, X_r$. Here k_1, k_2 are any numbers.

Results

(i) If a set of vectors is L.D., then at least one member of the set can be expressed as a linear combination of the remaining members.

(ii) If a set of vectors is L.I., then no member of the set can be expressed as a linear combination of the remaining members.

Ex Test if the vectors $(1,2,1), (2,1,0), (1,-1,2)$ are L.I. or not.

Solⁿ: $x(1,2,1) + y(2,1,0) + z(1,-1,2) = (0,0,0)$

$x + 2y + z = 0$

$2x + y - z = 0$ Solving $x = y = z = 0$

$x + 2z = 0$ $\therefore$ They are L.I.

Theorem

p vectors $x_1, x_2, \ldots x_p$ (with n components each) are L.I. if the matrix with row vectors $x_1, x_2, \ldots x_p$ has rank p; they are L.D. if the rank is less than p.

Theorem

p vectors with n $(<p)$ components are always L.D.

$(2,3), (5,1), (4,2)$ are L.D.