ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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DOUBLE INTEGRALS (Cont.)

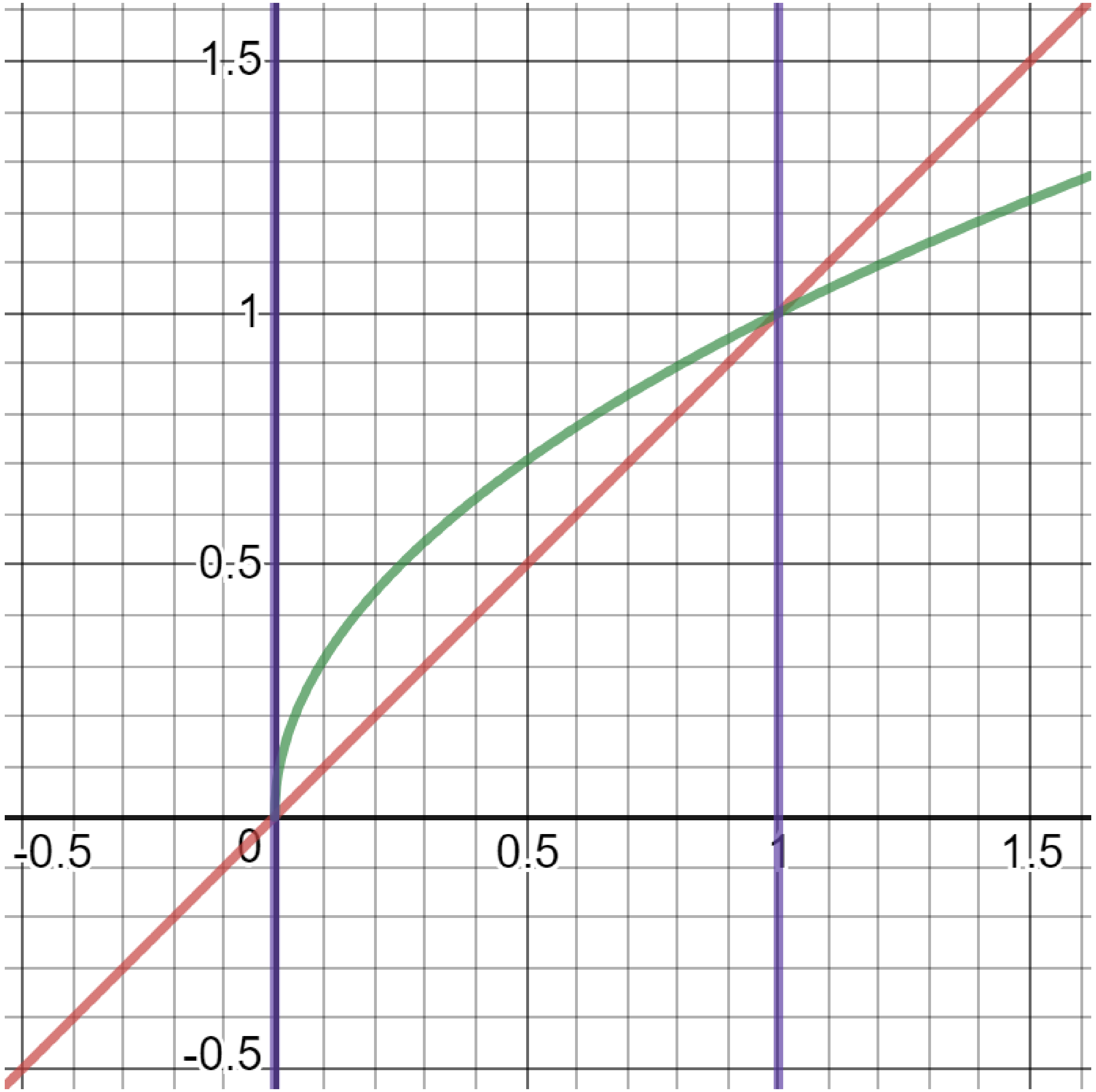
- Double Integrals - Change of Order

Problem - 3

Change the order of integration

$$\int_0^1 \int_x^{\sqrt{x}} f(x,y)\, dy\, dx$$

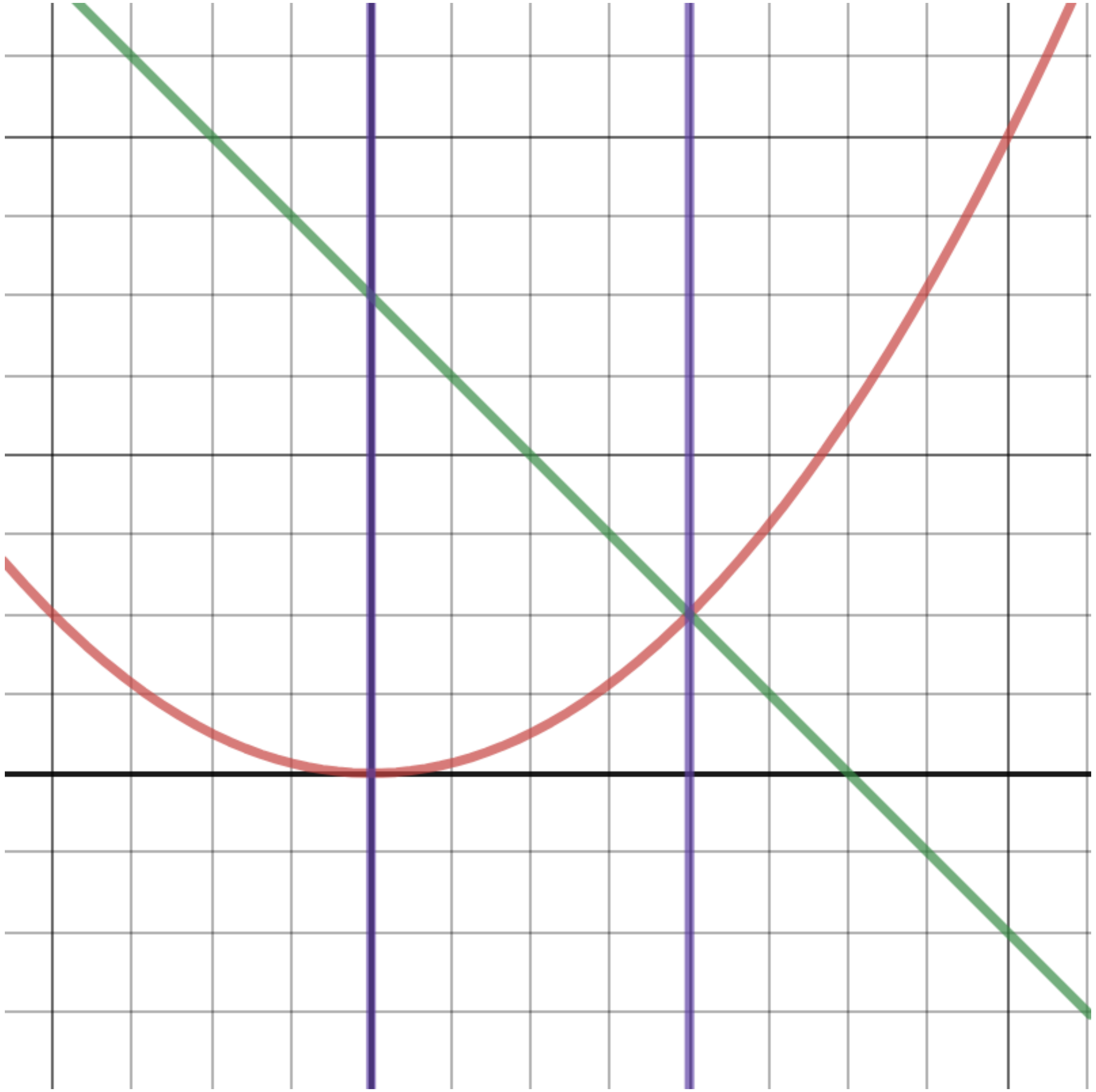
ANS: $\int_0^1 \int_{y^2}^{y} f(x,y) dx dy$



Problem - 4 Change the order of integration

$$\int_0^{2a}\int_{\frac{x^2}{4a}}^{3a-x} f(x,y)\,dy\,dx$$

$$\int_0^{a}\int_0^{2\sqrt{ay}} f(x,y)\,dx\,dy+\int_a^{3a}\int_0^{3a-y} f(x,y)\,dx\,dy$$



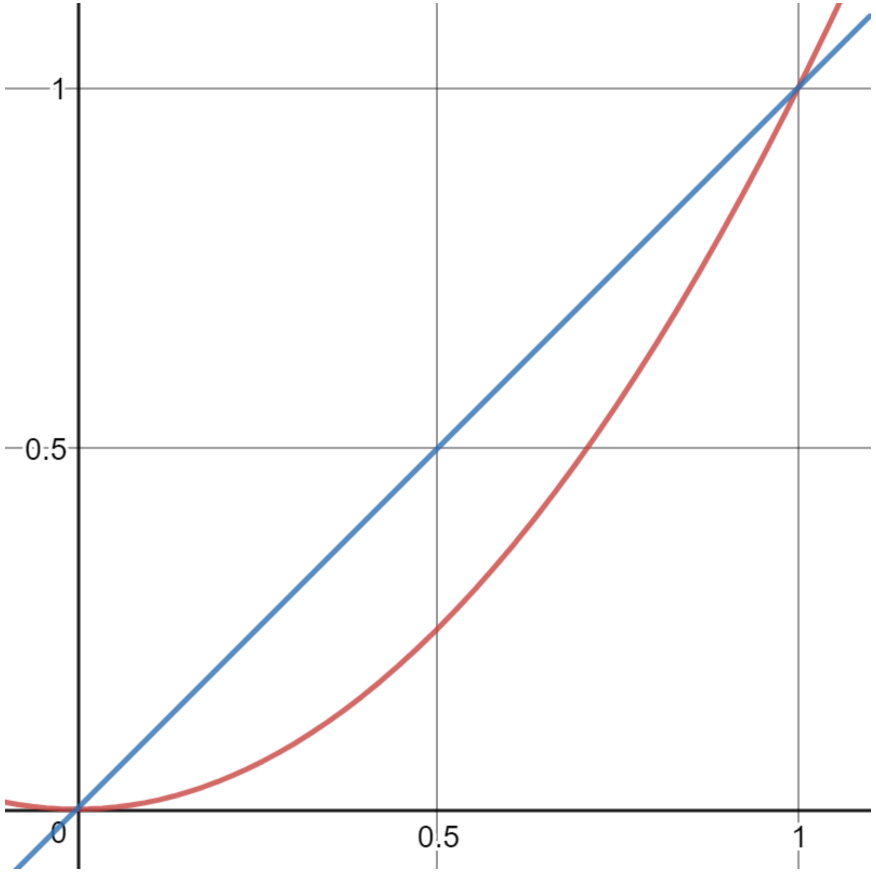
Conclusion:

- Sketching the region of integration
- Limit of integration

DOUBLE INTEGRALS (Cont.)

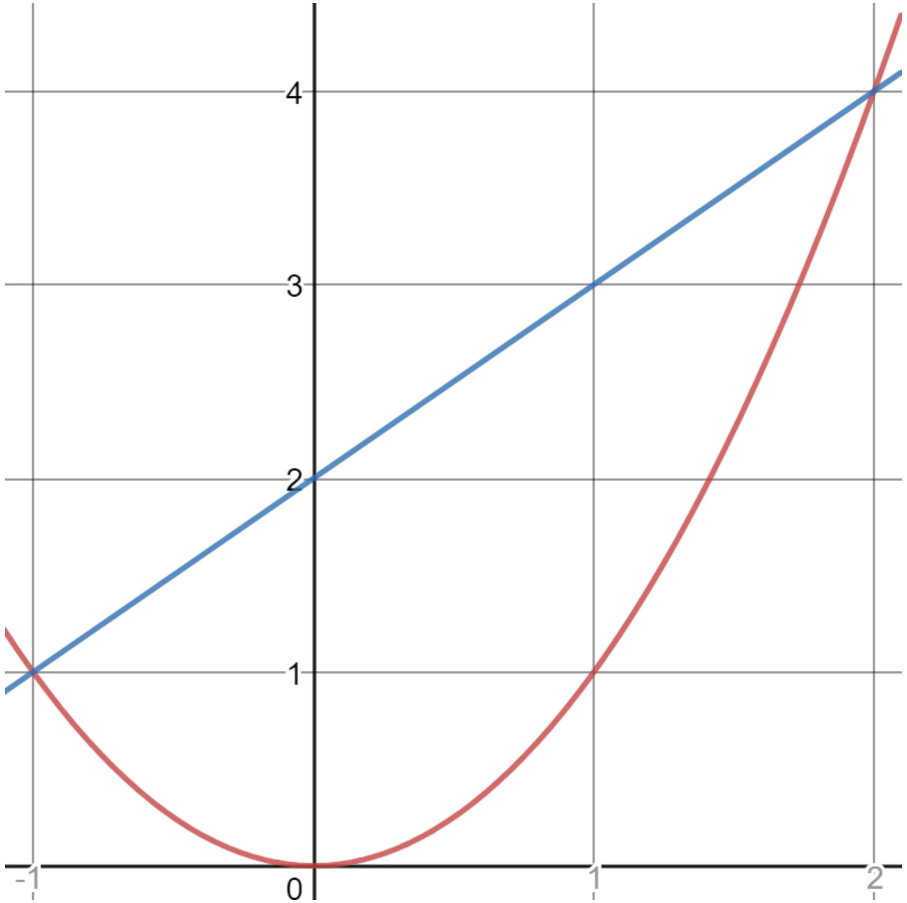
Applications

Problem - 1 Using a double integral find the area of the region enclosed by the parabola $y = x^2$ and the line $y = x$.



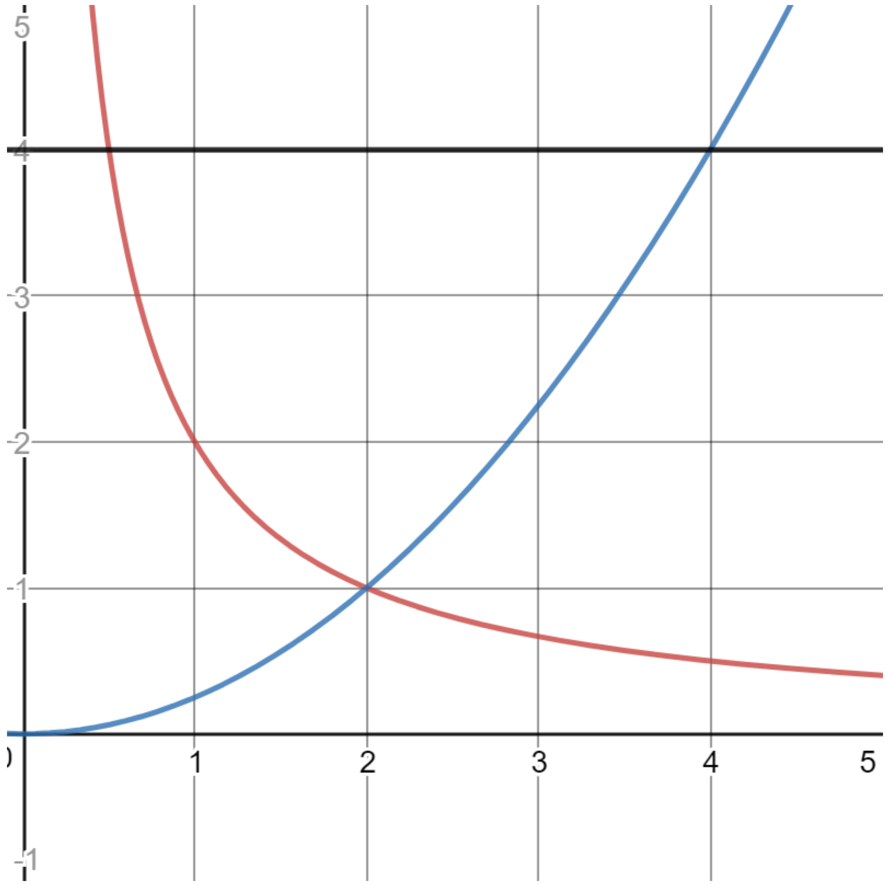
$$\int_0^1 \int_{x^2}^{x} \mathrm{d}y \, \mathrm{d}x = \frac{1}{6}$$

Problem - 2 Using a double integral find the area of the region enclosed by parabola $y = x^2$ and the line $y = x + 2$.



$$\int_{-1}^{2}\int_{x^2}^{x+2} \mathrm{d}y\, \mathrm{d}x = \frac{9}{2}$$

Problem - 3 Using a double integral, determine the area bounded by the curves $xy = 2$, $y = \frac{x^2}{4}$ and $y = 4$.

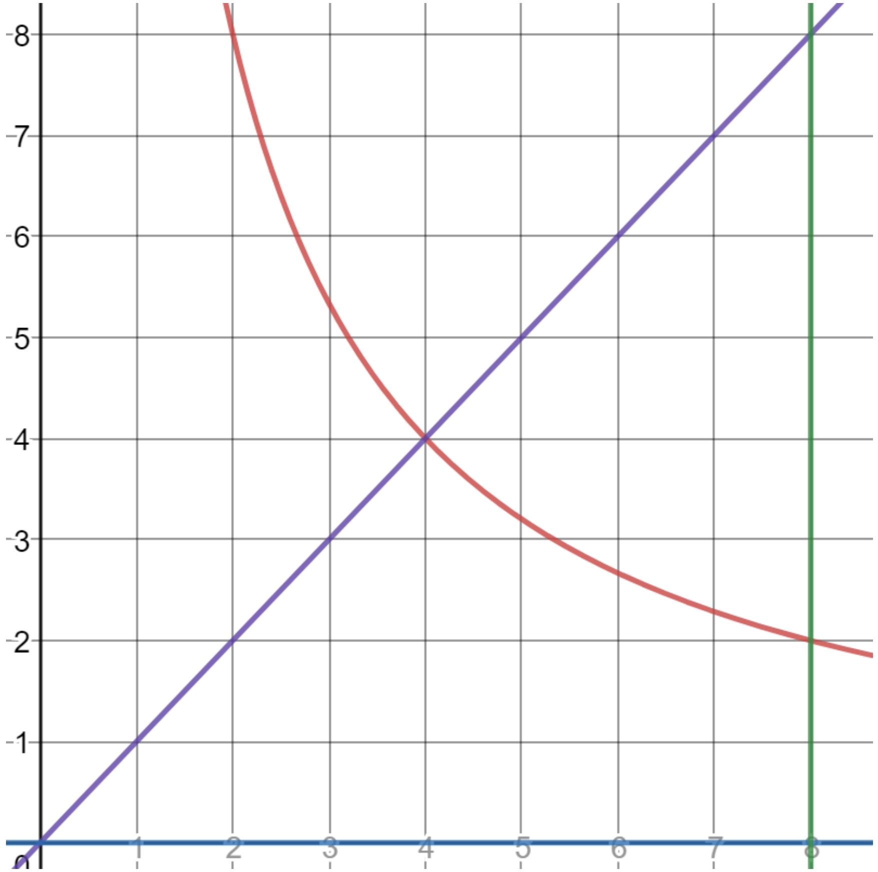


$$\int_{y=1}^{4} \int_{x=\frac{2}{y}}^{2\sqrt{y}} dx\, dy = \frac{28}{3} - 2 \ln 4$$

Problem - 4 Using double integrals find the volume of the solid below the $z = xy$ over the region enclosed by $y = 4 - x^2, x = 1, x = 2$ and the x-axis.

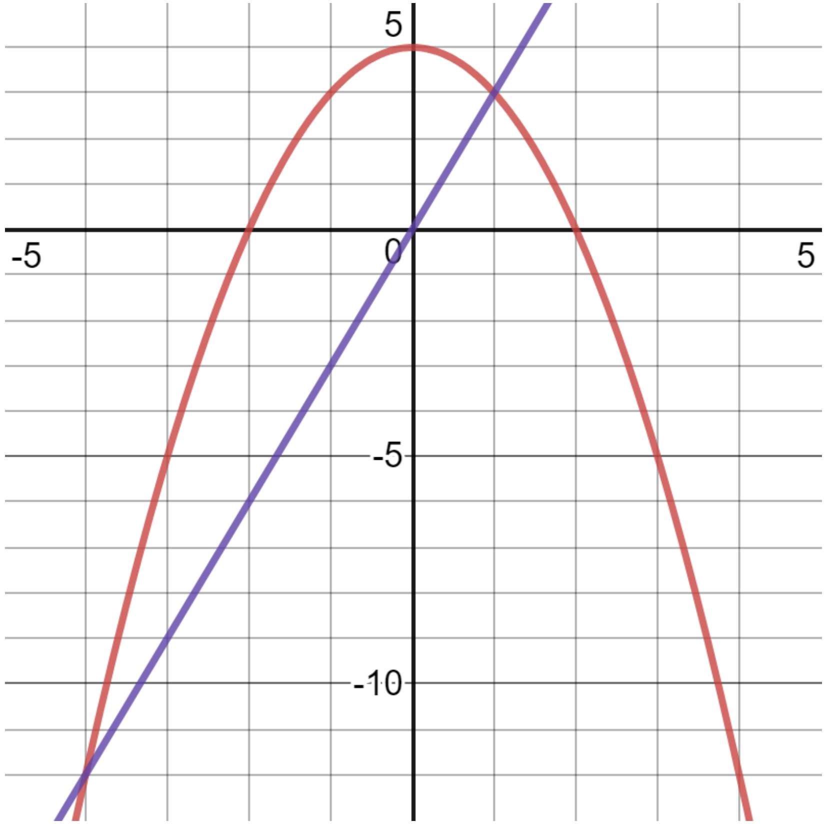
$$V = \int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} xy \;\, \mathrm{d}x \, \mathrm{d}y = \frac{9}{4}$$

Problem - 5 Calculate the volume of a solid whose base is in a $xy-$plane and is bounded by the curve $xy = 16$ and the line $y = x$, $y = 0$, $x = 8$ while the top of the solid is in the plane $z = x$.



$$\int_0^4 \int_0^x x\, dy\, dx + \int_4^8 \int_0^{16/x} x\, dy\, dx = \frac{256}{3}$$

Problem - 6 Calculate the volume of a solid whose base is in a $xy-\text{plane}$ and is bounded by the parabola $y = 4 - x^2$ and the straight line $y = 3x$ while the top of the solid is in the plane $z = x + 4$.



$$V = \int_{-4}^{1} \int_{3x}^{4-x^2} (x+4)\, \mathrm{d}y\, \mathrm{d}x = \frac{625}{12}$$

Conclusion:

Some Applications of Double Integrals

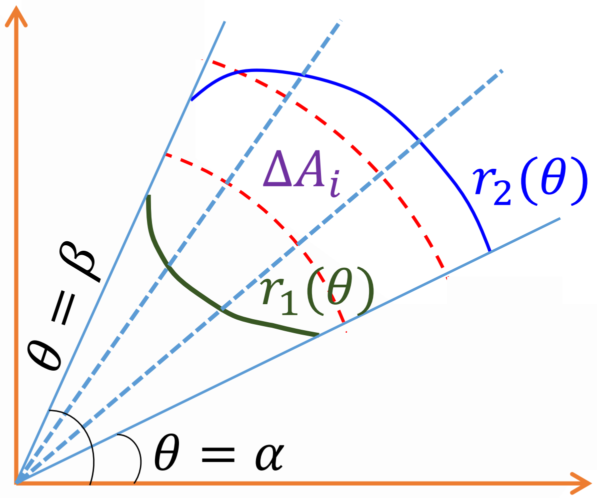
- Computation of Area
- Computation of Volume

DOUBLE INTEGRALS (Cont.)

Double Integrals in Polar Form

Double Integrals: Change of Variables

Double Integrals in Polar Forms



$$\Delta A_i = (r_i + \Delta r_i)^2 \frac{\Delta\theta_i}{2} - r_i^2 \frac{\Delta\theta_i}{2}$$

$$= \left(2r_i\Delta r_i + \Delta r_i{}^2\right)\frac{\Delta\theta_i}{2}$$

$$= \left(r_i + \frac{\Delta r_i}{2}\right)\Delta r_i\,\Delta\theta_i$$

$$= r_i^*\Delta r_i\Delta\theta_i$$

$$I = \lim_{n\to\infty}\sum_{j=1}^{n} f(r_j^*, \theta_j^*)\,\Delta A_j = \int_{\theta=\alpha}^{\beta}\int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Changing Cartesian integral to polar integrals

$$\iint\limits_{R} f(x,y)\,dx\,dy = \iint\limits_{G} f(r\cos\theta\,, r\sin\theta)\,r\,dr\,d\theta$$

- Substitute $x = r\cos\theta\,, y = r\sin\theta$
- Replace $dx\,dy$ by $r\,dr\,d\theta$
- G is same as R but described in polar corrdinates

Example: Compute area of first quadrant of a circle of radius a.

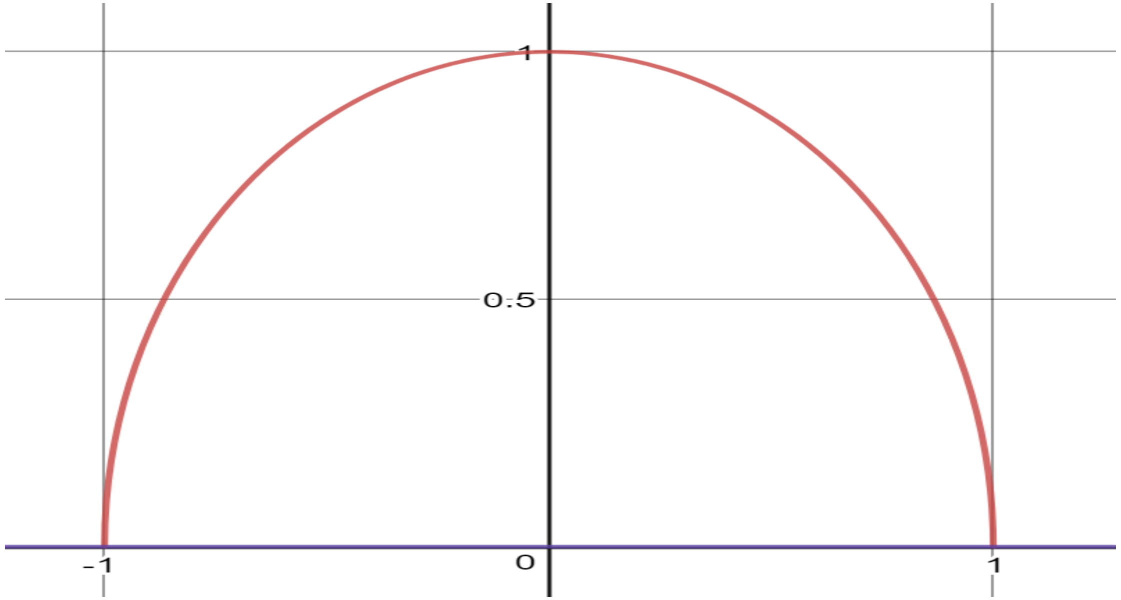
$$A = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r dr d\theta$$

$$= \frac{a^2}{2} \frac{\pi}{2}$$

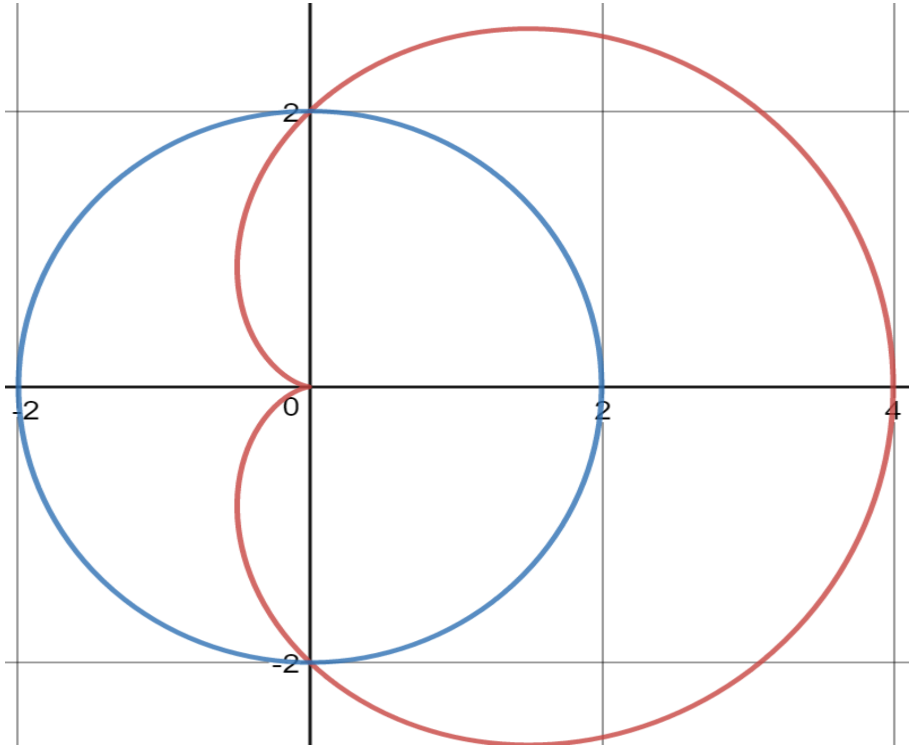
$$= \frac{\pi a^2}{4}$$

Problem -1: Evaluate $\displaystyle\iint_R e^{x^2+y^2}\,dy\,dx$

where R is the semicircular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$



$$\int_0^{\pi}\int_0^{1} e^{r^2}\,r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi} e^{r^2}\Big|_0^1\,d\theta = \frac{1}{2}\int_0^{\pi}(e-1)\,d\theta = \frac{\pi}{2}(e-1)$$



Problem -2:

Calculate the area which is inside the cardioid $r = 2(1 + \cos\theta\ )$ and outside the circle $r = 2.$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2}^{2(1+\cos\theta\ )} r\, dr\, d\theta = \pi + 8$$

Problem - 3: Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\,r\,dr\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,d\theta = \frac{\pi}{4}$$

Note: $$I = \int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-y^2} dy$$

$$I^2 = \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy$$

$$= \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$$

$$= \frac{\pi}{4}$$

$$\Rightarrow \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Conclusion:

Double Integrals in Polar form

- Some integrals become easier by changing to polar coordinate due to
  - Integrands
  - Domain

QUIZ QUESTION ?

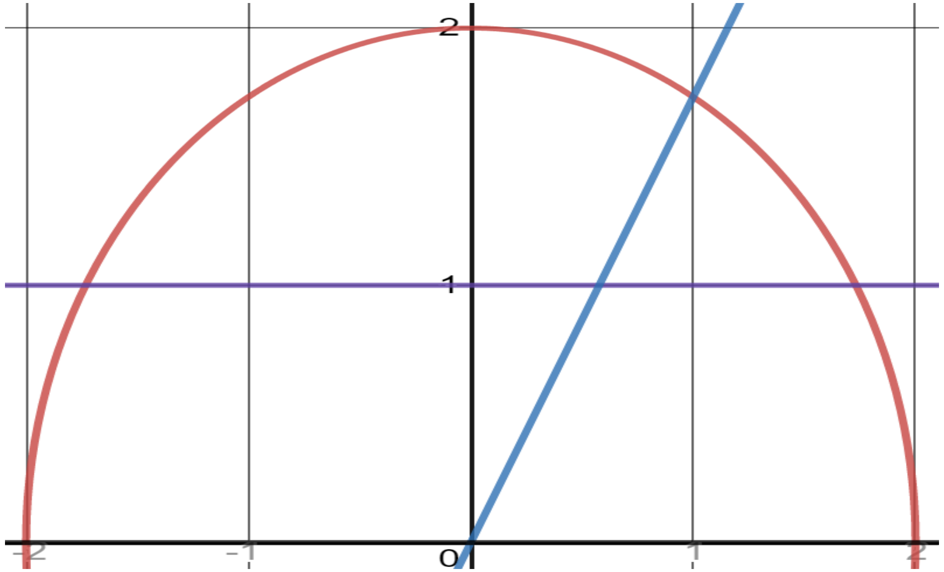
LINK FOR RESPONSES: http://www.facweb.iitkgp.ac.in/~jkumar/teach/MA11003.html

QUIZ QUESTION:

If the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 = 4$ above the line $y = 1$ and below the $y = \sqrt{3}\,x$ is

$$\frac{\pi - a}{b}$$

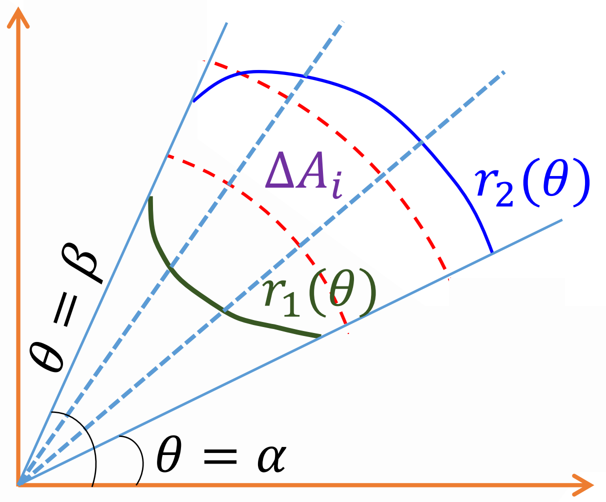
Then $a^2 b$ is___________



$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{\operatorname{cosec}\theta}^{2} r\,dr\,d\theta = \frac{\pi - \sqrt{3}}{3}$$

ANS: 9

Double Integrals in Polar Forms (Previous Lecture)



$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\; r\,dr\,d\theta$$

Double Integrals – Change of Variable

$$\int_a^b f(x)\,dx = \int_c^d f\big(g(t)\big)\,g'(t)\,dt$$

Substitution: $x = g(t)$.

where $a = g(c)$ and $b = g(d)$

Double Integrals – Change of Variables

$$\iint_R f(x, y)\, dx\, dy$$

Substitution $x = \Phi(u, v), y = \psi(u, v)$

$$\iint_{R'} f\big(\Phi(u, v), \psi(u, v)\big)\, |J|\, dudv$$

where $J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

R' is the region in uv plane which corresponds to the region R in the xy-plane.

Double Integrals – Change of Variables (Special Case)

$$\iint_R f(x,y)\,dx\,dy$$

Cartesian to polar co-ordinates:

$$x = r\cos\theta\,,\quad y = r\sin\theta\,;$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & \mathrm{r}\cos\theta \end{vmatrix} = r$$

$$\Rightarrow \quad \iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta\,, r\sin\theta)\ r\,dr\,d\theta$$

Example -1 Find the volume in one octant of a sphere of radius a.

$$V = \iint_S \sqrt{a^2 - x^2 - y^2}\, dx\, dy$$

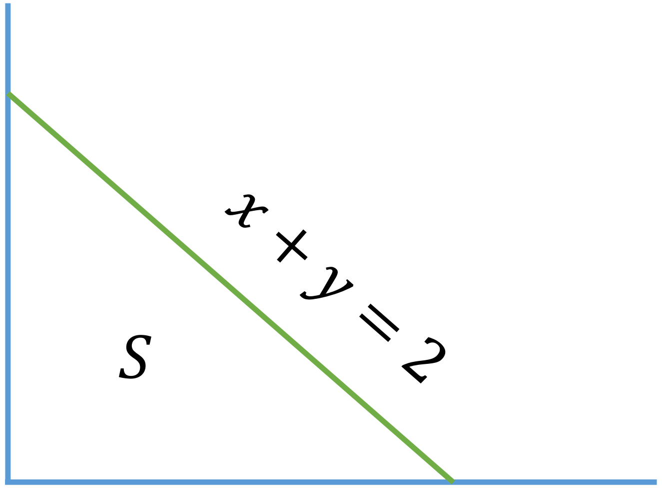
S is the first quadrant of the circular disc $x^2 + y^2 \leq a^2$

Change of variables $x = r\cos\theta$, $y = r\sin\theta$, $|J| = r$

$$\int\int_S \sqrt{a^2 - x^2 - y^2}\, dx\, dy = \int\int_R \sqrt{a^2 - r^2}\; r\, dr\, d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} \sqrt{a^2 - r^2}\; r\, dr\, d\theta$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(a^2 - r^2)^{\frac{3}{2}}\Big|_0^a = \frac{\pi}{6}a^3$$

Example -2 $\int\int_S e^{\frac{y-x}{y+x}}\,dx\,dy$

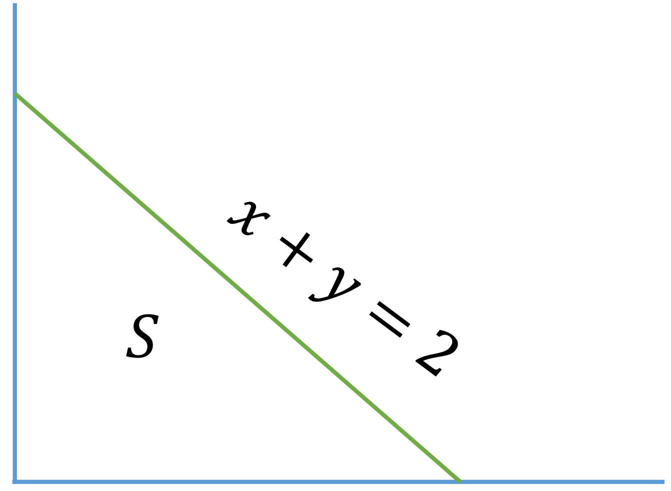


Change of variables $y-x=u,\quad y+x=v$ implies

$$x=\frac{v-u}{2},\qquad y=\frac{v+u}{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{vmatrix}=-\frac{1}{4}-\frac{1}{4}=-\frac{1}{2}$$

$$\int\int_S e^{\frac{y-x}{y+x}}\ dx\,dy$$

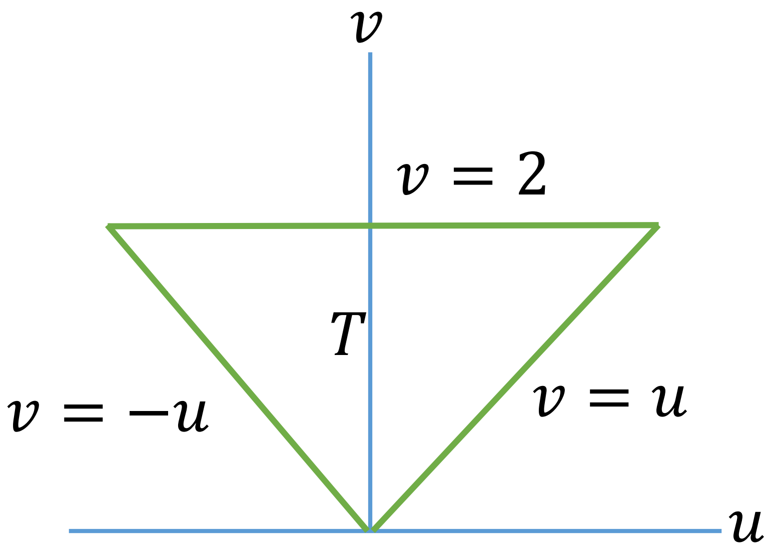


Change of variables

$$y-x=u, \quad y+x=v$$

$$x=\frac{v-u}{2}, \quad y=\frac{v+u}{2}$$

Domain in the uv-plane.

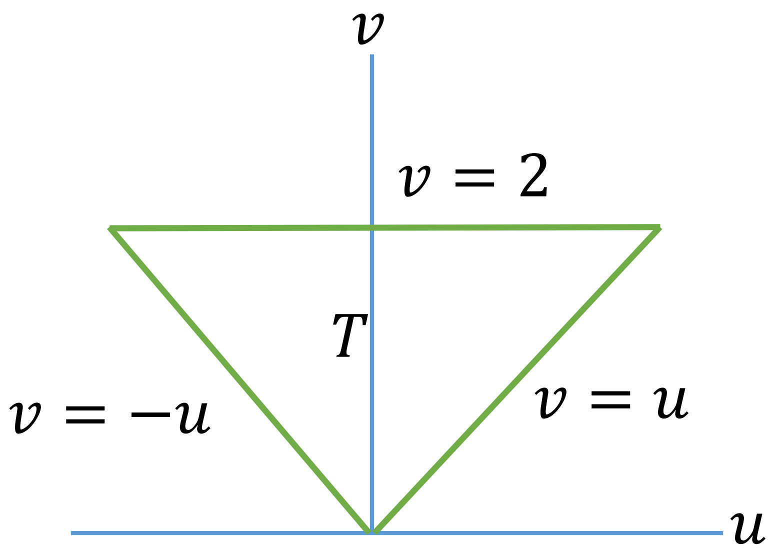


Line $x=0$ maps to

Line $y=0$ maps to

Line $x+y=2$ maps to

$$\iint_S e^{\frac{y-x}{y+x}}\, dx\, dy \qquad \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$



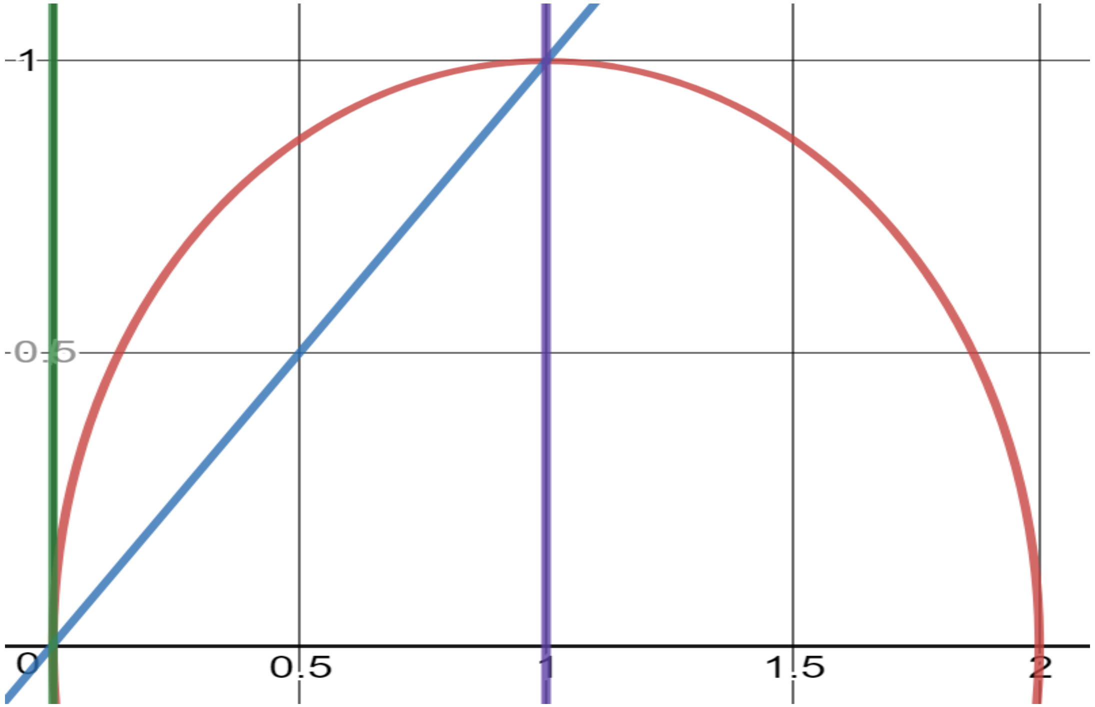
$$\iint_S e^{\frac{y-x}{y+x}}\, dx\, dy = \iint_T e^{\frac{u}{v}} \frac{1}{2}\, du\, dv$$

$$= \frac{1}{2}\int_{v=0}^{2}\int_{u=-v}^{v} e^{\frac{u}{v}}\, du\, dv$$

$$= \frac{1}{2}\int_0^2 v\left(e - \frac{1}{e}\right) dv = e - \frac{1}{e}$$

Problem -1: Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$ by changing to polar coordinates.

The region of integration is bounded by $y = x$, $y = \sqrt{2x - x^2}$, $x = 0$ and $x = 1$

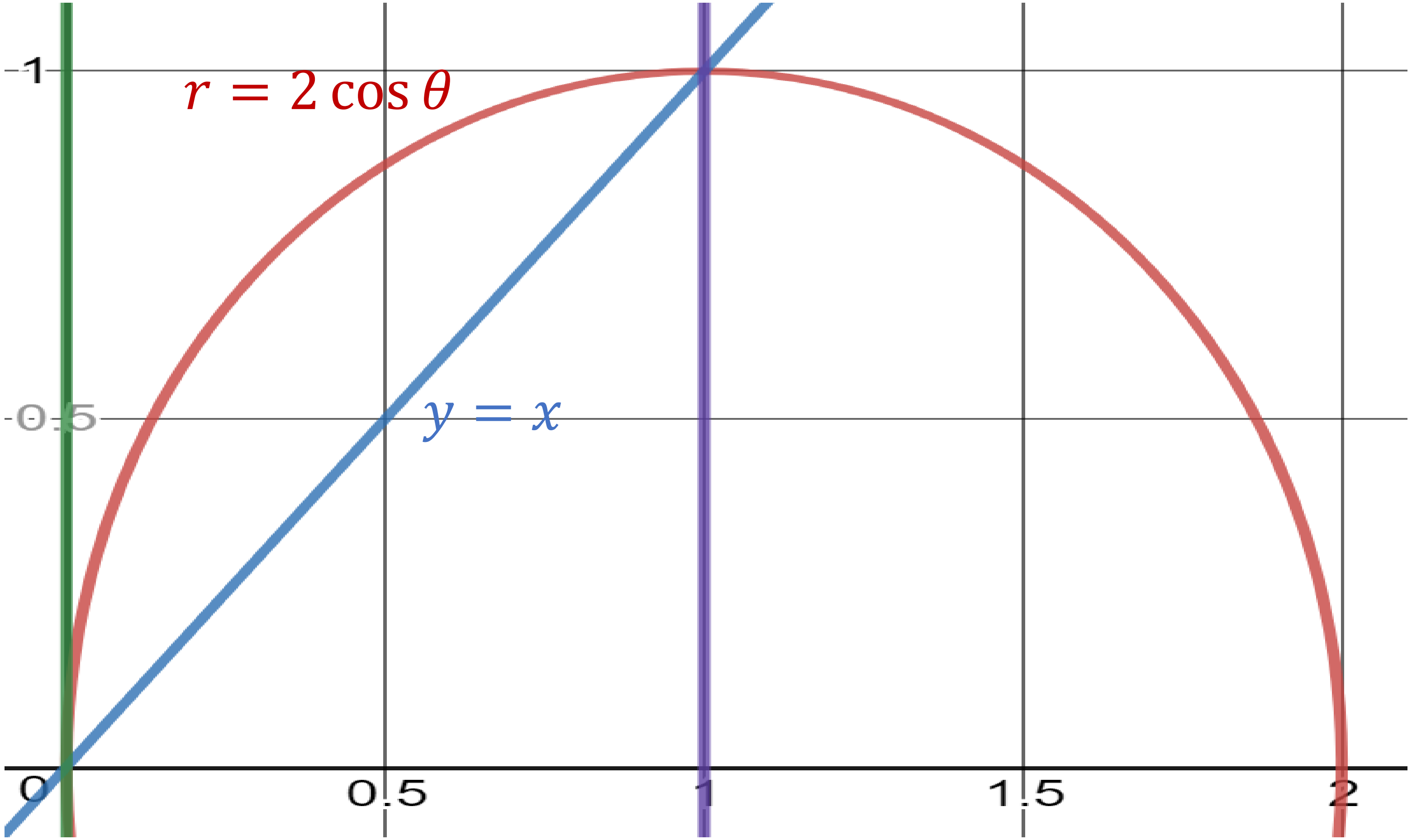


Polar equation of the circle

$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1,$

$r^2 - 2r\cos\theta = 0,$

$r = 2\cos\theta$



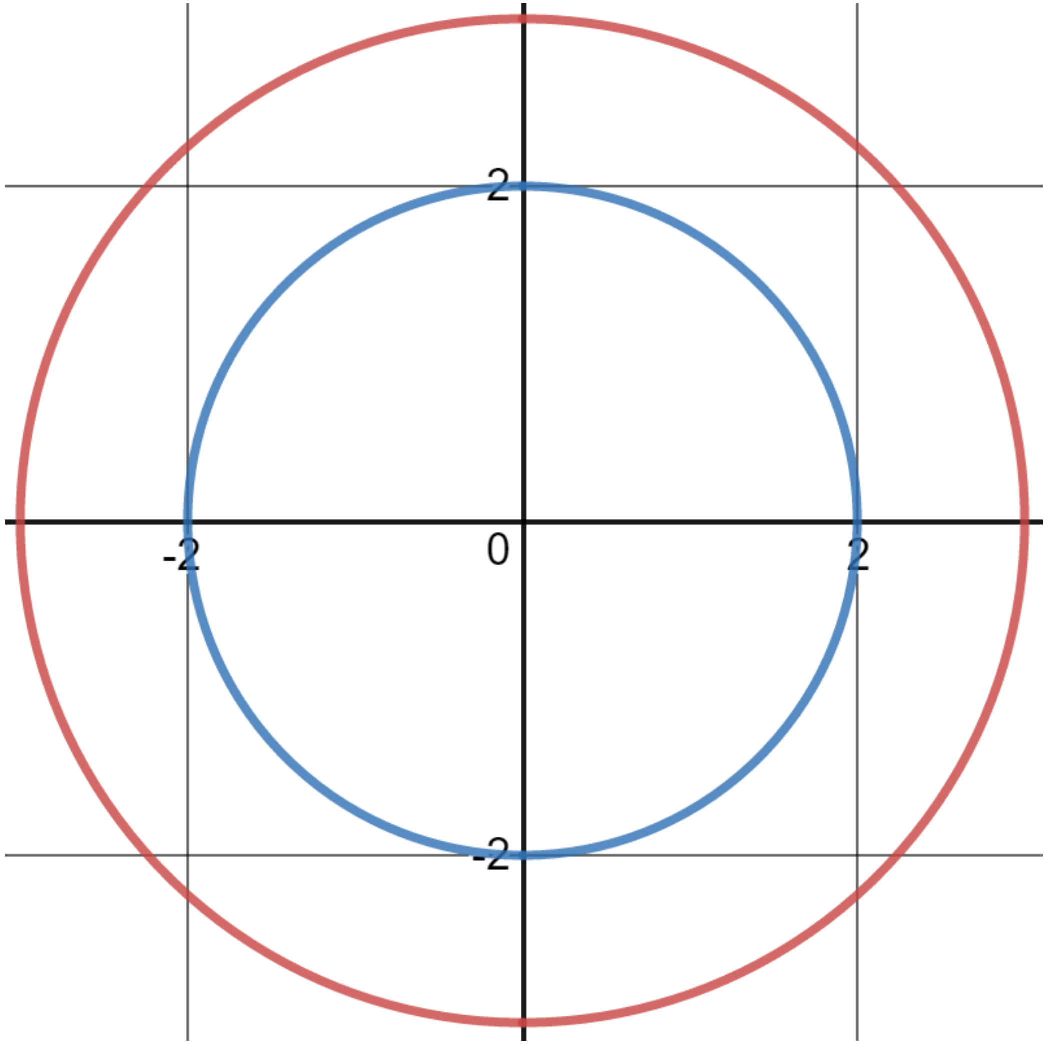
$$\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)dydx$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{r=0}^{2\cos\theta} r^2\, r\, dr\, d\theta$$

$$\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{2\cos\theta} r^2\, r\, dr\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^4\theta\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos^2\theta)^2 d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos 2\theta)^2\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos^2 2\theta + 2\cos 2\theta) d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1 + \frac{1}{2}(1+\cos 4\theta) + 2\cos 2\theta\right) d\theta = \frac{1}{8}(3\pi - 8)$$



Problem - 2: Evaluate $\iint_R \sqrt{x^2+y^2}\,dx\,dy$ by changing to polar coordinates, where R is the region in the xy plane bounded by the circles $x^2+y^2=4$ and $x^2+y^2=9$

$x = r\cos\theta\,,\quad y = r\sin\theta\,,\quad |J| = r$

$$I = \int_0^{2\pi}\int_2^3 r\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^3}{3}\right]_2^3 d\theta$$

$$= \left(\frac{27-8}{3}\right)2\pi = \frac{38}{3}\pi$$

Conclusion:

Double Integrals – Change of Variables

- Important for evaluation of integrals
- Changing to polar coordinate is a particular case

Thank You