Eigen values and Eigen vectors

Matric polynomials

An expression of the form

$$F(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_{m-1}\lambda^{m-1} + A_m\lambda^m$$

where $A_0, A_1, A_2, \cdots, A_m$ are all square matrices of the same order is called a matric polynomial of degree m provided A_m is not a null matrix. The λ is called indeterminate.

Characteristic values and characteristic vectors of a matrix

Let $A = [a_{ij}]_{n \times n}$ be a given n-rowed square matrix. Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

be a column vector. Consider the vector eqn. $AX = \lambda X$ --(1)

where λ is a scalar (number). It is obvious that the zero vector $X = O$ is a soln. of (1) for any value of λ. Now let us see whether $\exists$ scalars λ and non-zero vectors X which satisfy (1). If I denotes the unit matrix of order n, then the eqn. (1) may be written as $AX = \lambda I X$ or $(A - \lambda I)X = O$ ...(2). The matrix eqn. (2) represents the following system of n homogeneous eqns. in n unknowns

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}\right\} \quad \cdots (3)$$

The co-eff. matrix of (3) is $A - \lambda I$.

The necessary and sufficient conditions for eqns. (3) to possess a non-zero soln. $X \neq 0$ is that the coefficient matrix $A - \lambda I$ should be of rank less than the no. of unknowns n. But this will be so if and only if the matrix $A - \lambda I$ is singular i.e. if and only if $|A - \lambda I| = 0$. Thus the values of λ for which $|A - \lambda I| = 0$ are of special importance.

Definitions

Let $A = [a_{ij}]_{n \times n}$ be any square matrix and λ an indeterminate. The matrix $A - \lambda I$ is called the characteristic matrix of A where I is the unit matrix of order n. The determinant

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & -- & a_{1n} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ a_{n1} & a_{n2} & & a_{nn}-\lambda \end{vmatrix}$$

which is a polynomial in λ of degree n, is called the characteristic polynomial of A. The eqn. $|A - \lambda I| = 0$ is called the characteristic eqn. of A and the roots of this eqn. are called the " roots or characteristic values or eigen values of the matrix A. If λ is a " root of a $n \times n$ matrix A, then a non-zero vector X such that $AX = \lambda X$ is called a characteristic vector or eigen vector of A corresponding to the " root λ. An $n \times n$ matrix has at least one eigen value and at most n numerically different eigen values.

Theorem If X is a characteristic vector of a matrix A corresponding to the characteristic value λ, then kX is also a c. vector of A " " " same " " λ. Here k is any non-zero scalar.

Theorem

If X is a characteristic vector of a matrix A, then X cannot correspond to more than one characteristic values of A.

Theorem

The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.

The set of the eigen values is called the spectrum of A. The largest of the absolute values of the eigenvalues of A is called the spectral radius of A.

Ex. Determine the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

Solⁿ. The characteristic eqn. of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \quad \Rightarrow (5-\lambda)(2-\lambda) - 4 = 0 \text{ i.e. } \lambda^2 - 7\lambda + 6 = 0$$

$$\lambda_1 = 6, \lambda_2 = 1$$

∴ The eigen values of A are $6, 1$.

The eigen vectors $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of A corresponding to the eigenvalue 6 are given by the non-zero solⁿs. of the eqn. $(A - 6I)X = 0$

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad R_2 \to R_2 + R_1$$

The co-eff. matrix of these eqns. is of rank 1.

Therefore these eqns. have $2-1$ i.e. 1 linearly independent soln. These eqns. reduce to the single eqn. $-x_1+4x_2=0$. If $x_2=k$, then $x_1=4k$. So the set of all eigenvectors of A corresponding to the eigenvalue 6 is given by $k\begin{bmatrix}4\\1\end{bmatrix}$ where k is a non-zero scalar.

Similarly the eigenvectors X of A corresponding to the eigen value 1 are given by the non-zero soln's. of the eqn.

$$(A-1I)X=0$$

$$\Rightarrow \begin{bmatrix}4 & 4\\1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

From these $x_1=-x_2$

If $x_2=k$ $x_1=-k$

$\therefore$ Eigen vector $k\begin{bmatrix}-1\\1\end{bmatrix}$.

Ex Show that 0 is a characteristic root of a matrix if and only if the matrix is singular.

Soln: 0 is an eigenvalue of A $\Rightarrow \lambda=0$ satisfies $|A-\lambda I|=0$

$\Rightarrow |A|=0 \Rightarrow$ A is singular.

Conversely, A is singular $\Rightarrow |A|=0$

$\Rightarrow \lambda=0$ satisfies $|A-\lambda I|=0$

$\Rightarrow 0$ is an eigenvalue of A.

Ex Show that the matrices A and A' have the same eigen values.

$(A-\lambda I)'=A'-\lambda I'=A'-\lambda I \quad \therefore |(A-\lambda I)'|=|A'-\lambda I| \quad \therefore |A-\lambda I|=|A'-\lambda I|$

$\therefore |A-\lambda I|=0$ iff $|A'-\lambda I|=0$