Indian Institute of Technology Kharagpur
Department of Mathematics
MA11004 - Linear Algebra, Numeircal and Complex Analysis
Problem Sheet - 3
Spring 2021

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1. Prove the following statements:

   (a) If λ is an eigenvalue of a non-singular matrix A, then $\dfrac{|A|}{\lambda}$ is an eigenvalue of $\operatorname{adj} A$, where $|A|$ denotes the determinant of the matrix A.

   (b) If A and B are two invertible matrices, then AB and BA have same characteristic roots.

   (c) If λ is an eigenvalue of algebraic multiplicity r of A, then 0 is an eigenvalue of algebraic multiplicity r of the matrix $A - \lambda I_n$.

2. Find all the eigenvalues and the corresponding eigenvectors of the following matrices:

$$(a)\ \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}\ (b)\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ (c)\ \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}\ (d)\ \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

3. Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. Use Cayley-Hamilton theorem to express $2A^5 - 3A^4 + A^2 - 5I$ as a linear polynomial in A.

4. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Show that for every integer $(n \geq 3)$ $A^n = A^{n-2} + A^2 - I$. Hence evaluate A^{50}.

5. Let $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$. If $A = P^{-1}DP$, then find the diagonal matrix D.

6. Let $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$. Find a diagonal matrix D and an invertible matrix P such that $A = P^{-1}DP$.

7. Find two different 2×2 matrices A and B such that both have the same eigenvalues $\lambda_1 = \lambda_2 = 2$ and both have the same eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponding to 2.

8. (a) Show that the matrix $A = \begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$ is skew-Hermitian.

   (b) Diagonalize the matrix $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ and compute A^{2020}.

9. Let $a+b=c+d$. Show that $\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector of $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and find the eigenvalues.

10. Prove the following statements:

    (a) If $0<\theta<\pi$, then $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ has no real eigenvalues.

    (b) If λ is an eigenvalue of an orthogonal matrix, then $\dfrac{1}{\lambda}$ is also an eigenvalue of it.

11. Examine whether the matrices A and B are similar or not, where

    (a) $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}6 & -1\\ 4 & -1\end{bmatrix}$.

    (b) $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}$.

12. If A and B are two unitary matrices, show that AB is a unitary matrix.

13. Express the matrix $A=\begin{bmatrix}i & 2-3i & 4+5i\\ 6+i & 0 & 4-5i\\ -i & 2-i & 2+i\end{bmatrix}$ as the sum of a Hermitian and a skew Hermitian matrix.

14. If $N=\begin{bmatrix}0 & 1+2i\\ -1+2i & 0\end{bmatrix}$, then show that $(I-N)(I+N)^{-1}$ is a unitary matrix, where I is the identity matrix of order 2.

15. If $M=\begin{pmatrix}1 & 1 & 1\\ 1 & a^2 & a\\ 1 & a & a^2\end{pmatrix}$ where $a=e^{\frac{2i\pi}{3}}$, then prove that $M^{-1}=\frac{1}{3}\bar{M}$.

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