ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

Dr. Jitendra Kumar

Professor
Department of Mathematics
Indian Institute of Technology Kharagpur
West Bengal 721302, India

Webpage: http://www.facweb.iitkgp.ac.in/~jkumar/

Concepts Covered

Differential Calculus

Functions of Several Variables

- Derivative & Differentiability – One Variable

Derivative

Let $y = f(x)$ be a function of single variable.

If the ratio

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}, \qquad \Delta x \neq 0$$

tends to a definite limit as Δx tends to 0.

Then this limit is called the derivative of $f(x)$ at the point x.

It is usually denoted by $f'(x)$ or $y'(x)$ or $\dfrac{dy}{dx}$

Differentiability & Differentials

A function $f(x)$ is said to be *differentiable* at the point x, if when x is given the increment Δx (arbitrary increment), the increment Δy can be expressed in the form

$$\Delta y = A\,\Delta x + \epsilon\,\Delta x$$

where A is independent of Δx and $\epsilon \to 0$ as $\Delta x \to 0$.

The first term on the right hand side ($A\,\Delta x$) is called **differential** (or Total differential) of y and is denoted by dy. Thus

$$dy = A\,\Delta x$$

Differentiability & Derivative

*The necessary and sufficient condition that the function $y = f(x)$ is **differentiable** at the point x is that it possesses a finite definite **derivative** at this point.*

Differentiability $\Longrightarrow$ Existence of Derivative

Suppose the function $y = f(x)$ is differentiable. This implies $\Delta y = A\,\Delta x + \epsilon\,\Delta x$.

Taking limit $\Delta x \to 0$, we get $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = A + \lim_{\Delta x \to 0} \epsilon \implies f'(x) = A$

$\Longrightarrow$ if $f(x)$ is differentiable then $f'(x)$ exists and has definite value A

Existence of Derivative $\Longrightarrow$ **Differentiability**

Conversely, if $f'(x)$ has definite value A then

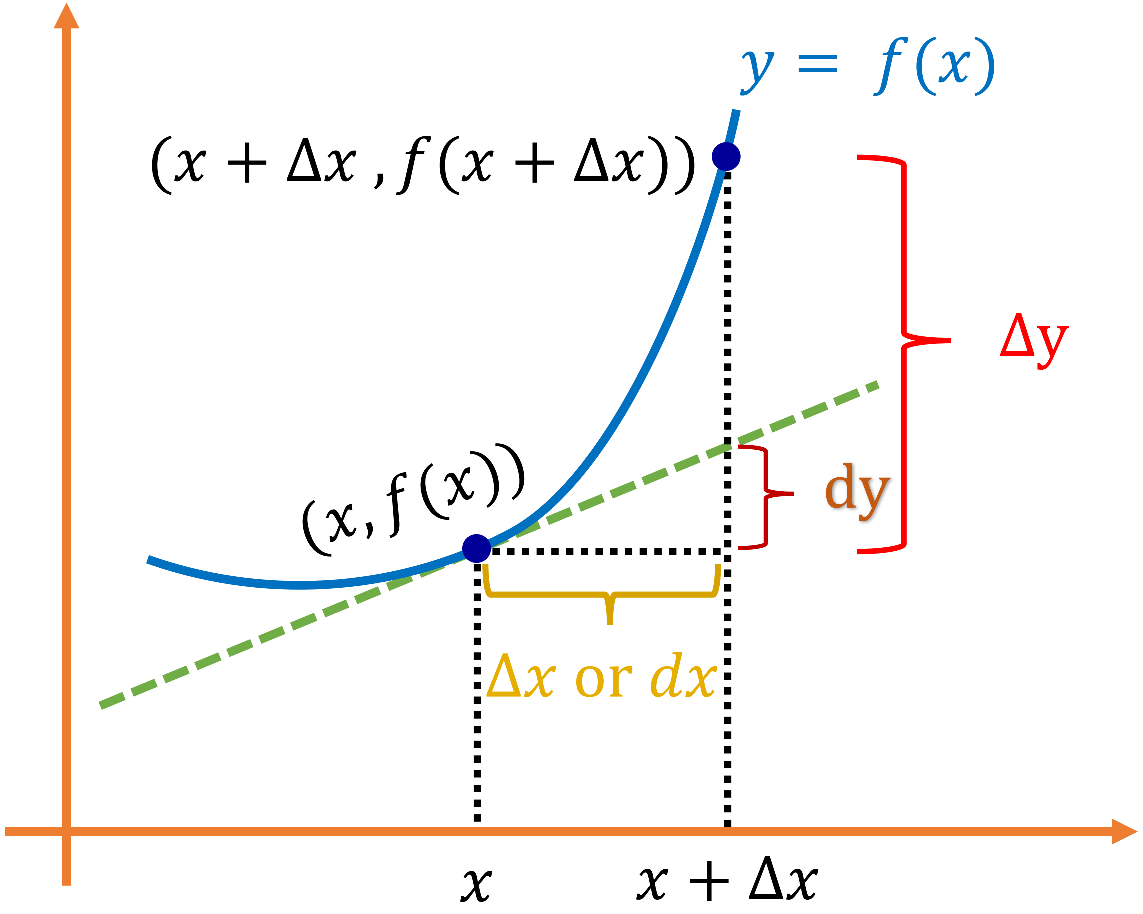
$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = A \quad \Longrightarrow \quad \frac{f(x+\Delta x)-f(x)}{\Delta x} = A + \epsilon, \qquad \epsilon \to 0, \text{as } \Delta x \to 0$$

$$\Longrightarrow \quad f(x+\Delta x)-f(x) = A\,\Delta x + \epsilon\,\Delta x, \qquad \epsilon \to 0,$$

This implies, f is differentiable

REMARK: The differential of a function is the product of its derivative and an (arbitrary) increment Δx of the independent variable x, i.e., $dy = f'(x)\,\Delta x$

Geometrical Interpretation of Differentials



$$\Delta y = A\,\Delta x + \epsilon\,\Delta x$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = A$$

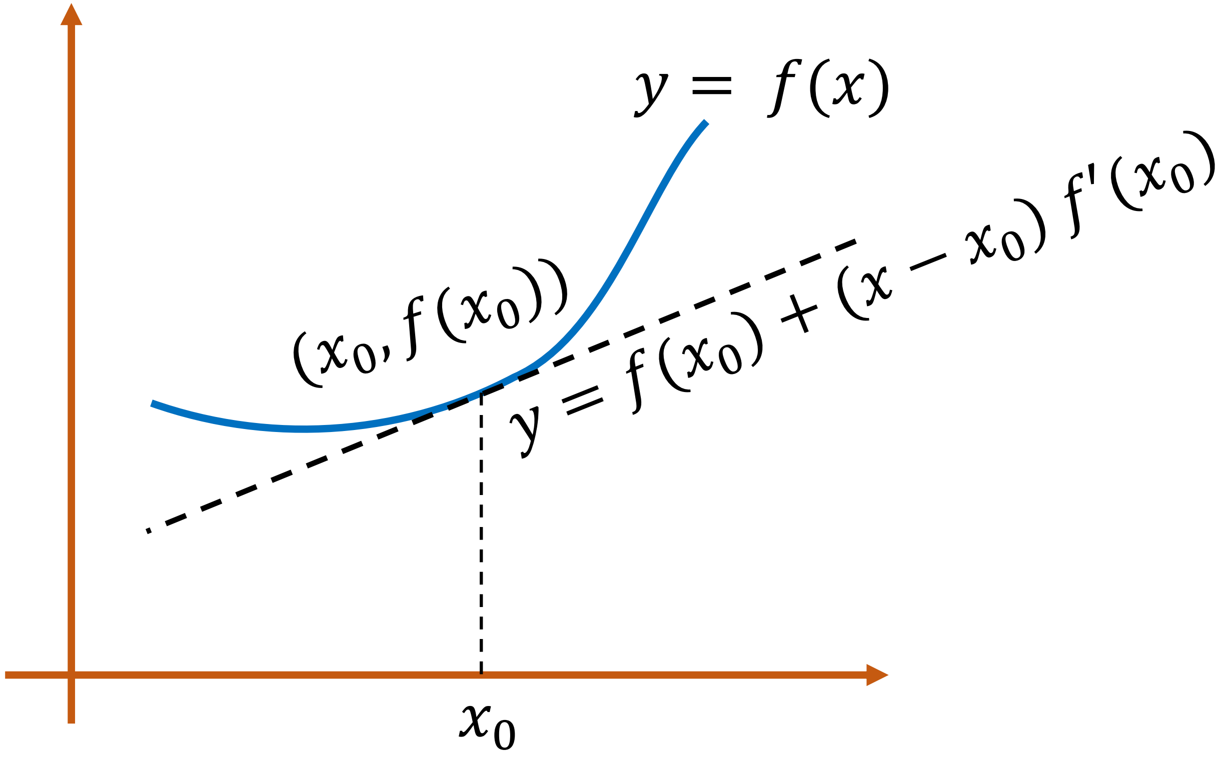
$$dy = A\,dx$$

Note: dy and dx measure changes along the tangent line

While Δy and Δx measure changes for the function $f(x)$

Geometrical Interpretation of Differentiability

A function $y = f(x)$ is said to be differentiable at the point $P(x_0, y_0)$ if it can be approximated in the neighborhood of this point by a linear function.



Mathematically,

$$f(x) = \underbrace{f(x_0) + (x - x_0)\ A}_{\text{linear function of } x} + \epsilon\,(x - x_0)$$

Equation of the tangent to the curve $y = f(x)$ at $\big(x_0, f(x_0)\big)$

Testing Differentiability

- Existence of $\lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} =: f'(x)$

- $\Delta y = dy + \epsilon\, \Delta x, \qquad dy = A\, \Delta x$

- $\lim_{\Delta x \to 0} \dfrac{\Delta y - dy}{\Delta x} = 0$

Example 1: Show that the function $f(x) = x^2$ is differentiable.

Let $y = f(x) = x^2$

$$\Delta y = f(x + \Delta x) - f(x) = \underbrace{2x}_{f'(x)}\ \Delta x + \underbrace{\Delta x}_{\epsilon}\ \Delta x$$

This implies the given function is differentiable and its derivative is $2x$.

Alternatively,

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 2x \qquad \text{OR} \qquad \lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = 0$$

Example 2: Given the function $y = x^2$, find Δy and dy at $x = 2$ and $\Delta x = 1, \Delta x = 0.1, \Delta x = 0.01$.

$$\Delta y = f(x + \Delta x) - f(x) \qquad \& \quad dy = f'(x)dx$$

$\boldsymbol{\Delta x}$	$\boldsymbol{\Delta y}$	$\boldsymbol{dy}$
1	5	4
0.1	0.41	0.40
0.01	0.0401	0.0400

Example 3: Test the differentiability of $f(x) = 1 + \sqrt[3]{(x-1)^2}$ at $x = 1$.

$$\Delta y = f(1+\Delta x) - f(1) = \sqrt[3]{\Delta x^2}$$

Now we check whether it is possible to find a number A such that

$$\lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y - A\,\Delta x}{\Delta x} = 0$$

$$\frac{\Delta y - A\,\Delta x}{\Delta x} = \frac{1}{(\Delta x)^{\frac{1}{3}}} - A \to \begin{cases} \infty \\ -\infty \end{cases}$$ as $\Delta x \to 0$ for any constant A

$\Longrightarrow$ the function f is not differentiable at $x = 1$.

KEY TAKEAWAY

The function $y = f(x)$ is said to be differentiable at the point (x, y) if, at this point

$$\Delta y = A\,\Delta x + \epsilon\,\Delta x$$

where A is independent of Δx and ϵ is a function of Δx such that $\epsilon \to 0$ as $\Delta x \to 0$.

The linear function $A\,\Delta x$ is called the total differential of y at the point (x, y) and is denoted by dy.

The value of A is the derivative of f at x.

KEY TAKEAWAY

We call a function $y = f(x)$ differentiable at the point $P(x, y)$ if

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{exists.}$$

The value of the above limit is called the derivative of f at x.

Remark: *Note that* $\frac{dy}{dx}$ *is not just a notation for* $f'(x)$ *but it is a ratio of the two differentials. Therefore writing* dx *and* dy *alone makes sense.*

Concepts Covered

Differential Calculus

Functions of Several Variables

Differentiability – Two Variables

- ❑ Definition
- ❑ Necessary Conditions of Differentiability
- ❑ Sufficient Conditions of Differentiability

Differentiability of Single Variable (Previous Lecture)

- Existence of $\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

- $\Delta y = dy + \epsilon\, \Delta x, \qquad dy = A\, \Delta x$

- $\lim_{\Delta x \to 0} \dfrac{\Delta y - dy}{\Delta x} = 0$

Differentiability of Two Variables

The function $z = f(x, y)$ is said to be differentiable at the point (x, y), if at this point

$$\Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$$

where a and b are independent of $\Delta x, \Delta y$ and ϵ_1 and ϵ_2 are functions of Δx and Δy such that

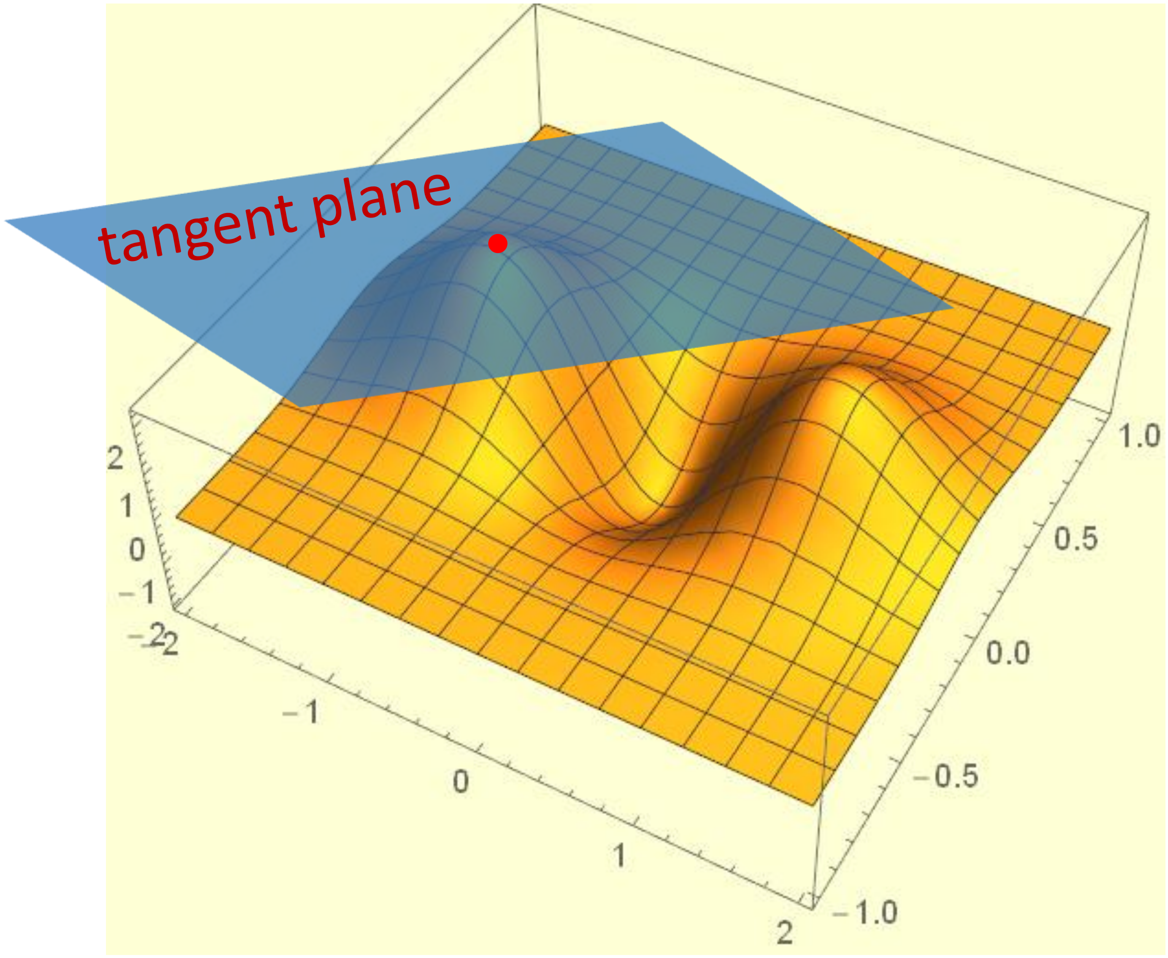
$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_1 = 0 \qquad \text{and} \qquad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_2 = 0$$

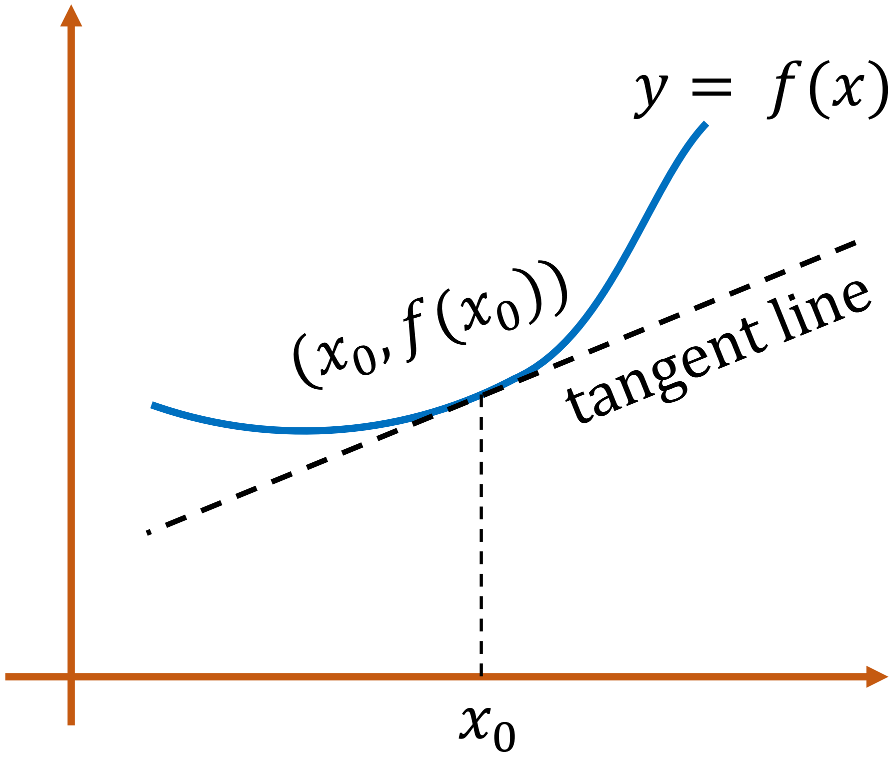
The linear function of Δx and Δy, $a\,\Delta x + \mathrm{b}\,\Delta y$ is called the total differential of z at the point (x, y) and is denoted by dz

$$dz = a\,\Delta x + b\,\Delta y \;= a\,\mathrm{d}x + b\,\mathrm{d}y$$

If Δx and Δy are sufficiently small, dz gives a close approximation to Δz.

Geometrical Interpretation of Differentiability





Necessary Condition for Differentiability

If $z = f(x, y)$ is differentiable ($\Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$) then $f(x, y)$ is continuous and has partial derivatives with respect to x and y at the point (x, y) and that

$$a = f_x(x, y) = \frac{\partial z}{\partial x} \qquad b = f_y(x, y) = \frac{\partial z}{\partial y}$$

Let f be differentiable, then

$$f(x + \Delta x, y + \Delta y) - f(x, y) = a\,\Delta x + b\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$$

Taking limit as $\Delta x \to 0,\ \Delta y \to 0$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f(x + \Delta x, y + \Delta y) = f(x, y)$$

Thus f is continuous

Necessary Condition for Differentiability (cont.)

Let f be differentiable, then

$$f(x+\Delta x, y+\Delta y) - f(x,y) = a\,\Delta x + b\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$$

Setting $\Delta y = 0$ and dividing by Δx yield the relation

$$\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = a + \epsilon_1 \qquad \Longrightarrow \qquad f_x(x,y) = a$$

Similarly, setting $\Delta x = 0$ and dividing by Δy yield the relation

$$\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = b + \epsilon_2 \qquad \Longrightarrow \qquad f_y(x,y) = b$$

Sufficient Condition for Differentiability

If one of the partial derivatives of $z = f(x, y)$ **exist** and the other is **continuous** at a point (x, y), then the function is differentiable at (x, y).

Suppose f_y exists and f_x is continuous.

Consider $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$

$$= f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y)$$

Existence of f_y implies $\lim_{\Delta y \to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = f_y(x, y)$

$$f(x, y + \Delta y) - f(x, y) = \Delta y\, f_y(x, y) + \epsilon_2\, \Delta y, \qquad \epsilon_2 \to 0 \text{ as } \Delta y \to 0$$

Sufficient Condition for Differentiability (cont.)

Using Lagrange's Mean Value Theorem

$$f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = \Delta x\, f_x(x+\theta_1 \Delta x, y+\Delta y), \quad 0 < \theta_1 < 1$$

Continuity of f_x implies

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f_x(x+\theta_1 \Delta x, y+\Delta y) = f_x(x,y) \implies f_x(x+\theta_1 \Delta x, y+\Delta y) = f_x(x,y) + \epsilon_1$$

$$\epsilon_1 \to 0 \ as\ \Delta x, \Delta y \to 0$$

$$\implies \quad f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = \Delta x f_x(x,y) + \epsilon_1 \Delta x$$

Sufficient Condition for Differentiability (cont.)

$$\boxed{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = \Delta x f_x(x,y) + \epsilon_1 \Delta x}$$ Continuity of f_x

$$\boxed{f(x, y+\Delta y) - f(x,y) = \Delta y\, f_y(x,y) + \epsilon_2\, \Delta y}$$ Existence of f_y

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x,y)$$

$$= \Delta x\, f_x(x,y) + \Delta y\, f_y(x,y) + \epsilon_1\, \Delta x + \epsilon_2\, \Delta y$$

$$\epsilon_1, \epsilon_2 \to 0 \;\text{ as }\; \Delta x, \Delta y \to 0$$

Existence of f_y and continuity of f_x $\implies$ Differentiability of f

Remarks

- The function may not be differentiable at a point $P(x, y)$ even if the partial derivatives f_x and f_y exists at P.

  (Existence of partial derivatives is a necessary condition)

- A function may be differentiable even if f_x and f_y are not continuous.

  (Existence of one partial derivative and continuity of other are sufficient conditions)

Problem - 1

Find the total differential and the total increment of the function $z = xy$ at the point $(2, 3)$ for $\Delta x = 0.1, \Delta y = 0.2$.

Total Increment

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = (x + \Delta x)(y + \Delta y) - xy = y\,\Delta x + x\,\Delta y + \Delta x\,\Delta y$$

$$\Delta z = 3 \times 0.1 + 2 \times 0.2 + 0.1 \times 0.2 = 0.72$$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = y\,dx + x\,dy = y\,\Delta x + x\,\Delta y$$

$$dz = 3 \times 0.1 + 2 \times 0.2 = 0.7$$

Problem - 2

Show that $z = x^2 + xy + xy^2$ is differentiable and write down its total differential.

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= (x + \Delta x)^2 + (x + \Delta x)(y + \Delta y) + (x + \Delta x)(y + \Delta y)^2 - x^2 - xy - xy^2$$

$$= \Delta x^2 + 2\, x\, \Delta x + x\Delta y + y\Delta x + \Delta x\, \Delta y + 2xy\Delta y + 2y\Delta x\Delta y + x\Delta y^2 + \Delta x\Delta y^2 + \Delta x\, y^2$$

$$= \Delta x\,(2x + y + y^2) + \Delta y\,(x + 2xy) + \underbrace{(\Delta x + \Delta y + 2y\Delta y)}_{\epsilon_1}\Delta x + \underbrace{(x\Delta y + \Delta x\Delta y)}_{\epsilon_2}\Delta y$$

Total Differential

$$dz = (2x + y + y^2)\, dx + (x + 2xy)\, dy$$

KEY TAKEAWAY

The function $z = f(x, y)$ is said to be differentiable at the point (x, y), if at this point

$$\Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$$

Necessary conditions

- Continuity of f
- Existence of partial derivatives f_x & f_y

Sufficient conditions

- Continuity of the partial derivatives f_x & f_y

OR

- Existence of one and continuity of the other

Thank You