LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTION 1 and 2

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Linear Algebra

- ❑ System of Linear Equation - Introduction
- ❑ Solution – Geometrical Interpretation
- ❑ Solution of the System - Gauss Elimination
- ❑ Consistency of Solution

System of Linear Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

m equations and n unknowns

Matrix form:

$$A\,x = b \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A system of equation is **consistent** if it has at least one solution, and **inconsistent** if it has no solution.

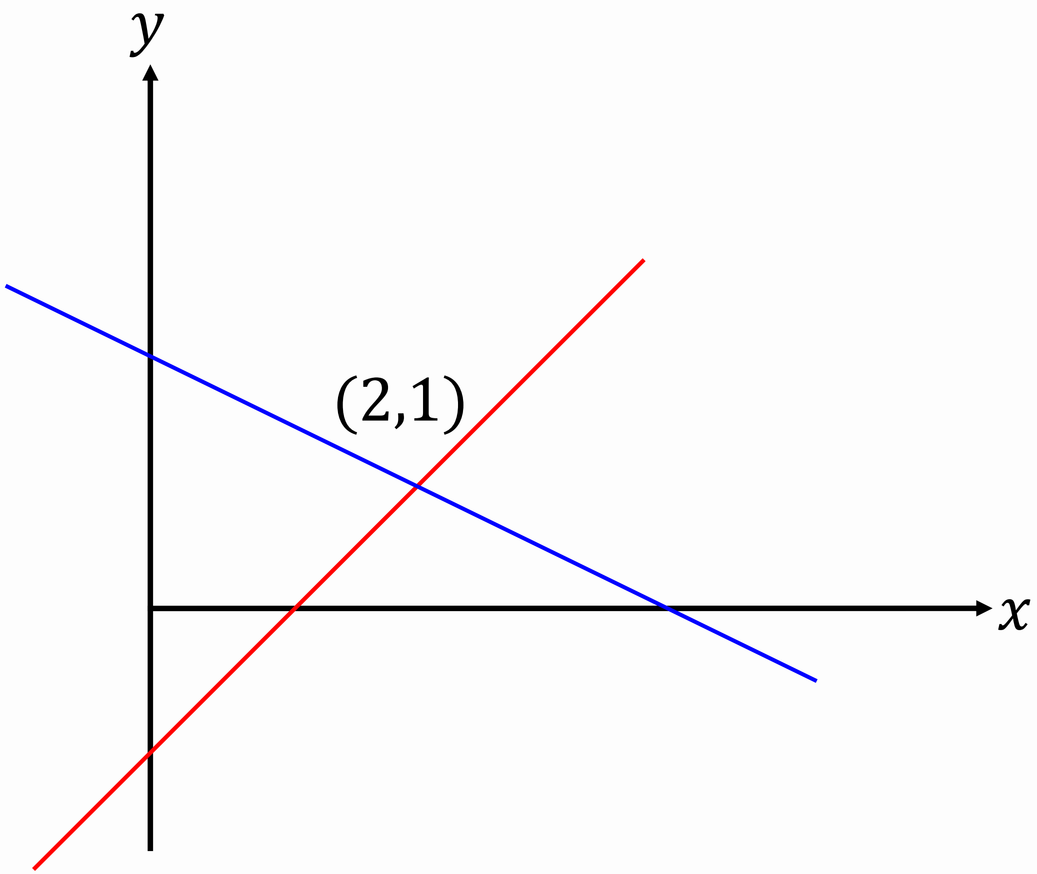
System of Linear Equations

Consider $x + 2y = 4 \quad (L_1)$

$x - y = 1 \quad (L_2)$

OR

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$



Case of Unique Solution

System of Linear Equations (vectors interpretation)

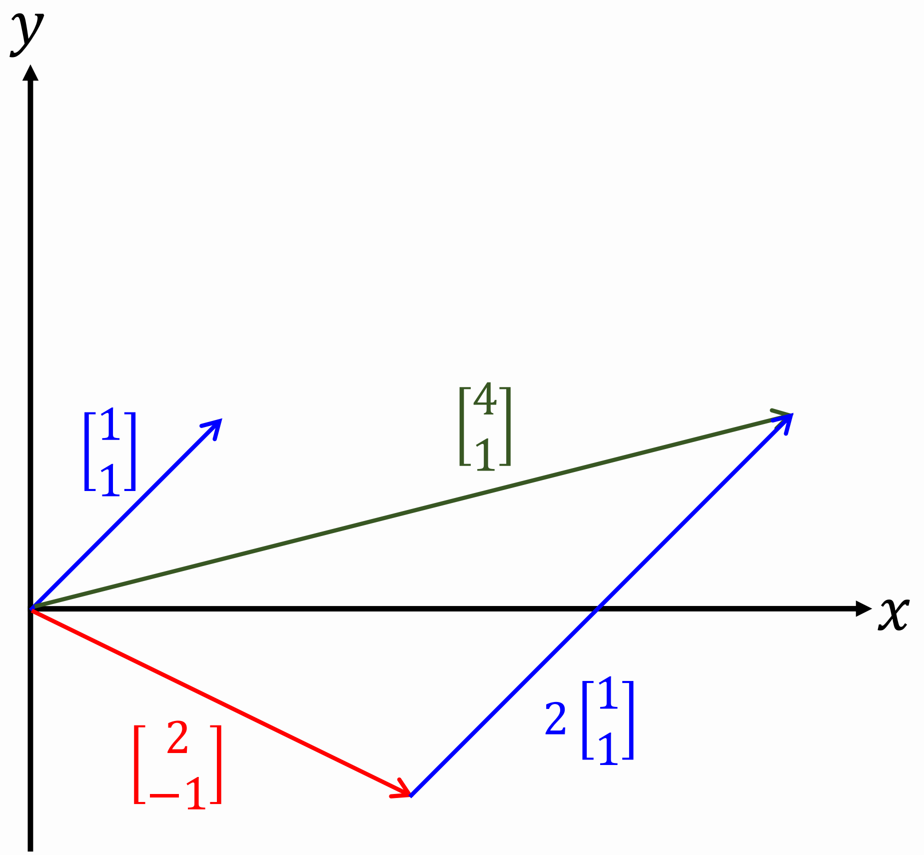
$x + 2y = 4 \qquad x - y = 1$

OR

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

OR

$$x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

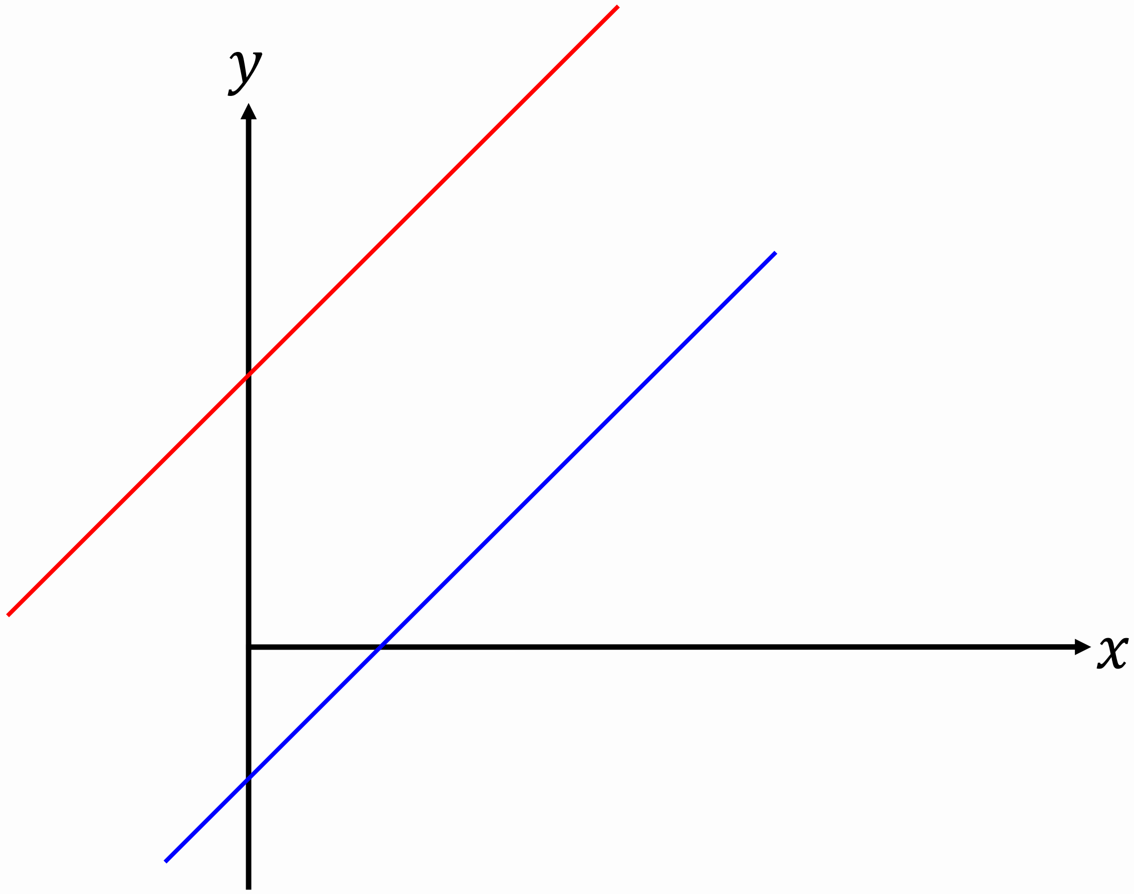


The vector $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ can only be produced by ONE linear combination of col-1 & col-2. Hence this is the case of unique solution.

System of Linear Equations

Consider $\quad x - y = 1 \qquad (L_1)$

$\qquad\qquad -x + y = 2 \qquad (L_2)$



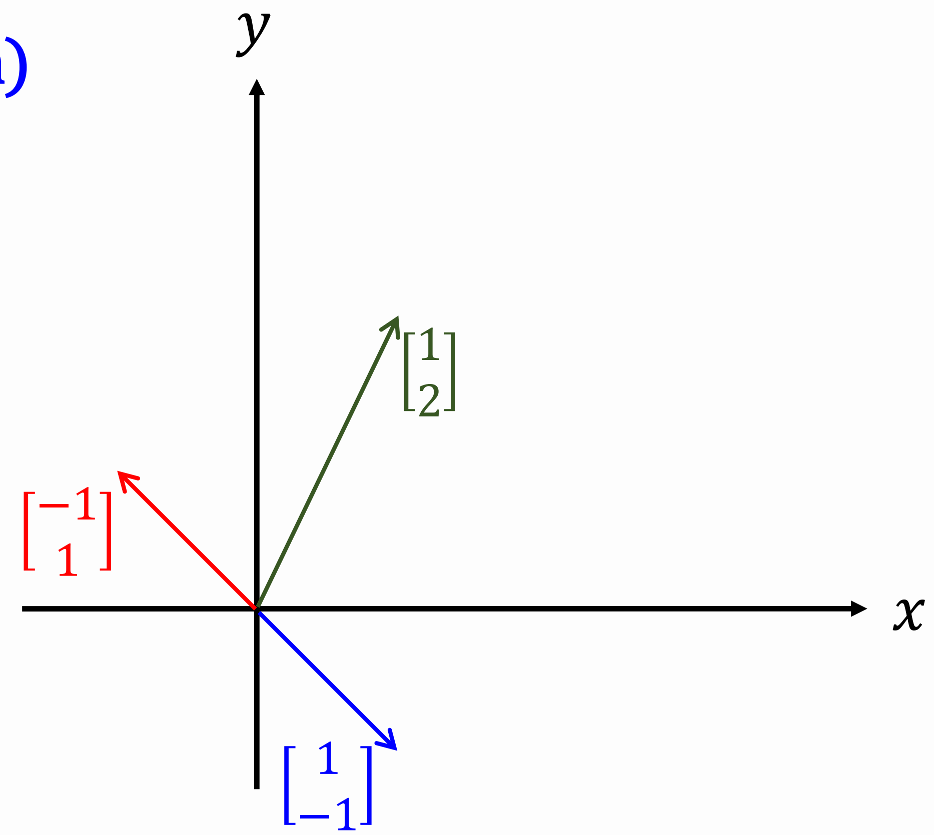
Lines do not intersect.

Case of No Solution

System of Linear Equations (vectors interpretation)

Consider $\quad x - y \ = 1 \qquad -x + y \ = 2$

$$x\begin{bmatrix}1\\-1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$



$$\begin{bmatrix}1\\2\end{bmatrix} \quad \begin{bmatrix}-1\\1\end{bmatrix} \quad \begin{bmatrix}1\\-1\end{bmatrix}$$

It is not possible to produce $\begin{bmatrix}1\\2\end{bmatrix}$ by any linear combination of col-1 & col-2. Hence this is the case of no solution.

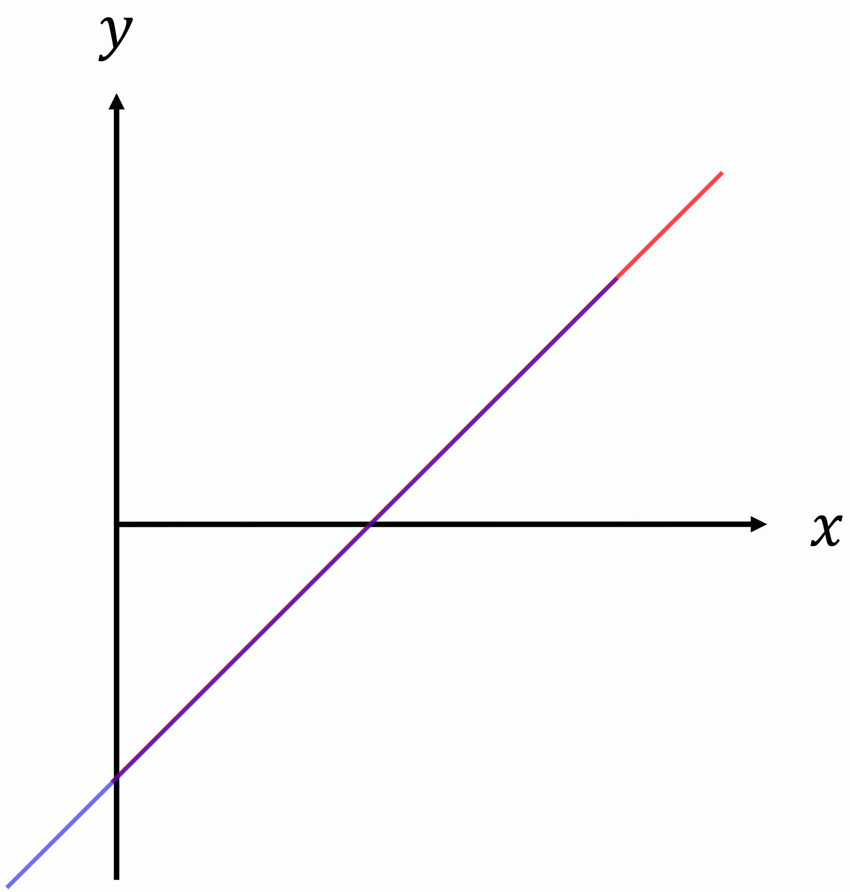
System of Linear Equations

Consider $\quad x - y = 2 \qquad (L_1)$

$\qquad\qquad -x + y = -2 \qquad (L_2)$

Both are the same equation. Any point on the line is a solution of the given system

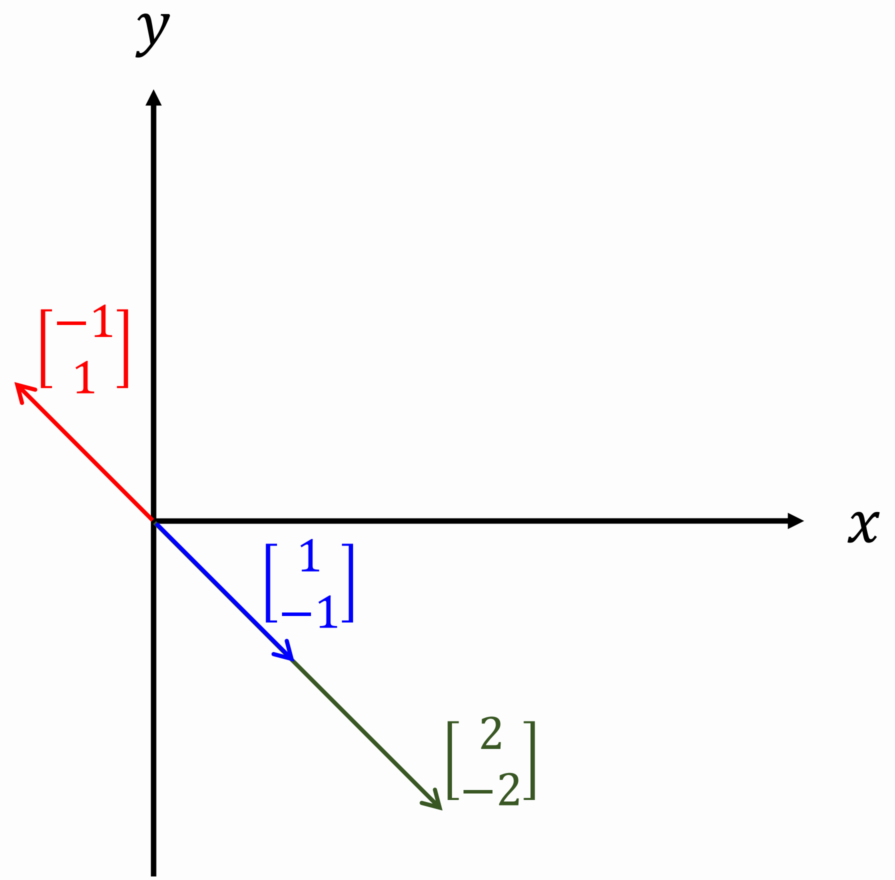
Case of Infinitely Many Solutions



System of Linear Equations (vectors interpretation)

$$x - y = 2 \qquad -x + y = -2$$

$$x\begin{bmatrix}1\\-1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix}$$



The vector $\begin{bmatrix}2\\-2\end{bmatrix}$ can be produced by many linear combinations of col-1 & col-2. Hence this is the case of infinitely many solution.

Summary:

System of Linear Equations

- Unique Solution
- Infinitely Many Solutions
- No Solution

System of Linear Equations: Solution Methods

- Method of Determinants: Cramer's rule
- Matrix Inversion Method: $Ax = b \Rightarrow x = A^{-1}b$
- Gauss Elimination Method

direct method (exact solution)

- Iterative Method – Jacobi & Gauss-Seidel method

approximate solution

System of Linear Equations: Gauss Elimination Method

Elementary Row Operations

- Interchange of i-th and j-th rows $\left(R_i \ \leftrightarrow R_j\right)$
- Multiplication of the i-th row by a nonzero number λ $(R_i \leftarrow \lambda R_i)$
- Addition of λ times the j-th row to the i-th row $\left(R_i \leftarrow R_i + \lambda R_j\right)$

Gauss Elimination: Example - 1

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + 3x_2 + x_3 = 7$$
$$x_1 + 2x_2 + 3x_3 = 9$$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 7 \\ 1 & 2 & 3 & 9 \end{array}\right]$$

Augmented Matrix

$$R_2 \leftarrow R_2 - 2R_1$$
$$R_3 \leftarrow R_3 - R_1$$

$$x_1 + x_2 + x_3 = 4$$
$$x_2 - x_3 = -1$$
$$x_2 + 2x_3 = 5$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_2$$

$$x_1 + x_2 + x_3 = 4$$
$$x_2 - x_3 = -1$$
$$3x_3 = 6$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

Gauss Elimination: Example - 1

$$x_1 + x_2 + x_3 = 4$$
$$x_2 - x_3 = -1$$
$$3x_3 = 6$$

$$[A|b] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

Echelon form

Solution:

$$x_3 = 2$$
$$x_2 = -1 + 2 = 1$$
$$x_1 = 4 - 1 - 2 = 1$$

Back substitution:

$$x_3 = 2$$
$$x_2 = -1 + x_3 = 1$$
$$x_1 = 4 - x_2 - x_3 = 1$$

Number of Pivots = Number of Unknowns $\Longrightarrow$ Unique Solution

OR every column has a pivot

Gauss Elimination: Example - 2

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 7 \\ 1 & 2 & 0 & 9 \end{array}\right]$$

$$R_2 \leftarrow R_2 - 2R_1$$
$$R_3 \leftarrow R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 5 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

Equations are inconsistent and hence the solution does not exist.

right-most column has a pivot

$\Longrightarrow$ No Solution

Gauss Elimination: Example - 3

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 7 \\ 1 & 2 & 0 & 3 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$
$R_3 \leftarrow R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Equations are Consistent

x_3 Free variable

number of pivots (r)

$<$ numer of unknowns (n)

number of free variable $= (n - r)$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 Free variable

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 + x_3 &= 7 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

Choose $x_3 = \alpha$

$x_2 = -1 + \alpha$

$x_1 = 5 - 2\alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

particular solution

solution of $Ax = 0$

(Null Space)

$$x = x_p + x_h$$

Solution of System of Linear Equations: $A_{m\times n}x_{n\times 1} = b_{m\times 1}$

Echelon Form:

$$[\tilde{A}|\tilde{b}] = \left(\begin{array}{ccccccccc|c}
\boxtimes & * & * & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & \boxtimes & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & \boxtimes & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
0 & \cdots & 0 & 0 & 0 & \boxtimes & * & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes \\
\vdots & & & & & & & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} \text{pivot rows } (r) \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{zero rows } (m-r)
\end{array}$$

$\boxtimes$ − pivot element $\neq 0$ $\otimes$ & $*$ − other elements (may be zero)

$$[A|b] \sim \begin{array}{c} \left(\begin{array}{ccccccccc|c}
\boxtimes & * & * & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & \boxtimes & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & \boxtimes & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
0 & \cdots & 0 & 0 & 0 & \boxtimes & * & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes \\
\vdots & & & & & & & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes
\end{array}\right) \end{array}
\begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} (r) \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} (m-r) \end{array}$$

Def. Rank (A) = *r* (number of pivots)

- If $\otimes \neq 0$ the equations become inconsistent and hence the system has no solution

OR in terms of rank: $\text{Rank }(A) \neq \text{Rank}([A|b])$

$$[A|b] \sim \left(\begin{array}{ccccccccc|c}
\boxtimes & * & * & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & \boxtimes & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & \boxtimes & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
0 & \cdots & 0 & 0 & 0 & \boxtimes & * & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes \\
\vdots & & & & & & & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} (r) \\
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} (m-r)
\end{array}$$

Def. Rank (A) = *r* (number of pivots)

If $\otimes = 0$

The system will be consistent

($Ax = 0$ is always consistent)

- If $\otimes = 0$ and number of pivot elements (r) = number of unknowns (n)

OR each column has a pivot

Then the system has a unique solution

OR in terms of rank: $\text{Rank }(A) = \text{Rank}([A|b]) = n$

$$
[A|b]\sim\left(\begin{array}{ccccccccc|c}
\boxtimes & * & * & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & \boxtimes & * & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & \boxtimes & * & \cdots & \cdots & \cdots & * & * \\
0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * \\
0 & \cdots & 0 & 0 & 0 & \boxtimes & * & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes \\
\vdots & & & & & & & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l}\\ \\ \\ \\ \\ \\ \end{array}\right\}(r) \\
\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}(m-r)
\end{array}
$$

Def. Rank (A) = *r* (number of pivots)

- If $\otimes = 0$ and number of pivot elements $(r) <$ number of unknowns (n)

  Then the system has infinitely many solutions

  OR in terms of rank: $\text{Rank }(A) = \text{Rank}([A|b]) < n$

Problem -1 Solve the system of equations $Ax = b$ with

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 2 & 4 & -4 & 0 & 3 & 2 \\ -1 & -2 & 3 & 3 & 4 & 3 \\ 3 & 6 & -7 & 1 & 1 & \beta \end{array}\right]; \qquad \beta \in \mathbb{R}$$

$R_2 \rightarrow R_2 - 2R_1$ $\qquad R_3 \rightarrow R_3 + R_1$ $\qquad R_4 \rightarrow R_4 - 3R_1$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & -1 & 4 & -2 & \beta - 3 \end{array}\right]$$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & -1 & 4 & -2 & \beta - 3 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 4 & -2 & \beta - 3 \end{array}\right]$$

$R_4 \rightarrow R_4 + R_2$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 6 & 3 & \beta + 1 \end{array}\right]$$

$R_4 \rightarrow R_4 - 3R_3$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta + 1 \end{array}\right]$$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta+1 \end{array}\right]$$

Case – I: $\beta \neq -1 \quad \Longrightarrow \text{No Solution}$

Case – II: $\beta = -1$

$$\begin{array}{cccccc} x_1 & & x_3 & x_4 & & \text{dependent variables} \end{array}$$

$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccccc} & x_2 & & & x_5 & \text{free variables} \end{array}$$

Case – II: $\beta = -1$

Take $x_2 = \alpha_1$, $x_5 = \alpha_2$, then

$$\begin{array}{cc} & \begin{array}{ccccc} x_1 & & x_3 & x_4 & \end{array} \\ [A|b] = & \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ & \begin{array}{ccccc} & x_2 & & & x_5 \end{array} \end{array}$$

dep. variables x_1, x_3, x_4

free variables x_2, x_5

$$x_4 = -\tfrac{1}{2}\alpha_2 \qquad x_3 = 4 - 4\alpha_2$$

$$x_1 = 9 - 2\alpha_1 - 9.5\alpha_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 9 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{\alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix}}_{x_h}$$

$$x = x_p + x_h$$

$$Ax_p = b \qquad Ax_h = 0$$

Solution of nonhomogeneous linear system are of the form $x = x_p + x_h$, where x_p is any fixed solution of $Ax = b$ and x_h runs through all the solutions corresponding to homogeneous system $Ax = 0$.

Remarks:

- Free variable(s) is (are) responsible for infinitely many solutions
- An invertible matrix has no free variable $(Ax = b \Longrightarrow x = A^{-1}b)$ unique solution
- Vectors that generate solutions of $Ax = 0$ are $[-2, 1, 0, 0, 0]^T$ & $[-9.5, 0, -4, -0.5, 1]^T$
- These generators are called BASIS of solution space of $Ax = 0$ (NULL Space)
- NULL Space is a vector space (next lecture)

SUMMARY:

System of Linear Equations

- Gauss Elimination
- Echelon form
- Solution (Consistency & Inconsistency)
- Free variables – Infinitely many solutions
- $x = x_p + x_h$

Thank You