ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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Integral Calculus

Improper Integrals

- Introduction
- Evaluation

Proper Integral

The Integral $\int_a^b f(x)dx$ is **proper** if

the range of integration is finite and the integrand is bounded.

Improper Integral

The Integral $\int_a^b f(x)dx$ is **improper** if

- $a = -\infty$ and/or $b = \infty$ and $f(x)$ is bounded. (First kind)
- $f(x)$ is unbounded at one or more points of $a \leq x \leq b$. (Second kind)
- Both 1 and 2 type. (Third kind or mixed kind)

Examples - Proper Integrals

$$\int_0^2 \sqrt{(x^2+1)}\,\mathrm{d}x \qquad \int_0^1 \frac{\sin x}{x}\,\mathrm{d}x$$

Examples - Improper Integrals

$$\int_0^\infty \cos x\, dx$$ (First Kind)

$$\int_0^1 \frac{1}{x-1}\, dx$$ (Second Kind)

$$\int_0^\infty \frac{1}{(1-x)^2}\, dx$$ (Third Kind)

Evaluation of Improper Integrals of First Kind

- $$\int_a^{\infty} f(x)dx = \lim_{R\to\infty}\int_a^R f(x)dx$$
- $$\int_{-\infty}^{b} f(x)dx = \lim_{R\to\infty}\int_{-R}^{b} f(x)dx$$
- $$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty}\int_{-R_1}^{c} f(x)dx + \lim_{R_2\to\infty}\int_{c}^{R_2} f(x)dx$$

$$= \lim_{\substack{R_1\to\infty \\ R_2\to\infty}}\int_{-R_1}^{R_2} f(x)dx$$

Evaluation of Improper Integrals of First Kind

- $$\int_2^\infty \frac{2x^2}{x^4-1}\,dx = \frac{\pi}{2} - \tan^{-1} 2 + \frac{1}{2}\ln 3$$

$$\frac{2x^2}{x^4-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) + \frac{1}{x^2+1}$$

- $$\int_0^\infty \sin x\,dx = \lim_{R\to\infty}(1-\cos R)$$

Does not exist

Evaluation of Improper Integrals of Second Kind

$$\int_a^b f(x)\,dx \qquad f(x) \text{ is unbounded}$$

- If $f(x) \to \infty$ as $x \to b$ then

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0^+} \int_a^{b-\varepsilon} f(x)dx$$

- If $f(x) \to \infty$ as $x \to a$ then

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0^+} \int_{a+\epsilon}^{b} f(x)dx$$

Evaluation of Improper Integrals of Second Kind

- If $f(x) \to \infty$ as $x \to c$ where $a < c < b$, then

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_a^{c-\varepsilon} f(x)\, dx + \lim_{\varepsilon \to 0^+} \int_{c+\varepsilon}^b f(x)\, dx$$

- If $f(x) \to \infty$ as $x \to a$ and $x \to b$, then

$$\int_a^b f(x)dx = \lim_{\substack{\varepsilon_1 \to 0^+ \\ \varepsilon_2 \to 0^+}} \int_{a+\varepsilon_1}^{b-\varepsilon_2} f(x)dx$$

Evaluation of Improper Integrals of Second Kind

$$\int_0^1 \frac{dx}{\sqrt{(1-x)}} = \lim_{\epsilon\to 0^+} \int_0^{1-\epsilon} \frac{dx}{\sqrt{(1-x)}} = \lim_{\epsilon\to 0^+} \left[-2\sqrt{1-x}\right]_0^{1-\epsilon}$$

$$= -2 \lim_{\epsilon\to 0^+} \left(\sqrt{\epsilon} - 1\right)$$

$$\int_0^1 \frac{dx}{\sqrt{(1-x)}} = 2$$

Integral converges

Evaluation of Improper Integrals of Second Kind

$$\int_0^2 \frac{dx}{(2x-x^2)} = \lim_{\epsilon_1\to 0^+}\int_{\epsilon_1}^{1} \frac{dx}{(2x-x^2)} + \lim_{\epsilon_2\to 0^+}\int_{1}^{2-\epsilon_2} \frac{dx}{(2x-x^2)}$$

$$= \lim_{\epsilon_1\to 0^+} \frac{1}{2}\left[\ln\frac{x}{2-x}\right]_{\epsilon_1}^{1} + \lim_{\epsilon_2\to 0^+} \frac{1}{2}\left[\ln\frac{x}{2-x}\right]_{1}^{2-\epsilon_2}$$

$$= -\lim_{\epsilon_1\to 0^+} \frac{1}{2}\left[\ln\frac{\epsilon_1}{2-\epsilon_1}\right] + \lim_{\epsilon_2\to 0^+} \frac{1}{2}\left[\ln\frac{2-\epsilon_2}{\epsilon_2}\right]$$

$$\int_0^2 \frac{dx}{(2x-x^2)} = \infty$$ Integral Diverges

Test Integral - I

$$\int_a^R \frac{1}{x^p}\,dx = \begin{cases} \ln\left(\frac{R}{a}\right), & p = 1 \\ \frac{1}{1-p}\left[\frac{1}{R^{p-1}} - \frac{1}{a^{p-1}}\right], & p \neq 1 \end{cases}$$

$$a > 0$$

$$\boxed{\int_a^\infty \frac{1}{x^p}\,dx = \lim_{R\to\infty} \int_a^R \frac{1}{x^p}\,dx = \begin{cases} \infty, & p \leq 1 \\ \frac{1}{p-1}\frac{1}{a^{p-1}}, & p > 1 \end{cases}}$$

Test Integral - II

$$\int_{a+\epsilon}^{b} \frac{1}{(x-a)^p}\,dx = \begin{cases} \frac{1}{1-p}\left[\frac{1}{(b-a)^{p-1}} - \frac{1}{\epsilon^{p-1}}\right], & p \neq 1 \\ \ln(b-a) - \ln\epsilon\,, & p = 1 \end{cases}$$

$$\boxed{\int_{a}^{b} \frac{1}{(x-a)^p}\,dx = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b} \frac{1}{(x-a)^p}\,dx = \begin{cases} \infty, & p \geq 1 \\ \frac{1}{1-p}\frac{1}{(b-a)^{p-1}}, & p < 1 \end{cases}}$$

KEY TAKEAWAY

$$\text{Improper Integral } \int_a^b f(x)dx$$

1. $a = -\infty$ and/or $b = \infty$ and $f(x)$ is bounded.

2. $f(x)$ is unbounded at one or more points of $a \leq x \leq b$.

Evaluation of Improper Integrals

Integral Calculus

Improper Integrals

- ❑ **Convergence: Type-I Integrals**

Recall (Previous Lecture)

Test Integral

$$\int_a^\infty \frac{1}{x^p}\,dx\,, \qquad a > 0, \qquad \text{converges for } p > 1 \quad \& \quad \text{diverges if } \; p \leq 1$$

Convergence: Type - I Integrals

$$\int_a^b f(x)dx$$

$a = -\infty$ and/or $b = \infty$ and $f(x)$ is bounded

Comparison Test-I:

Suppose f and g are integrable over $[a, c]$, $\forall\, c \geq a$ and that $0 \leq f \leq g$, $\forall x > a$, then

i. $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges

ii. $\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges

Comparison Test-II (limit Comparison test):

Suppose f and g are integrable over $[a, c]$, $\forall\, c \geq a$ and $f \geq 0, g > 0$ $\forall x > a$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = k(\neq 0)$$

Then both the integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge together

Further, If $k = 0$ and $\int_a^\infty g(x)dx$ converges then $\int_a^\infty f(x)dx$ converges

If $k = \infty$ and $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)dx$ diverges

REMARK: $\boldsymbol{\mu-\text{test}}$ Comparison test (II) with $g(x)=\dfrac{1}{x^{\mu}}$

Let $f(x)\geq 0$ in the interval $[a,\infty)$, $a>0$. (OR $f(x)\leq 0$)

a) If $\exists\ \mu>1$ such that $\lim\limits_{x\to\infty} x^{\mu} f(x)$ exists then $\int_a^{\infty} f(x)\ dx$ is convergent.

b) If $\exists\ \mu\leq 1$ such that $\lim\limits_{x\to\infty} x^{\mu} f(x)$ exists and $\neq 0$ then $\int_a^{\infty} f(x)\ dx$ is divergent and the same is true if $\lim\limits_{x\to\infty} x^{\mu} f(x)$ is $+\infty$. (OR $-\infty$)

Problem – 1: Test the convergence of $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$

Note that $\displaystyle\frac{1}{x\sqrt{x^2+1}} \sim \frac{1}{x^2}$

Let $\displaystyle f(x) = \frac{1}{x\sqrt{x^2+1}}$ and $\displaystyle g(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1 \ (\neq 0)$$

As $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges $\Longrightarrow$ $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ converges

(OR apply $\mu -$ test as $\mu = 2$)

Problem – 2: Test the convergence of $\int_1^{\infty} \frac{x^2}{\sqrt{x^5+1}}\,dx$

Let $f(x) = \frac{x^2}{\sqrt{x^5+1}} \left(\sim \frac{1}{\sqrt{x}}\right)$ and $g(x) = \frac{1}{\sqrt{x}}$

Note that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2\sqrt{x}}{\sqrt{x^5+1}} = 1$

As $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ diverges, by comparison test $\int_0^{\infty} \frac{x^2}{\sqrt{x^5+1}}\,dx$ diverges

(OR apply $\mu-$test as $\mu = 0.5$)

Problem – 3: Show that the integral $\int_0^\infty \mathrm{e}^{-x^2}\,dx$ converges

Exp.

$$\int_0^\infty \mathrm{e}^{-x^2}\,dx = \int_0^1 \mathrm{e}^{-x^2}\,dx + \int_1^\infty \mathrm{e}^{-x^2}\,dx$$

Note that: $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots > x^2, \quad \forall x > 0 \;\&\; x < 0 \quad \Rightarrow e^{-x^2} < \frac{1}{x^2}$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, the integral $\int_1^\infty \mathrm{e}^{-x^2}\,dx$ converges

Problem – 4: Show that the integral $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ converges

Note that $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^1 \frac{\sin^2 x}{x^2}\,dx + \int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

Since $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\,dx$ converges

$\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges

Problem – 5: Show that the integral $\int_1^\infty \frac{x\tan^{-1}x}{(1+x^4)^{\frac{1}{3}}}\mathrm{d}x$ diverges

Let $f(x) = \frac{x\tan^{-1}x}{(1+x^4)^{\frac{1}{3}}} = \frac{\tan^{-1}x}{x^{\frac{1}{3}}(1+x^{-4})^{\frac{1}{3}}}$ $\left(\sim x^{-\frac{1}{3}}\ \text{at}\ \infty\right)$

Let $g(x) = \frac{1}{x^{\frac{1}{3}}}$ Note that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\pi}{2}$

This follows the result. (OR apply μ – test as $\mu = 1/3$)

KEY TAKEAWAY

Comparison Test -I: Let $0 \le f(x) \le g(x)$

$$\int_a^{\infty} g(x)dx \ \text{ conveges } \Longrightarrow \int_a^{\infty} f(x)dx \ \text{ conveges}$$

$$\int_a^{\infty} f(x)dx \ \text{ diverges } \Longrightarrow \int_a^{\infty} g(x)dx \ \text{ diverges}$$

KEY TAKEAWAY

Comparison Test -II: Let $0 \le f(x) \le g(x)$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \mathrm{k}$$

$$\text{if } k \ne 0 \ \text{ then } \int_a^\infty f(x)dx \ \text{ and } \int_a^\infty g(x)dx \ \text{ behave the same}$$

$$\text{if } k = 0 \ \& \ \int_a^\infty g(x)dx \ \text{ conveges } \Longrightarrow \int_a^\infty f(x)dx \ \text{ conveges}$$

$$\text{if } k = \infty \ \& \ \int_a^\infty g(x)dx \ \text{ diverges } \Longrightarrow \int_a^\infty f(x)dx \ \text{ diverges}$$

QUIZ QUESTION ?

LINK FOR RESPONSES: http://www.facweb.iitkgp.ac.in/~jkumar/teach/MA11003.html

Let

$$I_1 = \int_1^{\infty} \frac{x^2}{9x^4 + 36}\,dx \quad \& \quad I_2 = \int_1^{\infty} \frac{\cos^2 x}{x^2}\,dx$$

Which of the following statements is correct?

(A)	I_1 converges and I_2 diverges
(B)	I_1 diverges and I_2 converges
(C)	Both I_1 and I_2 diverges
(D)	Both I_1 and I_2 converges

Integral Calculus

Improper Integrals

- Convergence Test: Dirichlet's Test

Dirichlet's Test: Let $f, g: [a, \infty) \to \mathbb{R}$ be such that

- f is integrable on each interval $[a, b]$, $b > a$
- The integrals $\int_a^b f(x)dx$ are uniformly bounded

$$\left\{ \exists\, C > 0, \text{ s.t. } \left|\int_a^b f(x)dx\right| \le C \text{ for all } b > a\ (b < \infty) \right\}$$

- g is monotone and bounded on $[a, \infty)$ and $\lim_{x\to\infty} g(x) = 0$

Then, the improper integral $\int_a^\infty f(x)\, g(x)\, dx$ converges

Problem – 1: The Integral $\displaystyle\int_1^\infty \frac{\sin x}{x^p}\,dx$ is convergent for $p > 0$.

Let $f(x) = \sin x$ and $g(x) = \dfrac{1}{x^p}$

Note that $\displaystyle\left|\int_1^b \sin x \; dx\right| = |\cos 1 - \cos b| \leq |\cos 1| + |\cos b| < 2,$ for $1 \leq b < \infty$.

Also note that

$g(x) = \dfrac{1}{x^p}$ is monotone decreasing function tending to 0 as $x \to \infty$, for $p > 0$.

Using Dirichlet's test $\displaystyle\int_1^\infty \frac{\sin x}{x^p}\,dx$ converges for $p > 0$.

Problem – 2: Test the convergence of $\int_0^\infty \frac{\sin x}{x}\, e^{-x}\, dx$

$$\int_0^\infty \frac{\sin x}{x}\, e^{-x}\, dx = \int_0^1 \frac{\sin x}{x}\, e^{-x}\, dx + \int_1^\infty \frac{\sin x}{x}\, e^{-x}\, dx$$

$\left|\int_1^b \sin x\, dx\right| < 2$ for $1 \le b < \infty$.

Note that e^{-x}/x is monotone and bounded as well as $\lim_{x\to\infty} e^{-x}/x = 0$

Hence by Dirichlet's test $\int_0^\infty \frac{\sin x}{x}\, e^{-x}\, dx$ converges

Thank You