Indian Institute of Technology Kharagpur
Department of Mathematics
MA11004 - Linear Algebra, Numeircal and Complex Analysis
Hints and answers - Problem Sheet - 3
Spring 2021

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1. (a) Show that $\det(\operatorname{adj} A - \frac{|A|}{\lambda} I_n) = 0$.
   (b) Show that $\det(B^{-1}(BA)B - \lambda I_n) = \det(BA - \lambda I_n)$.
   (c) Show that $\det(A - \lambda I_n - xI_n) = x^r \mu(x)$, where $\mu(x)$ is some function of x.

2. To find eigenvalues solve the characteristic equation of the matrix A, $\det(A - \lambda I_n) = 0$ where λ's are eigenvalues of A. To find eigenvector solve the equation $AX = \lambda X$, where X is the eigenvector of A corresponding to the eigenvalue λ.

   (a) Eigenvalues are : -1 and -6.
   Eigenvector corresponding to the eigenvalue -1 and -6 are $k\begin{pmatrix}1\\2\end{pmatrix}$ and $k_1\begin{pmatrix}-2\\1\end{pmatrix}$ respectively, where k, k_1 are non-zero real numbers.

   (b) Eigenvalues are : i, $-i$.
   Eigenvectors corresponding to the eigenvalues i and $-i$ are $k\begin{pmatrix}i\\1\end{pmatrix}$, where k is a non-zero complex number and $k_1\begin{pmatrix}1\\i\end{pmatrix}$, where k_1 is a non-zero complex number respectively.

   (c) Eigenvalues are: 3, 2, 2.
   Eigenvectors corresponding to the eigenvalues 3 and 2 are $4k\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ and $3k_1\begin{pmatrix}-5\\-2\\5\end{pmatrix}$ respectively, where k, k_1 are non-zero real numbers.

   (d) Eigenvalues are: 4, 1, 1.
   Eigenvectors corresponding to the eigenvalues 4, 1 and 1 are $c\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $c\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $c\begin{pmatrix}-1\\1\\2\end{pmatrix}$ respectively , where c is a non-zero real number.

4. Use Cayley-Hamilton theorem and mathematical induction to prove the formula and then use it to get $A^{50} = \begin{bmatrix}1 & 0 & 0\\25 & 1 & 1\\25 & 0 & 1\end{bmatrix}$.

5. The matrix D is the diagonal matrix having eigenvalues as diagonal entry.

6. Write the eigenvectors column wise to get $P = \begin{bmatrix}0 & 1 & 2\\1 & 3 & -1\\1 & 1 & 1\end{bmatrix}$, and $D = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 8\end{bmatrix}$ is a diagonal matrix with eigenvalues as diagonal entry.

7. Find the matrix of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for both A and B: using the formula $AX = \lambda X$ and trace of A = sum of the eigenvalues = 4. Then use the condition that A and B has only one eigenvector.

8. (a) To show M is a Skew-Hermitian matrix satisfy the equation $(\bar{M})^t = -M$.

   (b) Find an invertible matrix P placing the eigenvectors column-wise. Then compute diagonal matrix D from the formula $D = P^{-1}AP$. Hence deduce the formula $A^n = PD^nP^{-1}$ and put $n = 2020$.

9. Use the condition $a + b = c + d$ and formula $AX = \lambda X$ and trace = sum of diagonal entries to compute eigenvalues in terms of a, b, c and d.

10. (a) solve the characteristic equation to compute eigenvalues which are imaginary.

    (b) Use the property for orthogonal matrix and show that $A^{-1}X = \lambda^{-1}X$. Then show that $\det(A^T - \lambda I) = \det(A - \lambda I)$

11. Matrices A and B are similar if there exists an invertible matrix P such that $PA = BP$. Solve the equation $PA = BP$ and find the matrix A and show that A is non-singular.

13. Any matrix A can be written as $\frac{1}{2}(A+(\bar{A})^t)+\frac{1}{2}(A-(\bar{A})^t) = P+Q$, where P is a Hermitian matrix and Q is a skew-Hermitian matrix.

14. If A is unitary then it satisfies the relation $A(\bar{A})^t = I$.

15. Show that $M\bar{M} = 3I$.