Numerical Analysis

Jacobi's iteration method

Let us consider a system of n linear equations containing n variables:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n \end{array}\right\} \quad \text{— (A)}$$

The above equations can be written as

$$\left.\begin{array}{l} x_1 = -\dfrac{1}{a_{11}}\left(a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n\right) + \dfrac{b_1}{a_{11}} \\ x_2 = -\dfrac{1}{a_{22}}\left(a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n\right) + \dfrac{b_2}{a_{22}} \\ \cdots\cdots\cdots\cdots\cdots \\ x_n = -\dfrac{1}{a_{nn}}\left(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn-1}x_{n-1}\right) + \dfrac{b_n}{a_{nn}} \end{array}\right\} \quad \text{— (B)}$$

Let $x_1^{(0)}, x_2^{(0)}, \ldots x_n^{(0)}$ be the initial guess to the variables $x_1, x_2, \ldots x_n$ respectively (initial guess may be taken as zeros). Substituting these values in the right hand side of (B), which yields the first approximation as follows

$$\left.\begin{array}{l} x_1^{(1)} = -\dfrac{1}{a_{11}}\left(a_{12}x_2^{(0)} + a_{13}x_3^{(0)} + \cdots + a_{1n}x_n^{(0)}\right) + \dfrac{b_1}{a_{11}} \\ x_2^{(1)} = -\dfrac{1}{a_{22}}\left(a_{21}x_1^{(0)} + a_{23}x_3^{(0)} + \cdots + a_{2n}x_n^{(0)}\right) + \dfrac{b_2}{a_{22}} \\ \cdots\cdots\cdots\cdots\cdots \\ x_n^{(1)} = -\dfrac{1}{a_{nn}}\left(a_{n1}x_1^{(0)} + a_{n2}x_2^{(0)} + \cdots + a_{nn-1}x_{n-1}^{(0)}\right) + \dfrac{b_n}{a_{nn}} \end{array}\right\} \quad \text{— (C)}$$

Again substituting $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$ in the right hand side of (C) and obtain the second approximation $x_1^{(2)}, x_2^{(2)}, \ldots x_n^{(2)}$.

In general, if $x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}$ be the kth approximate roots, then the next approximate roots are given by

$$x_1^{(k+1)} = -\frac{1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \cdots + a_{1n}x_n^{(k)}\right) + \frac{b_1}{a_{11}}$$

$$x_2^{(k+1)} = -\frac{1}{a_{22}}\left(a_{21}x_1^{(k)} + a_{23}x_3^{(k)} + \cdots + a_{2n}x_n^{(k)}\right) + \frac{b_2}{a_{22}}$$

$$\cdots\cdots\cdots\cdots$$

$$x_n^{(k+1)} = -\frac{1}{a_{nn}}\left(a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{nn-1}x_{n-1}^{(k)}\right) + \frac{b_n}{a_{nn}}$$

$$k = 0, 1, 2, \ldots$$

The iteration process is continued until all the roots converge to the required number of significant figures i.e.

$|x_i^{(k+1)} - x_i^{(k)}| < \varepsilon \ \forall\ i = 1, 2, \ldots n$, $\varepsilon > 0$ is the error tolerence.

Sufficient condition for convergence

The Jacobi's iteration method surely converges if the coefficient matrix is strictly diagonally dominant by rows i.e.

$$|a_{ii}| > \sum_{j=1}^{n} |a_{ij}| \text{ for } j \neq i \text{ and } i = 1, 2, \ldots, n$$

In matrix form, the method can be written as

$$X^{(k+1)} = -D^{-1}(L+U)X^{(k)} + D^{-1}b$$

$$= HX^{(k)} + C, \quad k = 0, 1, 2, \ldots$$

where $H = -D^{-1}(L+U)$, $C = D^{-1}b$, L and U are lower and upper triangular matrices with zero diagonal entries, D is the diagonal matrix such that $A = L+D+U$.

Ex Solve the system of equations

$$4x_1 + x_2 + x_3 = 2$$
$$x_1 + 5x_2 + 2x_3 = -6$$
$$x_1 + 2x_2 + 3x_3 = -4$$

Using the Jacobi iteration method with the initial approximation as $x^{(0)} = (0.5, -0.5, -0.5)^T$ and perform two on three iterations.

Sol^n.

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H = -D^{-1}(L+U) = -\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{5} & 0 & -\frac{2}{5} \\ -\frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix}$$

$$c = D^{-1}b = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{6}{5} \\ -\frac{4}{3} \end{bmatrix}$$

$\therefore$ Jacobi iteration method becomes

$$x^{(k+1)} = \begin{bmatrix} 0 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{5} & 0 & -\frac{2}{5} \\ -\frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix} x^{(k)} + \begin{bmatrix} 1/2 \\ -\frac{6}{5} \\ -\frac{4}{3} \end{bmatrix}, \quad k \geq 0,$$

Starting with $x^{(0)} = [0.5, -0.5, -0.5]^T$, we obtain

$$x^{(1)} = \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.8500 \end{bmatrix}, \quad x^{(3)} = \begin{bmatrix} 0.933 \\ -1.073 \\ -1.100 \end{bmatrix}$$

Or directly

$$x_1^{(k+1)} = \frac{1}{4}\left[2 - x_2^{(k)} - x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{5}\left[-6 - x_1^{(k)} - 2x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{3}\left[-4, - x_1^{(k)} - 2x_2^{(k)}\right]$$

Starting with $x_1^{(0)} = 0.5$

$x_2^{(0)} = -0.5$

$x_3^{(0)} = -0.5$,

$$x^{(1)} = [0.75, -1.1, -1.1667]^T$$

$$x^{(2)} = [1.0667, -0.8833, -0.8500]^T$$

$$x^{(3)} = [0.9333, -1.0733, -1.1000]^T$$

Ex Solve the following system of linear equations by Gauss-Jacobi's method correct upto four decimal places

$$27x + 6y - z = 54$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Soln: Obviously, the system is diagonally dominant as

$$|6| + |-1| < |27|, \quad |6| + |2| < |15|, \quad |1| + |1| < |54|$$

The Jacobi's iteration scheme is

$$x^{(k+1)} = \frac{1}{27}\left(54 - 6y^{(k)} + z^{(k)}\right)$$
$$y^{(k+1)} = \frac{1}{15}\left(72 - 6x^{(k)} - 2z^{(k)}\right)$$
$$z^{(k+1)} = \frac{1}{54}\left(110 - x^{(k)} - y^{(k)}\right)$$

Let the initial soln be $(0,0,0)$. The next iterations are shown in the following table

Soln.

$x \simeq 1.1664$

$y \simeq 4.0748$

$z \simeq 1.9400$

k	x	y	z
0	0	0	0
1	2.00000	4.80000	2.03704
2	1.00878	3.72839	1.91111
3	1.24225	4.14167	1.94931
4	1.15183	4.04319	1.93733
5	1.17327	4.08096	1.94083
6	1.16500	4.07191	1.93974
7	1.16697	4.07537	1.94006
8	1.16614	4.07454	1.93996
9	1.16640	4.07488	1.93999
10	1.16632	4.07477	1.93998
11	1.16635	4.07481	1.93998

Gauss-Seidel's iteration method

A simple modification of Jacobi's iteration sometimes gives faster convergence. The modified method is known as the Gauss-Seidel's iteration method. Here we use on the RHS all the available values from the present iteration.

$$x_1^{(k+1)} = -\frac{1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \dots + a_{1n}x_n^{(k)}\right) + \frac{b_1}{a_{11}}$$

$$x_2^{(k+1)} = -\frac{1}{a_{22}}\left(a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} + \dots + a_{2n}x_n^{(k)}\right) + \frac{b_2}{a_{22}}$$

$$\dots \quad \dots \quad \dots$$

$$x_n^{(k+1)} = -\frac{1}{a_{nn}}\left(a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} + \dots + a_{n,n-1}x_{n-1}^{(k+1)}\right) + \frac{b_n}{a_{nn}}$$

(The method is repeated until $|x_i^{(k+1)} - x_i^{(k)}| < \varepsilon \ \forall\, i = 1, 2, \dots, n$, where $\varepsilon > 0$ is the error tolerance).

which may be written in the form

$$a_{11}x_1^{(k+1)} = -\sum_{i=2}^{n} a_{1i}x_i^{(k)} + b_1$$

$$a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} = -\sum_{i=3}^{n} a_{2i}x_i^{(k)} + b_2$$

$$a_{n1}x_1^{(k+1)} + \dots + a_{nn}x_n^{(k+1)} = b_n$$

In matrix notation, $(D+L)\,x^{(k+1)} = -Ux^{(k)} + b$.

or,
$$x^{(k+1)} = -(D+L)^{-1}Ux^{(k)} + (D+L)^{-1}b$$
$$= Hx^{(k)} + c \qquad k = 0, 1, 2, \dots$$

where $H = -(D+L)^{-1}U$ and $c = (D+L)^{-1}b$

Ex Solve by Gauss-Seidel method [matrix format] taking the initial approximation as $x^{(0)} = 0$ and perform 3 iterations.

$$
\begin{aligned}
2x_1 - x_2 \qquad &= 7 \\
-x_1 + 2x_2 - x_3 &= 1 \\
-x_2 + 2x_3 &= 1
\end{aligned}
$$

Solⁿ: $D + L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$, $U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$$x^{(k+1)} = -(D+L)^{-1} U x^{(k)} + (D+L)^{-1} b$$

$$(D+L)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}$$

$$(D+L)^{-1} U = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & 0 \\ 0 & -1/4 & -1/2 \\ 0 & -1/8 & -1/4 \end{bmatrix}$$

$$(D+L)^{-1} b = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 9/4 \\ 13/8 \end{bmatrix}$$

$$x^{(k+1)} = \begin{bmatrix} 0 & +1/2 & 0 \\ 0 & +1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} x^{(k)} + \begin{bmatrix} 7/2 \\ 9/4 \\ 13/8 \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix} \qquad x^{(2)} = \begin{bmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{bmatrix} \qquad x^{(3)} = \begin{bmatrix} 5.3125 \\ 4.3125 \\ 2.6563 \end{bmatrix}$$

Exact solⁿ is $[6, 5, 3]^T$

Ex Solve the following system of eqns. by Gauss-Seidel's iteration method correct upto four decimal places.

$$27x + 6y - z = 54$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Solⁿ: The iteration scheme is

$$x^{(k+1)} = \frac{1}{27}\left(54 - 6y^{(k)} + z^{(k)}\right)$$
$$y^{(k+1)} = \frac{1}{15}\left(72 - 6x^{(k+1)} - 2z^{(k)}\right)$$
$$z^{(k+1)} = \frac{1}{54}\left(110 - x^{(k+1)} - y^{(k+1)}\right)$$

Let $y=0$, $z=0$, be the initial approximation. The successive iterations are shown below

k	x	y	z
0	–	0	0
1	2·00000	4·00000	1·92593
2	1·18244	4·07023	1·93977
3	1·16735	4·07442	1·93997
4	1·16642	4·07477	1·93998
5	1·16635	4·07480	1·93998
6	1·16634	4·07480	1·93998

The solⁿ correct upto 4 decimal places is $x = 1{\cdot}1663$, $y = 4{\cdot}0748$, $z = 1{\cdot}9400$

Theorem A necessary and sufficient condition for convergence of an iterative method is that the eigenvalues of the iteration matrix satisfy $|\lambda_i(H)| < 1$, $i \ni (1)n$.

Ex Consider the 3x3 linear systems of the form $A_i x = b_i$ where b_i is unit vector and the matrices A_i are

$$A_1 = \begin{bmatrix} 3 & 0 & 4 \\ 7 & 4 & 2 \\ -1 & 1 & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -3 & 3 & -6 \\ -4 & 7 & -8 \\ 5 & 7 & -9 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 4 & 1 & 1 \\ 2 & -9 & 0 \\ 0 & -8 & -6 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 7 & 6 & 9 \\ 4 & 5 & -4 \\ -7 & -3 & 8 \end{bmatrix}$$

It can be checked that the Jacobi method fails to converge for A_1 $[\rho(H_J) = 1.33]$, while the Gauss-Seidal scheme is convergent. Conversely, in the case of A_2, the Jacobi method is convergent, while the Gauss-Seidel method fails to converge $[\rho(H_{GS}) = 1.1]$. In A_3, G-S method is faster than Jacobi $[\rho(H_{GS}) = 0.018,\ \rho(H_J) = 0.44]$ and the converse is true for A_4 $[\rho(H_J) = 0.64,\ \rho(H_{GS}) = 0.77]$