# ADVANCED CALCULUS
# MA11003

## SECTION 11, 12, & 15CD


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# Concepts Covered

## Differential Calculus

### Functions of Several Variables

- Composite Functions

- Homogeneous Functions

## Composite Functions

Consider $z = f(x, y) \quad \Big\} \quad (1)$

Let $\left.\begin{aligned} x &= \phi(t) \\ y &= \psi(t) \end{aligned}\right\} \quad (2)$ or $\left.\begin{aligned} x &= \phi(u, v) \\ y &= \psi(u, v) \end{aligned}\right\} \quad (2')$

The equations $(1\ \&\ 2)$ or $(1\ \&\ 2')$ are said to define $z$ as composite function of $t$ or $u\ \&\ v$.

## Differentiation of Composite Functions

Let $z = f(x, y)$ posses continuous partial derivatives (differentiable) and let $x = \phi(t)$, $\mathrm{y} = \psi(t)$ posses continuous derivatives (differentiable). Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

**Proof:** Let $z = f(x, y)$, $x = \phi(t)$, $y = \psi(t)$ be a composite function of $t$.

Assuming $z, \phi, \psi$ to be differentiable

$$\Delta\, z = z_x \Delta x + z_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

Dividing by $\Delta t$

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}$$

Taking limit $\Delta t \to 0$ $(\Delta x \to 0,\ \ \Delta y \to 0)$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

## Differentiation of Composite Functions

For the case $z = f(x, y) \qquad x = \phi(u, v), \qquad y = \psi(u, v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

**Problem - 1** Given $z = xy;\quad x = \cos t, \qquad y = \sin t.$ Find $\frac{dz}{dt}.$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= y(-\sin t) + x\cos t$$

$$= -\sin^2 t + \cos^2 t$$

$$= \cos 2t$$

**Problem - 2** Let $z$ be a function of $x$ & $y$. Further, it is given that

$$x = e^{u} + e^{-v} \qquad y = e^{-u} + e^{v}$$

Then show that $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^{u} + \frac{\partial z}{\partial y}(-e^{-u})$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -\frac{\partial z}{\partial x}e^{-v} + \frac{\partial z}{\partial y}e^{v}$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u}+e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} + e^{v}) = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

## Homogeneous Functions

An expression in $(x, y)$ is homogeneous of order $n$ if it can be expressed as

$$x^n f\left(\frac{y}{x}\right)$$

OR

A function $f(x, y)$ is said to be homogeneous of order $n$ if it satisfies

$$f(tx, ty) = t^n f(x, y)$$

## Example of Homogeneous Functions

- $f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$

$$= x^n \underbrace{\left(a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \cdots + a_n\left(\frac{y}{x}\right)^n\right)}_{g\left(\frac{y}{x}\right)}$$

Homogeneous function of order $n$

- $f(x,y) = \dfrac{\sqrt{y}+\sqrt{x}}{y+x} = \dfrac{\sqrt{x}}{x}\dfrac{\sqrt{\frac{y}{x}}+1}{\frac{y}{x}+1} = x^{-\frac{1}{2}}g\left(\frac{y}{x}\right)$

Homogeneous function of order $-\frac{1}{2}$

## Euler’s Theorem on Homogeneous Functions

If $z = f(x, y)$ be a homogeneous function of $x$ & $y$ of order $n$, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz, \qquad \forall x, y \in D$$

Given $\quad z = f(x, y) = x^n g\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = n\, x^{n-1} g\left(\frac{y}{x}\right) + x^n\left(-\frac{y}{x^2}\right) g'\left(\frac{y}{x}\right) = n\, x^{n-1} g\left(\frac{y}{x}\right) - y\, x^{n-2}\, g'\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^n\left(\frac{1}{x}\right) g'\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\, x^n g\left(\frac{y}{x}\right) - y\, x^{n-1}\, g'\left(\frac{y}{x}\right) + y x^{n-1} g'\left(\frac{y}{x}\right) = nz$$

**Problem - 3** If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}, x \neq y.$ Then, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2x$

Let $z = \tan u = \dfrac{x^3+y^3}{x-y} \quad = x^2\left(\dfrac{1+\left(\frac{y}{x}\right)^3}{1-\frac{y}{x}}\right)$ Homogeneous function of order 2

$$\text{Euler's Theroem:} \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

Subst. $z = \tan u$ gives

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\ \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

# Euler's Theorem on Homogeneous Functions (Generalization)

If $z = f(x, y)$ be a homogeneous function of $x$ & $y$ of order $n$, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz, \qquad \forall x, y \in D$$

If $z = f(x, y)$ be a homogeneous function of $x$ & $y$ of order $n$, then

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\, \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z, \qquad \forall\, x, y \in D$$

**QUIZ QUESTION ?**

LINK FOR RESPONSES: http://www.facweb.iitkgp.ac.in/~jkumar/teach/MA11003.html

## Euler's Theorem on Homogeneous Functions (Generalization)

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x\, \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z, \qquad \forall\, x, y \in D$$

**Question**: Let $z = xy\, f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, where $f$ & $g$ are 2 times differentiable functions.

Then $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x\, \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$ equals

| | | | |
|---|---|---|---|
| (A) | $xy\, f\left(\frac{y}{x}\right)$ | (B) | $2\, xy\, f\left(\frac{y}{x}\right)$ |
| (C) | $2\, xy\, g\left(\frac{y}{x}\right)$ | (D) | $xy\, g\left(\frac{y}{x}\right)$ |

**Problem - 4** Let $z = xy\, f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, where $f$ & $g$ are 2 times differentiable functions.

Then evaluate $$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x\, \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$$

Let $z = u_1 + u_2$, where

$$\underbrace{u_1 = xy\, f\left(\frac{y}{x}\right)}_{\text{Hom. function of order 2}} \qquad u_2 = \underbrace{g\left(\frac{y}{x}\right)}_{\text{Hom. function of order 0}}$$

$u_1$: Hom. function of order 2

$u_2$: Hom. function of order 0

$z = u_1 + u_2$

Euler's Theorem on $u_1$ & $u_2$

$$u_1 = xy\, f\left(\frac{y}{x}\right) \qquad u_2 = g\left(\frac{y}{x}\right)$$

$$x^2 \frac{\partial^2 u_1}{\partial x^2} + 2xy \frac{\partial^2 u_1}{\partial x\, \partial y} + y^2 \frac{\partial^2 u_1}{\partial y^2} = 2\, u_1$$

$$x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x\, \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x\, \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 2\, xy\, f\left(\frac{y}{x}\right)$$

# KEY TAKEAWAY

## Differentiation of Composite Functions

$$z = f(x, y), x = \phi(t),\ y = \psi(t) \qquad \frac{\mathrm{dz}}{\mathrm{dt}} = \frac{\partial z}{\partial x}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\partial z}{\partial y}\frac{\mathrm{dy}}{\mathrm{dt}}$$

$$z = f(x, y), \quad x = \phi(u, v),\ y = \psi(u, v)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\text{Euler's Theroem:} \qquad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

# *Thank You*