Ex Find a root of the equation $x^2+x-7=0$ by bisection method, correct upto 2 decimal places.

Solⁿ: $f(x) = x^2+x-7$

$f(2) = -1 \quad f(3) = 5$

So a root lies between 2 and 3

n	Left end pt a_n	Right end pt b_n	Midpt. $x_{n+1} = \frac{a_n+b_n}{2}$	$f(x_{n+1})$
0	2	3	2.5	1.750
1	2	2.5	2.250	0.313
2	2	2.250	2.125	-0.359
3	2.125	2.250	2.188	-0.027
4	2.188	2.250	2.219	0.143
5	2.188	2.219	2.204	0.062
6	2.188	2.204	2.196	0.018
7	2.188	2.196	2.192	-0.003
8	2.192	2.196	2.194	0.008
9	2.192	2.194	2.193	0.002
10	2.192	2.193	2.193	0.002

Therefore the root is 2.19 correct upto 2 decimal places.

Iteration method or fixed point iteration

Let $f(x)$ be a f^n. continuous on the interval $[a,b]$ and the eqn. $f(x)=0$ has at least one root on $[a,b]$. The eqn. $f(x)=0$ can be written in the form

$$x=\varphi(x) \quad \text{—(i)}$$

Suppose $x_0 \in [a,b]$ be an initial guess to the desired root ξ. Then $\varphi(x_0)$ is evaluated and this value is denoted by x_1. It is the first approximation of the root ξ. Again x_1 is substituted for x to the RHS of (i) and obtained a new value $x_2=\varphi(x_1)$. This process is continued to generate the sequence of numbers $x_0, x_1, x_2, \dots, x_n, \dots$ those are defined by the following relation:

$$x_{n+1}=\varphi(x_n) \qquad n=0,1,2,\dots$$

This successive iterations are repeated till the approximate numbers x_n's converges to the root with desired accuracy i.e. $|x_{n+1}-x_n|<\varepsilon$, where ε is the error tolerance.

Note: There is no guarantee that this sequence $x_0, x_1, \dots$ will converge. The f^n. $f(x)=0$ can be written as $x=\varphi(x)$ in many different ways. The f^n. $\varphi(x)$ is very important for convergence.

For example, $x^3+x^2-1=0$ can be rewritten as

$$x=\frac{1-x^2}{x^2}; \quad x=(1-x^2)^{1/3}; \quad x=\sqrt{\frac{1-x^2}{x}}; \quad x=\sqrt{1-x^3}; \quad x=\frac{1}{\sqrt{1+x}} \text{ etc}$$

Sufficient condition for convergence

Theorem

Let ξ be a root of the eqn. $f(x)=0$ and it can be written as $x=\phi(x)$ and further that

(i) the f^n. $\phi(x)$ is defined and differentiable on $[a,b]$

(ii) $|\phi'(x)|<1 \quad \forall x \in [a,b]$

Then the sequence $\{x_n\}$ in $x_{n+1}=\phi(x_n)$, $n=0,1,2,\dots$ converges to the desired root ξ irrespective of the choice of the initial approximation $x_0 \in [a,b]$ and the root ξ is unique

Order of convergence

Let x_n converges to the exact root ξ so that $\xi=\phi(\xi)$

Thus $\quad x_{n+1}-\xi = \phi(x_n)-\phi(\xi)$

Let

$$\begin{aligned}
\varepsilon_{n+1} &= x_{n+1}-\xi \\
&= \phi(x_n)-\phi(\xi) \\
&= \phi(\xi+\varepsilon_n)-\phi(\xi) \quad \left[\because \varepsilon_n = x_n-\xi \quad \therefore x_n = \xi+\varepsilon_n\right] \\
&= \varepsilon_n \phi'(\xi) + \frac{1}{2}\varepsilon_n^2 \phi''(\xi) + \cdots \\
&= \varepsilon_n \phi'(\xi) + O(\varepsilon_n^2)
\end{aligned}$$

$$\therefore \varepsilon_{n+1} \simeq \varepsilon_n \phi'(\xi)$$

$\therefore$ Order of convergence is linear.

[The exponent of ε_n in the first non vanishing term is called the order of the iteration process. Order measures the speed of convergence.]

Q. Consider the eqn. $5x^3 - 20x + 3 = 0$ Find the root lying on the interval $[0,1]$ with an accuracy of 10^{-4} [Initial guess 0.5]

Solⁿ:

$$x = \frac{5x^3+3}{20} = \phi(x)$$

$$|\phi'(x)| = \left|\frac{15x^2}{20}\right| = \left|\frac{3x^2}{4}\right| < 1 \text{ on } [0,1]$$

Let $x_0 = 0.5$

n	x_n	$\phi(x_n) = x_{n+1}$
0	0.5	0.18125
1	0.18125	0.15149
2	0.15149	0.15087
3	0.15087	0.15086
4	0.15086	0.15086

$\therefore \xi = 0.1509$ is taken as the reqd. root.

Newton - Raphson method

Let x_0 be an approximate root of the equation $f(x)=0$. Suppose $x_1 = x_0 + h$ be the exact root of the eqn., where h is the correction of the root (error). Then $f(x_1)=0$. Using Taylor's series, $f(x_1) = f(x_0+h)$ is expanded in the following form

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots = 0$$

Neglecting the second and higher order derivatives, the above eqn. reduces to

$$f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{——} \quad (1)$$

To compute the value of h, the second and higher powers of h are neglected, so the value of $h = -\frac{f(x_0)}{f'(x_0)}$ is not exact, it is an approximate value, So the x_1 obtained from (1) is not a root of the eqn. But it is a better approximation of x than x_0. In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Ex Use Newton-Raphson method to find a root of the eqn. $x^3+x-1=0$. [Initial guess $x_0=0$]

Soln:
$$f(x) = x^3+x-1$$
$$f(0) = -1 < 0$$
$$f(1) = 1 > 0$$

$\therefore$ One root lies between 0 and 1. Let $x_0=0$ be the initial root. The iteration scheme is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^3+x_n-1}{3x_n^2+1} = \frac{2x_n^3+1}{3x_n^2+1}$$

The sequence $\{x_n\}$ for different values of n is shown below.

n	x_n	x_{n+1}
0	0	1
1	1	0·7500
2	0·7500	0·6861
3	0·6861	0·6823
4	0·6823	0·6823

Therefore a root of the eqn. is 0·682 correct upto three decimal places.

Interpolation

Sometimes we have to compute the value of the dependent variable for a given independent variable, but the explicit relation between them is not known. In these cases, we can use interpolation.

(The general interpolation problem can be states as: Let $y = f(x)$ be a f^n. whose analytic expression is not known, but a table of values of y is known only at a set of values $x_0, x_1, x_2, \ldots x_n$ of x. i.e. $f(x_i) = y_i$, $i = 0, 1, \ldots, n$. The problem of interpolation is to find the value of $y (= f(x))$ for an argument, say x' when the value of y at x' is not available in the table.)

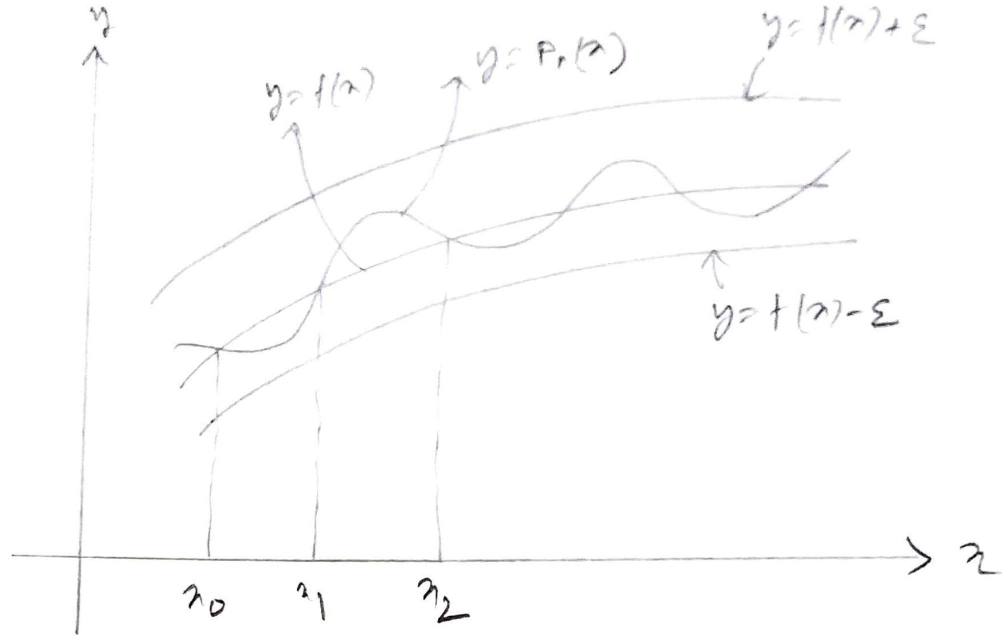
A large number of different techniques are used to determine the value of y at $x = x'$. One common technique is to find an approximate f^n. say $\phi(x)$ corresponding to the given f^n. $f(x)$ depending on the tabulated value.

polynomial.

Def^n. of interpolating polynomial

Let $x_0, x_1, \ldots x_n$ be $n+1$ distinct points in the interval $[a, b]$, where the values of $f(x)$ are known. Then $P_n(x)$ is an interpolating polynomial to $f(x)$ if $P_n(x_i) = f(x_i), i = 0, \ldots n$.

In all practical cases, $P_n(\bar{x})$ is considered as an approximation to $f(\bar{x})$ at a pt. $x = \bar{x}$ provided the error $|f(\bar{x}) - P_n(\bar{x})|$ is small i.e. $|f(\bar{x}) - P_n(\bar{x})| < \varepsilon$ where the no. $\varepsilon > 0$ is called an error tolerance.



Interpolation of a f^n.

The following theorem justifies the approximation of an unkno f^n - $f(x)$ to a polynomial $P_n(x)$.

Theorem

If the f^n. $f(x)$ is continuous in $[a,b]$, then for any pre-assigned no. $\varepsilon > 0$, $\exists$ a polynomial $P_n(x)$ such that

$$|f(x) - P_n(x)| < \varepsilon \quad \forall\ x \in [a,b].$$

Error in interpolating polynomial

It is obvious that if $f(x)$ is approximated by a polynomial $\phi(x)$, then there should be some error at the non-tabular points. The following theorem gives the amount of error in interpolating polynomial.

Theorem

Let I be an interval containing all interpolating points $x_0, x_1, \cdots, x_n$. Let $f(x)$ be continuous and have continuous derivatives of order $n+1$ for all x in I and $p(x)$ be a polynomial of degree $\leq n$ that interpolates $f(x)$ at $(n+1)$ distinct points $x_0, \cdots x_n$. Then the error at any point x is given by

$$f(x) - p(x) = E_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

where $\xi \in I$

The expression gives the error at any point x. But practically it has little utility, because, in many cases $f^{(n+1)}(\xi)$ cannot be determined.

If M_{n+1} be the upper bound of $f^{(n+1)}(\xi)$ in I, i.e. if $|f^{(n+1)}(\xi)| \le M_{n+1}$ in I, then the upper bound of $E_n(x)$ is

$$|E_n(x)| \le \frac{M_{n+1}}{(n+1)!} |w(x)|$$

$$\le \frac{M_{n+1}}{(n+1)!} \max_{x_0 \le x \le x_n} |w(x)|$$

where $M_{n+1} = \max_{x_0 \le x \le x_n} |f^{(n+1)}(x)|$.

Ex If $p(x)$ is a polynomial that interpolates the f^n. $f(x) = \sin x$ at 10 points on the interval $[0,1]$. Find an upper bound for the error.

Proof: $n+1 = 10$

$$f^{(n+1)}(\xi) = f^{10}(\xi) = -\sin \xi \qquad \xi \in [0,1]$$

$$E_n(x) \le \max \left| \frac{1}{10!} f^{10}(\xi)(x-x_0) \cdots (x-x_{10}) \right|$$

$$\max_{0 \le x \le 1} |(x-x_0) \cdots (x-x_{10})| = 1$$

$$\max_{0 \le x \le 1} |f^{(n+1)}(\xi)| = \max_{0 \le x \le 1} |-\sin \xi| = 1$$

$$\le \frac{1}{10!} 1.1$$

[Error bound $|f(x) - p(x)| \le \dfrac{\max_{x \in [a,b]} |(x-x_0) \cdots (x-x_n)| \max_{x \in [a,b]} |f^{n+1}(\xi)|}{(n+1)!}$

Finite differences

Let a f^n. $y = f(x)$ be known as (x_i, y_i) at $(n+1)$ pts. x_i $i = 0, 1, \ldots n$ where x_i's are equally spaced i.e $x_i = x_0 + ih$, h is the spacing between any two successive pts x_i's

i.e. $y_i = f(x_i)$, $i = 0, 1, \ldots n$.

Forward differences

The 1st order forward difference of $f(x)$ is defined as

$$\Delta f(x) = f(x+h) - f(x) \qquad \Delta \equiv \text{forward diff. operator}$$

$$\therefore \Delta f(x_0) = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0)$$

$$\Delta y_0 = y_1 - y_0 \qquad \text{using } y_i = f(x_i)$$

Similarly $\Delta y_1 = y_2 - y_1$

$$\Delta y_2 = y_3 - y_2$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

The 2nd order differences are

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$$

$$= \Delta y_1 - \Delta y_0$$

$$= (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

Similarly, $\Delta^3 y_0 = y_3 - 2y_2 + y_1$

In general, $\Delta^k y_0 = y_k - {}^kC_1\, y_{k-1} + {}^kC_2\, y_{k-2} + \cdots (-1)^k y_0$

$$\Delta^k y_i = y_{k+i} - {}^kC_1\, y_{k+i-1} + {}^kC_2\, y_{k+i-2} - \cdots (-1)^k y_i$$