EIGENVALUES & EIGENVECTORS – CONT.

- Eigenvalues & Eigenvectors
- Properties

PRODUCT AND SUM OF EIGENVALUES OF A MATRIX

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Its characteristic equation: $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow (a-\lambda)(d-\lambda) - bc = 0$

$$\Rightarrow (-\lambda)^2 + (a+d)(-\lambda) + (ad - bc) = 0$$

For any $n \times n$ matrix, the characteristic polynomial is of the form:

$$\Rightarrow (-\lambda)^n + tr(A)(-\lambda)^{n-1} + \cdots + \det(A) = 0$$

- The determinant of A is the product of the eigenvalues.
- The trace of A is the sum of the eigenvalues.

If $A \in \mathbb{R}^{n\times n}$ is a square matrix whose column sums are all 1, then what can we say about the eigenvalue(s) of A?

- One of the eigenvalues of A is 1. OR
- The equation $Ax = x$ has nontrivial solutions

Consider

$$A^T \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

This implies that one of the eigenvalues of A^T is 1.

Since A^T and A have the same eigenvalues and therefore one of the eigenvalue of A is 1.

Algebraic Multiplicity:

Multiplicity of λ as a root of the characteristic equation.

Geometric Multiplicity:

Dimension of the eigenspace of λ (number of linearly independent eigenvectors corresponding to an eigenvalue λ).

- **Note:** Geometric Multiplicity $\leq$ Algebraic Multiplicity

Example 1: Find eigenvalue and eigenvectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Characteristic Equation: $\det(A - \lambda I) = 0$

$$\Rightarrow (2-\lambda)(\lambda-2)(\lambda-8) = 0 \quad \Rightarrow \lambda = 2, 2, 8$$

Algebraic multiplicity of $\lambda = 2$: 2

Algebraic multiplicity of $\lambda = 8$: 1

- Eigenvector corresponding to $\lambda = 8$: $(A - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \alpha \neq 0, \alpha \in \mathbb{R}.$$

Geometric multiplicity of $\lambda = 8$: 1

- Eigenvector corresponding to λ =2: $(A - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Geometric multiplicity of λ =2: 2

Example 2: Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Eigenvalues are $\lambda = 2, 2, 3.$

❖ **Note**: Eigenvalues of a triangular matrix are its diagonal elements.

- **Eigenspace** of $\lambda = 2$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Geometric multiplicity of $\lambda = 2$:

Algebraic multiplicity of $\lambda = 2$:

- **Eigenspace** of $\lambda = 3$:

$$\begin{bmatrix} -1 & 0 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Geometric multiplicity of $\lambda = 3$:

Algebraic multiplicity of $\lambda = 3$:

Conclusion:

Algebraic Multiplicity: The number of occurrence of an eigenvalue

Geometric Multiplicity: The number of linearly independent eigenvectors associated with that eigenvalue

Geometric Multiplicity $\leq$ Algebraic Multiplicity

DIAGONALIZATION

- Similarity of Matrices
- Diagonalization

Similarity of Matrices:

An $n \times n$ matrix B is called similar to an $n \times n$ matrix A if

$$B = P^{-1}AP$$

for some non-singular matrix P.

Theorem: If B is similar to A, then B has the same eigenvalues as A. If x is an eigenvector of A. Then $y = P^{-1}x$ is an eigenvector of B corresponding to the same eigenvalue.

$$\lambda x = Ax \;\; \Rightarrow \lambda P^{-1}x = P^{-1}Ax$$

$$\Rightarrow \lambda P^{-1}x = P^{-1}A(PP^{-1})x$$

$$\Rightarrow \lambda(P^{-1}x) = B(P^{-1}x)$$

$\Rightarrow \lambda$ is an eigenvalue of B and $P^{-1}x$ is an eigenvector corresponding to the eigenvalue λ.

Theorem: If A and B are square matrices similar to each other, then they have the same characteristic polynomial.

Proof: $B = P^{-1}AP$

$$\det(B - \lambda I) = \det(P^{-1}AP - P^{-1}(\lambda I)P)$$

$$= \det(P^{-1}(A - \lambda I)P)$$

$$= \det(P^{-1})\det(A - \lambda I)\det(P)$$

$$= \det(A - \lambda I)$$

Diagonalization of a Matrix:

A square matrix A is said to be **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is a **diagonal matrix** (i.e., A is similar to a diagonal matrix).

Let A be an $n \times n$ matrix. Then A is diagonalizable iff A has n linearly independent eigenvectors.

Let A be an $n \times n$ matrix. Then A is diagonalizable iff A algebraic multiplicity is equal to geometric multiplicity of A for each eigenvalue.

If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Note: The matrix P which diagonalizes A is called Model Matrix of A whose columns are the eigenvectors corresponding to different eigenvalues.

Example 1: $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Eigenvalues: $1 \ \& \ 6$ **Eigenvectors:** $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \ \& \ \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \Longrightarrow P^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

Example 2: $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Eigenvalues: $2, 2 \,\&\, 8$

Eigenvectors: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \& \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Example 3: $A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Eigenvalues: $\underbrace{2, 2}$ & 3

Eigenvectors: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\Rightarrow$ The given matrix is not diagonalizable.

Applications of Diagonalization

- **Power of Matrices**

$P^{-1}AP = D \quad \Rightarrow A = PDP^{-1}$

Then $A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}$

Similarly $A^3 = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1}$

$\Longrightarrow A^n = PD^nP^{-1}$

Example: Find A^5 for $A = \begin{bmatrix} 1 & 4 \\ \frac{1}{2} & 0 \end{bmatrix}$

Eigenvalues: -1 & 2

Eigenvectors: $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Take $P = \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}$ $\Rightarrow$ $P^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$

Then $A^5 = PD^5P^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^5 & 0 \\ 0 & 2^5 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow A^5 = \begin{bmatrix} 21 & 44 \\ 5.5 & 10 \end{bmatrix}$

➢ Solution of System of Linear Differential Equations

Consider the system of linear differential equations

$$\dot{X}(t) = A\,X(t)$$

Let us assume that A is diagonalizable. Then $D = P^{-1}AP \quad \Rightarrow A = PDP^{-1}$

$$\therefore\ \dot{X}(t) = PDP^{-1}\,X(t) \quad \Rightarrow P^{-1}\,\dot{X}(t) = DP^{-1}\,X(t) \quad \Rightarrow [P^{-1}\,X(t)]' = D[P^{-1}\,X(t)]$$

Substituting $P^{-1}\,X(t) = Y(t)$ we get

$$\dot{Y}(t) = D\,Y(t)$$

$$\Rightarrow \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{y}_n(t) \end{bmatrix} = \mathrm{Diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

$$\Rightarrow \dot{y}_i(t) = \lambda_i y_i(t), \qquad \forall\, i$$

$$\Rightarrow y_i(t) = C_i e^{\lambda_i t}$$

where C_i is constant, and $i = 1,2,\dots,n$.

$$P^{-1}X(t) = Y(t) \quad \Rightarrow X(t) = P\,Y(t)$$

$\begin{bmatrix} | \\ v_i \\ | \end{bmatrix}$ is the eigenvector corresponding to λ_i

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = C_1 \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} e^{\lambda_1 t} + C_2 \begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} e^{\lambda_2 t} + \cdots + C_n \begin{bmatrix} | \\ v_n \\ | \end{bmatrix} e^{\lambda_n t}$$

Example: Solve the following system of equations

$$\frac{dx_1}{dt} = 3x_1 + 2x_2$$

$$\frac{dx_2}{dt} = 7x_1 - 2x_2$$

Rewrite the system of differential equations in matrix notation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies x' = Ax$$

Eigenvalues: $A = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix}$

$$\det(A - \lambda I) = 0 \;\Longrightarrow\; \lambda^2 - \lambda - 20 = 0$$

$$\Longrightarrow (\lambda + 4)(\lambda - 5) = 0 \;\Longrightarrow\; \lambda_1 = -4 \quad \& \quad \lambda_2 = 5$$

Eigenvectors: $\begin{bmatrix} 2 \\ -7 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ -7 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

Conclusion

Diagonalization of a Matrix

- Power of Matrices
- Solution of System of Linear Differential Equations

Thank You