ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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Concepts Covered

Differential Calculus

Functions of Several Variables

- Method of Lagrange’s Multiplier

Method of Lagrange's Multiplier

Find the Maxima/Minima of the function

$u = f(x, y)$ with the constraint $\phi(x, y) = 0$

Using chain rule

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

At the point of extrema $\frac{du}{dx} = 0 \quad \Longrightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$

At the point of extrema $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

The equation $\phi(x, y) = 0$ is satisfied at any point and so at the point of extrema

$$\boxed{\phi(x, y) = 0}$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0$$

$$\boxed{\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0}$$

We eliminate $\frac{dy}{dx}$ from the above equations.

$$\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}\right)\underbrace{\left(-\frac{f_y}{\phi_y}\right)}_{=:\lambda} + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \boxed{\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0}$$

$$\boxed{\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0} \qquad \boxed{\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0} \qquad \boxed{\phi(x, y) = 0}$$

Method of Lagrange's Multiplier (Working Rule)

max/min $u = f(x, y)$ with the constraint $\phi(x, y) = 0$

Define an auxiliary function $F(x, y, \lambda) = f(x, y) + \lambda\, \phi(x, y)$

Necessary conditions for extrema of F

$$F_x = 0 \Longrightarrow \quad f_x + \lambda \phi_x = 0$$

$$F_y = 0 \Longrightarrow \quad f_y + \lambda \phi_y = 0$$

$$F_\lambda = 0 \Longrightarrow \quad \phi = 0$$

REMARK:

Using the method of Lagrange's multiplier, we obtain stationary points (candidates for extrema). We do not determine the nature of the stationary points. In practice we usually are interested in finding max/min value of the function under some given constraint.

Usually there are a few candidates (critical points), so we can evaluate f at all of them and choose the largest and smallest values. Hence, no further test is required if we wish to find only absolute maximum and minimum.

Problem - 1: Find maximum/minimum of the function $x^2 - y^2 - 2x$ in the region $x^2 + y^2 \leq 1$

I. Local extrema in the interior $x^2 + y^2 < 1$

Let $f(x, y) = x^2 - y^2 - 2x$

$f_x = 0 \implies x = 1$

Critical Point: $(1,0)$

$f_y = 0 \implies y = 0$

However this point lies on the boundary so no critical point lies in the interior.

II. Local extrema on the boundary $x^2 + y^2 = 1$

Problem $\quad \max/\min \quad x^2 - y^2 - 2x \quad \text{subjet to} \quad x^2 + y^2 = 1$

Auxiliary function for the Lagrange multiplier

$$F(x, y, \lambda) = (x^2 - y^2 - 2x) + \lambda(x^2 + y^2 - 1)$$

Critical Point:

$$F_x = 0 \implies \quad x(1 + \lambda) = 1$$

$$F_y = 0 \implies \quad y(\lambda - 1) = 0$$

$$F_\lambda = 0 \implies \quad x^2 + y^2 = 1$$

II. Local extrema on the boundary $x^2 + y^2 = 1$

$$x(1+\lambda) = 1 \quad\quad y(\lambda - 1) = 0 \quad\quad x^2 + y^2 = 1$$

$$x(1+\lambda) = 1 \Downarrow \lambda = 0, -2 \Downarrow x = \frac{1}{2}$$

$$y(\lambda - 1) = 0 \Downarrow y = 0 \ \& \ \lambda = 1 \Downarrow y = 0 \Downarrow \lambda = 1$$

$$x^2 + y^2 = 1 \Downarrow x = \pm 1 \Downarrow y = \pm\frac{\sqrt{3}}{2}$$

Candidates for extrema $(\pm 1, 0)$ $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

III. Function Values:

Candidates for extrema $(\pm 1, 0)$ $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

$$f(x, y) = x^2 - y^2 - 2x$$

Points	$(1, 0)$	$(-1, 0)$	$\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$
Function Value	-1	3	$-\frac{3}{2}$

Maximum value of the function: 3

Minimum value of the function: $-\frac{3}{2}$

QUIZ QUESTION ?

LINK FOR RESPONSES: http://www.facweb.iitkgp.ac.in/~jkumar/teach/MA11003.html

Consider the problem

$$\text{Maximize } f(x,y) = x^2 + y^2 \text{ in the domain}$$

$$(x-2)^2+(y-1)^2 \leq 20$$

Critical points are

(A)	$(0,0),(2,1),(6,3)$
(B)	$(0,0),(2,-1),(6,3)$
(C)	$(0,0),(-2,1),(6,3)$
(D)	$(0,0),(-2,-1),(6,3)$

Problem - 2: Find maximum and minimum values of the function $f(x,y) = x^2 + y^2$ in the region $(x-2)^2+(y-1)^2 \leq 20$

I. Local extrema in the interior $(x-2)^2+(y-1)^2 < 20$

$$f_x = 0 \implies x = 0$$

$$f_y = 0 \implies y = 0$$

Critical Point: $(0,0)$

II. Local extrema on the boundary $(x-2)^2+(y-1)^2 = 20$

Problem $\max/\min \quad x^2 + y^2 \quad \text{subjet to} \quad (x-2)^2+(y-1)^2 = 20$

Auxiliary function for the Lagrange multiplier

$$F(x,y,\lambda) = (x^2 + y^2) + \lambda\left((x-2)^2 + (y-1)^2 - 20\right)$$

$$F(x,y,\lambda) = (x^2+y^2) + \lambda\left((x-2)^2 + (y-1)^2 - 20\right)$$

Critical Point:

$$F_x = 0 \implies \quad x + \lambda(x-2) = 0 \quad \implies \quad (x-2) = -\frac{2}{1+\lambda} \quad \implies x = -2, 6$$

$$F_y = 0 \implies \quad y + \lambda(y-1) = 0 \quad \implies \quad (y-1) = -\frac{1}{1+\lambda} \quad \implies y = -1, 3$$

$$F_\lambda = 0 \implies \quad (x-2)^2 + (y-1)^2 = 20 \quad \implies (1+\lambda) = \pm\frac{1}{2} \implies \lambda = -\frac{1}{2}, -\frac{3}{2}$$

Points	$(0,0)$	$(-2,-1)$	$(6,3)$
Function Value	0	5	$\mathbf{45}$

Minimum value : 0 Maximum value: 45

KEY TAKEAWAY

Method of Lagrange's Multiplier

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Necessary conditions for extrema of F

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Thank You