LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

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Determination of Roots of Algebraic and Transcendental Equations

- ➢ Bisection Method
- ➢ Fixed Point Iteration Method
- ❑ Newton-Raphson Method
- ❑ Secant Method

Bisection Method

It is based on the following theorem for zeroes of continuous functions:

Theorem: Given a continuous function $f: [a, b] \to \mathbb{R}$ such that $f(a)f(b) < 0$, then $\exists\ \alpha \in (a, b)$ such that $f(\alpha) = 0$.

Outline of the Algorithm

Choosing $I_0 = [a, b]$, so that $f(a)f(b) < 0$.

The bisection method generates a sequence of subinterval $I_k = \left[a^{(k)}, b^{(k)}\right], k \geq 0$

such that $I_k \subset I_{k-1}, k \geq 1$ and the property $f\left(a^{(k)}\right)f\left(b^{(k)}\right) < 0$.

Pseudocode

Set $a^{(0)} = a\,, b^{(0)} = b$ and $x^{(0)} = \dfrac{a+b}{2}$

For $k \geq 0$

if $f\left(a^{(k)}\right) f\left(x^{(k)}\right) < 0$

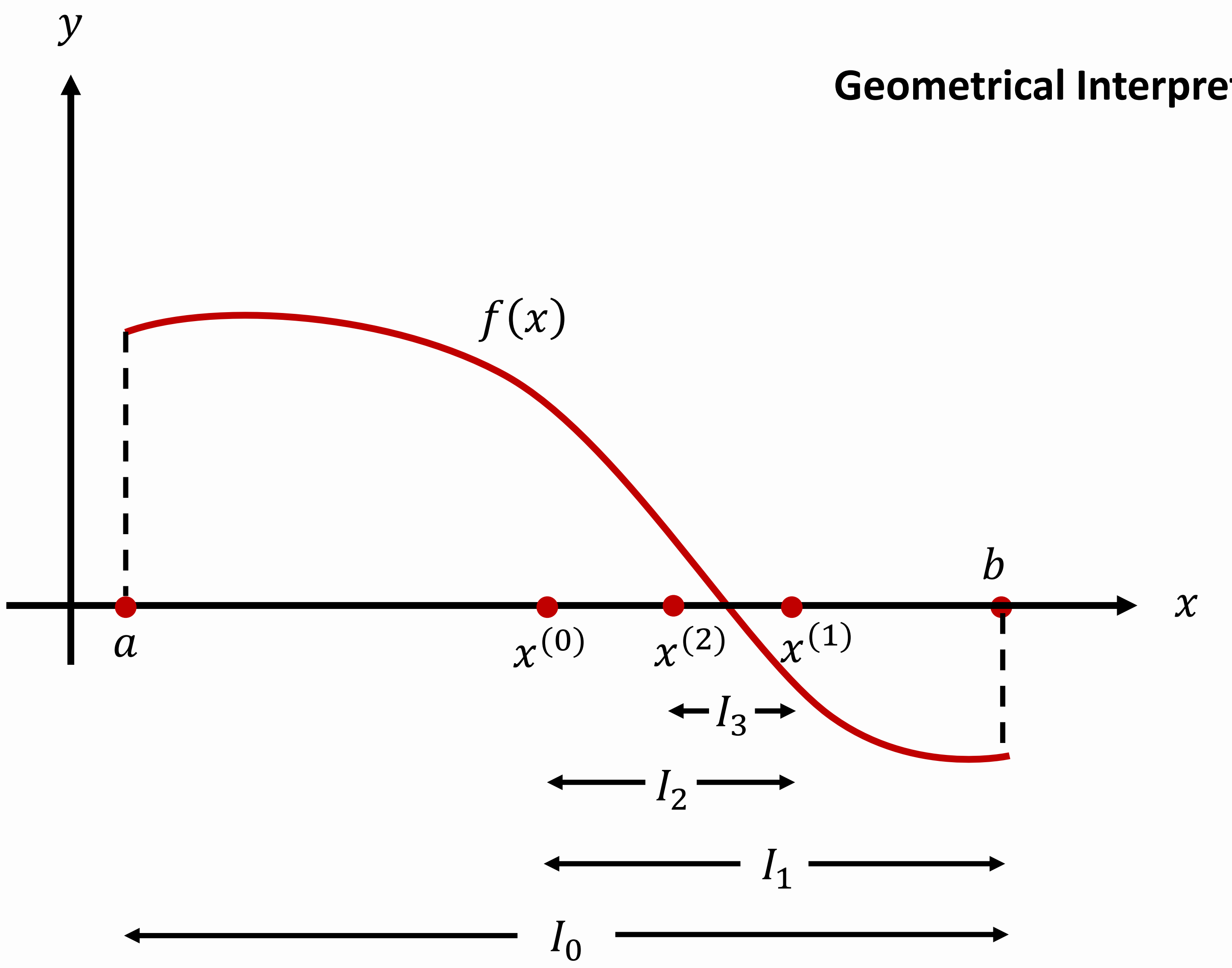
set $a^{(k+1)} = a^{(k)} \qquad b^{(k+1)} = x^{(k)}$

if $f\left(x^{(k)}\right) f\left(b^{(k)}\right) < 0$

set $a^{(k+1)} = x^{(k)} \qquad b^{(k+1)} = b^{(k)}$

Set $x^{(k+1)} = \dfrac{a^{(k+1)} + b^{(k+1)}}{2}$

Geometrical Interpretation of Bisection Method



Convergence of Bisection Method

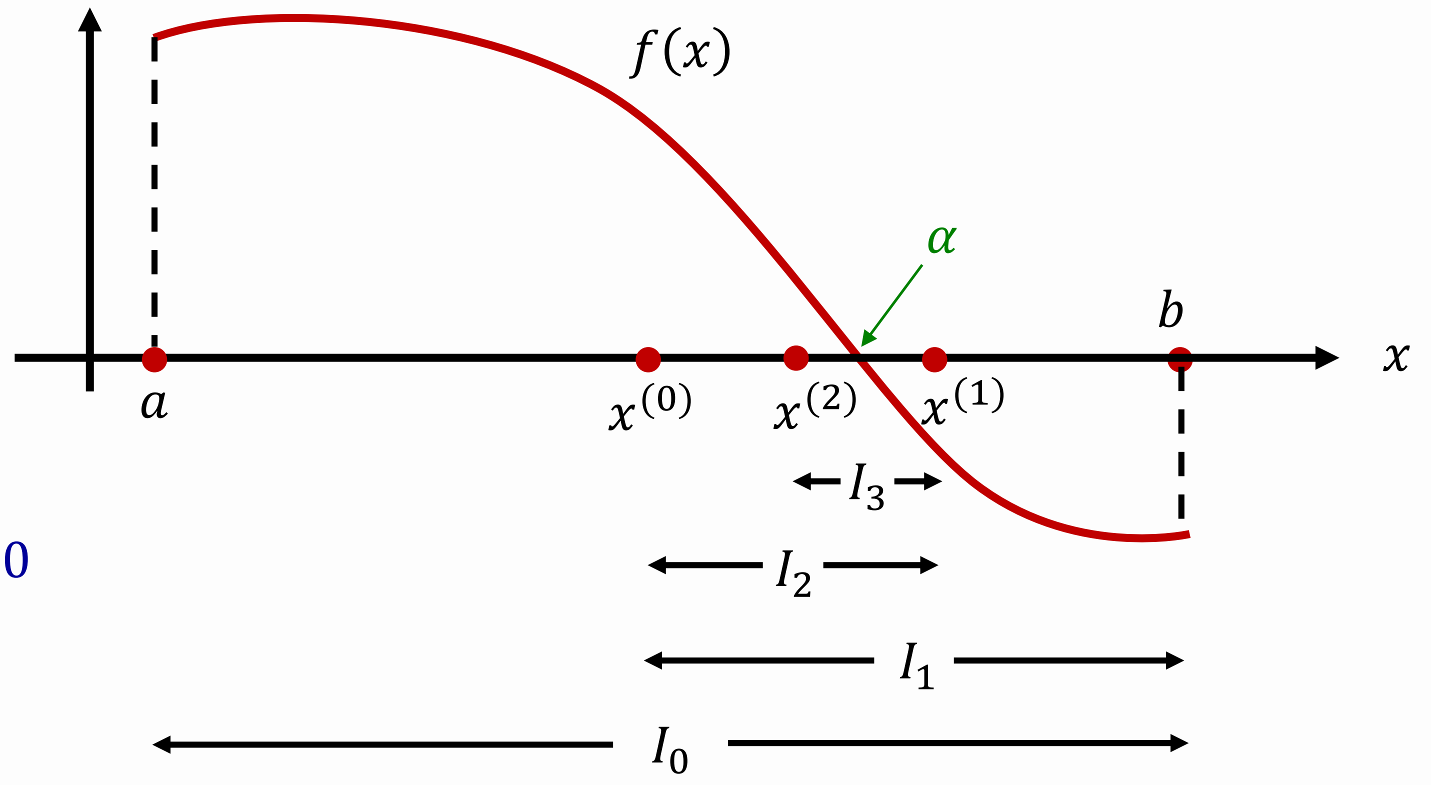
Let $|I_k| = \left|b^{(k)} - a^{(k)}\right|$ $f(\alpha) = 0$

Note that $|I_k| = \dfrac{|I_{k-1}|}{2}$

$$\implies |I_k| = \frac{|I_0|}{2^k};\ k \geq 0 \implies |I_k| = \frac{b-a}{2^k};\ k \geq 0$$

Denoting error $e^{(k)} = x^{(k)} - \alpha$

$$\implies \left|e^{(k)}\right| < \frac{|I_k|}{2} = \frac{(b-a)}{2^{(k+1)}};\ k \geq 0 \implies \lim_{k\to\infty}\left|e^{(k)}\right| = 0$$



The bisection method is globally convergent!

Example : Perform five iterations of the bisection method to obtain he smallest positive root of the equation

$$f(x) := x^3 - 5x + 1 = 0$$

Actual Roots:

$2.12841, \quad -2.33005, \quad 0.20163$

Solution : $f(0) = 1 \;\&\; f(1) = -3 \quad \Rightarrow f(0)f(1) < 0$

Initialization $\quad a^{(0)} = 0 \qquad b^{(0)} = 1 \qquad x^{(0)} = \dfrac{1+0}{2} = 0.5$

Observe $\; f\left(a^{(0)}\right) f\left(x^{(0)}\right) < 0$

$a^{(0)} = 0 \quad b^{(0)} = 1 \quad x^{(0)} = \frac{1+0}{2} = 0.5 \quad f\left(a^{(0)}\right) f\left(x^{(0)}\right) < 0$

Iteration	$a^{(k)}$	$x^{(k)}$	$b^{(k)}$	Observation
1	0 $(f > 0)$	0.25 $(f < 0)$	0.5 $(f < 0)$	$f\left(a^{(k)}\right) f\left(x^{(k)}\right) < 0$
2	0 $(f > 0)$	0.125 $(f > 0)$	0.25 $(f < 0)$	$f\left(x^{(k)}\right) f\left(b^{(k)}\right) < 0$
3	0.125 $(f > 0)$	0.1875 $(f > 0)$	0.25 $(f < 0)$	$f\left(x^{(k)}\right) f\left(b^{(k)}\right) < 0$
4	0.1875 $(f > 0)$	0.21875 $(f < 0)$	0.25 $(f < 0)$	$f\left(a^{(k)}\right) f\left(x^{(k)}\right) < 0$

$f(x) = x^3 - 5x + 1$

Root lies in (0.1875,0.21875)

Approximate root after 5 iterations:

$x^{(5)} = 0.203125$

Fixed Point Iteration Method:

Idea of general iteration method:

Rewrite $f(x) = 0$ to the form $x = g(x)$ and set up the iterations

$$x^{(k+1)} = g\left(x^{(k)}\right), \qquad k = 0, 1, 2, \ldots$$

Convergence of the method will depend on the function $g(x)$.

Remark: The point x^* is called a fixed point of the function g is $x^* = g(x^*)$.

$$f(x) = 0 \Leftrightarrow x = g(x)$$

Note that the choice of g is not unique. For instance, we me take:

$$g(x) = x - f(x)$$

$$g(x) = x + 2f(x)$$

$$g(x) = x - \frac{f(x)}{f'(x)} \quad \text{assuming } f'(x) \neq 0$$

Sufficient condition for convergence

If $g(x)$ is continuous in some interval $[a, b]$ that contains the root and $|g'(x)| \leq \rho < 1$ in this interval, then for any choice of $x^{(0)}$ from $[a, b]$ the sequence $x^{(k)}$ will converge to the root of the equation $f(x) = 0$.

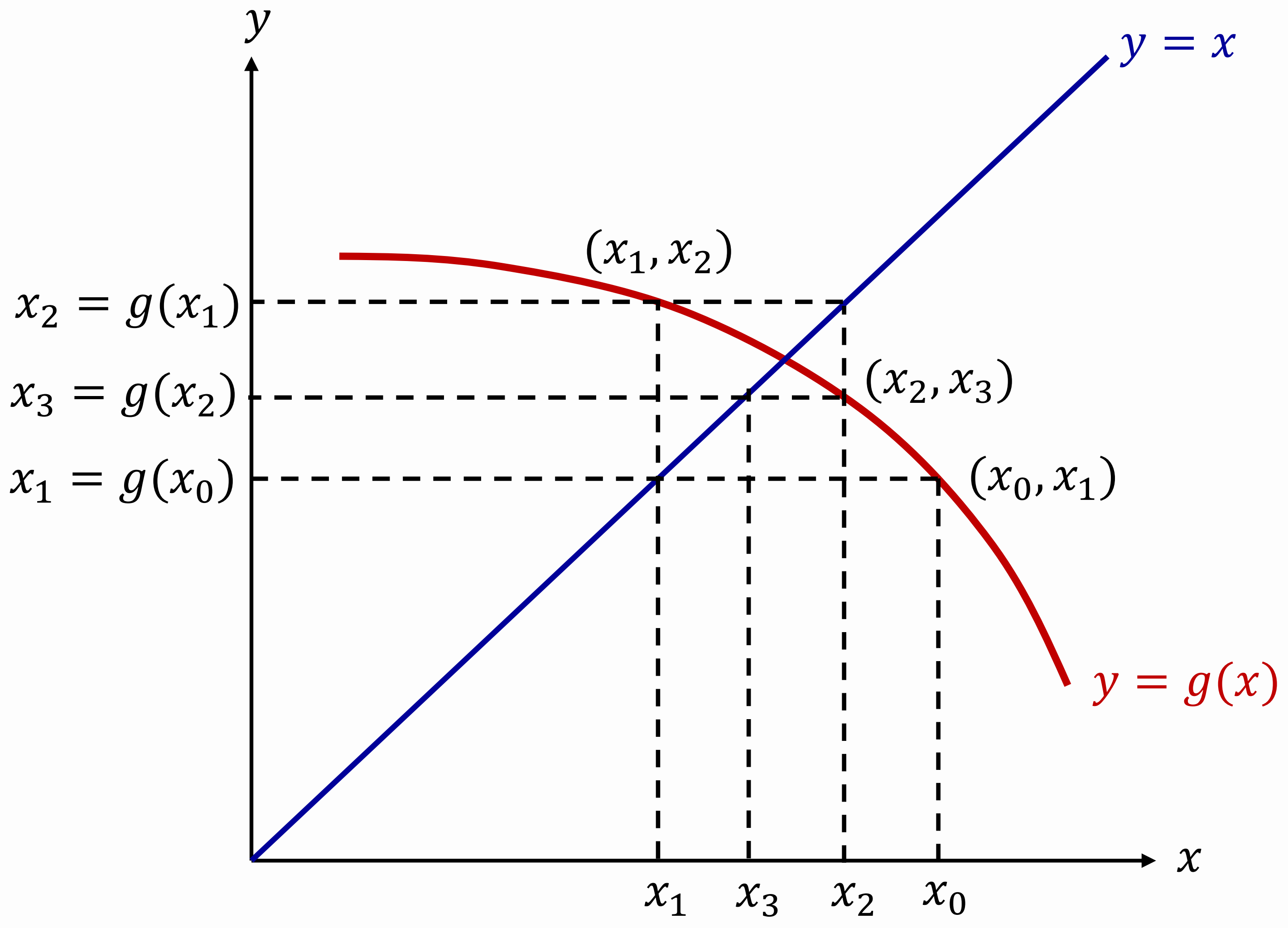
Proof: Consider $\left|x^{(k+1)} - x^*\right| = \left|g\left(x^{(k)}\right) - g(x^*)\right| = \left|g'(\xi)\left(x^{(k)} - x^*\right)\right|, \quad \xi \in \left(x^{(k)}, x^*\right)$ using MVT

Since $|g'(x)| \leq \rho$, we get

$$\left|x^{(k+1)} - x^*\right| \leq \rho\left|x^{(k)} - x^*\right| \leq \rho^2\left|x^{(k-1)} - x^*\right| \leq \cdots \leq \rho^{k+1}\left|x^{(0)} - x^*\right|$$

Since $\rho < 1$, we have $\rho^k \to 0$ as $k \to \infty$.

Geometrical Interpretation $x^{(k+1)} = g\left(x^{(k)}\right)$



Example : Consider $x^3 - 5x + 1 = 0$

Case 1: Rewrite the equation $x = g(x) = \dfrac{(1 + x^3)}{5}$

Iteration method becomes:

$$x^{(k+1)} = \frac{\left(1 + \left(x^{(k)}\right)^3\right)}{5} \qquad g'(x) = \frac{3x^2}{5}$$

Root lies in the interval $(0, 1)$ so we can choose $x^{(0)} = 0.5$ as initial guess

$x^{(0)} = 0.5$ $\quad$ $x^{(1)} = 0.2250$ $\quad$ $x^{(2)} = 0.2023$

$x^{(3)} = 0.2017$ $\quad$ $x^{(4)} = 0.2016$ $\quad$ $x^{(5)} = 0.2016$

Case 2: Now take initial guess $\boldsymbol{x^{(0)} = 2.5}$ in the above example.

$$x^{(k+1)} = \frac{\left(1 + \left(x^{(k)}\right)^3\right)}{5}$$

$x^{(1)} = 3.325$ $\quad$ $x^{(2)} = 7.552$

$x^{(3)} = 86.3419$ $\quad$ $x^{(4)} = 1.2873 \times 10^5$ $\quad$ $x^{(5)} = 4.2669 \times 10^{14}$

The iterations are diverging toward plus infinity.

Remark : Note that $g'(x) = \dfrac{3x^2}{5}$ in above both the cases.

- In case 1, in the interval containing the root and initial guess, $|g'| < 1$ and hence convergence is guaranteed.
- In case 2, in the interval containing the root and initial guess, $|g'| > 1$ and hence convergence is NOT guaranteed.

Case 3: Rewrite the equation as $x = g(x) = \dfrac{-1}{x^2 - 5}$

Now taking the initial guess $x^{(0)} = 2.5$, we get

$x^{(0)} = 2.5$ $\quad$ $x^{(1)} = -0.80$ $\quad$ $x^{(2)} = 0.2294$

$x^{(3)} = 0.2021$ $\quad$ $x^{(4)} = 0.2016$ $\quad$ $x^{(5)} = 0.2016$

Remark :

Note that, $|g'| = \dfrac{2|x|}{(x^2 - 5)^2}$

In the interval containing the root and initial guess $|g'| > 1$ but the sequence converges as this is the sufficient condition for convergence not necessary.

CONCLUSIONS

- **Bisection Method**

  The Bisection Method is an iterative approach that narrows down an interval that contains a root of the function $f(x)$. Convergence is always guaranteed.

- **Fixed Point Iteration Method** $\boldsymbol{f(x) = 0 \Leftrightarrow x = g(x)}$

  $$x^{(k+1)} = g\left(x^{(k)}\right), \qquad k = 0, 1, 2, \ldots$$

  Convergence is guaranteed if $|g'(x)| \leq \rho < 1$