LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

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Determination of Roots of Algebraic and Transcendental Equations

- ❑ **Bisection Method**
- ❑ **Fixed Point Iteration Method**
- ➢ **Newton-Raphson Method**
- ➢ **Secant Method**

Newton-Raphson Method

In the triangle $x_1 P x_0$:

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

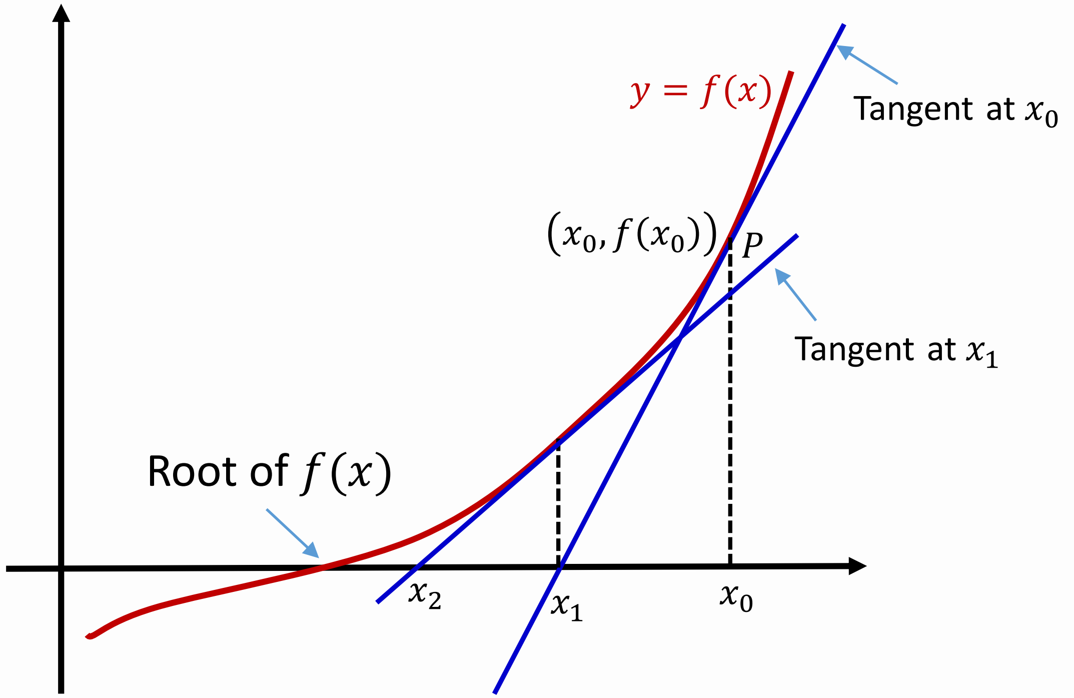
$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, the second step :

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\vdots$$

$(k+1)th$ step: $\quad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)};\ \ k = 0, 1, 2, \ldots$



Alternative Formulation:

Let x_k be an approximation to the solution of $f(x) = 0$.

Let Δx be an increment in x_k such that $x_k + \Delta x$ is an exact root,

i.e.

$$f(x_k + \Delta x) = 0$$

$$\Rightarrow f(x_k) + \Delta x f'(x_k) + \frac{1}{2!}\Delta x^2 f''(x_k) + \cdots = 0$$

Neglecting 2nd and higher order terms of Δx, we get

$$f(x_k) + \Delta x f'(x_k) \approx 0 \quad \Rightarrow \Delta x \approx -\frac{f(x_k)}{f'(x_k)}$$

Hence, the iteration method becomes

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}; \quad k = 0, 1, 2, \ldots$$

Recall Fixed Point Iteration Method:

$$f(x) = 0 \Leftrightarrow x = g(x)$$

$$x_{k+1} = g(x_k), \qquad k = 0, 1, 2, \ldots$$

Convergence is guaranteed if $|g'(x)| \leq \rho < 1$

If we take $g(x) = x - \dfrac{f(x)}{f'(x)}$ assuming $f'(x) \neq 0$

Then $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$ **Newton-Raphson Method**

Convergence of Newton-Raphson Method

Let $x_{k+1} = g(x_k)$ define the newton's method. Let s be a root of $f(x) = 0$, i.e., $s = g(s)$.

Note that $g(x) = x - \dfrac{f(x)}{f'(x)} \implies g'(x) = 1 - \dfrac{\left(f'(x)\right)^2 - f(x)f''(x)}{\left(f'(x)\right)^2}$

$$g'(s) = 1 - \left(\frac{\left(f'(s)\right)^2 - f(s)f''(s)}{\left(f'(s)\right)^2}\right) = \frac{f(s)f''(s)}{(f'(s))^2} = 0 \text{ as } f(s) = 0.$$

Using Taylor's formula for expanding $g(x_k)$ around s in the scheme $x_{k+1} = g(x_k)$:

$$x_{k+1} = g(x_k) = g(s) + g'(s)(x_k - s) + \frac{1}{2}g''(\xi)(x_k - s)^2 \quad \text{where } \xi \in (x_k,\ s)$$

Convergence of Newton-Raphson Method

$$x_{k+1} = g(x_k) = g(s) + g'(s)(x_k - s) + \frac{1}{2} g''(\xi)(x_k - s)^2 \quad \text{where } \xi \in (x_k,\ s)$$

$$\Rightarrow x_{k+1} - s = \frac{1}{2} g''(\xi)(x_k - s)^2$$

$$\Rightarrow e_{k+1} = \frac{1}{2} g''(\xi)\, {e_k}^2$$

Each successive error term is proportional to the square of the previous error.

Hence, Newton-Raphson method converges quadratically.

Note: In the case of fixed point iteration method $g'(x) \neq 0$ (in general), and hence the method converges linearly.

Moreover the size of $|g'(x)|$ matters and it has to be less than 1 for convergence. Note that $g'(s) = 0$ in the case of Newton's method and therefore convergence is guaranteed for x_0 sufficiently close to s.

As discussed, in the case of Newton's, the method converges quadratically for x_0 sufficiently close to s.

Example : Perform four iterations of the Newton-Raphson method to find the smallest positive root of the equation $f(x) = x^3 - 5x + 1 = 0$.

Solution: Take $x_0 = 0.5$

$$f'(x) = 3x^2 - 5$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{(x_k^3 - 5x_k + 1)}{3x_k^2 - 5}$$

$$= \frac{2x_k^3 - 1}{3x_k^2 - 5}; \quad k = 0, 1, 2, \ldots$$

$x_0 = 0.5$

$x_1 = 0.176470588$

$x_2 = 0.2015680743$

$x_3 = 0.2016396750$

$x_4 = 0.2016396757$

Example : Apply Newton-Raphson method to determine a root of the equation $f(x) = \cos x - xe^x = 0$ such that $|f(x^*)| < 10^{-8}$ where x^* is the approximation to the root. Take the initial approximation as $x_0 = 1$.

Iteration Scheme : $$x_{k+1} = x_k - \frac{(\cos x_k - x_k e^{x_k})}{(-\sin x_k - e^{x_k} - x_k e^{x_k})}$$

k	0	1	2	3	4	5
x_k	1	0.6531	0.5313	0.5179	0.5178	0.5178
$f(x_k)$	-2.1780	-0.4606	-0.0418	-4.6×10^{-4}	-5.9×10^{-8}	-8.8×10^{-16}

Secant Method :

Note that the newton's method is very powerful but it has the disadvantage of evaluating f' which may be computationally very expensive.

The Secant method is a variant of Newton's method where $f'(x_k)$ is replaced by the following differences:

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

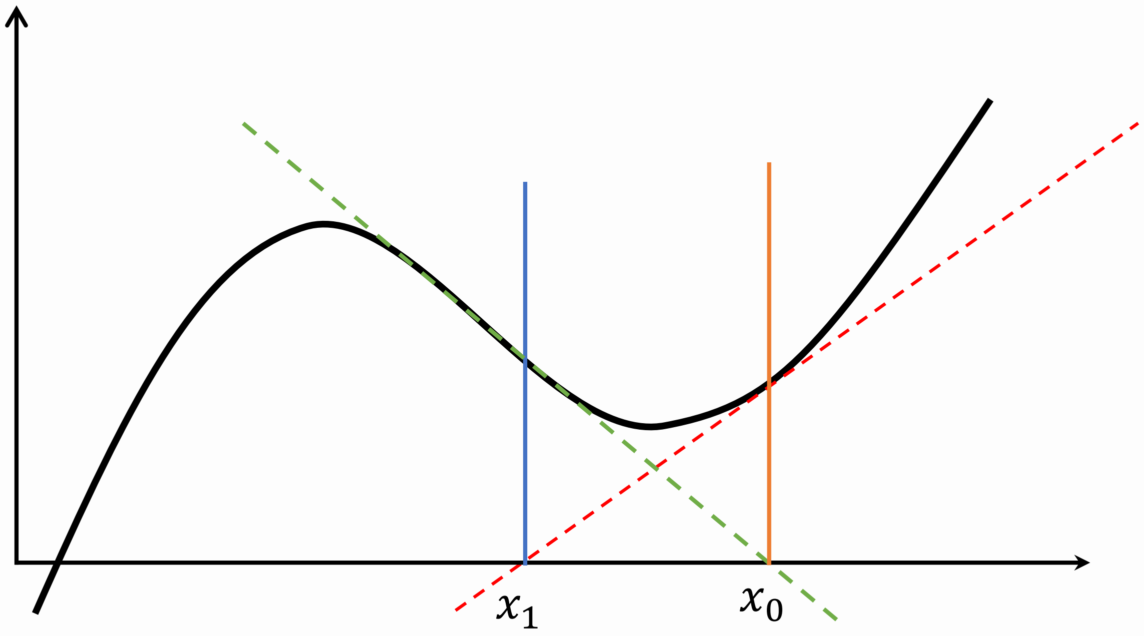
$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}; \quad k = 0, 1, 2, \ldots$$

Pitfalls: Newton-Raphson Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}; \quad k = 0,1,2, \dots .$$

1. The method fails if f' becomes 0 at any approximation $x_k, k = 0,1,2, \dots$

2. Cycling behavior leads to complete failure of the method



Pitfalls: Newton-Raphson Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)};\quad k = 0,1,2,\dots.$$

3. Consider $f(x) = x^3 - 23x^2 + 135x - 225$

Actual Roots are $3, 5, 15,$

Initial Guess	**4**	**4.2**	**3.9**
Iteration 1	15	6.1636	−5.0341
Iteration 2	15	5.2223	−1.3851
Iteration 3		5.0159	0.8586
Iteration 4		5.0001	2.1420
Iteration 5		5.0000	2.7697
Iteration 6		5.0000	2.9749
Iteration 7			2.9996
Iteration 8			3.0000
Iteration 9			3.0000

Pitfalls: Newton-Raphson Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}; \quad k = 0,1,2, \dots.$$

4. Consider $f(x) = x^3 + 94x^2 - 389x + 294$

Actual Roots are: $1, 3, -98$

Initial Guess	$\mathbf{2}$	$\mathbf{2.4}$	$\mathbf{3.9}$
Iteration 1	-98	3.4611	0.2061
Iteration 2	-98	3.0742	0.8282
Iteration 3		3.0026	0.9877
Iteration 4		3.0000	0.9999
Iteration 5		3.0000	1.0000
Iteration 6			1.0000

CONCLUSIONS

- **Newton-Raphson Method**

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}; \quad k = 0, 1, 2, \ldots$$

- **Secant Method**

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}; \quad k = 0, 1, 2, \ldots$$