ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

Dr. Jitendra Kumar

Professor
Department of Mathematics
Indian Institute of Technology Kharagpur
West Bengal 721302, India

Webpage: http://www.facweb.iitkgp.ac.in/~jkumar/

Integral Calculus

Improper Integrals

- Absolute Convergence

Absolute Convergence

The integral $\int_0^\infty f(x)\,dx$ converges absolutely $\Leftrightarrow$ $\int_0^\infty |f(x)|\,dx$ converges

The integral $\int_0^\infty f(x)\,dx$ converges conditionally $\Leftrightarrow$ It converges but not absolutely

Problem – 3: The Integral $\int_1^\infty \frac{\sin x}{x^p}\,dx$ converges absolutely for $p > 1$.

Note that $\frac{|\sin x|}{x^p} \leq \frac{1}{x^p}, \qquad p > 1$

Recall that $\int_1^\infty \frac{1}{x^p}\,dx$ converges

By comparison test $\int_1^\infty \left|\frac{\sin x}{x^p}\right|\,dx$ converges

Theorem: $\int_a^{\infty} f(x)dx$ converges if $\int_a^{\infty} |f(x)|dx$ converges but the converse is not true.

Example: $\int_0^{\infty} \frac{\sin x}{x} dx$ converges conditionally

Note that $$\int_0^{\infty} \frac{\sin x}{x} dx = \underbrace{\int_0^{1} \frac{\sin x}{x} dx}_{\text{Proper}} + \underbrace{\int_1^{\infty} \frac{\sin x}{x} dx}_{\text{Example -1}}$$

$\Rightarrow$ The integral $\int_0^{\infty} \frac{\sin x}{x} dx$ conveges

Now we will show that $\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx$ does not converge

$$\sin(n\pi + y) = (-1)^n \sin y$$

$$\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx = \sum_{n=0}^{\infty} \int_0^{\pi} \frac{|\sin(n\pi + y)|}{n\pi + y} dy$$

Subst. $x = n\pi + y$

$$= \sum_{n=0}^{\infty} \int_0^{\pi} \frac{|(-1)^n \sin y|}{(n\pi + y)} dy = \sum_{n=0}^{\infty} \int_0^{\pi} \frac{\sin y}{(n\pi + y)} dy \geq \sum_{n=0}^{\infty} \int_0^{\pi} \frac{\sin y}{(n\pi + \pi)} dy = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n+1}$$

divergent series

Hence the improper integral $\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx$ diverges

KEY TAKEAWAY

Dirichlet's Test:

$$\left|\int_a^b f(x)dx\right| \le C \quad \text{for all } b > a,$$

g is monotone decreasing to zero as $x \to \infty$

$$\int_a^\infty f(x)g(x)dx \quad \text{converges.}$$

Absolute Convergence

$$\int_0^\infty \frac{\sin x}{x} dx \ \text{ does not convege absolutely}$$

REMARKS

Integral of the type: $$\int_{-\infty}^{b} f(x)\,dx$$

Substitute $x = -t$:

$$\int_{-b}^{\infty} f(-t)\,dt$$

Improper Integrals

- **Convergence: Type-II Integrals**

Recall (Previous Lectures)

Test Integral

$$\int_a^b \frac{1}{(x-a)^p}\,dx \quad \text{converges for } p<1 \quad \& \quad \text{diverges if } p \geq 1$$

Convergence: Type - II Integrals

$$\int_{a^+}^{b} f(x)dx$$

$f(x)$ becomes unbounded at $x = a$

For the case

$$\int_{a}^{b^-} f(x)\,dx$$

We can set $x = b - t$ and get

$$\int_{0+}^{b-a} f(b-t)dt$$

Comparison Test-I

Suppose $0 \leq f \leq g, \quad a < x \leq b$, then

- $\int_{a^+}^{b} f(x)dx$ converges if $\int_{a^+}^{b} g(x)\,dx$ converges

- $\int_{a^+}^{b} g(x)dx$ diverges if $\int_{a^+}^{b} f(x)\,dx$ diverges

Comparison Test-II (limit Comparison test):

Suppose $0 \leq f \leq g, \quad a < x \leq b$ $\qquad \lim_{x \to a^+} \frac{f(x)}{g(x)} = k$

If $k \neq 0$ then both the integrals $\int_{a^+}^{b} f(x)dx$ and $\int_{a^+}^{b} g(x)\, dx$ behave the same

Further, if $k = 0$ and $\int_{a^+}^{b} g(x)dx$ converges then $\int_{a^+}^{b} f(x)dx$ converges

If $k = \infty$ and $\int_{a^+}^{b} g(x)dx$ diverges then $\int_{a^+}^{b} f(x)dx$ diverges

$\boldsymbol{\mu}$ − test Comparison test (II) with $g(x) = \dfrac{1}{(x-a)^{\mu}}$

- if $\exists\ 0 < \mu < 1$ such that $\lim_{x \to a+} (x-a)^{\mu} f(x)$ exsits then $\int_{a+}^{b} f(x)\,dx$ conveges absolutely

- if $\exists\ \mu \geq 1$ such that $\lim_{x \to a+} (x-a)^{\mu} f(x)$ exsits $(\neq 0, \text{it may be} \pm \infty)$ then $\int_{a+}^{b} f(x)\,dx$ diverges

Dirichlet's Test:

- $$\left| \int_{a+\epsilon}^{b} f(x)dx \right| < C, \qquad \forall \quad b > a,$$

- g is monotone, bounded and $\lim_{x \to a^+} g(x) = 0$

Then $\int_{a^+}^{b} f(x)g(x)\, dx$ conveges

Problem – 1: Test the convergence of $\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$

Note that the integrand is unbounded at upper end.

Set $3 - x = t$ implies $dx = -dt$

$$\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}} = \int_0^3 \frac{dt}{t\sqrt{(3-t)^2+1}}$$

Convergence of $\displaystyle\int_0^3 \frac{dt}{t\sqrt{(3-t)^2+1}}$

Take $g(t) = \dfrac{1}{t}$

Note that $$\lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{t\to 0}\frac{1}{\sqrt{(3-t)^2+1}} = \frac{1}{\sqrt{10}}$$

$\Rightarrow \displaystyle\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$ diverges since $\displaystyle\int_0^3 \frac{1}{t}\,dt$ diverges.

Problem – 2: Test the convergence of $\displaystyle\int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}\,dx$

Notice: $$\left|\frac{\sin x}{\sqrt[3]{x-\pi}}\right| \le \frac{1}{\sqrt[3]{x-\pi}}$$

and $\displaystyle\int_{\pi}^{4\pi} \frac{1}{\sqrt[3]{x-\pi}}\,dx$ converges

$\Rightarrow \displaystyle\int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}\,dx$ converges absolutely.

Note: Improper integrals of the third kind can be expressed in terms of improper integrals of the first and second kind.

Problem – 3: Test the convergence of $\int_1^{\infty} \frac{1}{x\sqrt{x-1}}\,dx$

$$\int_1^{\infty} \frac{1}{x\sqrt{x-1}}\,dx = \int_1^{2} \frac{1}{x\sqrt{x-1}}\,dx + \int_2^{\infty} \frac{1}{x\sqrt{x-1}}\,dx$$

Functions for comparison

$$g_1 = \frac{1}{\sqrt{x-1}} \qquad g_2 = \frac{1}{x^{3/2}}$$

Both converge by comparison test

Remark: One needs to be careful to evaluate the improper integral where the integrand is not defined or not bounded at an interior point of the of the range of the integral.

Consider $\int_a^b f(x)\,dx$ Suppose $f(x)$ is unbounded at a point c, where $a < c < b$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{\epsilon \to 0^+}\left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx\right]$$

OR $$= \lim_{\epsilon_1 \to 0^+} \int_a^{c-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2 \to 0^+} \int_{c+\epsilon_2}^b f(x)\,dx$$

Consider $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\, dx = \int_{-1}^{0} \frac{1}{x^3}\, dx + \int_{0}^{1} \frac{1}{x^3}\, dx$

$$= \lim_{\epsilon \to 0^+} \left[\int_{-1}^{-\epsilon} \frac{1}{x^3} dx + \int_{\epsilon}^{1} \frac{1}{x^3} dx \right] = \lim_{\epsilon \to 0^+} \left[\left(-\frac{1}{2}\right)\left(\frac{1}{\epsilon^2} - 1\right) + \left(-\frac{1}{2}\right)\left(1 - \frac{1}{\epsilon^2}\right) \right] = 0$$

$$= \lim_{\epsilon_1 \to 0} \int_{-1}^{-\epsilon_1} \frac{1}{x^3} dx + \lim_{\epsilon_2 \to 0} \int_{\epsilon_2}^{1} \frac{1}{x^3} dx$$

Both improper integrals do not exist!

Conclusion:

Comparison Test -I: Let $0 \leq f(x) \leq g(x), a < x \leq b$

$$\int_{a^+}^{b} g(x)dx \ \ \text{conveges} \Longrightarrow \int_{a^+}^{b} f(x)dx \ \ \text{conveges}$$

$$\int_{a^+}^{b} f(x)dx \ \ \text{diverges} \Longrightarrow \int_{a^+}^{b} g(x)dx \ \ \text{diverges}$$

Conclusion:

Comparison Test -II: Let $0 \le f(x) \le g(x), a < x \le b$

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \mathrm{k}$$

if $k \neq 0$ then $\int_{a^+}^{b} f(x)dx$ and $\int_{a^+}^{b} g(x)dx$ behave the same

if $k = 0$ & $\int_{a^+}^{b} g(x)dx$ conveges $\Longrightarrow \int_{a^+}^{b} f(x)dx$ conveges

if $k = \infty$ & $\int_{a^+}^{b} g(x)dx$ diverges $\Longrightarrow \int_{a^+}^{b} f(x)dx$ diverges

QUIZ QUESTION ?

LINK FOR RESPONSES: http://www.facweb.iitkgp.ac.in/~jkumar/teach/MA11003.html

QUIZ QUESTION ?

Let

$$\int_{1}^{\infty} \frac{1}{x\sqrt{x-1}}\, dx = \alpha\,\pi, \qquad \alpha \in \mathbb{R}$$

The value of α is______________

Thank You