ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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Concepts Covered

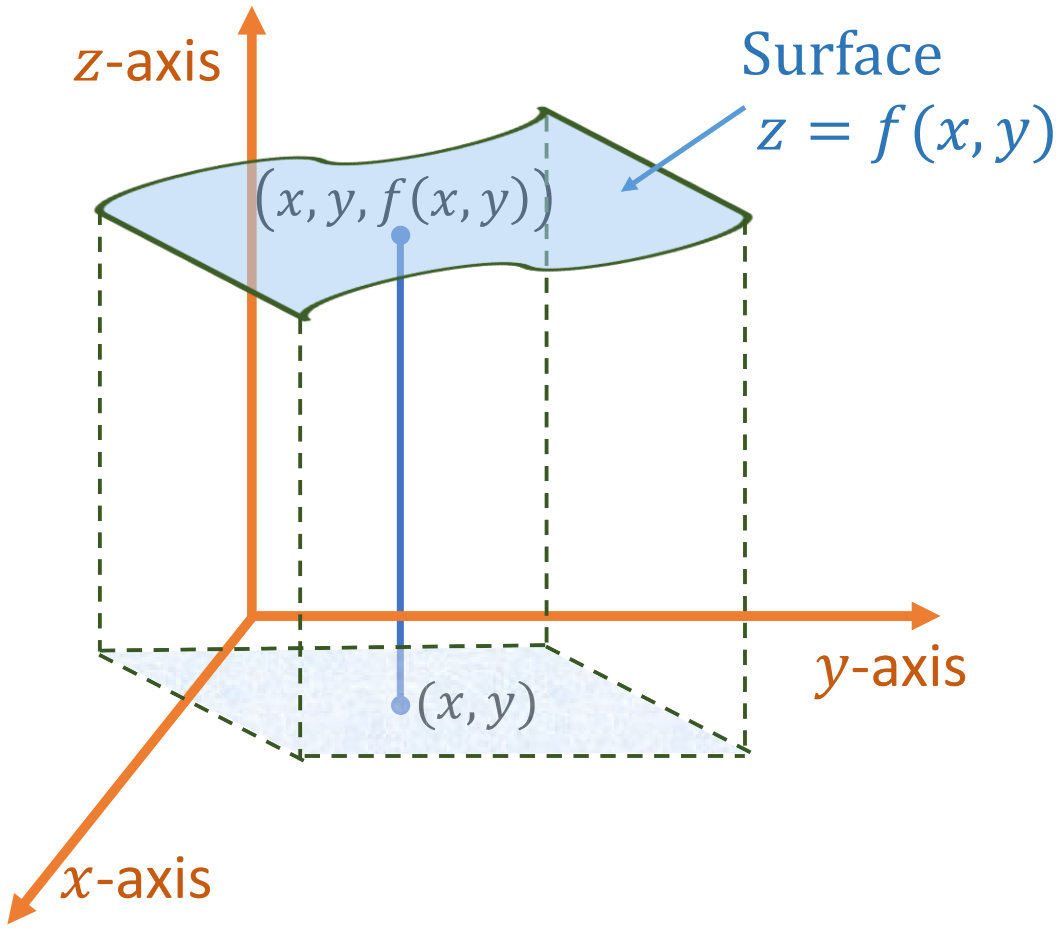
Differential Calculus

Functions of Several Variables

- Introduction
- Limit

Functions of Two Variables

A function $z = f(x,y)$ is a real valued function of two variables x & y if to each point (x,y) of a certain part of x-y plane corresponds to a real value z according to some given rule $f(x,y)$.



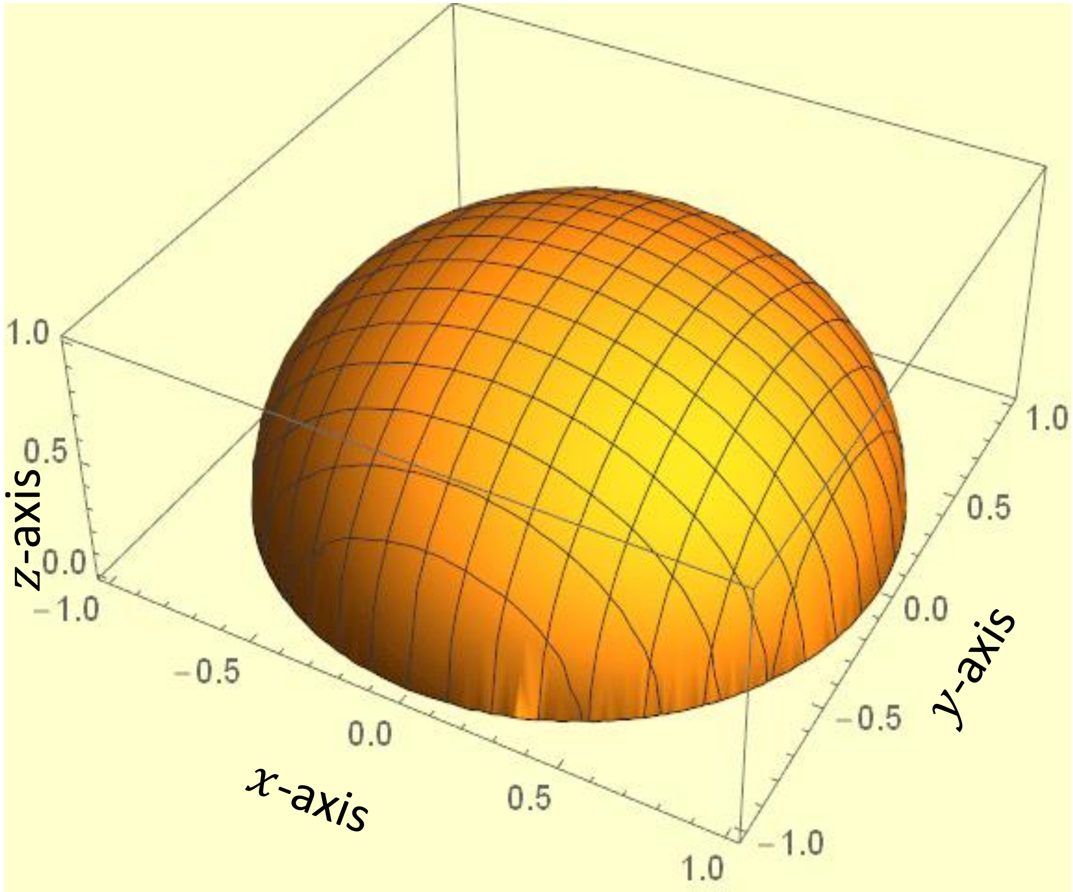
Domain: The set of points (x,y) in the x-y plane for which $f(x,y)$ is defined

Range: Collection of all possible value of z corresponding to the points (x,y)

$x,y \rightarrow$ independent variables

$z \rightarrow$ dependent variable

Functions of Two Variables



Example: $z = \sqrt{1 - x^2 - y^2}$

Since z is real, we must have $(1 - x^2 - y^2) \geq 0$

$\Longrightarrow x^2 + y^2 \leq 1$

Therefore, Domain:

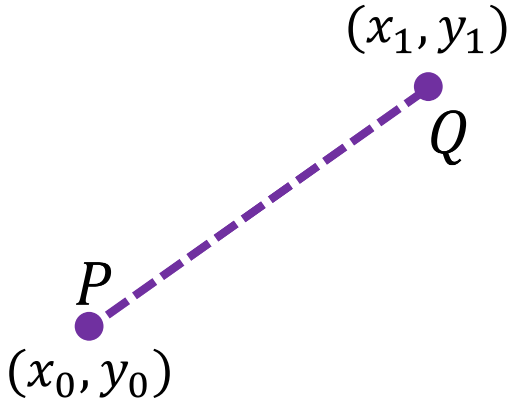
$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

Range:

R$= \{z \in \mathbb{R}, 0 \leq z \leq 1\}$

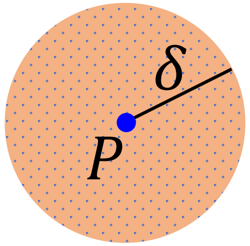
Distance between the two points

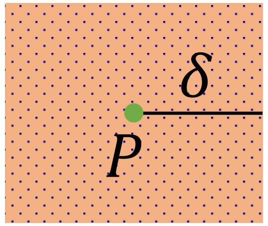
Distance $|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$



Neighborhood of a point $P(x_0, y_0)$

δ-neighborhood of P $(N_\delta(P)\;\; OR\; N(P, \delta))$





$$N_\delta(P) := \left\{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}$$

$$N_\delta(P) := \{(x, y) : x_0 - \delta < x < x_0 + \delta, \;\; y_0 - \delta < y < y_0 + \delta\}$$

Limit of a Function of One Variable (Recall)

We say $\lim_{x \to x_0} f(x) = L$, if for every $\epsilon > 0$,

there exists $\delta > 0$, such that $\forall\, x$,

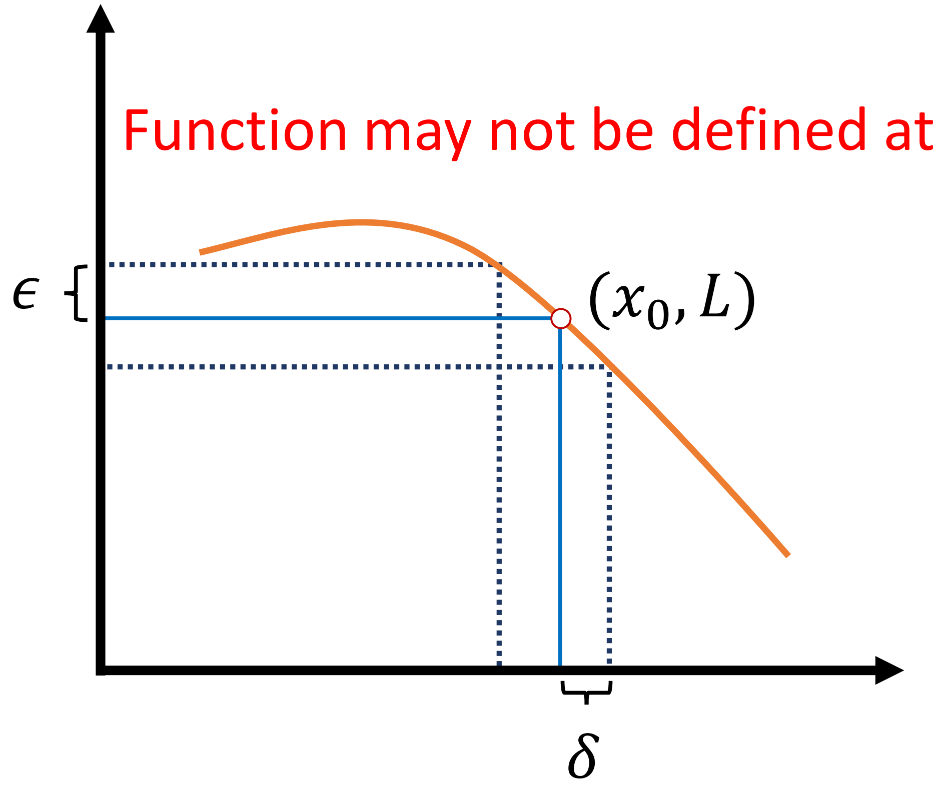
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$

In other words,

If we can make the difference $|f(x) - L|$ as small as we like by considering a small enough neighborhood around x_0, then we say that

$$\lim_{x \to x_0} f(x) = L$$

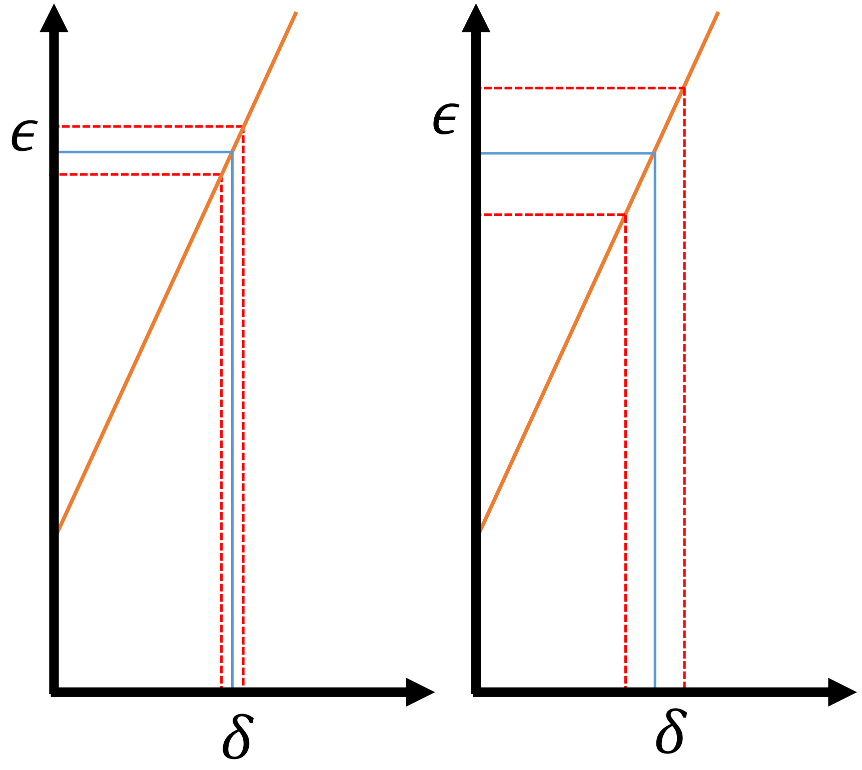
Function may not be defined at $x = x_0$



Example: $\lim_{x \to 1}(3x+1) = 4$

show that for a given $\epsilon > 0$, there exist a δ so that

$$|x-1| < \delta \quad \Longrightarrow \quad |(3x+1)-4| < \epsilon$$



We start with the difference

$$|(3x+1)-4| = |3x-3| = 3|x-1| < 3\delta \leq \epsilon$$

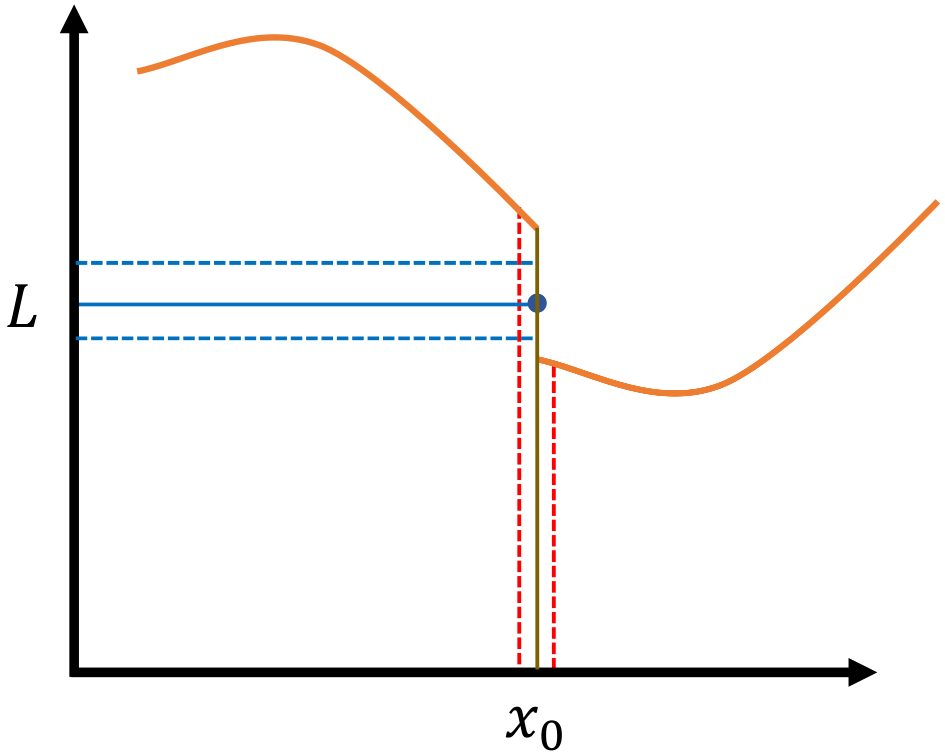
If we choose $\delta \leq \frac{\epsilon}{3}$ Then for any given ϵ, we have

$$|(3x+1)-4| < \epsilon \quad \text{whenever} \quad |x-1| < \delta$$

Non-Existence of Limit

For a given ϵ, there does not exist any δ such that

$|f(x) - L| < \epsilon$ whenever $0 < |x - x_0| < \delta$



Existence of Limit

$\lim_{x \to x_0} f(x) = L$ means that every neighborhood $N_\epsilon(L)$ of L there some neighborhood $N_\delta(x_0)$ s.t.

$f(x) \in N_\epsilon(L)$ whenever $x \in N_\delta(x_0), x \neq x_0$

Limit of Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables defined in a domain D. Let $P(x_0, y_0)$ be a point in D. If for a given real number $\epsilon > 0$, we can find a real number $\delta > 0$ such that for every point (x, y) in the δ-neighborhood of $P(x_0, y_0)$ satisfies $|f(x, y) - L| < \epsilon$, i.e.,

$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

(function may not be defined at (x_0, y_0))

Then the real number L is called the limit of the function $f(x, y)$ as $(x, y) \to (x_0, y_0)$

Symbolically, $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$

Problem - 1 $\lim_{(x,y)\to(0,0)}(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)=0$

For $(x, y) \neq (0,0)$, consider

$$\left|(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)-0\right| = (x^2+y^2)\left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \leq (x^2+y^2) < \delta^2 \leq \epsilon$$

Neighborhood of $(0,0)$: $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$

For given ϵ if we choose $\delta^2 \leq \epsilon$, then $\left|(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)-0\right| < \epsilon$

Problem - 2 $\lim_{(x,y)\to(0,0)}\left(\frac{xy}{\sqrt{x^2+y^2}}\right)=0$

For $(x,y)\neq(0,0)$, consider

$$\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|=\left|\frac{xy}{\sqrt{x^2+y^2}}\right|=\frac{|xy|}{\sqrt{x^2+y^2}}\leq\frac{x^2+y^2}{\sqrt{x^2+y^2}}\leq\sqrt{x^2+y^2}<\delta\leq\epsilon$$

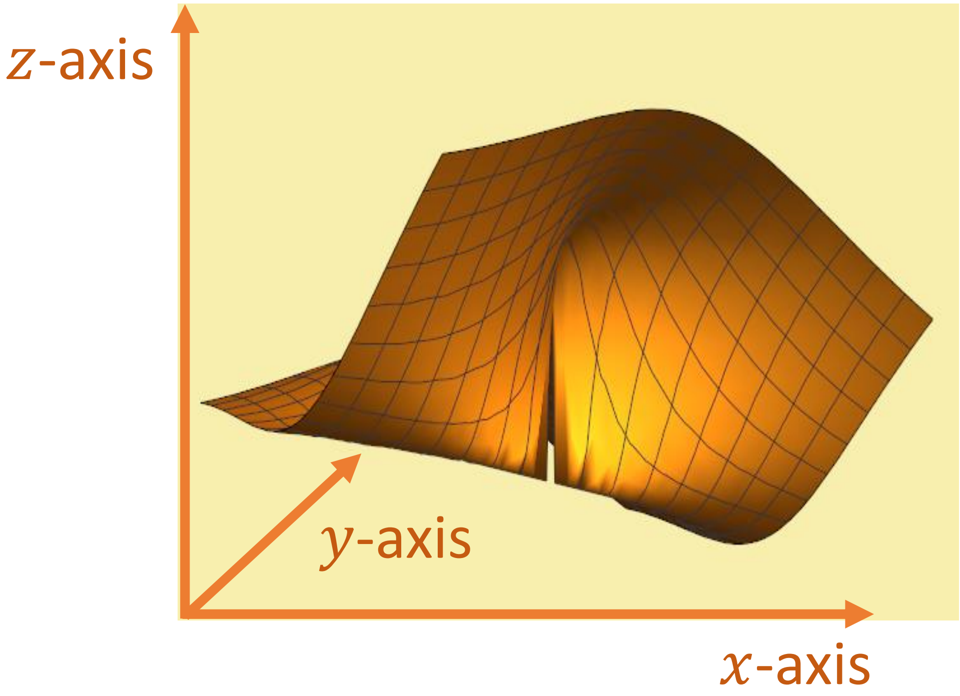
Neighborhood of $(0,0)$: $0<\sqrt{(x-0)^2+(y-0)^2}<\delta$

For given ϵ if we choose $\delta\leq\epsilon$, then $\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|<\epsilon$

Conclusion:

- Functions of Two Variables

$$Z = f(x, y)$$



- Definition of limit $(\epsilon - \delta)$

We need to have some idea about the limit L and then it may be used to verify that L is the limit

Concepts Covered

Differential Calculus

Functions of Several Variables

- ❑ Evaluation of Limit

Limit (Previous Lecture)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

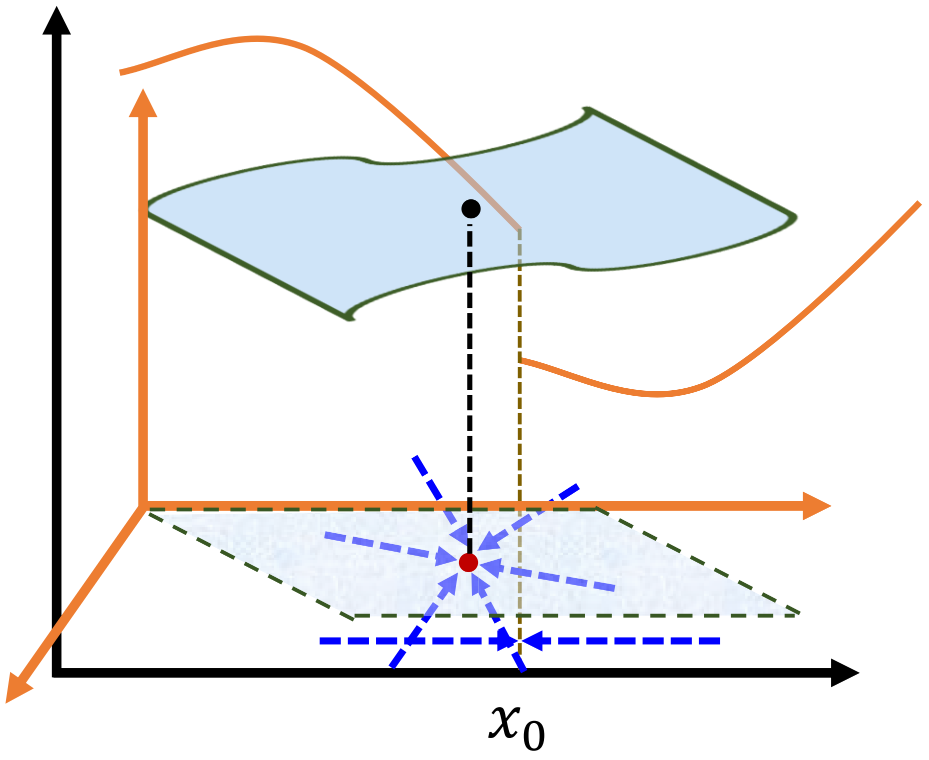
If for a given real number $\epsilon > 0$, we can find a real number $\delta > 0$ such that

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

Note: $\epsilon - \delta$ approach is useful for verifying that the given number L is the limit

Evaluation of Limit

Remark: Note that $(x, y) \to (x_0, y_0)$ in the two dimensional plane, there are infinite number of paths joining (x, y) to (x_0, y_0).



Since the limit, if exists, is unique, the limit should be the same along all the paths. Thus, the limit cannot be obtained by approaching the point P along a particular path and finding the limit of $f(x, y)$.

If the limit is dependent on a path, then the limit does not exist.

Example 1: $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} \dfrac{x^2y}{x^4+y^2}$ $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=x^2}} \dfrac{x^2y}{x^4+y^2}$

Along $y = x$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = 0$$

Along $y = x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \frac{1}{2}$$

Limit does not exist in this case!

Example 2: $\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right)$

Fix $y = 1$ and approach along x to 0

$$\lim_{x\to 0-0} \tan^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2}$$

$$\lim_{x\to 0+0} \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

The limit depends on path and hence it does not exist.

Example 3: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Along $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{m}{1+m^2}$$

The limit depends on path and hence it does not exist.

Example 4: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ **Alternative Approach**

Change of coordinate system from Cartesian to Polar

$$x = r\cos\theta \quad \& \quad y = r\sin\theta$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \cos\theta\sin\theta$$

The limit depends on the angle θ and hence it does not exist.

Example 5: $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$

Change of coordinate system from Cartesian to Polar

$$x = r\cos\theta \qquad \& \qquad y = r\sin\theta$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$$ No dependency on θ

Hence the limit exists in this case.

Example 6: $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(3x+6y)}$

Set $(x+2y)=t$

$$\lim_{(x,y)\to(0,0)} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(3x+6y)} = \lim_{t\to 0} \frac{\sin^{-1}(t)}{\tan^{-1}(3t)}$$

Using L'Hospital's rule $$= \lim_{t\to 0} \frac{\frac{1}{\sqrt{1-t^2}}}{\left(\frac{3}{1+9t^2}\right)} = \frac{1}{3}$$

Working with Limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$$

$$\lim_{(x,y)\to(x_0,y_0)} [k\, f(x,y)] = k\, L_1$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L_1 \pm L_2$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\, g(x,y)] = L_1\, L_2$$

$$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L_1}{L_2} \qquad \text{Provided} \quad \mathrm{L}_2 \neq 0$$

Working with Limits (generalization)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \infty \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = \infty.$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)g(x,y)] = \infty \qquad \lim_{(x,y)\to(x_0,y_0)} [f(x,y)+g(x,y)] = \infty$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \infty \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = -\infty.$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)g(x,y)] = -\infty$$

Working with Limits (generalization)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \infty \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L \text{ (finite real number).}$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = \infty$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \infty \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L\ (> 0).$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)g(x,y)] = \infty$$

Working with Limits (generalization)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \infty \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L\ (< 0).$$

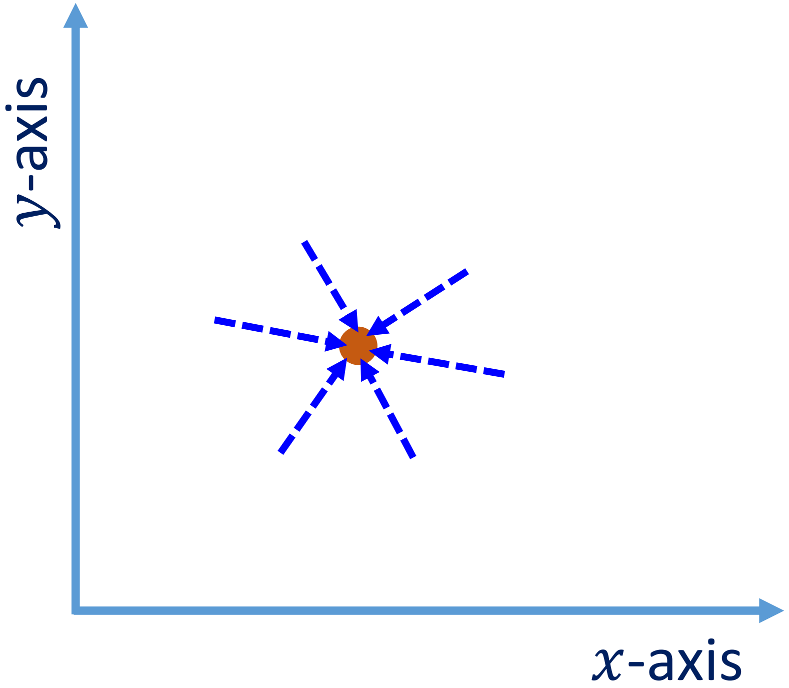
$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)g(x,y)] = -\infty$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \infty \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L\,.$$

$$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{g(x,y)}{f(x,y)}\right] = 0$$

Conclusion:

LIMIT



Changing to polar coordinate is often useful for evaluation of limit

Thank You