Algebraic and geometric multiplicity of a characteristic root

If λ_1 be a c. root of order t of the c. eqn. $|A-\lambda I|=0$ then t is called the algebraic multiplicity of λ_1. If s be the no. of l.i. eigenvectors corresponding to the eigenvalue λ_1, then s is called the geometric multiplicity of λ_1. In this case no of l.i. soln. of $(A-\lambda_1 I)X=0$ will be s and the matrix $A-\lambda_1 I$ will be of rank $n-s$.

The geometric multiplicity of a c. root cannot exceed its algebraic multiplicity i.e. $s \le t$.

Special matrices

Conjugate of a matrix $\overline{A}$

Transposed conjugate of a matrix A^{θ} or A^{*}

Hermitian matrix (i,j)th element of A = complex conjugate of (j,i)th element of A i.e. $a_{ij}=\overline{a_{ji}}$

Necessary & sufficient condition for a matrix A to be Hermitian is that $A=A^{\theta}$ Ex - $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$

Skew-Hermitian matrix $a_{ij}=-\overline{a_{ji}}$

Ex $\begin{pmatrix} 0 & -2-i \\ 2-i & 0 \end{pmatrix}$ $\begin{pmatrix} -i & 3+4i \\ -3+4i & 0 \end{pmatrix}$

Necessary & sufficient cond^n. for a matrix A to be skew Hermitian is that $A^{\theta}=-A$.

Orthogonal matrix $AA^T=I \Rightarrow A^T=A^{-1}$

Unitary matrix $AA^{\theta}=I$

Nature of eigenvalues of special types of matrices

Theorem

The eigenvalues of

(i) A Hermitian matrix are real

(ii) A skew-Hermitian matrix are zero or purely imaginary

(iii) An unitary matrix are of modulus 1

Result

The eigenvalues of

(i) a symmetric matrix are real

(ii) a skew-symmetric matrix are purely imaginary or zero.

(iii) an orthogonal matrix are of modulus 1 and are real or complex conjugate pairs.

Ex Find the characteristic roots of the orthogonal matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and verify that they are of unit modulus.

Solⁿ: $|A - \lambda I| = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$

$|A - \lambda I| = 0$ gives

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\Rightarrow \cos\theta - \lambda = \pm i\sin\theta$$

$$\Rightarrow \lambda = \cos\theta \pm i\sin\theta$$

$\therefore \cos\theta \pm i\sin\theta$ are the characteristic roots of A.

We have $|\cos\theta + i\sin\theta| = \sqrt{(\cos^2\theta + \sin^2\theta)} = 1$

Similarly $|\cos\theta - i\sin\theta| = 1$

Ex Determine the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solⁿ: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \qquad \lambda = 2, 2, 8$$

For eigenvalue 8

$(A - 8I)X = 0$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$-3x_2 - 3x_3 = 0$$

Set $x_3 = 1$ $\therefore x_2 = -1$ $x_1 = 2$ $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

[Or Set $x_3 = k$ $\therefore x_2 = -k$ $x_1 = -x_2 + x_3 = k + k = 2k$

$X_1 = k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, k is a non-zero scalar]

Eigen vector corresponding to eigenvalue 8 is $c_1 X_1$ where c_1 is a non-zero scalar.

$(A-2I)X = 0$

$-2x_1 + x_2 - x_3 = 0$ rank $= 1$

$x_2 = 0, \; x_3 = 2$

$$X_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$x_3 = 0, \; x_2 = 2$ $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

X_2 and X_3 are two L.I. solⁿ. of this eqn. $\therefore X_2$ and X_3 are two LI eigenvectors of A corresponding to eigen value 2. If c_2, c_3 are scalars not both zero, then $c_2X_2 + c_3X_3$ gives all the eigenvectors of A corresponding to the eigen value 2.

Characteristic subspace of a matrix

Suppose λ is an eigen value of a square matrix of order n. Then every non-zero vector X satisfying the eqn. $(A-\lambda I)X = 0$ —(1) is an eigenvector of A corresponding to the eigen value λ. If the matrix $A-\lambda I$ is of rank r, then (1) will possess $n-r$ l.i sol^n. Each non-zero linear combination of these sol^{n}s. is also a sol^n. of (1) and it will be an eigenvector of A. The set of all these linear combinations is a subspace of V_n provided we add zero vector also to this set. This subspace of V_n is called characteristic subspace of A corresponding to the eigenvalue λ. Its dimension $n-r$ is the geometric multiplicity of the eigen value λ.

The eigenvalues of a triangular matrix are just the diagonal elements of the matrix.

Sol^n: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

be a triangular matrix of order n.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & a_{nn}-\lambda \end{vmatrix}$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

$\therefore$ eigenvalues are $\lambda = a_{11}, a_{22}, \cdots, a_{nn}$

Note: Similarly it can be shown that the eigenvalues of a diagonal matrix are just the diagonal elements of the matrix.

Ex If A is non-singular, the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

Sol^n: Let λ be an eigenvalue of A and X be corresponding eigenvector

$\therefore AX = \lambda X \Rightarrow A^{-1}(AX) = \lambda(A^{-1}X)$ [$\because A$ is non-singular]

$\Rightarrow X = \lambda(A^{-1}X) \Rightarrow A^{-1}X = \frac{1}{\lambda}X$

$\therefore \frac{1}{\lambda}$ is an e.v. of A^{-1} and X corresponding e. vector.

Ex If the eigenvalues of A are $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2$.

Sol^n: $AX = \lambda X \Rightarrow A(AX) = A(\lambda X) \Rightarrow A^2X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(\lambda X)$

$\Rightarrow A^2X = \lambda^2 X$

Hence the result follows.

Similarity of matrices

Definition

Let A and B be square matrices of order n. Then B is said to be similar to A if $\exists$ a non-singular matrix P such that $B = P^{-1}AP$.

Theorem Similar matrices have the same determinant.

Proof: Suppose A and B are similar matrices. Then $\exists$ an invertible matrix P such that $B = P^{-1}AP$

$$
\begin{aligned}
\therefore \det B &= \det(P^{-1}AP) \\
&= \det(P^{-1}) \det(A) \det(P) \\
&= \det(P^{-1}) \det(P) \det(A) \\
&= \det(P^{-1}P) \det(A) \\
&= \det(I) \det(A) \\
&= 1(\det A) = \det A
\end{aligned}
$$

Theorem

Similar matrices have the same characteristic polynomial and hence the same eigenvalues. If X is an eigenvector of A corresponding to the eigenvalue λ, then $P^{-1}X$ is an eigenvector of B corresponding to the eigenvalue λ where

$$B = P^{-1}AP$$

Proof: Suppose A and B are similar matrices. Then $\exists$ an invertible matrix P such that $B = P^{-1}AP$. We have

$$B - \lambda I = P^{-1}AP - \lambda I$$
$$= P^{-1}AP - P^{-1}(\lambda I)P \quad [\because P^{-1}(\lambda I)P = \lambda P^{-1}P = \lambda I]$$
$$= P^{-1}(A - \lambda I)P$$

$$\therefore \det(B - \lambda I) = \det P^{-1} \det(A - \lambda I) \det P$$
$$= \det P^{-1} . \det P . \det(A - \lambda I)$$
$$= \det(P^{-1}P) . \det(A - \lambda I)$$
$$= \det I . \det(A - \lambda I)$$
$$= 1. \det(A - \lambda I)$$
$$= \det(A - \lambda I)$$

Thus the matrices A and B have the same characteristic polynomial and so they have the same eigenvalues.

If λ is an eigenvalue of A and X is a corresponding eigenvector, then $AX = \lambda X$, and hence

$$B(P^{-1}X) = (P^{-1}AP)P^{-1}X = P^{-1}AX = P^{-1}(\lambda X) = \lambda(P^{-1}X)$$

$\therefore P^{-1}X$ is an eigenvector of B corresponding to its eigenvalue λ.

Corollary: If A is similar to a diagonal matrix D, the diagonal elements of D are the eigenvalues of A.

Proof: We know that similar matrices have same eigenvalues. Therefore A and D have the same eigenvalues. But the eigenvalues of the diagonal matrix D are its diagonal elements. Hence the eigenvalues of A are the diagonal elements of D.

Diagonalizable matrix

Def^n. A matrix A is said to be diagonalizable if it is similar to a diagonal matrix.

Thus a matrix A is diagonalizable if $\exists$ an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. Also the matrix P is then said to diagonalize A or transform A to diagonal form.

Basis of eigenvectors

If an $n \times n$ matrix A has n distinct eigenvalues, then A has a basis of eigenvectors for R^n.

Theorem

If an $n \times n$ matrix A has a basis of eigenvectors, then $D = X^{-1}AX$ is diagonal with the eigenvalues of A as the entries on the main diagonal. X is the matrix with these eigenvectors as column vectors. Also $D^m = X^{-1}A^mX$, $m = 2, 3, \ldots$

Theorem

An $n \times n$ matrix is diagonalizable if and only if it possesses n linearly independent eigenvectors.

Theorem

If the eigenvalues of an $n \times n$ matrix are all distinct, then it is always similar to a diagonal matrix.

Proof Let A be a square matrix of order n and suppose it has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$. We know that eigenvectors of a matrix corresponding to distinct eigenvalues are L.I. $\therefore$ A has n L.I. eigenvectors and so it is similar to a diagonal matrix.

Theorem The necessary and sufficient condition for a square matrix to be similar to a diagonal matrix is that the geometric multiplicity of each of its eigenvalues coincides with the algebraic multiplicity.

Ex Show that the matrix

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

is diagonalizable. Find a diagonalizing matrix X.

Sol^n: The characteristic eqn. of A is

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 4 & 4 \\ -1-\lambda & 3-\lambda & 4 \\ -1-\lambda & 8 & 7-\lambda \end{vmatrix} = 0 \quad \text{by } C_1+C_2+C_3$$

$$\Rightarrow -(1+\lambda) \begin{vmatrix} 1 & 4 & 4 \\ 1 & 3-\lambda & 4 \\ 1 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+\lambda) \begin{vmatrix} 1 & 4 & 4 \\ 0 & -1-\lambda & 0 \\ 0 & 4 & 3-\lambda \end{vmatrix} = 0 \qquad \begin{matrix} R_2' \to R_2 - R_1 \\ R_3' \to R_3 - R_1 \end{matrix}$$

$$\Rightarrow (1+\lambda)(1+\lambda)(3-\lambda) = 0$$

$\therefore$ Eigen values are $-1, -1, 3$

The eigenvectors X of A corresponding to the eigen value -1 are given by $(A-(-1)I)X=0$ or $(A+I)X=0$

or,
$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.
$$\begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} R_2' \to R_2 - R_1 \\ R_3' \to R_3 - 2R_1 \end{array}$$

Rank 1 i.e. 2 L.I. solⁿ.

$$-2x_1 + x_2 + x_3 = 0$$

$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are two L.I. solⁿ.

$\therefore X_1$ and X_2 are two LI eigenvectors of A corresponding to the eigen value -1. Thus the geometric multiplicity of the eigenvalue -1 is equal to the algebraic multiplicity.

Now the eigenvectors of A corresponding to the eigenvalue 3 are given by $(A-3I)X=0$

i.e.
$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or,
$$\begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ -4 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} R_2' \to R_2 - R_1 \\ R_3' \to R_3 - R_1 \end{array}$$

or,
$$\begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_3' \to R_3 + R_2$$

The matrix has rank 2. $\therefore$ 1 L.I. sol^n.

$$-12x_1 + 4x_2 + 4x_3 = 0$$
$$4x_1 - 4x_2 = 0$$
$$x_1 = x_2 = 1 \text{ say} \quad x_3 = 2$$

$X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 3. The geometric multiplicity of the " " 3 is 1 and its algebraic multiplicity is also 1.

Since the geometric multiplicity of each eigenvalue of A is equal to its algebraic " , therefore A is similar to a diagonal matrix.

Let $X = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

Find X^{-1} by Gauss-Jordon or any other method. Find $X^{-1}AX$

$$X^{-1}AX = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D.$$