**Indian Institute of Technology Kharagpur**
**Department of Mathematics**
**MA11003 - Advanced Calculus**
**Problem Sheet - 4**
**Autumn 2020**

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1. Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

is continuous at $(0,0)$, but $f_x(0,0)$ and $f_y(0,0)$ do not exist.

2. Show that the following functions

(a) $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$

possess partial derivatives at $(0,0)$, though it is not continuous at $(0,0)$.

3. Find $f_x(x,y)$ and $f_y(x,y)$ using definition for the followings :

(a) $f(x,y) = x^2 + y^2$,

(b) $f(x,y) = \sin(3x+4y)$,

(c) $f(x,y) = ye^{-x} + xy$.

(d) $f(x,y) = x^2 + xy + y^3$,

(e) $f(x,y) = x\sin y + x^2$,

(f) $f(x,y) = e^{xy} + \dfrac{x}{y}$.

4. Find $f_x(0,0)$, $f_y(0,0)$, $f_x(0,y)$ and $f_y(x,0)$ for the followings :

(a) $f(x,y) = \begin{cases} \dfrac{xy}{x+y}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \log(1+xy)$,

(c) $f(x,y) = \begin{cases} 1, & \text{if } x=0 \text{ or } y=0 \text{ or both } x=0 \text{ and } y=0; \\ 0, & \text{Otherwise} \end{cases}$

(d) $f(x,y) = e^{x-y} - e^{y-x}$,

(e) $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x+y}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

5. Show that the following functions

(a) $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x^2+y^2) \neq 0; \\ 0, & \text{if } (x^2+y^2) = 0. \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & \text{if } (x^2+y^2) \neq 0; \\ 0, & \text{if } (x^2+y^2) = 0. \end{cases}$

have first order partial derivatives at $(0,0)$, and discuss the differentiability at $(0,0)$ .

6. Show that following function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous, possess first order partial derivatives but it is not differentiable at the origin.

7. Prove that the function $f(x,y) = \sqrt{|xy|}$ is not differentiable at $(0,0)$, but that $f_x$ and $f_y$ both exists at origin and have the value 0. Show that $f_x$ and $f_y$ are continuous everywhere except at the origin.

8. Test the differentiability of the following functions at $(0,0)$

(a) $f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{x^3-2y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

9. Let $f(x,y) = \begin{cases} y\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

Find $f_{xx}(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$ and $f_{yy}(x,y)$ at $(0,0)$. Also check the differentiability of the function $f(x,y)$ at the origin.

10. For the function $f(x,y) = \begin{cases} \dfrac{x^2y(x-y)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$
check that $f_{xy}(0,0) \neq f_{yx}(0,0)$. Also check the differentiability of $f(x,y)$ at the origin

11. Find $f_{yxx}(x,y)$ and $f_{xyx}(x,y)$ for the following functions:

(a) $f(x,y) = x^4 \sin 3y + 5x - 6y$.
(b) $f(x,y) = x^5y^3 + \log(xy) + 10x$.
(c) $f(x,y) = e^{xy} \tan x + x^3y^2$.
(d) $f(x,y) = x^3 \sin y + y^3 \cos x$
(e) $f(x,y) = e^x \ln y + \cos y \ln x$
(f) $f(x,y) = x^3y^2 + 2xy^3 + \cos(xy^2)$

12. Find the total differential of the following functions

(a) $w = x^2 + xy^2 + xy^2z^3$
(b) $z = \tan^{-1}(x/y)$,
(c) $u = e^{(x^2+y^2+z^2)}$,
(d) $w = \sin(3x+4y) + 5e^z$
(e) $w = z \ln y + y \ln z + xyz$,
(f) $u = \sqrt{x^2+y^2+z^2}$,
(g) $w = e^x \sin(y+2z) - x^2y^2$,
(h) $w = e^{\frac{x}{y}} + e^{\frac{z}{y}}$.