LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

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HARMONIC FUNCTIONS

A function $u(x, y)$ which satisfies the Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in a domain D is said to be harmonic in D.

THEOREM

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v satisfy Laplace equation

$$u_{xx} + u_{yy} = 0 \qquad \text{and} \qquad v_{xx} + v_{yy} = 0$$

Sketch of the Proof:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

C-R Equations

$$=\left(\frac{\partial^2 v}{\partial x\partial y}\right)+\left(-\frac{\partial^2 v}{\partial x\partial y}\right)=0$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=0$$

THEOREM Let u be harmonic on a domain D, then for some v, $u+iv$ defines an analytic function for $z=x+iy$ in D .

The functions u and v are called harmonic conjugate of each others.

CONSTRUCTION OF ANALYTIC FUNCTION

Example: Show that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic and find v such that $f(z) = u + iv$ is analytic

$$u_x = \frac{\partial u}{\partial x} = -e^{-x}\ (x\sin y - y\cos y) + e^{-x}\sin y$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = e^{-x}\ (x\sin y - y\cos y) - e^{-x}\sin y - e^{-x}\sin y$$

$$u_y = \frac{\partial u}{\partial y} = e^{-x}\ (x\cos y + y\sin y - \cos y)$$

$$u_{yy} = \frac{\partial^2\, u}{\partial y^2} = e^{-x}\ (-x\sin y + \sin y + y\cos y + \sin y)$$

Clearly it shows that u is harmonic.

$$\frac{\partial u}{\partial x} = -e^{-x}\,(x\sin y - y\cos y) + e^{-x}\sin y \qquad \frac{\partial u}{\partial y} = e^{-x}\,(x\cos y + y\sin y - \cos y)$$

Using C-R Equations:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -e^{-x}\,(x\sin y - y\cos y) + e^{-x}\sin y \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -e^{-x}\,(x\cos y + y\sin y - \cos y)$$

$$\Longrightarrow v = xe^{-x}\cos y + e^{-x}\left[y\sin y - \int \sin y\, dy\right] - e^{-x}\cos y + F(x)$$

$$= xe^{-x}\cos y + e^{-x}[y\sin y + \cos y] - e^{-x}\cos y + F(x)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y + F(x)$$

$$v = xe^{-x}\cos y + ye^{-x}\sin y + F(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -e^{-x}\,(x\cos y + y\sin y - \cos y)$$

$$\frac{\partial v}{\partial x} = -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y + F'(x)$$

$$\Longrightarrow \frac{\partial v}{\partial x} = -e^{-x}\,(x\cos y + y\sin y - \cos y) + F'(x) = -e^{-x}\,(x\cos y + y\sin y - \cos y)$$

$$\Longrightarrow F'(x) = 0 \;\Longrightarrow F(x) = c \text{ (constant)}$$

$$\boxed{v = x\,e^{-x}\cos y + y\,e^{-x}\sin y + c}$$

Example: Find an analytic function $f(z) = u(x, y) + iv(x, y)$ given that $u(x, y) = x^3 - 3xy^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial y} = -6xy$$

Using C-R Equations: $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = 6xy$

$$\Longrightarrow v = 3x^2y - y^3 + F(x) \Longrightarrow \frac{\partial v}{\partial x} = 6xy + F'(x) = 6xy$$

$$\Longrightarrow F'(x) = 0 \quad \Longrightarrow F(x) = c \quad \Longrightarrow v = 3x^2y - y^3 + c$$

$$\Longrightarrow f(z) = u(x, y) + iv(x, y) = x^3 - 3xy^2 + i(3x^2y - y^3 + c) = (x + iy)^3 + k$$

$$\boxed{f(z) = z^3 + k}$$

- CR equations are necessary conditions for analyticity (NOT SUFFICIENT)
- If $f(z) = u(x, y) + iv(x, y)$ is analytic then u and v satisfy Laplace equation
- Given u, how to find v and vice versa

Thank You