LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

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Polynomial Interpolation

- Polynomial Interpolation
- Existence and Uniqueness
- Error in Interpolating Polynomials

Interpolation

Interpolation is a process of estimating values between known data points or approximating complicated functions by simple polynomials or determining a polynomial that fits a set of given points.

Applications :

1. Constructing the function when it is not given explicitly and only the values of function are given at some points.

2. Replacing complicated function by an interpolating simpler function (usually polynomials) so that many operations such as determination of roots, differentiation, integrations or other such operations may be performed.

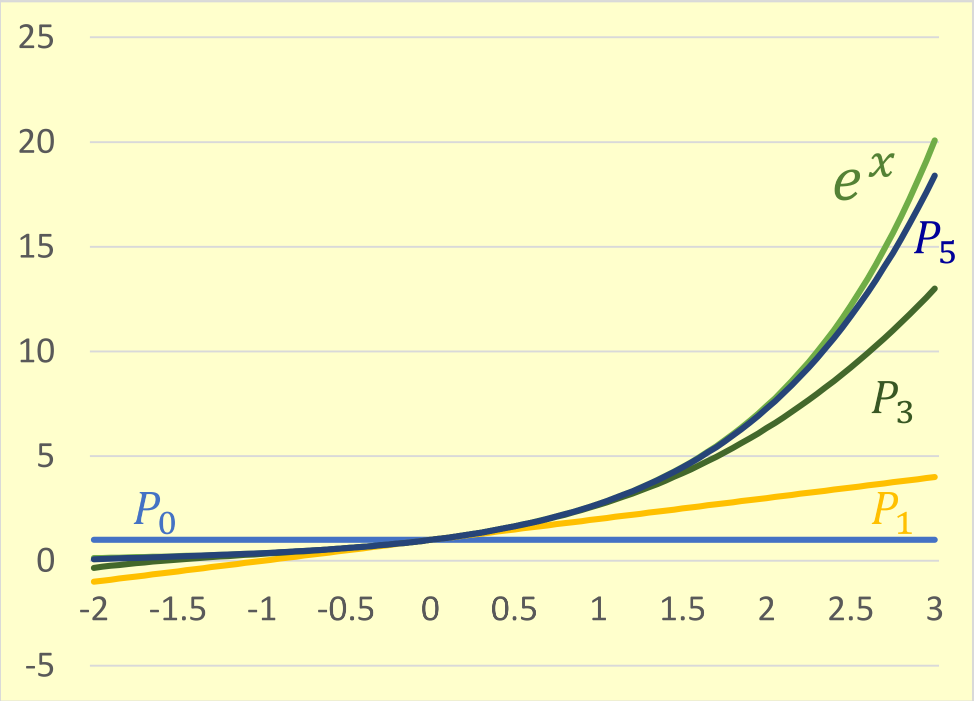
Fundamental principle behind polynomial interpolation

Weirstrass Approximation Theorem

Suppose that f is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$ with the property that

$$|f(x) - P(x)| < \epsilon; \quad \forall\, x \in [a, b].$$

Why not Taylor's Polynomial ? Consider Taylor's Polynomial of e^x around $x = 0$.



$$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$P_2(x) = 1 + x + \frac{x^2}{2} \qquad P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$P_0(x) = 1; \qquad P_1(x) = 1 + x$$

Taylor's polynomials agree as closely as possible with a given function at a specific point, so they concentrate their accuracy near that point.

For ordinary computation purposes it is more efficient to use methods that include information at various points.

Existence and uniqueness for polynomial interpolation

For $(n+1)$ data points there is one and only one polynomial of order $\leq n$ that passes through all the points.

For example, there is only one straight line (a first order polynomial) that passes through two points.

Consider for simplicity, a second order polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2$$

A straight forward method for computing the coefficients of a polynomial of degree n is based on the fact that $(n+1)$ data points are required (3 data points in this example) to determine $(n+1)$ unknowns (3 unknowns $a_0,\ a_1,\ a_2$ in this example)

Polynomial to be fitted with the given data $f(x) = a_0 + a_1 x + a_2 x^2$

Suppose that there are 3 given data points $(x_0,\ f(x_0)),\ (x_1,\ f(x_1)),\ (x_2,\ f(x_2))$.

If the given polynomial passes through the given data points then it must satisfy them, i.e.,

$$\left.\begin{aligned} f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 \\ f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 \\ f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 \end{aligned}\right\} \Leftrightarrow \begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix}$$

This system of equations has a unique solution as

$$\text{Det} \begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} = (x_0 - x_1)(x_1 - x_2)(x_2 - x_0) \neq 0 \ \text{ if } x_0,\ x_1,\ x_2 \text{ are distinct.}$$

In practice, it is observed that the above system of equations is *ill-conditioned*.

Whether they are solved with an elimination method or with a more efficient algorithm, the resulting coefficient can be highly inaccurate, in particular for, large n.

Therefore, we have some mathematical formats (interpolating formats) in which such calculation can be avoided.

Error in Interpolating Polynomials

Let $x_0,\ x_1,\ \dots,\ x_n$ be $(n+1)$ points and let x be a point belonging to the domain of a given function f.

Assume that $f \in C^{(n+1)}(I_x)$, where I_x is the smallest interval containing the nodes $x_0,\ x_1, \dots, x_n$ and x.

Then the interpolation error at the point x is given by

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n) \quad \text{where } \xi \in I_x.$$

Error in Interpolating Polynomials

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n) \quad \text{where } \xi \in I_x.$$

Proof : Note that the result is obviously true if x coincides with any of the interpolating nodes.

For simplicity, let us assume $w_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$

Now, define for any $x \in I_x$, the function

$$G(t) = E_n(t) - \frac{w_{n+1}(t)E_n(x)}{w_{n+1}(x)}; \quad t \in I_x$$

Since $f \in C^{(n+1)}(I_x)$ and w_{n+1} is a polynomial, then $G \in C^{(n+1)}(I_x)$.

$$G(t) = E_n(t) - \frac{w_{n+1}(t)E_n(x)}{w_{n+1}(x)}; \quad t \in I_x$$

$$E_n(x) = f(x) - P_n(x)$$

$$w_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$$

Note that $G(t)$ has $(n+2)$ distinct zeros in I_x since

$$G(x_i) = E_n(x_i) - \frac{w_{n+1}(x_i)E_n(x)}{w_{n+1}(x)} = 0; \quad i = 0, 1, 2, \dots, n.$$

$$G(x) = E_n(x) - \frac{w_{n+1}(x)E_n(x)}{w_{n+1}(x)} = 0$$

Then using **Rolle's theorem**, G' has atleast $(n+1)$ distinct zeros.

By recursion it follows that $G^{(j)}$ admits at least $(n+2) - j$ distinct zeros.

$\Rightarrow G^{(n+1)}$ has atleast one zero, which we denote by ξ, i.e., $G^{(n+1)}(\xi) = 0$

$$G(t) = E_n(t) - \frac{w_{n+1}(t)E_n(x)}{w_{n+1}(x)}; \ \ t \in I_x$$

$$w_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$$

$$G^{(n+1)}(\xi) = 0$$

$$\Longrightarrow G^{(n+1)}(t) = E_n^{(n+1)}(t) - \frac{w_{n+1}^{(n+1)}(t)E_n(x)}{w_{n+1}(x)}$$

Note that $E_n(t) = f(t) - P_n(t) \Longrightarrow E_n^{(n+1)}(t) = f^{(n+1)}(t)$ as $P_n^{(n+1)}(t) = 0$

$$w_{n+1}^{(n+1)}(t) = (n+1)! \quad \& \quad G^{(n+1)}(\xi) = 0$$

$$\Longrightarrow \ 0 = f^{(n+1)}(\xi) - \frac{(n+1)!\,E_n(x)}{w_{n+1}(x)} \ \Longrightarrow E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} w_{n+1}(x)$$

Polynomial Interpolation

Polynomial interpolation is the method of determining a polynomial that fits a set of given points

Existence and Uniqueness

For $(n+1)$ data points there is one and only one polynomial of order $\leq n$ that passes through all the points.

Error in Interpolating Polynomials

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1) \dots (x - x_n)$$

- Forward and Backward Difference Table
- Newton's Forward Interpolation Formula
- Newton's Backward Interpolation Formula

RECALL: Previous Lecture

- **Polynomial Interpolation**

  Polynomial interpolation is the method of determining a polynomial that fits a set of given points

- **Existence and Uniqueness**

  For $(n+1)$ data points there is one and only one polynomial of order $\leq n$ that passes through all the points.

- **Error in Interpolating Polynomials**

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

Different Methods of Determining Interpolating Polynomials

Although there is a unique nth order polynomial that fits $(n + 1)$ data points, there are a variety of mathematical formats in which this polynomial can be expressed.

- Newton's forward and backward interpolating polynomial
- Newton's Divided Difference Formula
- Lagrange Interpolation Formula

Newton's Forward and Backward Interpolation Formula

Let the tabular points $x_0, x_1, \dots, x_n$ be equally spaced, i.e., $x_i = x_0 + ih,\ \ i = 0, 1, \dots, n$

Finite Difference Operator

- The Shift operator: $Ef(x_i) = f(x_i + h)$
- The Forward difference operator: $\Delta f(x_i) = f(x_i + h) - f(x_i)$
- The Backward difference operator: $\nabla f(x_i) = f(x_i) - f(x_i - h)$
- The Central difference operator: $\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$

Newton's Forward and Backward Interpolation Formula

Let $f_i = f(x_i)$ $\qquad f_{i+1} = f(x_i + h)$ $\qquad f_{i+\frac{1}{2}} = f\left(x_i + \frac{h}{2}\right)$

It can easily be verified that

$$\Delta f_i = \nabla f_{i+1} = \delta f_{i+\frac{1}{2}}$$

Also, note that

$$\Delta \equiv E - 1 \qquad (\Delta f_i = E f_i - f_i)$$

$$\nabla \equiv 1 - E^{-1} \qquad (\nabla f_i = f_i - E^{-1} f_i)$$

$$\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}} \qquad \left(\delta f_i = E^{\frac{1}{2}} f_i - E^{-\frac{1}{2}} f_i\right)$$

Higher Order Differences

$$\Delta^2 f(x_i) = \Delta\big(\Delta f(x_i)\big)$$

$$= \Delta(f_{i+1} - f_i)$$

$$= f_{i+2} - f_{i+1} + (f_{i+1} - f_i)$$

$$= f_{i+2} - 2f_{i+1} + f_i$$

Similarly, $\nabla^2 f_i = \nabla(\nabla f_i) = \nabla(f_i - f_{i-1})$

$$= f_i - f_{i-1} - (f_{i-1} - f_{i-2})$$

$$= f_i - 2f_{i-1} + f_{i-2}$$

Forward Difference Table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	f_0			
		Δf_0		
x_1	f_1		$\Delta^2 f_0$	
		Δf_1		$\Delta^3 f_0$
x_2	f_2		$\Delta^2 f_1$	
		Δf_2		
x_3	f_3			

$$\Delta f_i = f_{i+1} - f_i$$

$$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$$

$$\Delta^3 f_i = \Delta^2 f_{i+1} - \Delta^2 f_i$$

Backward Difference Table

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
x_0	f_0			
		∇f_1		
x_1	f_1		$\nabla^2 f_2$	
		∇f_2		$\nabla^3 f_3$
x_2	f_2		$\nabla^2 f_3$	
		∇f_3		
x_3	f_3			

$$\nabla f_i = f_i - f_{i-1}$$

$$\nabla^2 f_i = \nabla f_i - \nabla f_{i-1}$$

$$\nabla^3 f_i = \nabla^2 f_i - \nabla^2 f_{i-1}$$

Difference Table – Numerical Example

x	$f(x)$	Δ/∇	Δ^2/∇^2
0	1		
		1	
1	2		-2
		-1	
2	1		

$\Delta f_0 = 1$ $\qquad$ $\Delta^2 f_0 = -2$

$\nabla f_2 = -1$ $\qquad$ $\nabla^2 f_2 = -2$

Newton's Forward Difference Formula

1. **Linear Interpolation :** The simplest way to connect two data points with a straight line.

Given that :

x_0	x_1
$f(x_0)$	$f(x_1)$

Consider a general equation of straight line

$$P_1(x) = b_0 + b_1(x - x_0)$$

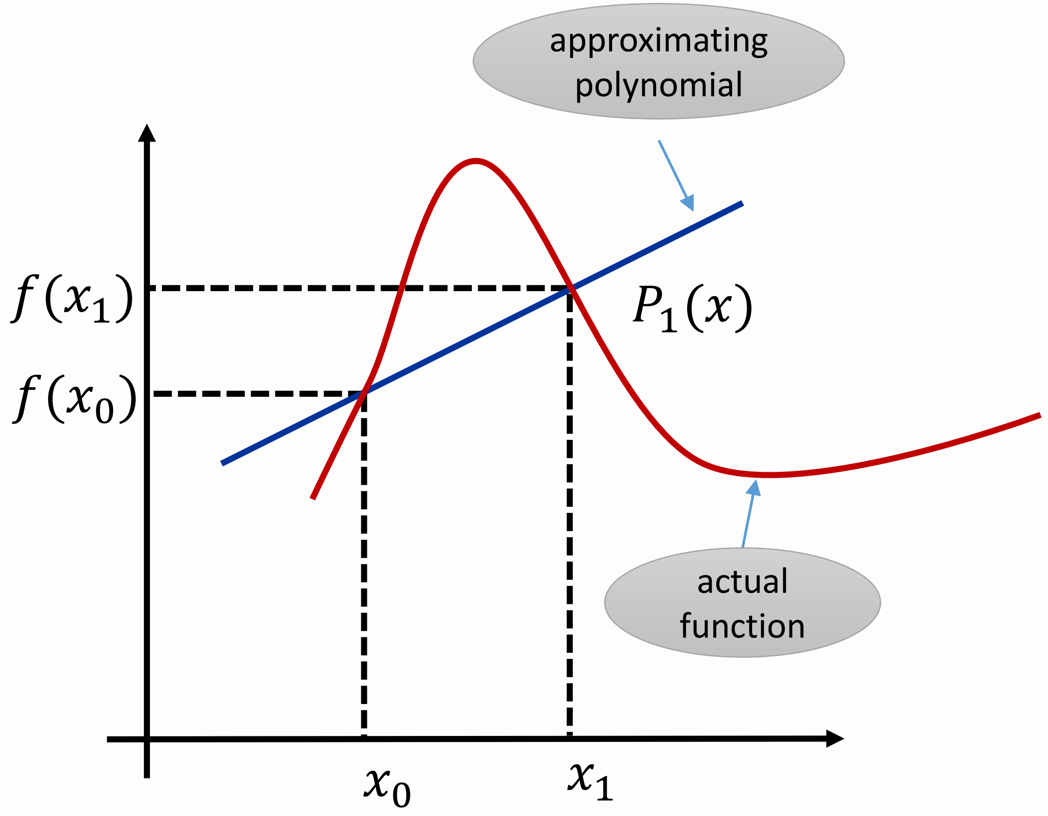
At the point $x = x_0$:

$$P_1(x_0) = f(x_0) = b_0$$

At the point $x = x_1$:

$$P_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$

$$\Rightarrow b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



$$P_1(x) = b_0 + b_1(x - x_0)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Let us consider the equidistant data points, then

$$b_1 = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{\Delta f_0}{h}$$

Interpolating Polynomial $\boxed{P_1(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h}}$

2. Quadratic Interpolation :

Suppose 3 data points are given:

x_0	x_1	x_2
$f(x_0)$	$f(x_1)$	$f(x_2)$

Consider a second order polynomial

$$P_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At the point $x = x_0$: $\quad b_0 = f(x_0)$

At the point $x = x_1$: $\quad b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

In the case of equidistant data points : $\quad b_1 = \dfrac{\Delta f_0}{h}$

$$P_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \qquad b_0 = f(x_0) \qquad b_1 = \frac{\Delta f_0}{h}$$

At the point $x = x_2$: $f(x_2) = f(x_0) + \dfrac{\Delta f_0}{h}(x_2 - x_0) + b_2 (x_2 - x_0)(x_2 - x_1)$

$$\Rightarrow f(x_2) = f(x_0) + 2f(x_1) - 2f(x_0) + 2!\, h^2 b_2$$

$$\Rightarrow \frac{f(x_2) - 2f(x_1) + f(x_0)}{2!\, h^2} = b_2 \qquad \Rightarrow b_2 = \frac{\Delta^2 f_0}{2!\, h^2}$$

Interpolating polynomial :

$$\boxed{P_2(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\, h^2}}$$

$$P_1(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h}$$

$$P_2(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2}$$

Generalized Formula :

We can now write the Newton's forward difference formula based on $(n+1)$ nodal points $x_0, x_1, \dots, x_n$ as:

$$P_n(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2} + \cdots + (x - x_0)(x - x_1)(x - x_{n-1})\frac{\Delta^n f_0}{n!\,h^n}$$

If we put $\dfrac{x - x_0}{h} = u$ then it takes the following form:

$$P_n(x_0 + hu) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \cdots + \frac{u(u-1)\cdots(u-n+1)}{n!}\Delta^n f_0$$

Alternative Derivation :

$$f(x) = f\left(x_0 + \frac{(x - x_0)}{h} h\right) = f(x_0 + uh)$$

$$= E^u f(x_0)$$

$$= (1 + \Delta)^u f(x_0)$$

$$= f_0 + u\,\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \cdots + \frac{u(u-1)\cdots(u-n+1)}{n!}\Delta^n f_0 + \cdots$$

Neglecting the difference $\Delta^{n+1} f_0$ and higher order differences , we get the above generalized formula.

Newton's Backward Difference Formula :

$$\boxed{P_n(x) = f_n + (x - x_n)\frac{\nabla f_n}{h} + \frac{(x - x_n)(x - x_{n-1})}{2!\,h^2}\nabla^2 f_n + \cdots + \frac{(x - x_n)(x - x_{n-1})\ldots(x - x_1)}{n!\,h^n}\nabla^n f_n}$$

$$f(x) = f\left(x_n + \frac{x - x_n}{h}h\right) = f(x_n + hu) = E^u f(x_n) = (1 - \nabla)^{-u} f(x_n) \qquad \nabla \equiv 1 - E^{-1}$$

$$= f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \cdots + \frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n f(x_n) + \cdots$$

Neglecting the difference $\nabla^{n+1} f(x_n)$ and higher order differences, we get:

$$\boxed{P_n(x_n + hu) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \cdots + \frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n f_n}$$

➢ Newton's Forward Interpolation Formula

$$P_n(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2} + \cdots + (x - x_0)(x - x_1)(x - x_{n-1})\frac{\Delta^n f_0}{n!\,h^n}$$

➢ Newton's Backward Interpolation Formula

$$P_n(x) = f_n + (x - x_n)\frac{\nabla f_n}{h} + (x - x_n)(x - x_{n-1})\,\frac{\nabla^2 f_n}{2!\,h^2} + \cdots$$

$$+(x - x_n)(x - x_{n-1}) \ldots (x - x_1)\,\frac{\nabla^n f_n}{n!\,h^n}$$

Example : Using Newton forward and backward interpolation formula, find the quadratic polynomial which takes the following values

x	0	1	2
$f(x)$	1	2	1

Solution : The difference table

x	$f(x)$	$\nabla f/\Delta f$	$\nabla^2 f/\Delta^2 f$
0	1		
		1	
1	2		-2
		-1	
2	1		

Newton's Forward Formula:

x	$f(x)$	$\nabla f/\Delta f$	$\nabla^2 f/\Delta^2 f$
0	1		
		1	
1	2		-2
		-1	
2	1		

$$P_2(x) = f_0 + \frac{(x-x_0)}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 f_0$$

$$= 1 + \frac{x-0}{1}.1 + \frac{(x-0)(x-1)}{2!\cdot 1}\cdot(-2)$$

$$= 1 + x - x^2 + x$$

$$= 1 + 2x - x^2$$

Newton's Backward Formula:

x	$f(x)$	$\nabla f/\Delta f$	$\nabla^2 f/\Delta^2 f$
0	1		
		1	
1	2		-2
		-1	
2	1		

$$P_2(x) = f_2 + \frac{(x - x_2)}{h}\nabla f_2 + \frac{(x - x_2)(x - x_1)}{2!\,h^2}\nabla^2 f_2$$

$$= 1 + \frac{(x - 2)}{1}\cdot(-1) + \frac{(x - 2)(x - 1)}{2!\cdot 1}\cdot(-2)$$

$$= 1 - x + 2 - (x^2 - 3x + 2)$$

$$= 1 + 2x - x^2$$

Note : If we wish to add another data point, say $\left(x_3,\ f(x_3)\right) = (3,\ 10)$ we need to add only another row to the difference table in order to apply forward difference formula.

x	$f(x)$	$\nabla f/\Delta f$	$\nabla^2 f/\Delta^2 f$	$\Delta^3 f/\nabla^3 f$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

$$P_2(x) = 1 + 2x - x^2 + \frac{(x-0)(x-1)(x-2)}{3!}(12)$$

$$= 1 + 2x - x^2 + 2x(x^2 - 3x + 2)$$

$$= 2x^3 - 7x^2 + 6x + 1$$

Example : Construct Newton's forward interpolation polynomial for the following table :

x	4	6	8	10
y	1	3	8	16

Hence evaluate the interpolating polynomial for $x = 5$.

Solution : Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

$$P(x) = 1 + \frac{(x-4)}{2} \cdot 2 + \frac{(x-4)(x-6)}{2!\,2^2} \cdot 3 + 0$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

$$= 1 + x - 4 + \frac{3}{8}(x^2 - 10x + 24)$$

$$= 6 - \frac{11}{4}x + \frac{3}{8}x^2$$

$$P(5) = 6 - \frac{11}{4} \times 5 + \frac{3}{8} \times 25 = 1.625$$

Note : In this example, given 4 data points we get only a second degree polynomial.

Example: Apply Newton's backward difference formula to the data below:

x	1	2	3	4	5
y	1	-1	1	-1	1

Difference Table:

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f_3$	$\nabla^4 f_3$
1	1				
		-2			
2	-1		4		
		2		-8	
3	1		-4		16
		-2		8	
4	-1		4		
		2			
5	1				

Backward Difference Formula

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f_3$	$\nabla^4 f_3$
1	1				
		-2			
2	-1		4		
		2		-8	
3	1		-4		16
		-2		8	
4	-1		4		
		2			
5	1				

$$P_4(x) = f_4 + (x - x_4)\frac{\nabla f_4}{h} + \frac{(x-x_4)(x-x_3)}{2!\,h^2}\nabla^2 f_4 + \frac{(x-x_4)(x-x_3)(x-x_2)}{3!\,h^3}\nabla^3 f_4 + \frac{(x-x_4)(x-x_3)(x-x_2)(x-x_1)}{4!\,h^4}\nabla^4 f_4$$

$$P_4(x) = 1 + (x-5)\,2 + \frac{(x-5)(x-4)}{2}4 + \frac{(x-5)(x-4)(x-3)}{6}\,8 + \frac{(x-5)(x-4)(x-3)(x-2)}{24}\,16$$

$$= \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31$$

Example : Approximate the function e^x on the interval $[0, 1]$ by using polynomial interpolation with $x_0 = 0,\ x_1 = 0.5,\ x_2 = 1$. Let $P_2(x)$ denote the interpolation polynomial.

a) Find the interpolating polynomial using any method.

b) Find an upper bound for the error magnitude $\max_{0 \le x \le 1} |e^x - P_2(x)|$.

c) Compare the actual error at different points of your choice with the error bound.

a) Difference table :

$$P_2(x) = f_0 + \frac{(x-x_0)}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 f_0$$

x	$f(x)$	Δf	$\Delta^2 f$
0	1		
		0.6487	
0.5	1.6487		0.4208
		1.0696	
1	2.7183		

$$P_2(x) = 1 + (x-0)\frac{0.6487}{0.5} + (x-0)\,(x-0.5)\,\frac{0.4208}{2\times 0.5^2}$$

$$= 1 + 1.2974\,x + \left(x^2 - \frac{1}{2}x\right) \times 0.8416$$

$$= 0.8416x^2 + 0.8766x + 1$$

$b)$ $$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

$$\max_{x\in[0,1]} |e^x - P_2(x)| \leq \frac{1}{6} \max_{t\in[0,1]} e^t \max_{x\in[0,1]} \left|(x-0)\left(x-\frac{1}{2}\right)(x-1)\right|$$

$$= \frac{1}{6} \times e \times \left(\left|(x-0)\left(x-\frac{1}{2}\right)(x-1)\right|\right)_{x=\frac{3\pm\sqrt{3}}{6}} \quad = 0.0218$$

Let $g = x\left(x-\frac{1}{2}\right)(x-1) \quad \Rightarrow g' = \left(x-\frac{1}{2}\right)(x-1) + x(x-1) + x\left(x-\frac{1}{2}\right) = 0$

$$\Rightarrow 3x^2 - 3x + \frac{1}{2} = 0 \quad \Rightarrow x = \frac{3 \pm \sqrt{9 - 4\times 3\times \frac{1}{2}}}{2\times 3} = \frac{3\pm\sqrt{3}}{6}$$

$b)$ Error bound

$$\max_{x\in[0,1]} |e^x - P_2(x)| \leq \ 0.0218$$

$c)$ Comparison with actual error

x	0.1	0.3	0.6	0.9
$\lvert e^x - P_2(x)\rvert$	0.0091	0.0111	0.0068	0.0110

- **Newton’s Divided - Difference Interpolating Polynomial**
- **Lagrange's Interpolation Formula**

Recall: Previous Lecture

- **Newton's Forward Interpolation Formula**

$$P_n(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2} + \cdots + (x - x_0)(x - x_1)(x - x_{n-1})\frac{\Delta^n f_0}{n!\,h^n}$$

- **Newton's Backward Interpolation Formula**

$$P_n(x) = f_n + (x - x_n)\frac{\nabla f_n}{h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 f_n}{2!\,h^2} + \cdots$$

$$+(x - x_n)(x - x_{n-1}) \ldots (x - x_1)\frac{\nabla^n f_n}{n!\,h^n}$$

Note that both formulas were derived taking equidistant nodal points!

Newton's Divided - Difference Interpolating Polynomial

Linear Polynomial : Given that :

x_0	x_1
$f(x_0)$	$f(x_1)$

$$P_1(x) = b_0 + b_1(x - x_0)$$

At the point $x = x_0$: $f(x_0) = b_0$

At the point $x = x_1$: $f(x_1) = b_0 + b_1(x_1 - x_0)$

$$\Rightarrow b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_1, x_0] \quad \text{divided difference}$$

$$\boxed{P_1(x) = f(x_0) + f[x_1, x_0](x - x_0)}$$

Quadratic polynomial: Given data points: $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$

In this case, we can fit a polynomial of degree 2: $P_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$

At the point $x = x_0 : b_0 = f(x_0)$

At the point $x = x_1 : b_1 = f[x_1, x_0]$

At the point $x = x_2$: $b_2 = \dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} = \dfrac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$

$= f[x_2, x_1, x_0]$

$$\boxed{P_2(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)}$$

Generalized formula for Newton's Divided-Difference Interpolating Polynomial

$$P_n(x) = f(x_0) + (x - x_0)f[x_1, x_0] + (x - x_0)(x - x_1)f[x_2, x_1, x_0] + \cdots$$

$$+(x - x_0)(x - x_1) \dots (x - x_{n-1})f[x_n, x_{n-1}, \dots, x_0]$$

This is called Newton's divided difference interpolating polynomial.

Lagrange Interpolating Polynomial

Linear Polynomial : Given data points : $\left(x_0, f(x_0)\right)$, $\left(x_1, f(x_1)\right)$

$$P_1(x) = \sum_{i=0}^{1} L_i(x) f(x_i)$$

$$P_1(x) = f(x_0) + f[x_1, x_0](x - x_0)$$

$$P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$= \underbrace{\frac{(x - x_1)}{(x_0 - x_1)}}_{L_0(x)} f(x_0) + \underbrace{\frac{(x - x_0)}{(x_1 - x_0)}}_{L_1(x)} f(x_1)$$

L_0 & L_1 are called Lagrange polynomials of degree 1.

Linear Polynomial :

$$\boxed{P_1(x) = \sum_{i=0}^{1} L_i(x) f(x_i)}$$

$$L_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} \qquad L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{1} \frac{x - x_j}{x_i - x_j}; \quad i = 0, 1$$

Generalized Lagrange Interpolating Polynomial

Given data points : $(x_0, f(x_0)),\ (x_1, f(x_1)),\ \ldots,\ (x_n, f(x_n))$.

$$\boxed{P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)}$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \frac{(x - x_0)(x - x_1) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n)}{(x_i - x_0)(x_i - x_1) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)}$$

Lagrange's polynomial of degree n

Example : Using Lagrange and the Newton-divided difference formulas, construct a polynomial of degree 2 or less with the following data:

x	1	2	4
$f(x)$	1	3	3

Solution : Lagrange interpolating polynomial: $P_2(x) = \sum_{i=0}^{2} L_i(x) f(x_i)$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2)(x-4)}{(1-2)(1-4)} = \frac{1}{3}(x-2)(x-4)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-4)}{(2-1)(2-4)} = -\frac{1}{2}(x-1)(x-4)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-2)}{(4-1)(4-2)} = \frac{1}{6}(x-1)(x-2)$$

$$L_0(x) = \frac{1}{3}(x-2)(x-4) \qquad L_1(x) = -\frac{1}{2}(x-1)(x-4) \qquad L_2(x) = \frac{1}{6}(x-1)(x-2)$$

Lagrange interpolating polynomial : $P_2(x) = \sum_{i=0}^{2} L_i(x) f(x_i)$

$$P_2(x) = \frac{1}{3}(x-2)(x-4) \times 1 - \frac{1}{2}(x-1)(x-4) \times 3 + \frac{1}{6}(x-1)(x-2) \times 3$$

$$= \frac{1}{3}(x^2 - 6x + 8) - \frac{3}{2}(x^2 - 5x + 4) + \frac{1}{2}(x^2 - 3x + 2)$$

$$= -\frac{2}{3}x^2 + 4x - \frac{7}{3}$$

Newton's Divided-Difference Formula:

Divided-Difference table

x	$f(x)$	$f[x_i, x_{i-1}]$	$f[x_i, x_{i-1}, x_{i-2}]$
1	1		
		2	
2	3		$-\frac{2}{3}$
		0	
4	3		

$$P_2(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

$$= 1 + 2(x - 1) - \frac{2}{3}(x - 1)(x - 2)$$

$$= -\frac{2}{3}x^2 + 4x - \frac{7}{3}$$

Example: Use the Lagrange and the Newton-divided difference formulas to derive interpolating polynomial from the following table:

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Divided difference table:

x	$f(x)$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot,\cdot]$
0	1					
		13				
1	14		-6			
		1		1		
2	15		-2		0	
		-5		1		0
4	5		2		0	
		1		1		
5	6		6			
		13				
6	19					

$$P_5(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

$$= 1 + 13(x) - 6(x)(x - 1) + 1(x)(x - 1)(x - 2)$$

$$= x^3 - 9x^2 + 21x + 1$$

Divided difference table:

x	$f(x)$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot,\cdot]$
0	1					
		13				
1	14		-6			
		1		1	0	
2	15		-2			0
		-5		1	0	
4	5		2			
		1		1		
5	6		6			
		13				
6	19					

Lagrange's Interpolation Formula

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

$$P_5(x) = \sum_{i=0}^{5} L_i(x) f(x_i)$$

$$= \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(-1)(-2)(-4)(-5)(-6)} \times 1 + \frac{(x)(x-2)(x-4)(x-5)(x-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14$$

$$+ \frac{(x)(x-1)(x-4)(x-5)(x-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15 + \frac{(x)(x-1)(x-2)(x-5)(x-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5$$

$$+ \frac{(x)(x-1)(x-2)(x-4)(x-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 6 + \frac{(x)(x-1)(x-2)(x-4)(x-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 19$$

$$= x^3 - 9x^2 + 21x + 1$$

CONCLUSIONS

Newton's Divided-Difference Interpolating Polynomial

$$P_n(x) = f(x_0) + (x - x_0)f[x_1, x_0] + (x - x_0)(x - x_1)f[x_2, x_1, x_0] + \cdots$$

$$+(x - x_0)(x - x_1) \ldots (x - x_{n-1})f[x_n, x_{n-1}, \ldots, x_0]$$

Lagrange Interpolating Polynomial

$$\boxed{P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)} \qquad L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$