Indian Institute of Technology Kharagpur
Department of Mathematics
MA11003 - Advanced Calculus
Problem Sheet - 2
Autumn 2020

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1. a) Find the value of a for which the limit

$$\lim_{x\to 0} \frac{\sin(ax) - \sin x - x}{x^3}$$

is finite and evaluate the limit.

b) Find the values of a and b such that

$$\lim_{x\to 0} \frac{\cos(ax) - b}{2x^2} = -1.$$

2. Use Taylor's theorem to prove that

a) $x - \frac{x^2}{2} < \log(1+x) < x$ for $x > 0$.

b) $\cos x \geq 1 - \frac{x^2}{2}$ for $-\pi < x < \pi$.

c) $1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ for $x > 0$.

3. Let $c \in \mathbb{R}$ and a real function f be such that f'' is continuous on some neighbourhood of c. Prove that

$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

4. Let $a \in \mathbb{R}$ and a real function f defined on some neighbourhood $N(a)$ of a such that f'' is continuous at a and $f''(a) \neq 0$. Prove that $\lim_{h\to 0} \theta = \frac{1}{2}$, where θ is given by $f(a+h) = f(a) + hf'(a+\theta h)$ $(0 < \theta < 1)$.

5. Each of the series in the following is the value of the Taylor series at $x = 0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

a) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \ldots$

b) $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \ldots$

c) $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \ldots$

6. Using Taylor series expansion, evaluate

a) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ *b)* $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

c) $\lim\limits_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$ *d*) $\frac{\cosh x - \cos x}{x \sin x}$

7. If f is continuous at x_0, and there are constants a_0 and a_1 such that

$$\lim_{x\to x_0} \frac{f(x) - a_0 - a_1(x - x_0)}{x - x_0},$$

then prove that $a_0 = f(x_0)$, f is differentiable at x_0, and $f'(x_0) = a_1$.

8. Find the Maclaurin's infinite series expansion for

a) $e^x, \quad x \in \mathbb{R}$
b) $\log(1 + x), \quad x \in (-1, 1]$
c) $e^x \cos x, \quad x \in \mathbb{R}$

9. Obtain the fourth degree Taylor's polynomial approximation to $f(x) = e^{2x}$ about $x = 0$. Find the maximum error when $0 \le x \le 0.5$.

10. Using Taylor's series find the approximate value of *a*) $\sqrt{1.5}$ and *b*) $\cos 31^\circ$.

11. Obtain the Maclaurin's series expansion of $f(x) = \sin(m \sin^{-1} x)$, where m is a constant.

12. For the Taylor's polynomial approximation of degree less than or equal to n about the point $x = 0$ for the function e^x, determine the value of n such that the error satisfies $|R_n(x)| \le 0.005$, when $-1 \le x \le 1$.

13. Can the function $f(x)$ defined by $f(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ be expanded by Maclaurin's theorem?

14. Find the Taylor's series expansion of $\sin^2 x$ upto five terms with Lagrange's form of remainder.

15. Find the Maclaurin's series expansion of $\tan^{-1} x$ upto four terms with Lagrange's form of remainder.

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