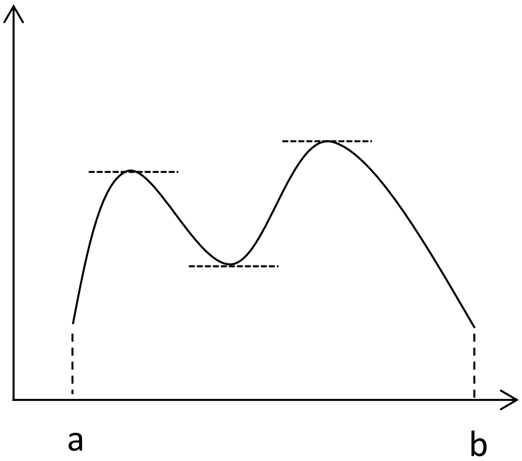
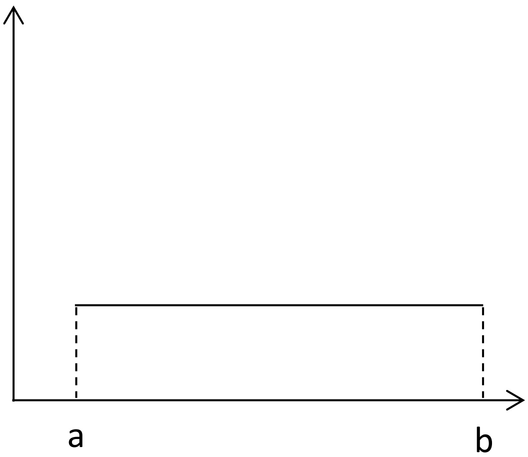
Rolle's Theorem:

If a function f is

a) continuous in [a, b]
b) differentiable in (a, b)
c) $f(a) = f(b)$

Then $\exists\ c \in (a, b)$ s.t. $f'(c) = 0$





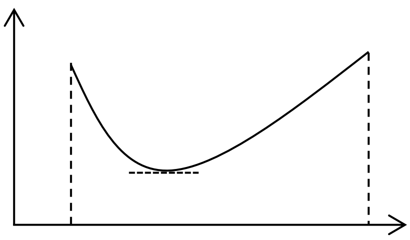
Proof: Suppose M & m are maximum and minimum of $f(x)$ in $[a, b]$.

(It will always exist because of Weierstrass extreme value theorem as f is continuous in [a, b])

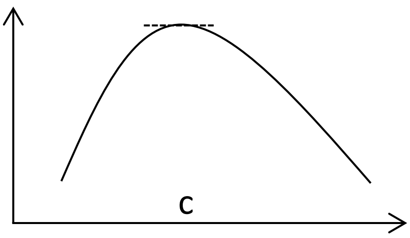
Case I: if $M = m$ i.e. $f(x) = M = m = constant$

This implies $f'(x) = 0 \quad \forall x \in (a, b)$

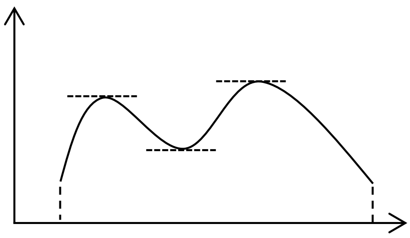
Case II: $M \neq m$. Then at least one of them must be different from equal values of $f(a)$ and $f(b)$.



Minimum is different from f(a) and f(b)



Maximum is different from f(a) and f(b)



Both are different from f(a) and f(b)

Let $M = f(c)$ be different. Since f is differentiable in (a, b), $f'(c)$ exists. Note that $f(c)$ is the maximum value, then

$f(c + \Delta x) - f(c) \leq 0$ for $\Delta x > 0$ or $\Delta x < 0$

This implies:

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} \leq 0, \quad \text{for } \Delta x > 0$$

and

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} \geq 0, \quad \text{for } \Delta x < 0$$

Since $f'(c)$ exists, passing limit as $\Delta x \rightarrow 0$, we get

$$\lim_{\substack{\Delta x \to 0 \\ (\Delta x > 0)}} \frac{f(c + \Delta x) - f(c)}{\Delta x} = f'(c) \leq 0 \tag{1}$$

and

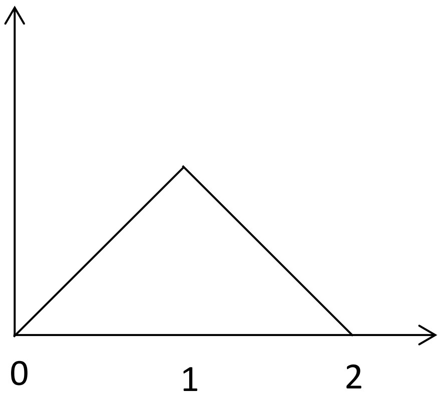
$$\lim_{\substack{\Delta x \to 0 \\ (\Delta x < 0)}} \frac{f(c + \Delta x) - f(c)}{\Delta x} = f'(c) \geq 0 \tag{2}$$

Inequality (1) and (2) implies $f'(c) = 0$.

Remark 1: The conclusion of Rolle 's Theorem may not hold for a function that does not satisfy any of its conditions.

Ex 1: Consider

$$f(x) = \begin{cases} x, & x \in [0, 1] \\ 2 - x, & x \in (1, 2] \end{cases}$$

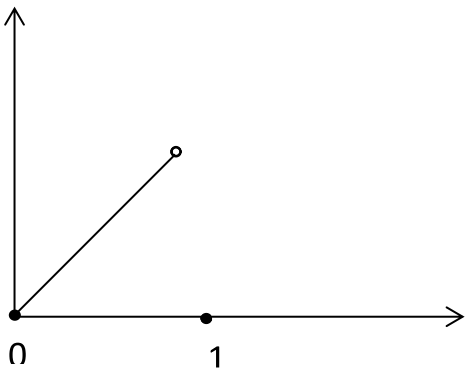


Note that $f'(x) \neq 0$ for any $x \in (1, 2)$. However, this does not contradict Rolle's Theorem, since $f'(1)$ does not exist.

Remark 2: The continuity condition for the function on the closed interval [a, b] is essential.

Ex: Consider

$$f(x) = \begin{cases} x, & x \in [0, 1) \\ 0, & x = 1 \end{cases}$$



Then, f is continuous and differentiable on (0, 1), and also $f(0) = f(1)$. But $f'(x) \neq 0$ for any $x \in (0, 1)$.

Remark 3: The hypotheses of Rolle's theorem are sufficient but not necessary for the conclusion. Meaning, if all three hypotheses are met then conclusion is guaranteed. Not necessary means if the hypotheses are not met then you may (or may not) reach the conclusion.

Example: Discuss the applicability of Rolle's theorem to the function

$$f(x) = \begin{cases} x^2 + 1\,, & x \in [0, 1] \\ 3 - x, & x \in (1, 2] \end{cases}$$

Solution:

1) Continuity

$$f(1+0) = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} 3 - (1 + \Delta x) = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} [2 - \Delta x] = 2 = f(1)$$

2) Differentiability

$$f'(1+0) = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{(2 - \Delta x) - 2}{\Delta x} = -1$$

$$f'(1-0) = \lim_{\substack{\Delta x \to 0 \\ \Delta x < 0}} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\substack{\Delta x \to 0 \\ \Delta x < 0}} \frac{(1+\Delta x)^2 + 1 - 2}{\Delta x}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta x < 0}} \frac{2\Delta x + \Delta x^2}{\Delta x} = 2$$

Thus $f'(1+0) \neq f'(1-0)$. This implies f is not differentiable.

Example: Using Rolle 's Theorem, show that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one real root in [0, 1].

Solution: Let $f(x) = x^{13} + 7x^3 - 5$ has two real roots, say α and β in [0, 1]. That is, we have $f(\alpha) = f(\beta) = 0$. All hypotheses of Rolle's theorem are satisfied in $[\alpha, \beta]$.

Rolle 's Theorem implies $f'(c) = 0$ for some $c \in (\alpha, \beta)$.

$\Rightarrow 13\, c^{12} + 21c^2 = 0$ for some $c \in (\alpha, \beta)$. Note that $c > 0$ as $\alpha \geq 0$. It contradicts our assumption of two real roots.

On the other hand $f(0) = -5$ and $f(1) = 3$. It confirms the existence of at least one root. Hence the function has exactly one root.

Lagrange's mean value theorem:

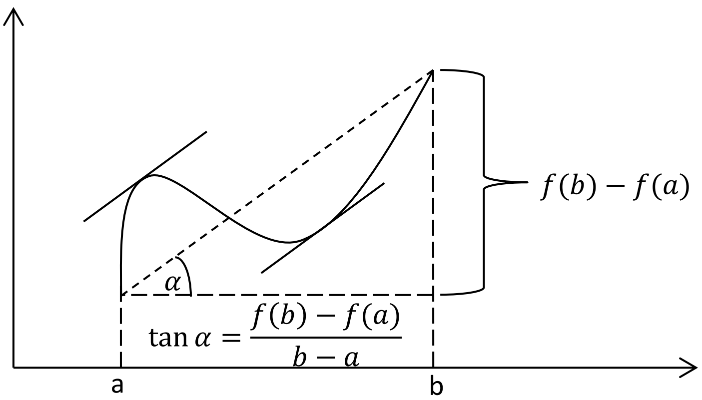
If a function f is

a) continuous in [a, b]
b) differentiable in (a, b)

then there exists at least one value $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

In other words, there is at least one tangent line in the interval that is parallel to the secant line that goes through the endpoints of the interval.



Proof: Define a function

$$\phi(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}\right] x$$

Note that the function $\phi(x)$ satisfies all the conditions of Rolle 's Theorem as $\phi(a) = \phi(b)$, and continuity and differentiability follows from the continuity and differentiability of $f(x)$. Rolle 's Theorem gives

$\phi'(c) = 0$ for some $c \in (a, b) \Rightarrow f'(c) - \frac{f(b)-f(a)}{b-a} = 0.$

Generalized mean value theorem (Cauchy mean value theorem):

If $f(x)$ and $g(x)$ are two functions continuous in [a, b] and differentiable in (a, b), and $g'(x)$ does not vanish anywhere inside the interval then ∃ a point c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Define

$$\phi(x) = \big(f(x)-f(a)\big) - \left[\frac{f(b)-f(a)}{g(b)-g(a)}\right]\big(g(x)-g(a)\big)$$

Note that $g(b) \neq g(a)$ because g' does not vanish in (a, b). If $g(b) = g(a)$ then Rolle's Theorem implies $g'(c) = 0$, which contradicts the assumption that $g'(x) \neq 0$.

$\phi(x)$ satisfies all hypotheses of the Rolle's theorem on the interval [a, b]. Then there exists a point $c \in (a, b)$ such that $c \in (a, b)$ and $\phi'(c) = 0$.

$$\Rightarrow f'(c) - \left[\frac{f(b)-f(a)}{g(b)-g(a)}\right] g'(c) = 0$$

$$\Rightarrow \left[\frac{f(b)-f(a)}{g(b)-g(a)}\right] = \frac{f'(c)}{g'(c)}\ .$$

Notice that:

$$\text{Generalized MVT} \xRightarrow{g(x)=x} \text{Lagrange MVT} \xRightarrow{f(a)=f(b)} \text{Rolle's Theorem}$$

Ex: Using mean value theorem show that

$|\cos e^{x} - \cos e^{y}| \leq |x - y|$ for $x, y \leq 0$ (equality holds for x = y)

Sol: Consider $f(t) = \cos e^{t}$ in the interval$[x, y]$. Using Lagrange mean value theorem

$$\frac{\cos e^{x} - \cos e^{y}}{x - y} = f'(c), \qquad c \in (x, y)$$

$$\Rightarrow |\cos e^{x} - \cos e^{y}| \leq |x - y| \max_{c \in (x,y)} f'(c) < |x - y|$$

as $f'(t) = -\mathrm{e}^{\mathrm{t}} \sin e^{t} \Rightarrow |f'(t)| = |e^{t}||\sin e^{t}| < 1 \text{ for } t < 0$

Ex: Using mean value theorem show that

$$\ln(1 + x) \leq \frac{x}{\sqrt{(1+x)}} \text{ for } x \geq 0.$$

Hint: Consider $f(t) = \ln(1 + t) - \frac{t}{\sqrt{(1+t)}}$ in the interval $[0, x]$.

TAYLOR'S FORMULA (Approximations of differentiable functions by polynomials)

Assume that the function f has all derivatives up to the $(n+1)$th order in some interval containing the point $x=a$.

We wish to find a polynomial $P_n(x)$ of degree n, such that

$$P_n(a)=f(a), \quad P_n'(a)=f'(a), \quad P_n''(a)=f''(a), \dots, P_n^{(n)}(a)=f^{(n)}(a) \quad -(i)$$

Note that its is expected that the polynomial is 'in some sense' close to the function f at least in the neighbourhood of $x=a$.

Polynomial Construction: consider a polynomial in powers of $(x-a)$ with undetermined coefficients:

$$P_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \dots + C_n(x-a)^n \quad -(ii)$$

We now define the coefficients $C_0, C_1, \dots C_n$ so that the conditions (i) are satisfied. First, we calculate the derivatives of $P_n(x)$ as

$$P_n'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \dots + nC_n(x-a)^{n-1}$$

$$P_n''(x) = 2C_2 + 3\cdot 2\cdot C_3(x-a) + \dots + n\cdot(n-1)(x-a)^{n-2}$$

$$\vdots$$

$$P_n^{(n)}(x) = n\cdot(n-1)\cdots 2\cdot 1\cdot (x-a)^0$$

Using conditions (i), we get

$$C_0 = f(a), \quad C_1 = f'(a), \quad C_2 = \frac{f''(a)}{2\cdot 1} \quad \dots \quad C_n = \frac{f^{(n)}(a)}{n(n-1)\cdots}$$

subst. in (ii), we obtain:

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\underline{|2}}f''(a) + \dots + \frac{(x-a)^n}{\underline{|n}}f^{(n)}(a)$$

Denoting $R_n(x)$ the difference between the values of the given function $f(x)$ and the constructed polynomial $P_n(x)$:

$$R_n(x) = f(x) - P_n(x)$$

How to evaluate the remainder $R_n(x)$?

Let us write the remainder in the form

$$R_n(x) = \frac{(x-a)^{n+1}}{\lfloor n+1} Q$$

Alternate proof
Advanced calculus
P.M. Fitzpatrik.

Now we define an auxiliary function of t as

$$F(t) = f(x) - f(t) - \frac{(x-t)}{\lfloor 1} f'(t) - \frac{(x-t)^2}{\lfloor 2} f''(t) - \cdots$$
$$- \frac{(x-t)^n}{\lfloor n} f^{(n)}(t) - \frac{(x-t)^{n+1}}{\lfloor n+1} Q \quad , \quad t \in [a, x]$$

Note that $F(a) = 0$ & $F(x) = 0$ and all other conditions of Rolle's theorem are satisfied for $F(t)$. Then we have

$F'(c) = 0$ for some $c \in (a, x)$

$$\Rightarrow \left[-\frac{(x-t)^n}{\lfloor n} f^{(n+1)}(t) + \frac{(x-t)^n}{\lfloor n} Q \right]_{t=c} = 0$$

$$\Rightarrow Q = f^{(n+1)}(c)$$

Therefore $R_n(x) = \dfrac{(x-a)^{n+1}}{\lfloor n+1} f^{(n+1)}(c)$, $c \in (a, x)$ LAGRANGE FORM OF REMAINDER

Finally
$$\boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \cdots + \frac{(x-a)^n}{\lfloor n} f^{(n)}(a) + R_n(x)}$$

This is called Taylor's formula of the function $f(x)$.

REMARKS:

1. Since c lies between x & a, the remainder may be represented in the following form:

$$R_n(x) = \frac{(x-a)^{n+1}}{\lfloor n+1} f^{(n+1)}\left(a+\theta(x-a)\right), \quad \theta \in (0,1)$$

2. If, we set $a=0$ in the Taylor's formula of the function $f(x)$, then it is called Maclaurin's formula.

3. In the Taylor's formula, if the remainder $R_n(x) \to 0$ as $n \to \infty$, then

$$f(x) = f(a) + (x-a) f'(a) + \cdots + \frac{(x-a)^n}{\lfloor n} f^{(n)}(a) + \cdots$$

is called Taylor's series. For $a=0$, it is called Maclaurin's serie

SLIDE (3')

EXAMPLE: Obtain the Taylor's formula for the function $f(x) = \sin x$ about the point $x=0$.
Show that the remainder term goes to zero as $n \to \infty$ and write down the Taylor's series exponsion of $f(x)$.
Approximate $\sin 30°$ with the Taylor's polynomial of degree 3 and estimate the error using remainder form.
Verify the error estimate with the exact error.

$F(x) = \exp(x)$

$P_0(x) = 1$

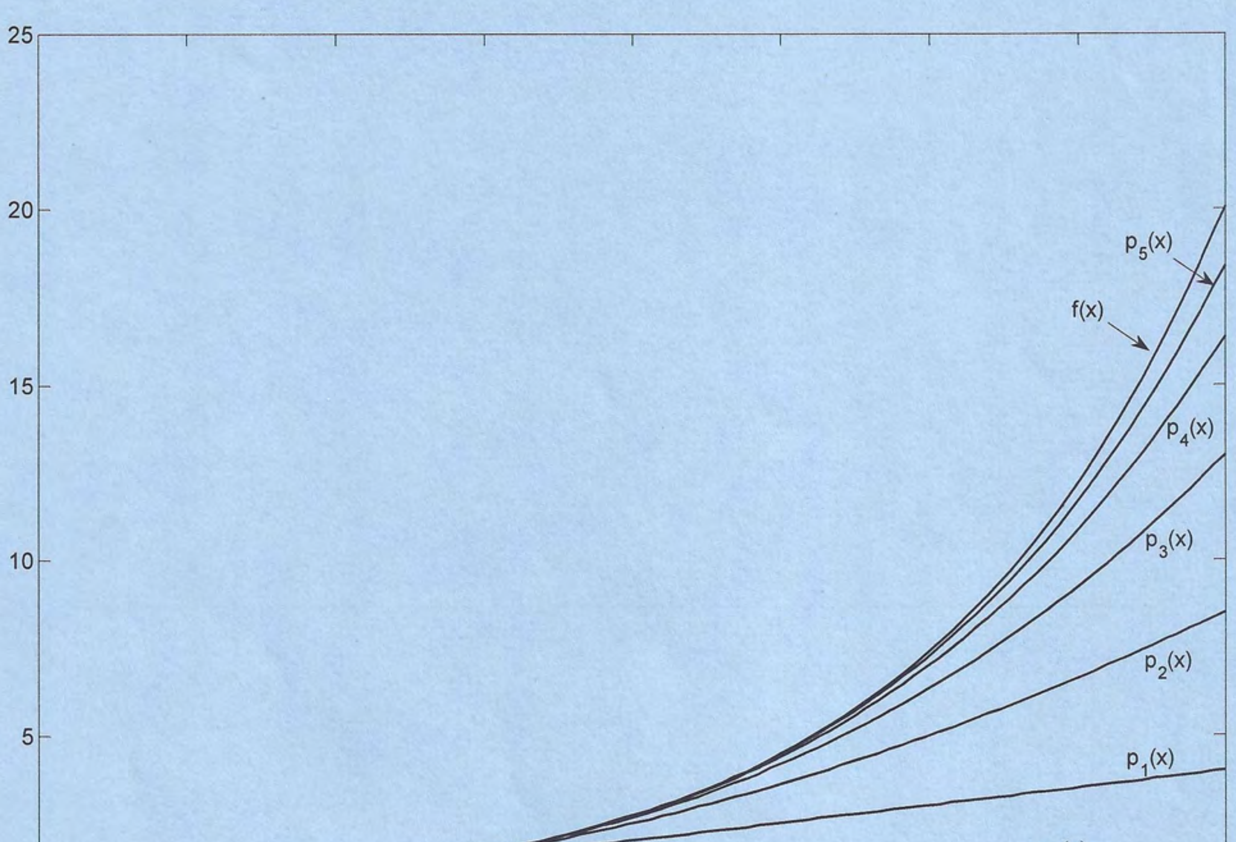
$P_1(x) = 1 + x$

$P_2(x) = 1 + x + x^2/2$

$P_3(x) = 1 + x + x^2/2 + x^3/6$

$P_4(x) = 1 + x + x^2/2 + x^3/6 + x^4/24$

$P_5(x) = 1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120$



SOLUTION:

TAYLOR'S FORMULA

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(IV)}(x) = \sin x \qquad f^{(IV)}(0) = 0$

$f^{(V)}(x) = \cos x \qquad f^{(V)}(0) = 1$

$\vdots$

$f^{(2n)}(x) = (-1)^n \sin x \qquad f^{(2n)}(0) = 0$

$f^{(2n+1)}(x) = (-1)^n \cos x \qquad f^{(2n+1)}(0) = (-1)^n$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor\underline{2}} f''(0) + \cdots + \frac{x^{2n}}{\lfloor\underline{2n}} f^{(2n)}(0) + \frac{x^{2n+1}}{\lfloor\underline{2n+1}} f^{(2n+}(0$$

$$+ \frac{x^{2n+2}}{\lfloor\underline{2n+2}} f^{(2n+2)}(c) \, , \quad c \in (0,$$

$$f(x) = x - \frac{x^3}{\lfloor\underline{3}} + \frac{x^5}{\lfloor\underline{5}} + \cdots + \frac{x^{2n+1}}{\lfloor\underline{2n+1}} (-1)^n + \frac{x^{2n+2}}{\lfloor\underline{2n+2}} f^{(2n+2)}(c) \, .$$

REMAINDER $\rightarrow 0$:

$$|R_n| = \left| \frac{x^{2n+2}}{\lfloor\underline{2n+2}} \cdot (-1)^{n+1} \sin c \right| \leq \left| \frac{x^{2n+2}}{\lfloor\underline{2n+2}} \right| = \frac{|x|^{2n+2}}{\lfloor\underline{2n+2}}$$

$$\lim_{n \to \infty} \frac{|x|^{2n+2}}{\lfloor\underline{2n+2}} = ?$$

$$\lim_{n\to\infty} \frac{|x|^{2n+2}}{\lfloor\underline{2n+2}}$$

For a fixed x we can always find a N such that

$$|x| < N$$

Consider $2n+2 > N$ and do the following:

$$\frac{|x|^{2n+2}}{\lfloor\underline{2n+2}} = \frac{|x|^{2n+2}}{1\cdot 2 \cdot \ldots \ldots \cdot (2n+2)} = \frac{|x|}{1}\cdot\frac{|x|}{2}\cdots\frac{|x|}{N-1}\cdot\frac{|x|}{N}\,\frac{|x|}{N+1}\cdots\frac{|x|}{(2n+}$$

Let $\frac{|x|}{N} = q < 1$

Then

$$\frac{|x|^{2n+2}}{\lfloor\underline{2n+2}} < \underbrace{\frac{|x|}{1}\,\frac{|x|}{2}\cdots\frac{|x|}{N-1}}\, q\cdot q\cdot\cdots q$$

$$= \frac{|x|^{N-1}}{\lfloor\underline{N-1}}\cdot q^{(2n+2)-(N-1)} = \frac{|x|^{N-1}}{\lfloor\underline{N-1}}\cdot \underbrace{q}_{<1}{}^{2n-N+3}$$

As $n \to \infty$

$$\frac{|x|^{2n+2}}{\lfloor\underline{2n+2}} \to 0 . \quad \text{Hence} \quad \lim_{n\to\infty} |R_n| = 0 .$$

The Taylor's series expansion is given as

$$\sin x = x - \frac{x^3}{\lfloor\underline{3}} + \frac{x^5}{\lfloor\underline{5}} - \cdots + (-1)^n \frac{x^{2n+1}}{\lfloor\underline{2n+1}} + \cdots$$

APPROXIMATION OF SIN 30°:

$$\sin 30^\circ = \sin\frac{\pi}{6} \simeq \frac{\pi}{6} - \left(\frac{\pi}{6}\right)^3 \frac{1}{\lfloor 3}$$

$$= 0.49967417$$

ERROR ESTIMATE:

$$\left|R_3(x)\right| = \left|\frac{x^4}{\lfloor 4} f^{(4)}(c)\right| \Rightarrow \left|R_3(x)\Big|_{x=\pi/6}\right| = \left|\frac{\pi^4}{6^4}\cdot\frac{1}{\lfloor 4}\cdot \sin c\right|$$

$$\leq \frac{\pi^4}{6^4}\frac{1}{\lfloor 4} = 0.00313.$$

In this case $f^{IV}(0) = 0$, so a better error bound may be obtained

$$\left|R_4(x)\right| = \left|\frac{x^5}{\lfloor 5} f^{(5)}(c)\right| \Rightarrow \left|R_4\left(\frac{\pi}{6}\right)\right| = \left|\left(\frac{\pi}{6}\right)^5 \cdot \frac{1}{\lfloor 5} \cos c\right|$$

$$\leq \frac{\pi^5}{6^5}\frac{1}{\lfloor 5} = 0.000327$$

Exact error:

$$= \sin\left(\frac{\pi}{6}\right) - 0.49967417$$

$$= 0.000325.$$

Ex. Find the number of terms that must be retained in the Taylor's polynomial approximation about the point $x=0$ for the function $\cosh x$ in the interval $[0,1]$ such that $|\text{error}| < 0.001$.

Sol:

$$f(x) = \cosh x \qquad \cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$f'(x) = \sinh x$$

$$f''(x) = \cosh x$$

$$\vdots$$

$$|R_n(x)| = \left|\frac{x^{n+1}}{\lfloor n+1} f^{(n+1)}(\xi)\right| \qquad \xi \in (0,1)$$

$$= \frac{|x|^{n+1}}{\lfloor n+1} |f^{(n+1)}(\xi)| \leq \frac{1}{\lfloor n+1} |f^{(n+1)}(\xi)|$$

Note that $|f^{(n+1)}(\xi)| \leq \frac{e^{\xi}+e^{-\xi}}{2} < \frac{e+e^{-1}}{2}$

Now set $\left(\frac{e+\frac{1}{e}}{2}\right)\frac{1}{\lfloor n+1} < 0.001 \Rightarrow n \geq 6$

$\Rightarrow$ Minimum six terms are required.

$$P_5 = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{(iv)}(0) + \frac{x^5}{\lfloor 5} f^{(v)}(0) + \frac{x^6}{\lfloor 6} f^{(vi)}(0).$$

FUNCTIONS OF SEVERAL REAL VARIABLES

DEF. If to each point (x,y) of a certain part of the x-y plane, there corresponds a real value z according to some given rule $f(x,y)$, then $f(x,y)$ is called a real valued function of two variables x & y. It is written as

$$z = f(x,y) \quad ; \quad (x,y) \in \mathbb{R}^2, \; z \in \mathbb{R}$$

$x, y \rightarrow$ independent variables

$z \rightarrow$ dependent variable

A real valued function of n-variables is defined as

$$z = f(x_1, x_2, \ldots, x_n) \; ; \; (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \; z \in \mathbb{R} \quad —($$

The function given in (1) is called an explicit function, whereas a function defined by $\Phi(z, x_1, x_2, \ldots, x_n) = 0$ is called an implicit function.

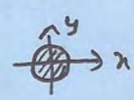
FUNCTION OF TWO VARIABLES:

$$z = f(x,y)$$

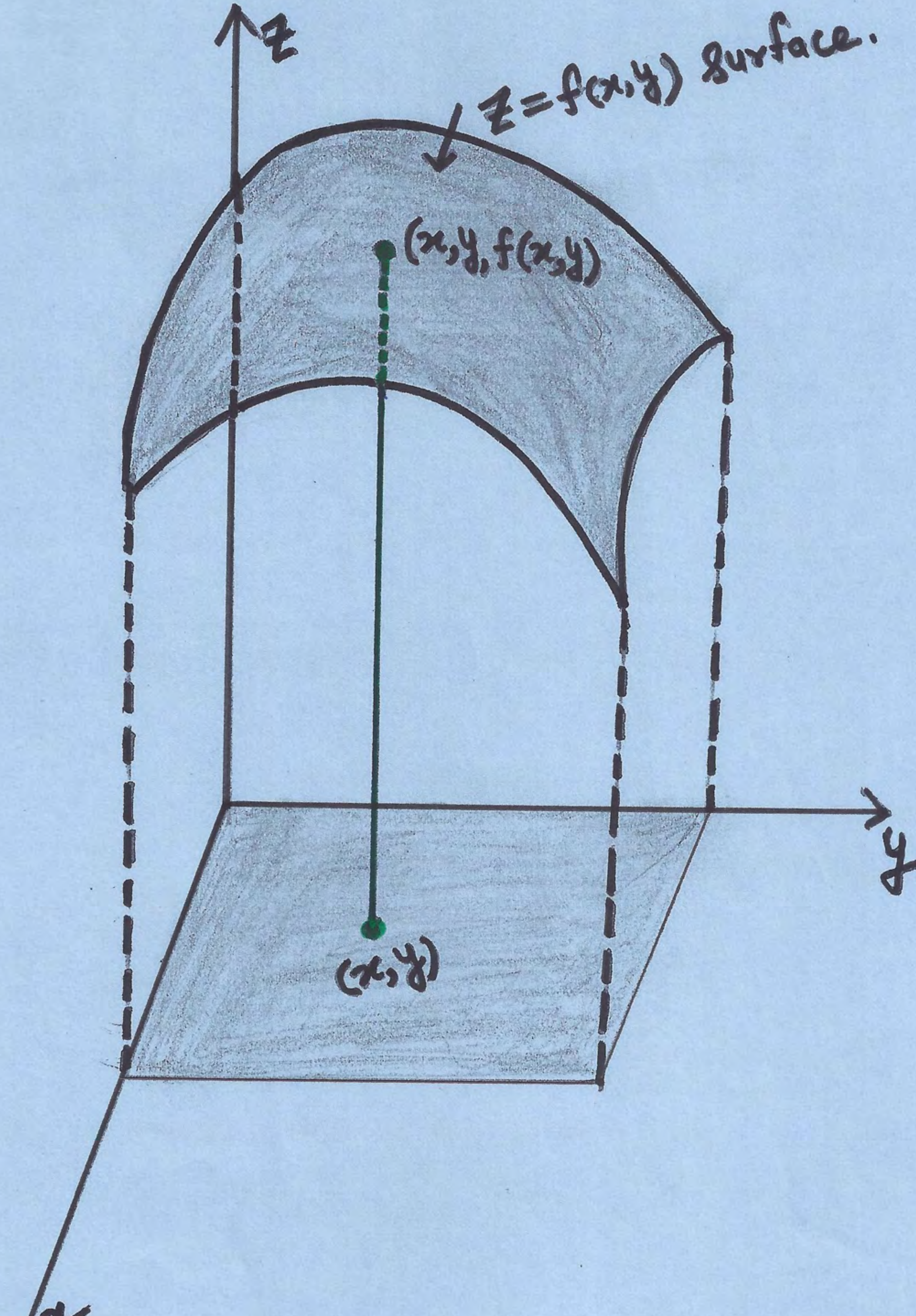
The set of point (x,y) in the x-y plane for which $f(x,y)$ is defined is called domain of definition of the function and is denoted by D. The collection of the corresponding values of z is called the range of the function f.

Ex. $\quad z = \sqrt{1-x^2-y^2}$

Since z is real, we have $\quad 1-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 1$.

$\Rightarrow$ Domain : $D = \{(x,y) : x^2+y^2 \leq 1\}$ 

Range: set of all real positive numbers between 0 & 1.



GEOMETRIC REPRESENTATION OF A FUNCTION OF TWO VARIABLES

DEF.

- DISTANCE BETWEEN THE TWO POINTS:

$$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

- NEIGHBOURHOOD OF A POINT:

Consider a point $P(x_0, y_0)$

δ-neighbourhood of P $\left(N_\delta(P) \text{ or } N(P, \delta)\right)$

$$:= \left\{ (x,y) : \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \right\}$$

OR

$$N_\delta(P) = \left\{ (x,y) : \underset{(<)}{x_0 - \delta \leq} x \underset{(<)}{\leq x_0 + \delta},\ \underset{(<)}{y_0 - \delta \leq} y \underset{(<)}{\leq y_0 + \delta} \right\}$$

- OPEN DOMAIN:

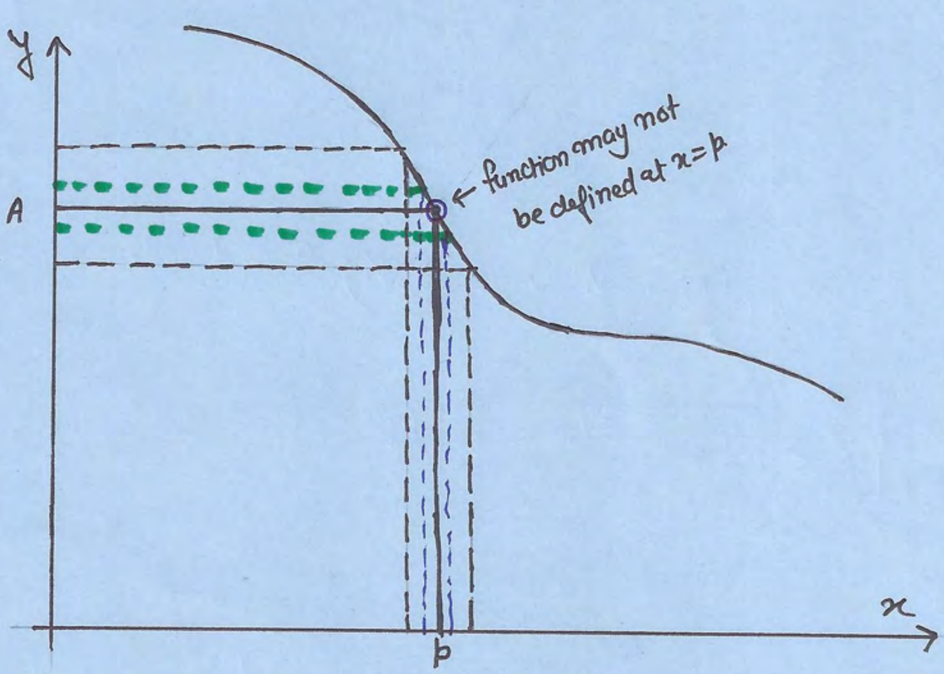
A domain D is open, if there exists a number $\delta > 0$ corresponding to every point p in D such that all point in δ- neighboured of p are in D.

- BOUNDED DOMAIN: D is bounded if $\exists$ a number (finite & positive) M such that D can be enclosed within a circle with radius M and centre at origin.
- CLOSED REGION: A closed region is a bounded domain with its boundary.
- BOUNDED FUNCTIONS: A function $f(x,y)$ defined in some domain D in $\mathbb{R}^2$ is bounded, if there exists a real number (finite) M, such that $|f(x,y)| \leq M$ for all $(x,y) \in D$..

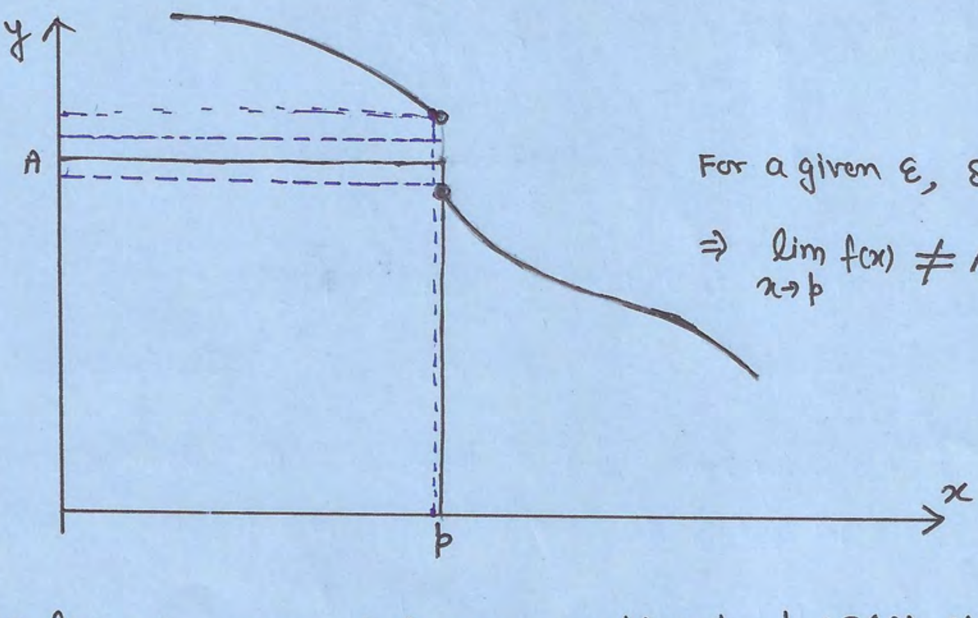
LIMIT OF A FUNCTION

- ONE VARIABLE (RECALL)

  - $\lim_{x \to p} f(x) = A$ means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ whenever $0 < |x - p| < \delta$.



$\lim_{x \to p} f(x) = A$



For a given ε, δ does not exist

$\Rightarrow \lim_{x \to p} f(x) \neq A$

- $\lim_{x \to p} f(x) = A$ means that every neighbourhood $N_\varepsilon(A)$ of A there is some neighbourhood $N_\delta(p)$ such that

$$f(x) \in N_\varepsilon(A) \quad \text{whenever} \quad x \in N_\delta(p) \text{ and } x \neq p.$$

Let $z = f(x,y)$ be a function of two variables defined in a domain

D. Let $P(x_0, y_0)$ be a point in D. If for a given real number $\varepsilon > 0$, however small, we can find a real number $\delta > 0$ such that for every point (x,y) in the δ-neighbourhood of $P(x_0,y_0)$

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

then the real number L is called the limit of the function $f(x,y)$ as $(x,y) \to (x_0,y_0)$. Symbolically,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

REMARK:
Note that for the limit to exist, the function $f(x,y)$ may or may not be defined at (x_0,y_0). If $f(x,y)$ is not defined at $P(x_0,y_0)$ then we write

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

EXAMPLE: Using ε-δ approach show that

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{x^2+y^2}}\right) = 0$$

Sol. For $(x,y) \neq (0,0)$,

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{2xy}{2\sqrt{x^2+y^2}}\right| \leq \frac{x^2+y^2}{2\sqrt{x^2+y^2}} = \underbrace{\frac{1}{2}\sqrt{x^2+y^2}}_{< \frac{1}{2}\delta} < \varepsilon$$

as $(x-y)^2 = x^2+y^2-2xy \geq 0 \Rightarrow x^2+y^2 \geq 2xy$

If we choose $\delta < 2\varepsilon$

then

$$\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta$$

Hence

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 .$$

EXAMPLE: Show that

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{xy}\right) = 0$$

Sol: For $(x,y) \neq (0,0)$

$$\left| (x^2+y^2)\sin\left(\frac{1}{xy}\right) - 0 \right| = \left| (x^2+y^2)\sin\left(\frac{1}{xy}\right) \right| \leq \underbrace{(x^2+y^2) \overset{!}{<} \varepsilon}_{\delta^2 < \varepsilon}$$

If we choose $\delta^2 < \varepsilon$ then

$$\left| (x^2+y^2)\sin\left(\frac{1}{xy}\right) \right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta .$$

Hence

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{xy}\right) = 0$$

LIMIT (Cont.)

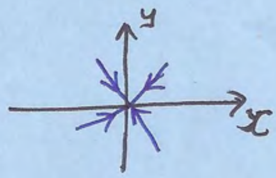
REMARK: Since $(x, y) \to (x_0, y_0)$ in the two dimensional plane, there are infinite number of paths joining (x, y) to (x_0, y_0). Since the limit, if exists, is unique, the limit should be the same along all the paths. Thus, the limit cannot be obtained by approaching the point P along a particular path and finding the limit of $f(x, y)$. If the limit is dependent on a path, then the limit does not exist.

EXAMPLE :

- $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = mx}} \dfrac{x^2 y}{x^4 + y^2}$

- $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = mx^2}} \dfrac{x^2 y}{x^4 + y^2}$

Along $y = mx$:

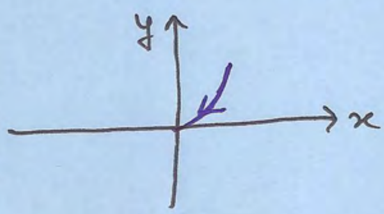


$$\lim_{x\to 0} \frac{x^2 . mx}{x^4 + m^2 x^2} = \lim_{x\to 0} \left(\frac{mx}{x^2 + m^2}\right) = 0$$

Along $y = mx^2$:

$$\lim_{x\to 0} \frac{x^2 . mx^2}{x^4 + m^2 x^4} = \frac{m}{1 + m^2}$$

In particular, $y = x^2$, we get $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4 + y^2} = \dfrac{1}{2}$



Hence limit does not exist in this case.

Ex. show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exist.

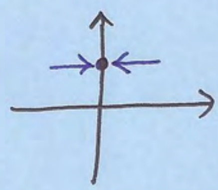
Sol. consider $y = mx$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

The limit depends on path. Hence the limit does not exist.

Ex. Show that the limit $\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right)$ does not exist.

Sol. Let us fix $y = 1$ and calculate



$$\left.\begin{aligned} &\lim_{x\to 0-0} \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(-\infty) = -\frac{\pi}{2} \\ &\lim_{x\to 0+0} \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(+\infty) = \frac{\pi}{2} \end{aligned}\right\} \Rightarrow \text{limit does not exist.}$$

Working with limits: suppose, we have

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$$

then

i) $\lim_{(x,y)\to(x_0,y_0)} [K f(x,y)] = K L_1$ for any real constant K.

ii) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L_1 \pm L_2$

iii) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\, g(x,y)] = L_1 \cdot L_2$

iv) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)/g(x,y)] = \frac{L_1}{L_2}$ provided, $L_2 \neq 0$.

CONTINUITY:

A function $z = f(x,y)$ is said to be continuous at a point (x_0, y_0) if

i) $f(x,y)$ is defined at the point (x_0, y_0)

ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

OR

A function $f(x,y)$ is said to be continuous at (x_0,y_0) if for a given $\varepsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x,y) - f(x_0,y_0)| < \varepsilon \quad \text{whenever} \quad \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

If $f(x_0,y_0)$ is defined and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ exists but $f(x_0,y_0) \neq L$, then the point (x_0,y_0) is called a point of removable discontinuity.

If a function $f(x,y)$ is continuous at every point in a domain D, then it is said to be continuous in D.

EXAMPLE: Show that the following functions are continuous

i) $f(x,y) = \begin{cases} \dfrac{2x^4+3y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

ii) $f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & ; (x,y) \neq (0,0) \\ \frac{1}{2} & (x,y) = (0,0) \end{cases}$

Sol: i) Change of coordinate system:

$$x = r\cos\theta \qquad y = r\sin\theta$$

Note that $r = \sqrt{x^2+y^2} \neq 0$ since $(x,y) \neq (0,0)$.

Consider $\left|f(x,y) - f(0,0)\right| = \left|\dfrac{2r^4\cos^4\theta + 3r^4\sin^4\theta}{r^2(\cos^2\theta + \sin^2\theta)}\right| = \left|2r^2\cos^4\theta + 3r^2\sin^4\theta\right|$

$$\leq 2r^2\cos^4\theta + 3r^2\sin^4\theta < 5r^2 < 5\delta^2 < \varepsilon$$

$\Rightarrow$ Choose $\delta < \sqrt{\dfrac{\varepsilon}{5}}$

$\Rightarrow \left|f(x,y) - f(0,0)\right| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

Hence $f(x,y)$ is continuous at $(0,0)$.

Alternative Approach: Change of coordinate $x = r\cos\theta$, $y = r\sin\theta$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^4+3y^4}{x^2+y^2} = \lim\limits_{\substack{(x,y)\to(0,0) \\ \text{or } r\to 0}} \dfrac{2\cdot r^4\cos^4\theta + 3r^4\sin^4\theta}{r^2} = 0$ (No dependency on θ) If it depends on θ then limit does not exist.

ii) $$f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & (x,y) \neq (0,0) \\ 1/2 & (x,y) = (0,0) \end{cases}$$

Choose $x+2y = t$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} = \lim_{t\to 0} \frac{\sin^{-1}(t)}{\tan^{-1}(2t)} = \lim_{t\to 0} \frac{\left(\frac{1}{\sqrt{1-t^2}}\right)}{\left(\frac{2}{1+4t^2}\right)} = \frac{1}{2}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2} = f(0,0)$$

$\Rightarrow$ The function is continuous.

Ex. Discuss the continuity of the functions

i) $$f(x,y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Choosing the path $y = mx$.

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x-mx)^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{(1-m)^2}{1+m^2} = \frac{(1-m)^2}{1+m^2}$$

The limit depends on the path. Therefore the limit does not exist and the function is not continuous at $(0,0)$.

ii) $$f(x,y) = \begin{cases} \dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{t\to 0} \frac{\sin\sqrt{t}}{\sqrt{t}} = 1 \neq f(0,0)$$

The function is discontinuous at $(0,0)$.

Note that the point $(0,0)$ is a point of removable discontinuity.

iii) $$f(x,y) = \begin{cases} \dfrac{e^{xy}}{x^2+1} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x^2+1} = \frac{1}{0+1} = 1 \neq f(0,0)$$

The function is not continuous at $(0,0)$

(iv) $$f(x,y) = \begin{cases} \dfrac{x^4 y^4}{(x^2+y^4)^3} & ; \quad (x,y) \neq (0,0) \\ 0 & ; \quad (x,y) = (0,0) \end{cases}$$

Choosing the path $y^2 = mx$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0}\left[\frac{x^4y^4}{(x^2+y^4)^3}\right]_{y^2=mx} = \lim_{x\to 0} \frac{x^4 \cdot m^2x^2}{(x^2+m^2x^2)^3}$$

$$= \frac{m^2}{(1+m^2)^3}$$

The limit depends on the path and therefore does not exist.

The function is discontinuous at $(0,0)$.

v)

$$f(x,y)=\begin{cases} \dfrac{x^2+y^2}{\tan xy} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

Sol:

Take path $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\tan xy} = \lim_{x\to 0} \frac{x^2+m^2x^2}{\tan(mx^2)} \qquad (m\neq 0)$$

$$= \lim_{x\to 0} \frac{(1+m^2)}{m\cdot\dfrac{\tan(mx^2)}{mx^2}} = \frac{1+m^2}{m}$$

as $\displaystyle\lim_{x\to 0} \frac{\tan(mx^2)}{mx^2} = 1.$

The limit depends on the path, hence it does not exist.

The function is discontinuous at $(0,0)$.

Do not follow as:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\tan xy} = \underbrace{\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}}_{\text{does not exist } (y=mx)} \cdot \underbrace{\lim_{(x,y)\to(0,0)} \frac{1}{\frac{\tan(xy)}{xy}}}_{=1} \Rightarrow \text{limit does not exist.}$$

$\lim(fg) = \lim f.\lim g$ is valid when both limits $\lim f$ & $\lim g$ exist!

Consider for example: $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x^2+y^2}{1+xy}\right)$, Clearly this limit exists and is equal to zero.

However, if we rewrite as $\displaystyle\underbrace{\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}}_{\text{does not exist}} \cdot \underbrace{\lim_{(x,y)\to(0,0)} \left(\frac{xy}{1+xy}\right)}_{=0}$

$\Rightarrow$ limit does not exist.

REMARK: Changing to polar coordinate (subst. $x = r\cos\theta$, $y = r\sin\theta$) and investigating the limit of the resulting expression as $r \to 0$ is often very useful. for example:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta = 0.$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0 \quad \text{(Useful in finding limit)}$$

Also, Note that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+y^2)} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta \quad \text{(depends on } \theta)$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+y^2)} \text{ does not exist.}$$

Shifting to polar coordinate does not always help, however, and may even tempt us to false conclusions.

For example:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta * r\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{r\cos^2\theta\sin\theta}{r^2\cos^4\theta+\sin^2\theta}$$

If we hold θ constant then

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = 0 ,$$ BUT this is not the limit, we should not fix θ. Taking the path $r\sin\theta = r^2\cos^2\theta$ ($y = x^2$):

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r\to 0} \frac{\cancel{r}\cos^2\theta\,\cancel{\sin\theta}\, r^2\cos^2\theta}{r^2\cos^4\theta + r^2\cos^4\theta} = \frac{1}{2}$$

$\Rightarrow$ The limit does not exist.

PARTIAL DERIVATIVE

DEF. The usual derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables as constan[t] is called the partial derivative of the function with respect to that variable.

Let $z = f(x, y)$; $(x, y) \in \mathbb{R}^2$, $z \in \mathbb{R}$

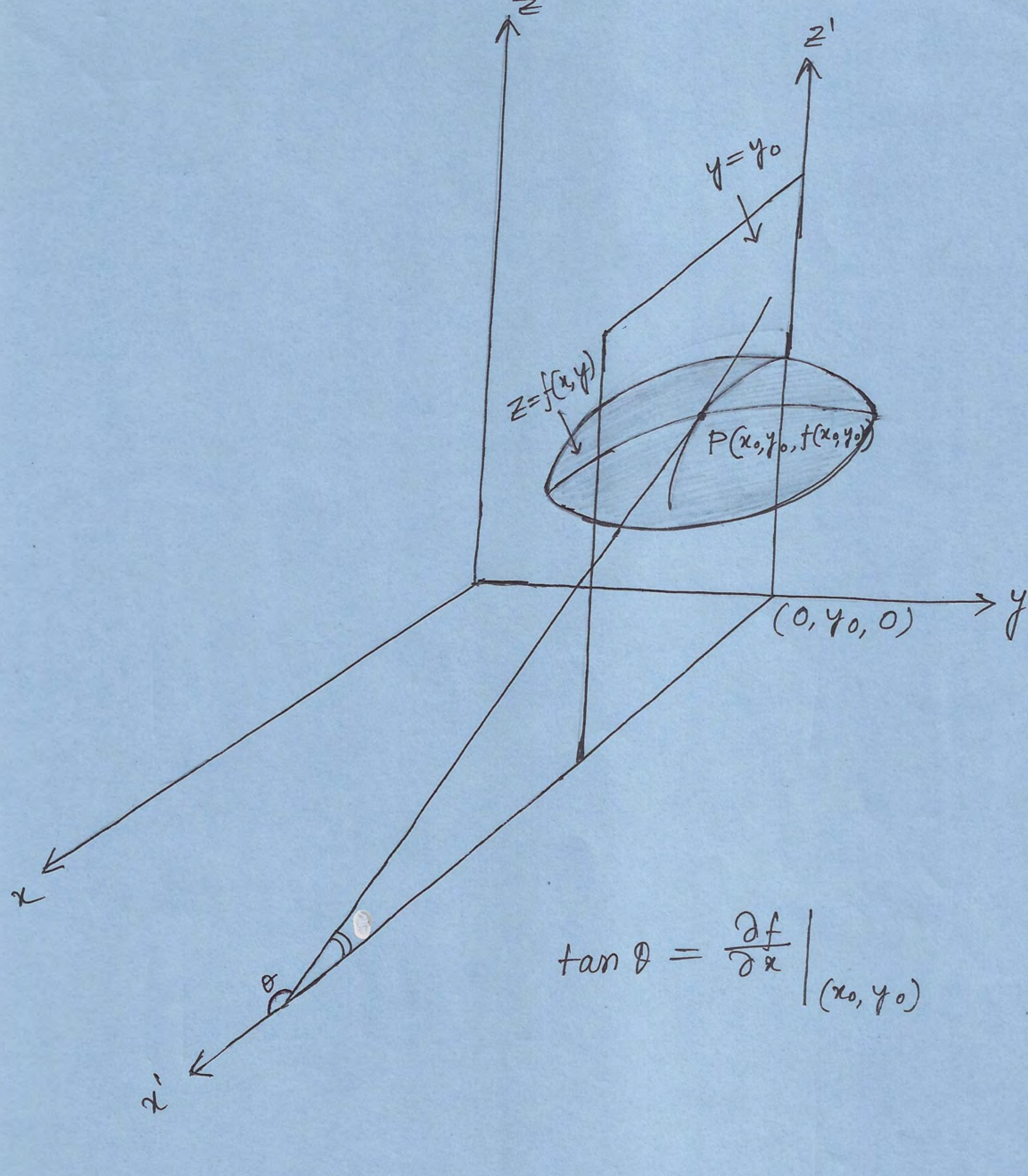
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$= \left.\frac{d}{dx} f(x, y_0)\right|_{x = x_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

$$= \left.\frac{d}{dy} f(x_0, y)\right|_{y = y_0}$$

GEOMETRIC INTERPRETATION:



$$\tan\theta = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}$$

Ex. Find the value of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (x, y) of the following function

(i) $f(x,y) = ye^{-x}$ (ii) $f(x,y) = \sin(2x+3$

from the first principles.

SOL: (i)

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{ye^{-(x+\Delta x)} - ye^{-x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{ye^{-x}\{e^{-\Delta x} - 1\}}{\Delta x}$$

$$= ye^{-x} \lim_{\Delta x \to 0} \frac{\{1 - \Delta x - \frac{\Delta x^2}{2!} - \cdots - 1\}}{\Delta x}$$

$$= -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(y+\Delta y)\, e^{-x} - y e^{-x}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} e^{-x}$$

$$= e^{-x}$$

ii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\sin\{2(x+\Delta x)+3y\} - \sin(2x+3y)}{\Delta x}$$

$\left[\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right.\right.$

$$= \lim_{\Delta x \to 0} \frac{2.\cos(2x+3y+\Delta x) \cdot \sin \Delta x}{\Delta x}$$

$$= 2\cos(2x+3y)$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{\sin(2x+3(y+\Delta y)) - \sin(2x+3y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2 \cdot \cos\left(2x+3y+\frac{3\Delta y}{2}\right) \cdot \sin\left(\frac{3}{2} \cdot \Delta y\right)}{\frac{2}{3} \cdot \left(\frac{3}{2}\Delta y\right)}$$

$$= 3\cos(2x+3y) \cdot \lim_{\Delta y \to 0} \frac{\sin\left(\frac{3}{2}\Delta y\right)}{\frac{3}{2}\Delta y}$$

$$= 3\cos(2x+3y)$$

III RELATIONSHIP BETWEEN CONTINUITY AND THE EXISTENCE OF PARTIAL DERIVATIVES

A function can have partial derivatives with respect to both x and y at a point without being continuous there. On the other hand a continuous function may not have partial derivatives.

EXAMPLE: Show that the function

$$f(x,y) = \begin{cases} (x+y)\,\mathrm{Sin}\left(\dfrac{1}{x+y}\right), & (x,y) \neq (0, \\ 0, & (x,y) = (0, \end{cases}$$

is continuous at $(0,0)$ but its partial derivatives do not exist at $(0,0)$.

SOLUTION:

$$|f(x,y) - f(0,0)|$$

$$= \left|(x+y)\sin\left(\frac{1}{x+y}\right)\right|$$

$$\le |x+y|$$

$$\left.\begin{aligned} &\le |x| + |y| \\ &\le \sqrt{2}\cdot\sqrt{x^2+y^2} \end{aligned}\right\}^{*}$$

$$< \epsilon$$

Choose $\delta < \frac{\epsilon}{\sqrt{2}}$, then

$|f(x,y) - f(0,0)| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} <$

$$\Rightarrow \boxed{\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)}$$

Hence the function is continuous at $(0,0)$

(*)

$$\left(|x| - |y|\right)^2 \geqslant 0$$

$$\Rightarrow \quad x^2 + y^2 \geqslant 2\,|x|\,|y|$$

$$\Rightarrow \quad 2\,(x^2 + y^2) \geqslant x^2 + y^2 + 2\,|x|\,|y|$$

$$\Rightarrow \quad 2\,(x^2 + y^2) \geqslant \left(|x| + |y|\right)^2$$

$$\Rightarrow \quad \left(|x| + |y|\right) \leq \sqrt{2}\sqrt{(x^2 + y^2)}$$

(III) ALTERNATIVE : 1

$$\lim_{(x,y) \to (0,0)} (x+y)\ \sin\left(\frac{1}{x+y}\right)$$

put $x + y = t$

Then $$\lim_{t \to 0} t\ \sin\left(\frac{1}{t}\right)$$

$$= 0$$

Now consider

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \ \sin\left(\frac{1}{\Delta x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \sin\left(\frac{1}{\Delta x}\right)$$

$\Rightarrow f_x(0, 0)$ does not exist.

Similarly $f_y(0, 0)$ does not exist.

EXAMPLE:

Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exist at $(0,0)$.

SOLUTION:

Choose the path $y = mx$

The limit

$$\lim_{x \to 0} \frac{x \cdot mx}{x^2 + 2m^2x^2}$$

$$= \frac{m}{1+m^2}$$

depends on the path.

Hence the function is not continuou at $(0,0)$.

Now consider

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0-0}{\Delta x}$$

$$= 0 = f_x(0,0)$$

$$\lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0-0}{\Delta y}$$

$$= 0 = f_y(0,0)$$

$\Rightarrow$ The partial derivatives f_x and f_y exist at $(0,0)$

THEOREM: [SUFFICIENT CONDITION FOR CONTINUITY AT (0, 0)]

One of the first order partial derivative exist and is bounded in the neighbourhood of (x_0, y_0) and the other exist at (x_0, y_0).

PARTIAL DERIVATIVES OF HIGHER ORDER

$$\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)}_{f_{xx}}, \quad \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}_{f_{yx}}, \quad \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}_{f_{xy}}$$

are called second order partial derivatives of f.

The derivatives f_{xy} and f_{yx} are called mixed derivatives.

If the mixed derivatives f_{yx} and f_{xy} are continuous in an open domain $\mathcal{D}$, then at any point $(x, y) \in \mathcal{D}$

$$f_{xy} = f_{yx}.$$

EXAMPLE: Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

SOLUTION:

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y}$$

$$= 0$$

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \cdot y^3}{\Delta x + y^2} \cdot \frac{1}{\Delta x}$$

$$= y$$

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{x \cdot \Delta y^{\cancel{3}\,2}}{x + \Delta y^2} \cdot \frac{1}{\cancel{\Delta y}}$$

$$= 0$$

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$$

$$= 0$$

$$\therefore \boxed{f_{xy}(0,0) = 0}$$

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(\Delta y - 0)}{\Delta y}$$

$$= 1$$

$$\therefore \boxed{f_{yx}(0,0) = 1}$$

Since $f_{xy}(0,0) \neq f_{yx}(0,0)$, f_{xy} and f_{yx} are not continuous at $(0,0)$.

III CHECK: For $(x,y) \neq (0,0)$

$$f_{yx}(x,y) = \frac{y^6 + 5xy^4}{(x+y^2)^3} = f_{xy}(x,y)$$

Along the path $x = my^2$ the limit $\lim_{(x,y)\to(0,0)} f_{yx}(x,y)$ depends on the path.

This implies, f_{yx} is not continuous at $(0,0)$.

TOTAL DIFFERENTIAL AND DIFFERENTIABILITY

ONE VARIABLE:

Def. 1: We call a function $y = f(x)$ differentiable at a point (x, y) if

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

exists. The value of the above limit is called the derivative of f at x.

Def. 2: The function $y = f(x)$ is said to be differentiable at the point (x, y) if, at this point

$$\Delta y = f(x+\Delta x) - f(x) = a\,\Delta x + \varepsilon \cdot \Delta x$$

where a is independent of Δx and $\lim\limits_{\Delta x \to 0} \varepsilon = 0$.
The value of a is the derivative of f at x.

REMARK: Not that Def 1 & 2 are equivalent as

$$f(x+\Delta x) - f(x) = a\,\Delta x + \varepsilon\,\Delta x$$

$$\Leftrightarrow \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} = a + \varepsilon$$

$$\Leftrightarrow \quad \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(a)}{\Delta x} = a \qquad (\varepsilon \to 0 \text{ as } \Delta x \to 0)$$

Def 1 is more practical for verifying differentiability of a function.

DIFFERENTIAL: The differential of the dependent variable y, written as dy, is defined to be

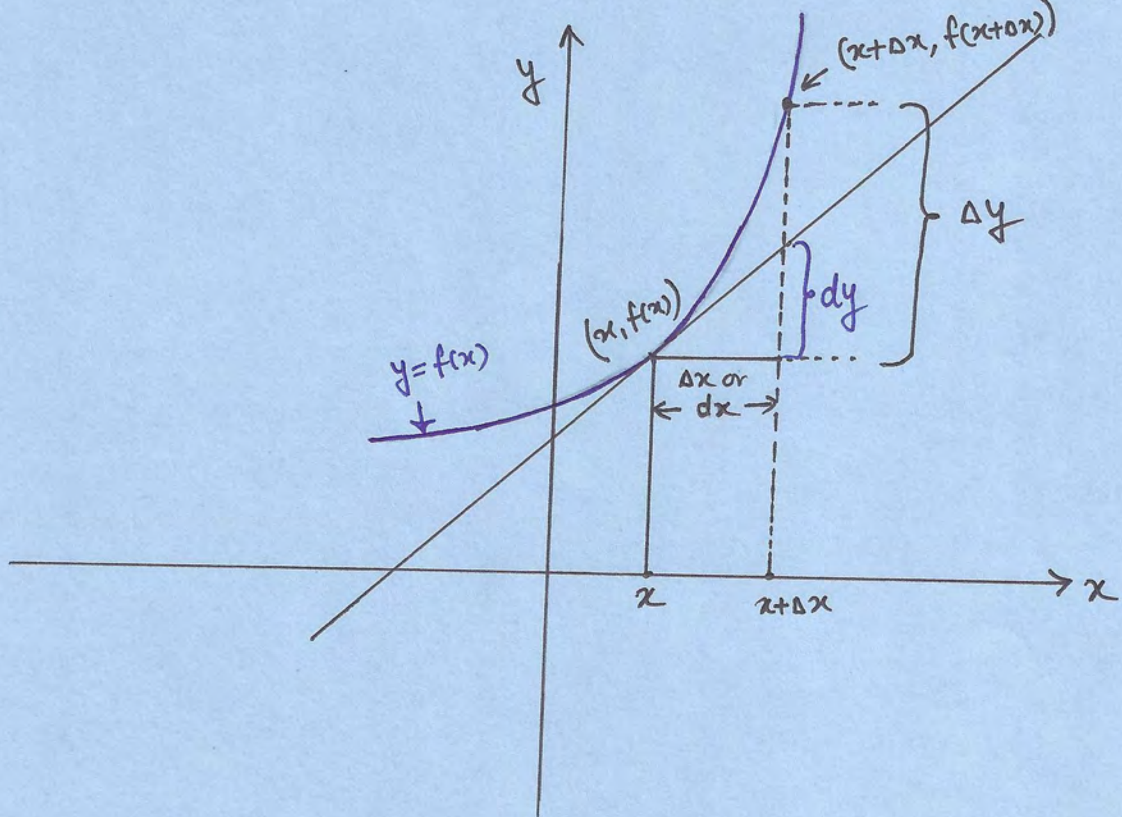
$$dy = f'(x)\,\Delta x, \quad \text{where } y = f(x)$$

$$\text{or } dy = f'(x)\,dx$$

$$\text{or } df = f'(x)\,dx$$

Differential of the independent variable x, written as dx, is same as Δx. one can also observed this by taking $y=x$ and using the above definition of differential as

$$dx = (x)' \Delta x \Rightarrow dx = \Delta x$$



Note that Δx (or dx) is an increment while dy is tolal differential. Δy is the change in y due to change in x by Δx or dx

Also Note that $\Delta y = \underbrace{f'(x)\,\Delta x}_{\text{linear part}} + \varepsilon\,\Delta x$

So the differential is a linear function of the increment Δx.

TWO VARIABLE :

The function $z = f(x, y)$ is said to be differentiable at the point (x, y) if, at this point

$$\Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1 . \Delta x + \epsilon_2 . \Delta y$$

where a and b are independent of Δx, Δy and ϵ_1 and ϵ_2 are functions of Δx and Δy such that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_1 = 0 \quad \text{and} \quad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_2 = 0$$

The linear function of Δx and Δy $a\,\Delta x + b\,\Delta y$ is called the total differential of z at the point (x, y) and is denoted by dz

$$\begin{aligned} dz &= a\,\Delta x + b\,\Delta y \\ &= a\,dx + b\,dy \end{aligned}$$

If Δx and Δy are sufficiently small, dz gives a close approximation to Δz.

EXAMPLE: Show that $z = x^2 + xy + xy^2$ is differentiable and write down its total differential.

SOLUTION:

$$\Delta z = f(x+\Delta x,\ y+\Delta y) - f(x,y)$$

$$= (x+\Delta x)^2 + (x+\Delta x)(y+\Delta y) + (x+\Delta x)(y+\Delta y)^2 - x^2 - xy - xy^2$$

$$= \Delta x\,(2x+y+y^2) + \Delta y\,(x+2x$$

$$+ (\Delta x + \Delta y\,(1+2y))\,\Delta x$$

$$+ (x.\Delta y + \Delta x\,\Delta y)\,\Delta y$$

hence the function is differentiabl

Total differential

$$dz = (2x + y + y^2)\,dx + (x + 2xy)\,dy.$$

NECESSARY CONDITION FOR DIFFERENTIABILITY

THEOREM: If $z = f(x,y)$ is differentiabl then $f(x,y)$ is continuous and has partial derivatives with respe to x and y at the point (x,y) and that

$$a = f_x(x,y) = \frac{\partial z}{\partial x}, \quad b = f_y(x,y) = \frac{\partial z}{\partial y}$$

PROOF: Let f be differentiable, the

$$f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= a \cdot \Delta x + b \cdot \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

then

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f(x+\Delta x, y+\Delta y) = f(x,y)$$

Thus f is continuous.

Setting $\Delta y = 0$ and deviding by Δx yield the relation

$$\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = a + \epsilon_1$$

$$\Rightarrow \boxed{f_x(x,y) = a}$$

Similarly $\boxed{f_y(x,y) = b}$

THEOREM: (SUFFICIENT CONDITION FOR DIFFERENTIABILITY)

If the function $z = f(x, y)$ has continuous first order partial derivatives at a point (x, y), then $f(x, y)$ is differentiable at (x, y).

PROOF:

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$$

$$= f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x, y)$$

Using mean value theorem

$$\Delta z = \Delta x \, f_x(x+\theta_1 \Delta x, y+\Delta y) + \Delta y \, f_y(x, y+\theta_2 \Delta y)$$

when $0 < \theta_1, \theta_2 < 1$

Since the partial derivatives f_x and f_y are continuous at the point (x, y), we can wri

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f_x(x + \theta_1 \Delta x,\ y + \Delta y) = f_x(x, y)$$

$$\Rightarrow f_x(x + \theta_1 \Delta x,\ y + \Delta y) = f_x(x, y) + \epsilon_1$$

$$[\ \epsilon_1 \to 0 \text{ as } \Delta x, \Delta$$
$$\to 0$$

and

$$f_y(x,\ y + \theta_2 \Delta y) = f_y(x, y) + \epsilon_2$$

$$[\ \epsilon_2 \to 0 \text{ as } \Delta x, \Delta y \to$$

Thus,

$$\Delta z = \Delta x\, f_x(x, y) + \Delta y\, f_y(x, y) + \epsilon_1 \Delta x + \epsilon_2$$

This implies differentiability of f.

THEOREM (V-2) (SUFFICIENT CONDITION FOR DIFFERENTIABILITY)

If one of the partial derivatives exists and the other is continuous, then the function is differentiable at (x, y).

PROOF: $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$$= f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x, y+\Delta y) - f(x,y) \quad \text{—(1)}$$

Suppose f_y exist and f_x is continuous, then

$$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = f_y(x,y) \quad \left(\text{existence of } f_y\right)$$

$$\Rightarrow \quad f(x, y+\Delta y) - f(x,y) = \Delta y\, f_y(x,y) + \varepsilon_2 \cdot \Delta y \quad \text{—(2)}$$

where $\varepsilon_2 \to 0$ as $\Delta y \to 0$

Also, By Lagrange mean value theorem

$$f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = \Delta x \cdot f_x\left(x+\theta_1 \Delta x, y+\Delta y\right) \quad \text{—(3)}$$

where $0 < \theta_1 < 1$

Continuity of f_x implies:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f_x\left(x+\theta_1 \Delta x, y+\Delta y\right) = f_x(x,y)$$

$$\Rightarrow f_x\left(x+\theta_1 \Delta x, y+\Delta y\right) = f_x(x,y) + \varepsilon_1 \quad \text{(4)} \quad (\varepsilon_1 \to 0 \text{ as } \Delta x, \Delta y \to 0)$$

(1), (2), (3), (4) $\Rightarrow$ $\Delta z = \Delta x\, f_x(x,y) + \Delta y\, f_y(x,y) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

$\Rightarrow$ Differentiability of f.

To test the differentiability at a point $P(x, y)$, we use

either

$$\Delta z = \underbrace{\frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y}_{dz} + \epsilon_1 \Delta x + \epsilon_2$$

or

$$\frac{\Delta z - dz}{\Delta \rho} = \epsilon_1 \frac{\Delta x}{\Delta \rho} + \epsilon_2 \frac{\Delta y}{\Delta \rho},$$

where $\Delta \rho = \sqrt{\Delta x^2 + \Delta y^2}$

$$\Rightarrow \lim_{\Delta \rho \to 0} \frac{\Delta z - dz}{\Delta \rho}$$

$$= \lim_{\Delta \rho \to 0} \left[\epsilon_1 \left(\frac{\Delta x}{\Delta \rho} \right) + \epsilon_2 \left(\frac{\Delta y}{\Delta \rho} \right) \right]$$

$$= 0$$

Since $\left| \frac{\Delta x}{\Delta \rho} \right| \leq 1$ and $\left| \frac{\Delta y}{\Delta \rho} \right| \leq 1$

and ϵ_1 and ϵ_2 tends to zero as $\Delta \rho \to 0$.

To test the differentiability we show that

$$\boxed{\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = 0}$$

REMARKS:

① The function may not be differentiable at a point $P(x, y)$, even if the partial derivatives f_x and f_y exists at P.
(Because existence of partial derivatives is a necessary condition)

② A function may be differentiable even if f_x and f_y

are not continuous. (Because continuity of the f_x and f_y is a sufficient condition).

③ Sufficient conditions of continuity can be relaxed. It is sufficient that one of the partial derivative exist and the other is continuous.

EXAMPLE: (CONTINUOUS, PARTIAL DERIVATIVES EXIST BUT NOT DIFFERENTIABLE)

$$f(x,y) = \begin{cases} \dfrac{x^3+2y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

i) CONTINUITY: $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{x^3+2y^3}{x^2+y^2}$ (Necessary for differentiability)

Changing to polar coordinates:

$$\lim_{r \to 0} \frac{r^3\cos^3\theta + 2r^3\sin^3\theta}{r^2}$$

$$= \lim_{r \to 0} r\left(\cos^3\theta + 2\sin^3\theta\right) = 0 = f(0,0)$$

ALTERNATIVE:

$$|f(x,y) - 0| = \left| \frac{r^3\cos^3\theta + 2r^3\sin^3\theta}{r^2} \right|$$ (subst. $x = r\cos\theta$, $y = r\sin\theta$)

$$\leq r\,|\cos^3\theta| + 2r\,|\sin^3\theta|$$

$$< 3r \overset{!}{<} \varepsilon$$

Choose $\delta < \dfrac{\varepsilon}{3}$ then

$$|f(x,y) - 0| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta$$

$\Rightarrow$ $f(x,y)$ is continuous at $(0,0)$.

ii) Existence of partial derivatives: (Necessary for differentiability)

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^3}{\Delta x^3} = 1.$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2\Delta y^3}{\Delta y^3} = 2$$

iii) Differentiability:

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} \neq 0$$

as.
$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0)$$

$$= \frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2 + \Delta y^2}$$

$$dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y$$

$$= \Delta x + 2\Delta y$$

Now:
$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho}$$

$$= \lim_{\Delta\rho \to 0} \left[\frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2 + \Delta y^2} - (\Delta x + 2\Delta y) \right] \frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$= \lim_{\Delta\rho \to 0} \frac{-\Delta x \Delta y^2 - 2\Delta x^2 \Delta y}{(\Delta x^2 + \Delta y^2)^{3/2}}$$

Along the path $\Delta y = m \Delta x$

$$\Rightarrow \lim_{\Delta x \to 0} - \frac{m^2 - 2m}{(1+m^2)^{3/2}}$$

limit depends on the path.

$\Rightarrow$ The given function is not differentiable.

EXAMPLE: (FUNCTION is DIFFERENTIAL BUT f_x & f_y are not continuous)

$$f(x,y) = \begin{cases} (x^2+y^2)\cos\left[\frac{1}{\sqrt{x^2+y^2}}\right] & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

i) Continuity: $\lim\limits_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) = 0 = f(0,0)$

ii) Existence of partial derivatives:

$$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \Delta x \cos\left(\frac{1}{|\Delta x|}\right) = 0$$

$$f_y(0,0) = \lim_{\Delta y\to 0} \frac{f(0,\Delta y)-f(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \Delta y \cos\left(\frac{1}{|\Delta y|}\right) = 0$$

iii) Differentiability:

$dz = z_x \Delta x + z_y \Delta y = 0$

$$\lim_{\Delta\rho\to 0}\left(\frac{\Delta z - dz}{\Delta \rho}\right) = \lim_{\Delta\rho\to 0} \frac{(\Delta x^2+\Delta y^2)\cos\left(\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\right)}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sqrt{\Delta x^2+\Delta y^2}\cos\left(\frac{1}{\Delta x^2+\Delta y^2}\right) = 0$$

Hence the function is differentiable.

iv) Continuity of f_x & f_y.

At $(x,y) \neq (0,0)$: $f_x(x,y) = -(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)\cdot\left(-\frac{1}{2}\frac{1\cdot 2x}{(x^2+y^2)^{3/2}}\right)$

$$+ 2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{x}{\sqrt{x^2+y^2}}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + 2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

Along x-axis: $\lim\limits_{x\to 0} f_x(x,y) = \lim\limits_{x\to 0}\left[\frac{x}{|x|}\sin\left(\frac{1}{|x|}\right) + 2x\cos\left(\frac{1}{|x|}\right)\right] \neq 0$

Hence f_x is not continuous at $(0,0)$. Similarly one can show that f_y is not continuous.

This example shows that continuity of partial 1st order derivatives is not a necessary condition for differentiabity. A function can be differentiable without having first order partial derivatives continuous.

Ex. For the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y(x-y)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Find $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$.

Sol:

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$$

where

$$f_y(\Delta x, 0) = \lim_{\Delta y \to 0} \frac{f(\Delta x, \Delta y) - f(\Delta x, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta x^2 \cancel{\Delta y}(\Delta x - \Delta y)}{(\Delta x^2 + \Delta y^2)\cancel{\Delta y}} = \Delta x$$

Hence:

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

because

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = 0$$

Now:

$$\left.\frac{\partial^2 f}{\partial y\,\partial x}\right|_{(0,0)} = \lim_{\Delta y\to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y}$$

where. $$f_x(0,\Delta y) = \lim_{\Delta x\to 0} \frac{f(\Delta x,\Delta y) - f(0,\Delta y)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\Delta x^2 \Delta y(\Delta x - \Delta y)}{\Delta x(\Delta x^2 + \Delta y^2)} = 0$$

& $$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} = 0$$

Hence

$$\left.\frac{\partial^2 f}{\partial y\,\partial x}\right|_{(0,0)} = \lim_{\Delta y\to 0} \frac{0-0}{\Delta y} = 0$$

Ex: Test the continuity and existence of f_x & f_y at the origin of the followin[g] function:

$$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

Sol. limit along $x=y$: $\lim_{x\to 0} f(x,y) = 0$

Since $f(0,0) = 1$, f is not continuous at $(0,0)$.

$$f_x|_{(0,0)} = \lim_{\Delta x\to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{1-1}{\Delta x} = 0$$

$$f_y|_{(0,0)} = \lim_{\Delta y\to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{1-1}{\Delta y} = 0.$$

$\Rightarrow$ First order partial derivatives exist at $(0,0)$

Ex. Test the differentiability of the following function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

at the origin.

Sol: Clearly the function is continuous at the origin as

$$\lim_{x\to 0, y\to 0} f(x,y) = \lim_{r\to 0} r\cos\theta\sin\theta = 0$$

Existence of partial derivatives:

$$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0}{\Delta x} = 0$$

Similarly: $f_y(0,0) = \lim_{\Delta y\to 0} \dfrac{f(0,\Delta y)-f(0,0)}{\Delta y} = 0$

So, $df = \Delta x . f_x + \Delta y\, f_y = 0$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta f - df}{\sqrt{\Delta x^2+\Delta y^2}} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x\, \Delta y}{\Delta x^2+\Delta y^2}$$

Along the path $\Delta y = m\,\Delta x$:

$$= \lim_{\Delta x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$ limit does not exist.

$\Rightarrow$ f is not differentiable.

Ex. Find the total differential and the total increment of the function $z = xy$ at the point $(2,3)$ for $\Delta x = 0.1$, $\Delta y = 0.2$.

Sol.
$$\Delta z = (x+\Delta x)(y+\Delta y) - xy$$
$$= x\Delta y + y\Delta x + \Delta x \Delta y$$

& $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = y\,dx + x\,dy = y\Delta x + x\Delta y$$

Consequently:
$$dz = 3.(0.1) + 2(0.2)$$
$$= 0.3 + 0.4 = 0.7$$

$$\Delta z = 0.7 + 0.1 \times 0.2$$
$$= 0.7 + 0.02 = 0.72$$

Q. Let
$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Discuss the continuity of f_{yx} at $(0,0)$.

Sol:
$$f_x = \frac{(x^2+y^2)\,2xy^2 - x^2y^2(2x)}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2}$$

$$f_{yx} = \frac{8x^3y^3}{(x^2+y^2)^3}$$

Along the path $y = mx$:
$$\lim_{x\to 0} f_{yx} = \lim_{x\to 0} \frac{8x^3m^3x^3}{(x^2+m^2x^2)^3} = \frac{8m^3}{(1+m^2)^3}$$

$\Rightarrow$ f_{yx} is not continuous at $(0,0)$.

COMPOSITE FUNCTIONS:

Consider
$$z = f(x, y) \quad -(1)$$

and let

$$\left.\begin{array}{l} x = \varphi(t) \\ y = \psi(t) \end{array}\right\} (2) \qquad \text{or} \qquad \left.\begin{array}{l} x = \Phi(u, v) \\ y = \Psi(u, v) \end{array}\right\} (2')$$

The equations (1 & 2) or (1 & 2') are said to define z as composite function of t or (u & v).

Differentiation of composite functions (Chain Rule)

let $z = f(x, y)$ posses continuous partial derivatives (or differentiabl

and let $x = \varphi(t)$, $y = \psi(t)$ posses continuous derivatives (differentia

Then,
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Proof: let $z = f(x, y)$, $x = \varphi(t)$, $y = \psi(t)$ be a composite function of t. Assuming z to be differentiable and also $\varphi(t)$ & $\psi(t)$ are differentiable function of t.

$$\Rightarrow \Delta z = z_x \Delta x + z_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\Rightarrow \frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x} \cdot \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \cdot \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta y}$$

Taking limit $\Delta t \to 0$; ($\Delta x \to 0$, $\Delta y \to 0$)

$$\Rightarrow \boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

For the case $x = \varphi(u, v)$ & $y = \psi(u, v)$, we have

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$

&

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$

Example: $z = xy$; $x = \cos t$ $y = \sin t$

Find $\frac{dz}{dt}$.

Sol. $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= y\cdot(-\sin t) + x\cos t$$

$$= -\sin^2 t + \cos^2 t$$

$$= \cos 2t$$

Ex. Let z be a function of x & y. Prove that if

$$x = e^{u} + e^{-v} \qquad y = e^{-u} + e^{v}$$

Then $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Sol. $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot e^{u} + \frac{\partial z}{\partial y}\cdot(-e^{-u})$$

$$= \frac{\partial z}{\partial x}e^{u} - \frac{\partial z}{\partial y}e^{-u} \quad \text{—(I)}$$

Similarly: $$\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x}\cdot e^{-v} + \frac{\partial z}{\partial y}e^{v} \quad \text{—(II)}$$

(I) − (II) $\Rightarrow$ $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u} + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} + e^{v})$$

$$= \frac{\partial z}{\partial x}x - y\frac{\partial z}{\partial y}.$$

Derivative of a function defined implicitely:

ONE VARIABLE:

Let the function y of x be defined by

$$F(x,y)=0$$

and let

$$z \equiv F(x,y)=0$$

$$\Rightarrow \frac{dz}{dx} \equiv \frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y} \quad \text{if } \frac{\partial F}{\partial y} \neq 0$$

TWO INDEPENDENT VARIABLE

$$F(x,y,z)=0$$

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} \quad \text{if } \frac{\partial F}{\partial z} \neq 0$$

and $$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z} \quad \text{if } \frac{\partial F}{\partial z} \neq 0$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of $x^2+y^2+z^2-c=0$

Sol:
$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} = -\frac{2x}{2z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z} = -\frac{2y}{2z} = -\frac{y}{z}$$

OR Directly from $x^2+y^2+z^2-c=0$

$$\Rightarrow 2x + 2z\cdot\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}$$

& Diff. w.r.t. y $$\Rightarrow 2y + 2z\cdot\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$$

HARMONIC FUNCTIONS

If a function of two variables $f(x,y)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

then we say that $f(x,y)$ is an harmonic function.

Example: i) $f(x,y) = x^3y - y^3x$

$$\frac{\partial f}{\partial x} = 3x^2y - y^3 \qquad \frac{\partial f}{\partial y} = x^3 - 3y^2x$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy \qquad \frac{\partial^2 f}{\partial y^2} = -6xy$$

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

ii) $f(x,y) = e^{-x}\cos y$

$$\frac{\partial f}{\partial x} = -e^{-x}\cos y \qquad \frac{\partial^2 f}{\partial x^2} = e^{-x}\cos y$$

$$\frac{\partial f}{\partial y} = -e^{-x}\sin y \qquad \frac{\partial^2 f}{\partial y^2} = -e^{-x}\cos y$$

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

HOMOGENEOUS FUNCTION

We say an expression in (x, y) is homogeneous of order n, if it can be expressed as

$$x^n f\left(\frac{y}{x}\right)$$

Examples:

i) $f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$

$$= x^n \underbrace{\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \dots + a_n\left(\frac{y}{x}\right)^n\right]}_{=: g\left(\frac{y}{x}\right)}$$

$\Rightarrow f(x,y)$ is a homo. func. of order n.

ii) $$f(x,y) = \frac{\sqrt{y} + \sqrt{x}}{y+x} = \frac{\sqrt{x}}{x}\left[\frac{\sqrt{\left(\frac{y}{x}\right)} + 1}{\frac{y}{x} + 1}\right]$$

$$= x^{-\frac{1}{2}} g(y/x)$$

$\Rightarrow f(x,y)$ is a homo. func. of order $-\frac{1}{2}$.

ALTERNATIVE DEF. A function $f(x,y)$ is said to be homogeneous of degree n if it satisfies

$$f(tx, ty) = t^n f(x,y).$$

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS:

If $z = f(x,y)$ be a homogeneous function of x & y of order n, then

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n z \qquad \forall\, x, y \in D.$$

D: Domain of the function f.

PROOF: Given $z = f(x,y) = x^n g\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = nx^{n-1} g\left(\frac{y}{x}\right) + x^n \cdot \left(-\frac{y}{x^2}\right) g'\left(\frac{y}{x}\right)$$

$$= nx^{n-1} g\left(\frac{y}{x}\right) - x^{n-2} \cdot y\, g'\left(\frac{y}{x}\right) \quad \text{---} (1)$$

$$\frac{\partial z}{\partial y} = x^n \cdot g'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad \text{---} (2)$$

from (1) and (2)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n g\left(\frac{y}{x}\right) - yx^{n-1} g'\left(\frac{y}{x}\right) + yx^{n-1} g'\left(\frac{y}{x}\right)$$

$$= nz$$

□

Theorem: If $z = f(x,y)$ is a homogeneous function of x & y of degree n. Then

$$\boxed{x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z}$$

Example: If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, $x \neq y$, Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2x$$

Sol: Let $z = \tan u = \frac{x^3+y^3}{x-y} = x^2\left[\frac{1+(y/x)^3}{1-y/x}\right]$

Clearly z is a homogeneous of degree 2.

$$\Rightarrow \quad x \cdot \frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

Subst. $z = \tan u$ gives,

$$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2 \cdot \tan u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2 . \sin u \cdot \cos u .$$

$$= \sin 2u.$$

Ex: If $u = z e^{ax+by}$ where z is a homogeneous function in x & y of degree n.

Show that $\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (ax+by+n)u$

Sol: Since z is a homogeneous function of degree n,

we have $\quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad$ (Euler's theorem)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[\frac{\partial z}{\partial x} \cdot e^{ax+by} + z \cdot e^{ax+by} \cdot a\right] + y\left[\frac{\partial z}{\partial y} e^{ax+by} + z \cdot e^{ax+by} \cdot b\right]$$

$$= e^{ax+by}\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right] + z \cdot \left[ax e^{ax+by} + by e^{ax+by}\right]$$

$$= (nz + azx + bzy)e^{ax+by}$$

$$= (n + ax + by)\, u:$$

Ex. Let $z = xy\,f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ where f & g are continuous and 2 times differentiable functions. Then, evaluate

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$$

Sol: Let $z = u_1 + u_2$ where $u_1 = \underbrace{xy\,f\left(\frac{y}{x}\right)}_{\substack{\text{homo fun.} \\ \text{of deg. 2}}}$ & $u_2 = \underbrace{g\left(\frac{y}{x}\right)}_{\substack{\text{homo. func.} \\ \text{of deg. 0}}}$

Applying Euler's theorem on u_1 & u_2 we get

$$x^2 \frac{\partial^2 u_1}{\partial x^2} + 2xy \frac{\partial^2 u_1}{\partial x \partial y} + y^2 \frac{\partial^2 u_1}{\partial y^2} = 2(2-1)\cdot u_1 \quad \text{— (1)}$$

$$\& \quad x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = 0 \quad \text{— (2)}$$

Adding (1) & (2):

$$x^2 \cdot \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 2\cdot u_1$$

$$= 2\cdot xy\,f\left(\frac{y}{x}\right)$$

Ex. If $z = y + f\left(\frac{x}{y}\right)$ where f is cont & differentiable function.

Find the value of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$.

Sol. Let $z = u_1 + u_2$ where $u_1 = y$ & $u_2 = f\left(\frac{x}{y}\right)$

Now. $x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} = y$ (Either by Euler's theorem or direct result

& $x\frac{\partial u_2}{\partial x} + y\frac{\partial u_2}{\partial y} = 0$

Adding the above two, we get: $\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = y}$

DERIVATIVE (GEOMETRICAL INTERPRETATION)

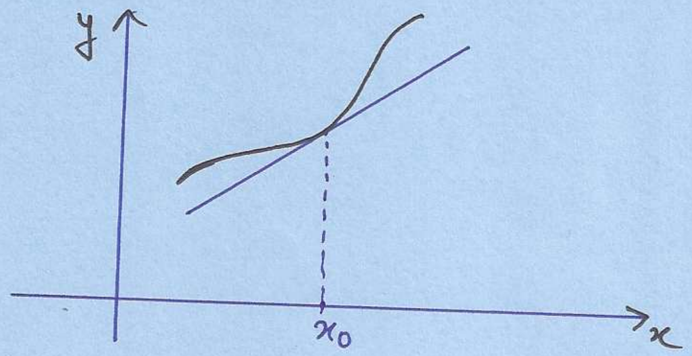
ONE VARIABLE:

$$f(x) - f(x_0) = (x - x_0) A + \varepsilon_1 (x - x_0)$$

where $\varepsilon_1 \to 0$ as $x \to x_0$

OR $$f(x) = \underbrace{f(x_0) + (x - x_0) A}_{\text{linear function, say } \Phi(x)} + \varepsilon_1 (x - x_0)$$

$\Phi(x) = f(x_0) + (x - x_0) A \rightarrow$ tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$.



GENERAL DEF.

A function $y = f(x)$ (or $z = f(x,y)$) is differentiable at the point P if it can be approximated in the neighbourhood of this point by a linear function.

Two variables:

$$f(x,y) = \underbrace{f(x_0,y_0) + (x - x_0) f_x(x_0,y_0) + (y - y_0) f_y(x_0,y_0)}_{\text{linear function, say } \psi(x,y)} + \varepsilon_1 (x - x_0) + \varepsilon_2 (y - y_0)$$

$$\psi(x,y) = f(x_0,y_0) + (x - x_0) f_x(x_0,y_0) + (y - y_0) f_y(x_0,y_0)$$

$\hookrightarrow$ tangent plane.

TAYLOR'S THEOREM FOR A FUNCTION OF TWO VARIABLES

Let a function $f(x,y)$ be defined in some domain D in $\mathbb{R}^2$ and have continuous partial derivatives up to $(n+1)$th order in some neighbourhood of a point $P(x_0, y_0)$ in D. Then,

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0,y_0) + \frac{1}{\underline{|2}}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(x_0,$$

$$+ \cdots + \frac{1}{\underline{|n}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0,y_0) + R_n$$

where the remainder is given by

$$R_n = \frac{1}{\underline{|(n+1)}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k), \quad 0<\theta<1.$$

Proof: For simplicity, $n=2$.

Let $x = x_0 + th$, $y = y_0 + tk$ where the parameter $t \in [0,1]$.

Define $\Phi(t) = f(x_0+th, y_0+tk)$

Using chain rule:

$$\Phi'(t) = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0+th, y_0+tk)$$

$$\Phi''(t) = h\left\{\frac{\partial^2 f}{\partial x^2}\cdot h + \frac{\partial^2 f}{\partial y\partial x}k\right\} + k\left\{\frac{\partial^2 f}{\partial x\partial y}h + \frac{\partial^2 f}{\partial y^2}k\right\}$$

$$= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0+th, y_0+tk)$$

$$\Phi'''(t) = h^2\left\{\frac{\partial^3 f}{\partial x^3}h + \frac{\partial^3 f}{\partial y\partial x^2}k\right\} + 2hk\left\{\frac{\partial^3 f}{\partial x^2\partial y}h + \frac{\partial^3 f}{\partial x\partial y^2}k\right\}$$
$$+ k^2\left\{\frac{\partial^3 f}{\partial x\partial y^2}h + \frac{\partial^3 f}{\partial y^3}k\right\}$$

$$= h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0+th, y_0+tk)$$

Using Taylor's theorem for a function ($\Phi(t)$) of one variable about the point 0 as:

$$\Phi(t) = \Phi(0) + t\,\Phi'(0) + \frac{t^2}{\underline{|2}}\Phi''(0) + \frac{t^3}{\underline{|3}}\Phi'''(\theta t)$$

$$0 < \theta < 1$$

For $t = 1$:

$$\Phi(1) = \Phi(0) + \Phi'(0) + \frac{1}{2}\Phi''(0) + \frac{t^3}{\underline{|3}}\Phi'''(\theta)$$

$\Rightarrow$

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$
$$+ \frac{1}{\underline{|2}}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) + \frac{1}{\underline{|3}}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(x_0+\theta h,$$
$$y_0+\theta k)$$

where $0<\theta<1$.

$\square$

Ex. : Find the quadratic Taylor's polynomial approximation to the function $f(x,y) = \frac{x-y}{x+y}$ about the point $(1,1)$.

Sol :
$$f_x = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\Rightarrow f_x(1,1) = \frac{1}{2}$$

$$f_y = \frac{-(x+y)-(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

$$\Rightarrow f_y(1,1) = -\frac{1}{2}.$$

$$f_{xx} = \frac{-4y}{(x+y)^3} \Rightarrow f_{xx}(1,1) = -\frac{1}{2}$$

$$f_{yy} = \frac{4x}{(x+y)^3} \Rightarrow f_{yy}(1,1) = \frac{1}{2}$$

$$f_{xy} = \frac{2x-2y}{(x+y)^3} \Rightarrow f_{xy}(1,1) = 0$$

$$P_2(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$
$$+ \frac{1}{2} f_{xx}(1,1)(x-1)^2 + f_{xy}(1,1)(x-1)(y-1) + \frac{1}{2} f_{yy}(1,1)(y-1)^2$$

$$= \frac{1}{2}(x-1) - \frac{1}{2}(y-1) - \frac{1}{4}(x-1)^2 + \frac{1}{4}(y-1)^2$$

Q: Let $f(x,y) = x^2 + xy + y^2$ be linearly approximated by the Taylor's polynomial about the point $(1,1)$. Find out the maximum error in this approximation at a point in the square $|x-1| \leq 0.1$, $|y-1| \leq 0.1$.

Sol:

$$f(x,y) = x^2 + xy + y^2$$

$$f_x = 2x + y \qquad f_{xx} = 2 \qquad f_{xy} = 1$$

$$f_y = 2y + x \qquad f_{yy} = 2$$

Remainder R_1:

$$R_1 = \frac{1}{2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0 + \theta h, y_0 + \theta k)$$

$$= \frac{1}{2}\left((x-1)^2 f_{xx} + 2(x-1)(y-1) f_{xy} + f_{yy}(y-1)^2\right)$$

$$= \frac{1}{2}\left[(x-1)\cdot 2 + 2\cdot(x-1)(y-1) + 2\cdot(y-1)\right]$$

$$= (x-1)^2 + (x-1)(y-1) + (y-1)^2$$

Maximum error:

$$R_1 = (0.1)^2 + (0.1)^2 + (0.1)^2$$

$$= 3 * 0.01$$

$$= 0.03$$

Ex. Obtain Taylor's formula $(n=2)$ for $f(x,y) = \cos(x+y)$ at $(0,0)$

Sol: $$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{\underline{|2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{\underline{|3}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(\theta x, \theta y)$$

$0 < \theta < 1.$

- $f(0,0) = 1$

- First order derivatives: $f_x = -\sin(x+y) \Rightarrow f_x(0,0) = 0$

  $f_y = -\sin(x+y) \Rightarrow f_y(0,0) = 0$

- Second order derivatives: $f_{xx} = f_{yy} = f_{xy} = -\cos(x+y)$

  At $(0,0)$, $f_{xx} = f_{yy} = f_{xy} = -1$

- Third order derivatives:

  $$f_{xxx} = f_{yyy} = f_{xxy} = f_{yyx}\Big|_{(\theta x, \theta y)} = \sin(\theta x + \theta y)$$

Taylor's Theorem:

$$f(x,y) = 1 + 0 - \frac{1}{\underline{|2}}\left(x^2 + 2xy + y^2\right) + \frac{1}{\underline{|3}}\left(x^3 + 3x^2y + 3xy^2 + y^3\right) \sin(\theta x + \theta y)$$

$$= 1 - \frac{1}{\underline{|2}}(x+y)^2 + \frac{1}{\underline{|3}}(x+y)^3 \sin(\theta x + \theta y).$$

$\square$

MAXIMA AND MINIMA OF A FUNCTION

DEF:

1. A function $z = f(x, y)$ has a maximum (or a minimum) at the point (x_0, y_0) if at every point in a neighbourhood of (x_0, y_0) the function assumes a smaller value (or a larger value) than at the point itself. Such a maximum or minimum is ofter called relative (or local) maximum or minimum respectively.

2. For a given closed and bounded domain, a function may also attain its greatest value (or least value) on the boundary of the domain.

   The smallest and the largest values attained by a function over the entire domain including the boundary are called the absolute (or global) minimum and absolute (or global) maximum, respectively.

3. The point (x_0, y_0) is called critical point (or stationary point) of $f(x, y)$ if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

4. A critical point where the function has no minimum or maximum is called a saddle point.

5. Minimum and maximum values together are called extreme values.

Theorem (Necessary conditions for a function to have extremum)

Let $f(x,y)$ be continuous and have first order partial derivatives at a point $P(a,b)$. Then necessary conditions for the existence of an extreme value of it at the point P are

$$f_x(a,b) = 0 \quad \& \quad f_y(a,b) = 0 .$$

OR

If the point (a,b) is a relative extrema of the function $f(x,y)$ then (a,b) is also a critical point of $f(x,y)$.

Proof: Let $(a+h, b+k)$ be a point in the neighbourhood of the point $P(a,b)$. Then P will be a point of maximum if

$$\Delta f = f(a+h, b+k) - f(a,b) \leq 0 \text{ for all sufficiently small } h \,\&\, k$$

and a point of minimum if

$$\Delta f = f(a+h, b+k) - f(a,b) \geq 0 \text{ for all sufficiently small } h \,\&$$

Taylor's series expansion about the point (a,b):

$$f(a+h, b+k) = f(a,b) + (hf_x + kf_y)_{(a,b)} + \frac{1}{2}(hf_x + kf_y)^2_{(a,b)} + \cdots$$

For sufficiently small h & k, we can neglect second and higher order terms, to set

$$\Delta f \simeq h f_x(a,b) + k f_y(a,b)$$

The sign of Δf depends on the sign of $h f_x(a,b) + k f_y(a,b)$.

Letting $h \to 0$ we find that Δf changes sign with k,

i.e., assuming $f_y(a,b) > 0$:

for $k > 0$; $\Delta f > 0$

for $k < 0$; $\Delta f < 0$

Therefore the function cannot have an extremum unless $f_y = 0$

Similarly, letting $k \to 0$, we find that the function f cannot have an extremum unless $f_x = 0$.

Therefore the necessary conditions for the existence of an extremum at the point (a,b) is that

$f_x(a,b) = 0$ & $f_y(a,b) = 0$.

$\square$.

SUFFICIENT CONDITIONS FOR A FUNCTION TO HAVE MINIMA/MAXIMA

For simplicity, we set

$$r = f_{xx}(a,b), \quad s = f_{xy}(a,b), \quad t = f_{yy}(a,b)$$

Let a function $f(x,y)$ be continuous and have first and second order partial derivatives at a point $P(a,b)$. If (a,b) is a critical point, then the point P is a point of

i) local maximum if $rt - s^2 > 0$ and $r < 0$ $(r<0)$

ii) local minimum if $rt - s^2 > 0$ and $r > 0$ $(r>0)$

iii) saddle point if $rt - s^2 < 0$

iv) may be a local minimum, local maximum or a saddle point if $rt - s^2 = 0$.

<u>Proof</u>: consider $\Delta f = f(a+h, b+k) - f(a,b)$

Note that $(a+h, b+k)$ is a point in the neighbourhood of (a,b)

By Taylor's series expansion

$$\Delta f = \left(h f_x + k f_y\right)_{(a,b)} + \frac{1}{2}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{(a,b)} + \cdots$$

As (a,b) is a critical point, meaning $f_x\big|_{(a,b)} = f_y\big|_{(a,b)} = 0$

$$\Rightarrow \Delta f = \frac{1}{2}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{(a,b)} + R$$

$$= \frac{1}{2}\left[h^2 r + 2hks + k^2 t\right] + R$$

$$= \frac{1}{2r}\left[h^2 r^2 + 2hkrs + k^2 rt\right] + R \quad (\text{Assuming } r \neq 0)$$

$$= \frac{1}{2r}\left[(hr+ks)^2 - k^2s^2 + k^2rt\right] + R$$

$$= \frac{1}{2r}\left[(hr+ks)^2 + k^2(rt-s^2)\right] + R \quad \left(\text{OR } \frac{1}{2t}\left[(hs+kt)^2 + h^2(rt-s^2)\right] + \dots\right)$$

same conclusions follow if $r=0$ & $t\neq 0$

Since $(hr+ks)^2$, the sufficient condition for the expression

$[(hr+ks)^2 + k^2(rt-s^2)]$ to be positive is that

$$rt - s^2 > 0$$

$\Rightarrow$ If $rt-s^2>0$, then

i) $\Delta f > 0$ if $r>0$

ii) $\Delta f < 0$ if $r<0$

$\Rightarrow$ The point (a,b) is a point of $\begin{cases} \text{minimum if } (rt-s^2)>0 \text{ \& } r>0 \\ \text{maximum if } (rt-s^2)>0 \text{ \& } r<0 \end{cases}$

iii) If $rt-s^2<0$, then the sign of Δf depends on h & k.

For example,

Let $k\to 0$ & $h\neq 0$ $\Rightarrow$ $\Delta f>0$ if $r>0$

and if $k\neq 0$ & we choose h such that $hr+ks=0$

$\Rightarrow \Delta f<0$ for $r>0$

Hence no maximum/minimum of f can occur at $P(a,b)$.

$\Rightarrow P(a,b)$ is a saddle point

(iv) If $rt-s^2=0$, then

$$\Delta f = \frac{1}{2r}\left[(hr+ks)^2\right] + R$$

If we take h & k such that $hr=-ks$ i.e, $\frac{h}{k} = -\left(\frac{s}{r}\right)$, then the whole second order terms of right hand side will vanish.

Therefore for these points in the neighbourhood we have to consider third order terms in the remainder. Other than these points we have

$\Delta f > 0$ for $r > 0$ and

$\Delta f < 0$ for $r < 0$.

Thus the conclusion will depend on the higher order terms.

$\Rightarrow$ A FURTHER INVESTIGATION IS REQUIRED.

WORKING RULES:

1) FIND CRITICAL POINTS OR STATIONARY POINTS $f_x = 0$ & $f_y = 0$.

2) FOR EACH CRITICAL POINT, EVALUATE

$$r = f_{xx}, \quad s = f_{xy}, \quad t = f_{yy}$$

3) IDENTIFICATION:

i) If $rt-s^2>0$ & $r<0$ $\rightarrow$ Maximum

ii) If $rt-s^2>0$ & $r>0$ $\rightarrow$ Minimum

iii) If $rt-s^2<0$ $\rightarrow$ Saddle point

iv) If $rt-s^2=0$ $\rightarrow$ Doubtful, needs further investigation

Ex. Discuss the local extrema of the function

$$f(x,y) = (4x^2+y^2)\, e^{-x^2-4y^2}$$

Sol:

$$f_x(x,y) = e^{-x^2-4y^2}\left[8x - 2x(4x^2+y^2)\right]$$

$$= e^{-x^2-4y^2}\left[8x - 8x^3 - 2xy^2\right]$$

$$= e^{-x^2-4y^2}(2x)\left[4 - 4x^2 - y^2\right]$$

$$f_y(x,y) = e^{-x^2-4y^2}\left[2y - 8y(4x^2+y^2)\right]$$

$$= e^{-x^2-4y^2}(2y)\left[1 - 16x^2 - 4y^2\right]$$

CRITICAL POINTS: $f_x = 0$ & $f_y = 0$

i) $x=0, \quad y=0$

ii) $x=0, \quad 1-4y^2=0 \Rightarrow y = \pm\frac{1}{2}$

$\Rightarrow (0, \frac{1}{2})$ & $(0, -\frac{1}{2})$

iii) Let $x \neq 0, \; y = 0$

$\Rightarrow 4 - 4x^2 = 0 \Rightarrow x = \pm 1.$

$(1,0) \quad (-1,0)$

iv) $x \neq 0, \; y \neq 0 \Rightarrow$ $\left.\begin{array}{l} 4x^2+y^2 = 4 \\ \& \; 4x^2+y^2 = \frac{1}{4} \end{array}\right\}$ NO SOLUTION

Hence the critical points are:

$P_1 = (0,0)$, $P_2 = (0, \frac{1}{2})$ $P_3 = (0, -\frac{1}{2})$ $P_4 = (1,0)$ $P_5 = (-1,0)$

Second order derivatives:

$$r = f_{xx} = e^{-x^2-4y^2}\left[8-24x^2-2y^2+(8x-8x^3-2xy^2)(-2x)\right]$$

$$= 2e^{-x^2-4y^2}\left[4-20x^2+8x^4-y^2+2x^2y^2\right]$$

$$t = f_{yy} = e^{-x^2-4y^2}\left[2-32x^2-24y^2+(2y-32x^2y-8y^3)(-8y)\right]$$

$$= 2e^{-x^2-4y^2}\left[1-20y^2-16x^2-128x^2y^2+32y^4\right]$$

$$s = f_{xy} = e^{-x^2-4y^2}\left[-4xy+(8x-8x^3-2xy^2)(-8y)\right]$$

$$= 4xy\,e^{-x^2-4y^2}\left[-17+16x^2+4y^2\right]$$

<u>Identification:</u>

<u>$P_1(0,0)$:</u> $r = 8$ $s = 0$ $t = 2$

$rt - s^2 = 16 > 0$ & $r > 0$

$\Rightarrow$ The point P_1 is a local minima.

<u>$P_2(0,\frac{1}{2})$ & $P_3(0,-\frac{1}{2})$:</u>

$$r = 2e^{-1}\left[4-\tfrac{1}{4}\right] = \frac{15}{2e}$$

$$S = 0$$

$$t = 2e^{-1}\left[1-5+2\right] = -\frac{4}{e}$$

$$rt - s^2 = -\frac{30}{e^2} < 0$$

$\Rightarrow$ P_2 & P_3 are saddle points.

$P_4(1,0)$ & $P_5(-1,0)$

$$r = 2e^{-1}[4-20+8] = -16e^{-1}$$

$$s = 0$$

$$t = 2e^{-1}[1-16] = -30e^{-1}$$

$$rt - s^2 = \frac{480}{e^2} > 0, \quad r < 0$$

Hence P_4 & P_5 are the point of local maximum.

EXAMPLE: $f(x,y) = y^2 + x^2y + x^4$.

Stationary points: $f_x = 0$ & $f_y = 0$

$$\Rightarrow 2xy + 4x^3 = 0 \quad \& \quad 2y + x^2 = 0$$

$$\Rightarrow x = 0 \ \& \ y = 0.$$

$$r = f_{xx}|_{(0,0)} = (2y + 12x^2)|_{(0,0)} = 0$$

$$s = f_{xy}|_{(0,0)} = 2x|_{(0,0)} = 0$$

$$t = f_{yy}|_{(0,0)} = 2|_{(0,0)} = 2.$$

$rt - s^2 = 0$ further investigation is required.

$$\Delta f = f(0+h, 0+k) - f(0,0) = k^2 + h^2k + h^4$$

$$= \left(\frac{k}{2}\right)^2 + h^2k + h^4 + \frac{3}{4}k^2$$

$$= \left(\frac{k}{2} + h^2\right)^2 + \frac{3}{4}k^2 > 0 \quad \forall \ h \neq 0, \ k \neq 0$$

$\Rightarrow (0,0)$ is a point of LOCAL MINIMUM.

Ex. Find local minima/maxima of the function

$$f(x,y) = 2x^4 - 3x^2y + y^2$$

Sol.

$$f_x = 8x^3 - 6xy$$

$$f_y = -3x^2 + 2y$$

Stationary points: $8x^3 - 6xy = 0$ & $-3x^2 + 2y = 0$

$$\Rightarrow 8x^3 - 3x(3x^2) = 0 \Rightarrow x = 0.$$

$$\Rightarrow y = 0$$

Stationary point $(0,0)$.

$$r = f_{xx}\big|_{(0,0)} = (24x^2 - 6y)\big|_{(0,0)} = 0$$

$$s = f_{xy}\big|_{(0,0)} = -6x\big|_{(0,0)} = 0$$

$$t = f_{yy}\big|_{(0,0)} = 2$$

$rt - s^2 = 0$ test fails,

$$\Delta f = f(h,k) - f(0,0)$$

$$= 2h^4 - 3h^2k + k^2$$

$$= 2h^4 - 2h^2k - h^2k + k^2$$

$$= 2h^2(h^2 - k) - k(h^2 - k)$$

$$= (2h^2 - k)(h^2 - k)$$

$\left.\begin{array}{ll} \underline{\text{For } k < 0:} & \Delta f > 0 \\ \underline{\text{For } h^2 < k < 2h^2:} & \Delta f < 0 \end{array}\right\}$ sign changes

$\Rightarrow (0,0)$ is a saddle point.

Ex. The function $f(x,y) = (y-x^2)^2 + x^5$ has a stationary point at the origin. Characterize the function at the point $(0,0)$.

Sol: $f_x = 2(y-x^2)(-2x) + 5x^4 \Rightarrow f_{xx} = -4[(y-x^2)+x(-2x)] + 20x^3$

$r = f_{xx}|_{(0,0)} = 0$ $\qquad f_{xy} = -4x.$

$f_y = 2(y-x^2) \Rightarrow f_{yy} = 2$

$s = f_{xy}|_{(0,0)} = 0$

$t = 2.$

$rt - s^2 = 0$ test fails!

However, we can readily see that the function has no extreme value there, as the function assumes both positive and negative values in the neighbourhood of the origin.

Ex. Find and characterize the extreme values of the function

$$f(x,y) = (x-y)^4 + (y-1)^4.$$

Sol. $f_x = 4(x-y)^3$ $\qquad f_{xx} = 12(x-y)^2$ $\qquad f_{xy} = -24(x-y)$

$f_y = -4(x-y)^3 + 4(y-1)^3$ $\qquad f_{yy} = +12(x-y)^2 + 12(y-1)$

Critical points: $(x-y)^3 = 0$ & $-(x-y)^3 + (y-1)^3 = 0$

$\Rightarrow x = 1,\ y = 1.$

$r = f_{xx}|_{(1,1)} = 0$ $\qquad s = f_{xy}|_{(1,1)} = 0$ $\qquad t = f_{yy}|_{(1,1)} = 0$

Criterion fails!

However, if we consider: $f(1+h, 1+k) - f(1,1)$

$= (1+h-1-k)^4 + (1+k-1)^4$

$= (h-k)^4 + k^4 > 0 \quad \forall\, h,k \neq 0$

$\Rightarrow$ f has a minimum at the point $x=1,\ y=1.$

LAGRANGE'S METHOD OF UNDETERMINED COEFFICIENTS

Method of Lagrange Multipliers

Find the maxima/minima of the function

$$u = f(x, y) \qquad \text{—} (1)$$

with the following constraint

$$\varphi(x, y) = 0 \qquad \text{—} (2)$$

From equation (1), we have using chain rule of composite function

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

(we can write because x & y are related from relation (2))

At the point of extremum

$$\frac{du}{dx} = 0$$

(one variable problem)

$$\Rightarrow \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \qquad \text{—} (3)$$

Also, equation (2) satisfies at any point; so at the point of extremum

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \cdot \frac{dy}{dx} = 0 \qquad \text{—} (4)$$

(Differentiation of implicit function)

In order to avoid calculation of $\frac{dy}{dx}$, aim is to eliminate $\frac{dy}{dx}$ from (3) and (4). We assume that an extremum point the two partial derivatives φ_x & φ_y do not both vanish. Assuming $\varphi_y \neq 0$, and multiplying (4) by $\lambda = -f_y/\varphi_y$ and add it to equation (3), we get

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0$$

By the definition of λ, the equation

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \qquad \text{holds}$$

Hence, at the extremum point, three equations are satisfied;

$$\left.\begin{aligned} \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} &= 0 \\ \varphi(x,y) &= 0 \end{aligned}\right\} \quad (5)$$

Out of these three equations, we determine x, y & λ.

LAGRANGE'S RULE:

We can write the system (5) using an auxiliary function of the form

$$F(x,y,\lambda) = f(x,y) + \lambda \varphi(x,y)$$

and now writting the necessary condition of an extreme value as

$$F_x = 0 \Rightarrow f_x + \lambda \varphi_x = 0$$
$$F_y = 0 \Rightarrow f_y + \lambda \varphi_y = 0$$
$$F_\lambda = 0 \Rightarrow \varphi = 0.$$

GENERAL CASE: Find extremum of $f(x_1,x_2,\dots,x_n)$ und the conditions $\varphi_i(x_1,x_2,\dots x_n) = 0 \quad i = 1,2,\dots k$.

Construct the auxiliary function

$$F(x_1,x_2,\dots x_n, \lambda_1,\lambda_2,\dots \lambda_k) = f(x_1,x_2,\dots,x_n) + \sum_{i=1}^{k} \lambda_i \varphi_i(x_1,x_2,\dots,x_n)$$

Find stationary points of F:

$$\frac{\partial F}{\partial x_1} = 0 = \frac{\partial F}{\partial x_2} = \cdots = \frac{\partial F}{\partial x_n} = \frac{\partial F}{\partial \lambda_1} = \cdots = \frac{\partial F}{\partial \lambda_k}$$

↳ $(n+k)$ equations and $(n+k)$ unknowns.

Note that, using method of Lagrange multiplier, we obtain stationary points. We donot determine the nature of the stationary point. The second derivative test for constrained problem is more theoretical importance than practical. In practice we usually are interested in finding max/min value of a function under some given constraints

Example: Find maximum/minimum of the function

$$x^2-y^2-2x$$

in the region $x^2+y^2 \leq 1$

Sol: I] local extrema in the interior domain $x^2+y^2<1$

Let $f(x,y) = x^2-y^2-2x$

$f_x = 0 \Rightarrow$

$2x-2=0 \Rightarrow x=1$

$f_y = 0 \Rightarrow -2y=0 \Rightarrow y=0$

Critical point $(1,0)$, however this point lies on the boundary

so no extrema in the interior.

II.] Auxiliary function for the problem Max/Min x^2-y^2-2x

subject to $x^2+y^2=1$.

$$F(x,y,\lambda) = (x^2-y^2-2x)+\lambda(x^2+y^2-1) = 0$$

$F_x = 0 \Rightarrow 2x-2+2\lambda x = 0$ — ①

$F_y = 0 \Rightarrow -2y+2\lambda y = 0 \Rightarrow 2y(\lambda-1)=0 \Rightarrow y=0, \lambda=1$

If $y=0$, then $x^2+y^2=1$ gives $x=\pm 1$, Points: $(1,0)$ & $(-1,0)$

If $\lambda=1$. then ① $\Rightarrow 4x-2=0 \Rightarrow x=\frac{1}{2}$

if $x=\frac{1}{2}$ then $x^2+y^2=1 \Rightarrow y^2 = 1-\frac{1}{4} = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$

Points: $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ & $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$

Function values at critical points:

1. $(1,0)$: $f(x,y) = -1$
2. $(-1,0)$: $f(x,y) = \boxed{3}$ ← MAX
3. $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$: $f(x,y) = \frac{1}{4}-\frac{3}{4}-1 = \boxed{-\frac{3}{2}}$ ← MIN

Ex. Find the maximum and minimum of

$$f(x,y) = x^2+2y^2$$

on the disk $x^2+y^2 \le 1$.

Sol: I] Find local maxima/minima in $x^2+y^2<1$?

$f_x = 2x$ & $f_y = 4y$

Critical point $(0,0)$.

Clearly $(0,0)$ is absolute (Global) minimum of the function $f(x,y)$.

II] Find max/min on the circle $x^2+y^2=1$.

auxiliary function: $F(x,y,\lambda) = (x^2+2y^2)+\lambda(x^2+y^2-1)$

Critical point: $F_x=0 \Rightarrow 2x+2x\lambda=0 \Rightarrow 2x(1+\lambda)=0$ — ①

$F_y=0 \Rightarrow 4y+2y\lambda=0 \Rightarrow 2y(\lambda+2)=0$ — ②

$F_\lambda=0 \Rightarrow x^2+y^2-1=0$ — ③

① $\Rightarrow \lambda=-1$, ② $\Rightarrow y=0$ ③ $\Rightarrow x=\pm 1$

① $\Rightarrow x=0$, ② $\Rightarrow \lambda=-2$ ③ $\Rightarrow y=\pm 1$

Critical point are $(\pm 1,0)$ & $(0,\pm 1)$.

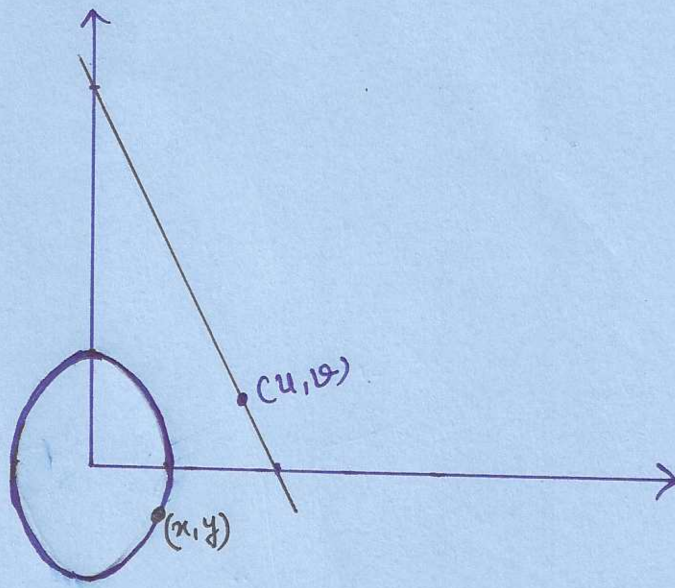
Functional value: $f(\pm 1,0) = 1$

$f(0,\pm 1) = 2$

Global maximum: $\boxed{2}$ at $(0,\pm 1)$

Global minimum: $\boxed{0}$ at $(0,0)$.

Ex. Find the shortest distance between the line $y = 10 - 2x$ and the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$



Sol. Shortest distance between the line and the ellipse :

Min

$$f(x,y,u,v) = \sqrt{(x-u)^2 + (y-v)^2}$$

subject to

$$\varphi_1(x,y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \quad — ①$$

$$\varphi_2(u,v) = 2u + v - 10 = 0 \quad — ②$$

Auxiliary function

$$F(x,y,u,v,\lambda_1,\lambda_2) = (x-u)^2 + (y-v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \lambda_2(2u + v - 10)$$

(for simplicity, we have taken $f(x,y,u,v) = (x-u)^2 + (y-v)^2$)

For critical points:

$$\left.\begin{aligned} F_x = 0 &\Rightarrow 2(x-u) + \frac{x}{2}\lambda_1 = 0 \Rightarrow -\lambda_1 x = 4(x-u) \\ F_y = 0 &\Rightarrow 2(y-v) + \frac{2y}{9}\lambda_1 = 0 \Rightarrow -\lambda_1 y = 9(y-v) \end{aligned}\right\} \Rightarrow 4(x-u)y = 9(y-v)x \quad — ③$$

$$\left.\begin{aligned} F_u = 0 &\Rightarrow -2(x-u) + 2\lambda_2 = 0 \Rightarrow \lambda_2 = (x-u) \\ F_v = 0 &\Rightarrow -2(y-v) + \lambda_2 = 0 \Rightarrow \lambda_2 = 2(y-v) \end{aligned}\right\} \Rightarrow x - u = 2(y-v) \quad — ④$$

$F_{\lambda_1} = 0 \Rightarrow \varphi_1(x,y) = 0$ & $F_{\lambda_2} = 0 \Rightarrow \varphi_2(u,v) = 0.$

From (3) & (4) $\quad 4y = \frac{9}{2}x \Rightarrow 8y = 9x$

(1) $\Rightarrow \quad \frac{x^2}{4} + \frac{1}{9} \cdot \frac{9^2 x^2}{8^2} - 1 = 0 \quad \Rightarrow \quad x = \pm \frac{8}{5}$

$y = \pm \frac{9}{5}$

For : $x = \frac{8}{5}, \; y = \frac{9}{5}$

(4) $\Rightarrow \quad \frac{8}{5} - u = 2\left(\frac{9}{5} - v\right) \Rightarrow 2v - 2 = u.$

(2) $\Rightarrow \quad 2(2v-2) + v - 10 = 0 \Rightarrow \quad v = \frac{14}{5}$

$u = \frac{18}{5}$

One critical point: $\quad (x, y) = \left(\frac{8}{5}, \frac{9}{5}\right) \; (u, v) = \left(\frac{18}{5}, \frac{14}{5}\right)$

The distance in this case: $\sqrt{\left(\frac{8}{5} - \frac{18}{5}\right)^2 + \left(\frac{9}{5} - \frac{14}{5}\right)^2} = \sqrt{5}$

For $x = -\frac{8}{5}, \; y = -\frac{9}{5}$

$$\left.\begin{aligned} &(4) \Rightarrow \quad u = 2v + 2 \\ &(2) \Rightarrow \quad v = \frac{6}{5} \end{aligned}\right\} \Rightarrow \quad (u, v) = \left(\frac{22}{5}, \frac{6}{5}\right)$$

The distance in this case: $\sqrt{\left[\left(-\frac{8}{5}\right) - \frac{22}{5}\right]^2 + \left[\left(-\frac{9}{5}\right) - \left(\frac{6}{5}\right)\right]^2} = 3\sqrt{5}$

Hence the shortest distance between the line and the ellipse is $\boxed{\sqrt{5}}$.