LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

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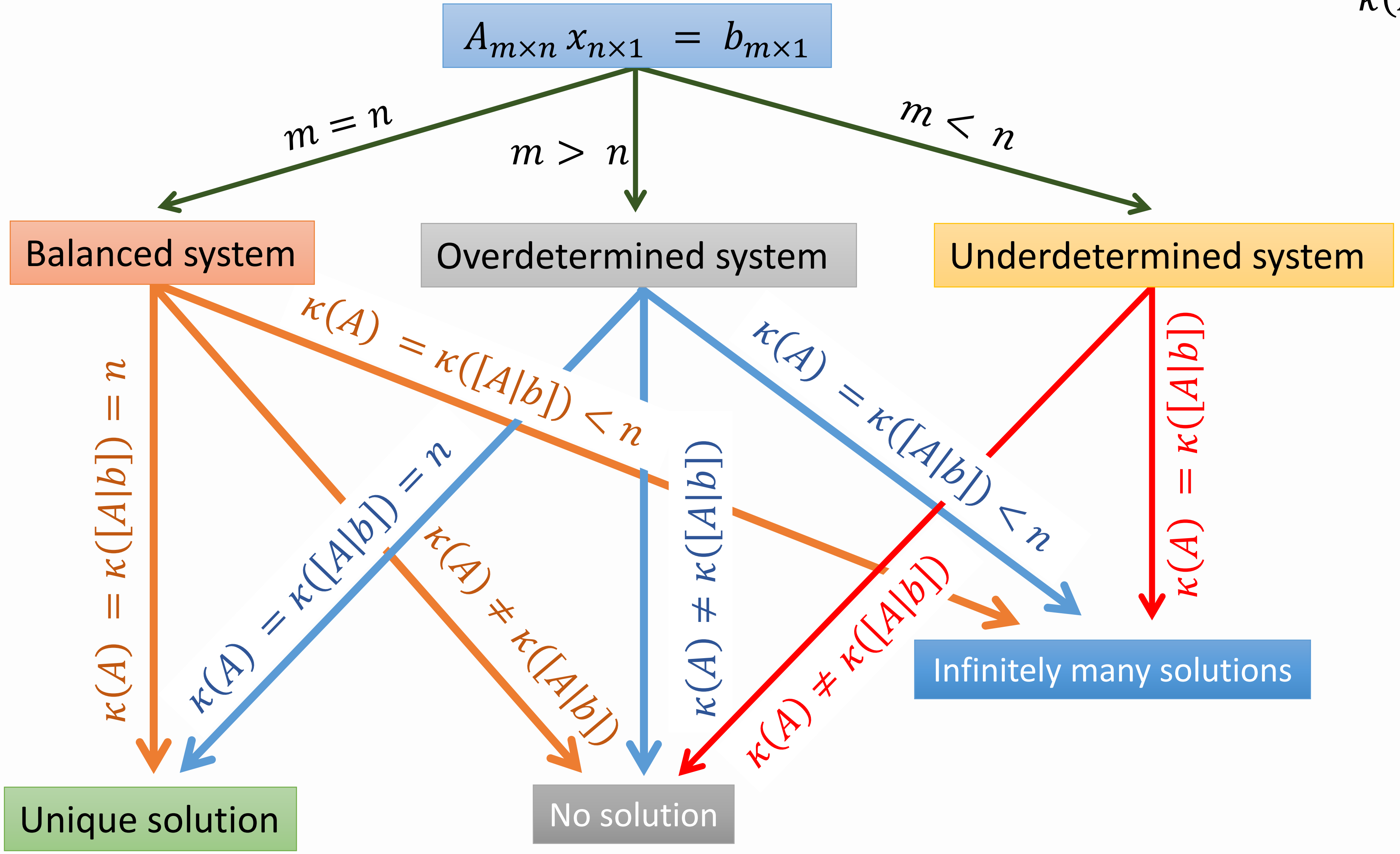
- **Iterative Methods for Solving System of Linear Equations**
    - **Jacobi Method**
    - **Gauss-Seidel Method**

$\kappa(A)$: Rank of A



System of Linear Equations $Ax = b$ $A \in \mathbb{R}^{n\times n}$ $x \in \mathbb{R}^{n\times 1}$ $b \in \mathbb{R}^{n\times 1}$

Solution Methods

Direct Methods

- Cramer Rule
- Gauss Elimination Method
- Gauss-Jordan Method
- Decomposition Methods

Iterative Methods

- Jacobi Method
- Gauss-Seidel Method
- Conjugate Gradient Method
- Conjugate Residual Method

Direct Methods:

- Deliver Exact Solution
  (in the absence of rounding errors)
- Very Expensive
  (especially for large systems)

Iterative Methods:

- Deliver Approximate Solution
- Less Expensive

Diagonally Dominant Matrix

A matrix $A \in \mathbb{R}^{n\times n}$ is called **diagonally dominant by rows** if

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}|, \qquad i = 1, 2, \dots, n$$

while it is called **diagonally dominant by columns** if

$$|a_{jj}| \geq \sum_{i=1, i\neq j}^{n} |a_{ij}|, \qquad j = 1, 2, \dots, n$$

If the above inequalities hold in a strict sense, A is called **strictly diagonally dominant**

(by rows or by columns respectively).

Matrix Norms

A number associated with a matrix that is often requires in analysis of Matrix based algorithm.

Matrix norms give some notion of “size” of a matrix or “distance” between the two matrices.

Some Example: Let $A \in \mathbb{R}^{n\times n}$

Frobenius Norm: $$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2}$$

Row Sum Norm: $$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

Column Sum Norm: $$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

ITERATIVE METHOD

A method for solving the linear system $Ax = b$ is called iterative if it is a numerical method computing a sequence of approximate solutions $x^{(k)}$ that converges to the exact solution x as the number of iterations k goes to ∞.

IDEA FOR DERIVING AN ITERATIVE METHOD

Consider a system of linear equations $Ax = b$

Idea of iterative schemes is based on the splitting $A = P - N$

where P is a non-singular matrix.

$$\text{Given } Ax = b \quad \Rightarrow (P - N)x = b \quad \Rightarrow Px = Nx + b$$

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Consider the iterations with a suitable guess $x^{(0)}$

$$Px^{(k+1)} = Nx^{(k)} + b$$

$$\Longrightarrow x^{(k+1)} = Gx^{(k)} + Hb$$

where $G = P^{-1}N$ is called **iteration matrix** and $H = P^{-1}$.

Definition (Convergence of an Iterative Method):

An iterative method is said to **converge** if for any choice of initial vector $x^{(0)} \in \mathbb{R}^n$, the sequence of approximate solutions $x^{(k)}$ converges to the exact solution x.

Definition (Error):

We call the vector $r_k = b - Ax^{(k)}$ **residual** (respectively **error** $e_k = x^{(k)} - x$) at the kth iteration.

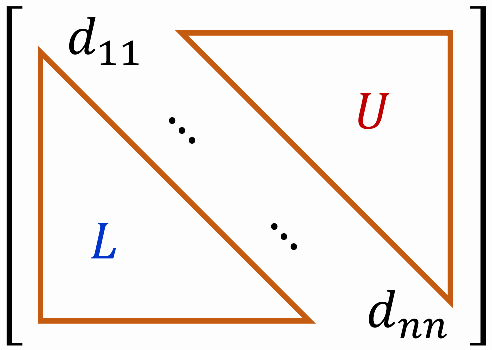
REMARK:

In general, we have no knowledge of e_k because the exact solution x is unknown. However, it is easy to compute the residual r_k, so convergence dedicated on the residual in practice.

Jacobi Iteration Method

Consider a system of linear equations $A_{n\times n}x_{n\times 1} = b_{n\times 1}$

Take splitting of A as $A = L + D + U$

$A =$ 

L: Lower triangular part of A

D : Diagonal entries of A

U : Upper triangular part of A

$$A = L + D + U \qquad Ax = b \implies (L + D + U)x = b \quad \implies Dx = -(L + U)x + b$$

Assume that D^{-1} exists, then $\Rightarrow x = -D^{-1}(L + U)x + D^{-1}b$

Introducing iterations, the iterative method known as Jacobi iteration method, becomes

$$\boxed{x^{(k+1)} = -D^{-1}(L + U)\, x^{(k)} + D^{-1}b}$$

In component form

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_j^{(k)}\right)} \quad i = 1, 2, \ldots, n$$