ADVANCED CALCULUS
MA11003

SECTION 11, 12

Dr. Jitendra Kumar

Professor
Department of Mathematics
Indian Institute of Technology Kharagpur
West Bengal 721302, India

Webpage: http://www.facweb.iitkgp.ac.in/~jkumar/

Concepts Covered

Differential Equations

- ❑ Introduction
- ❑ Formation of Differential Equation

Differential Equations

An equation involving **derivatives** or **differentials** of one or more dependent variables with respect to one or more independent variable is called a **differential equation**.

ODE
Ordinary derivatives

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = e^x$$

order-4 degree-1

ODE
Ordinary derivatives

$$y(y^2+1)dx + x(y^2-1)dy$$

order-1 degree-1

PDE
Partial derivatives

$$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2$$

order-3 degree-2

Order: order of the highest order derivatives involved

Degree: degree of the highest order derivatives involved

Further Classifications

Linear and nonlinear differential equation

A differential equation is called Linear if

i) Every dependent variable and every derivative occur in the first degree only, and

ii) No products of dependent variables and/or derivatives occur

If the differential equation is not linear then it is called nonlinear.

Note: Every linear equation is of first degree, but every first degree equation may not be linear

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0$$

1st degree but nonlinear

Solution of a Differential Equation

Any relation between the dependent and independent variables (no derivative terms) which satisfies the differential equation is called a solution or integral of the differential equation

$$y = \frac{A}{x} + B \quad \text{is a solution of} \quad y'' + \left(\frac{2}{x}\right) y' = 0$$

$$y' = -\frac{A}{x^2} \quad \Longrightarrow \quad y'' = \frac{2A}{x^3}$$

Note that y satisfies the given differential equation.

Family of curves: An n-parameter family of curves is a set of relations of the form

$$\{(x, y): f(x, y, c_1, c_2, \dots c_n) = 0\}$$

Example: $i)$ Set of concentric circles $x^2 + y^2 = c$

One parameter family if c takes non-negative real values

$ii)$ Set of circles $(x - c_1)^2 + (y - c_2)^2 = c_3$

Three parameters family if c_1, c_2 takes all real values

and c_3 takes all non-negative real values.

Note: Solution of a differential equation is a family of curves.

Formation of differential equations from a given n −parameters family of curves:

From a given family of curves containing n arbitrary constants, we can obtain an nth order differential equation whose solution is the given family.

- Differentiate the given equation n times to get n additional equations containing those arbitrary constants.
- Eliminate n arbitrary constant from the $(n+1)$ equations.
- Obtain a differential equation of the $n\text{th order}$.

Example: Obtain the differential equation satisfied by

$$xy = ae^{x} + be^{-x} + x^{2} \qquad \text{where } a \;\&\; b \text{ are arbitrary constants}$$

Given family of curves: $xy = ae^{x} + be^{-x} + x^{2}$

Differentiating w.r.t. x, we get: $xy' + y = ae^{x} - be^{-x} + 2x$

Differentiating again: $xy'' + 2y' = ae^{x} + be^{-x} + 2$

Using (1) we get: $xy'' + 2y' = xy - x^{2} + 2$

General and Particular Solution:

Let $F(x, y, y', y'', y''' \ldots y^n) = 0$ be an nth order ordinary differential equation

$i)$ **General solution**: Solution containing n-independent arbitrary constant.

$ii)$ **Particular solution**: Solution by giving particular values to one or more of the n-independent constants.

Remark: Observe that the number of arbitrary constants in a solution of a differential equation depends upon the order of the differential equation.

It is evident from the above example that a general solution of an nth order differential equation will contain n arbitrary constants.

Example 1: Consider $$\left(\frac{dy}{dx}\right)^2 - 4y = 0$$

General Solution: $y = (x - c)^2$

Particular Solution: $y = x^2 \ (c = 0)$

Example 1: Consider $yy' - x(y')^2 = 1$

General Solution: $y = cx + \dfrac{1}{c}$

Particular Solution: $y = x + 1 \ (c = 1)$

Explicit & Implicit Solutions

Explicit Solution: $y = y(x)$

Implicit Solution: $F(x, y) = 0$

Example: $y'' + k^2 y = 0$

Solution: $y = c_1 \cos kx + c_2 \sin kx$ explicit solution

Example: $x + 3yy' = 0$

Solution: $x^2 + 3y^2 = c$ implicit solution

Conclusion

Order: order of the highest order derivatives involved

Classification – linear/nonlinear

Formation of Differential Equations

General solution of an nth order differential equation will contain n arbitrary constants

Concepts Covered

Differential Equations

- First Order & First Degree Differential Equations
- Solution Techniques

Equation of First Order and First Degree

We shall consider two standard forms of differential equation

$i)\quad \dfrac{dy}{dx} = F(x, y)$

$ii)\quad M(x, y)\, dx + N(x, y)\, dy = 0$

Solution Methods

- **Separation of Variables**

If a differential equation can be written in the form

$$f_1(y)\frac{dy}{dx} = f_2(x)$$

Then we say variables are separable in the given differential equation.

Solution: $\int f_1(y)\,dy = \int f_2(x)\,dx + c$

Example: $$\frac{dy}{dx} = e^{x-2y} + x^2 e^{-2y}$$

Rewrite: $$e^{2y}\frac{dy}{dx} = \mathrm{e}^{x} + x^2$$

Integrating both side: $$\frac{e^{2y}}{2} = e^{x} + \frac{x^3}{3} + c_1$$

or $$e^{2y} = 2e^{x} + \frac{2}{3}x^3 + c$$

➢ Equation Reducible to Separation of Variables

Consider $\dfrac{dy}{dx} = f(ax + by + c)$ or $\dfrac{dy}{dx} = f(ax + by)$

Substitute $ax + by + c = v$ or $ax + by = v$

$$\Rightarrow a + b\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{1}{b}\left[\frac{dv}{dx} - a\right] = f(v) \quad \Rightarrow \quad \frac{dv}{dx} = bf(v) + a \quad \Rightarrow \int \frac{dv}{bf(v) + a} = \int dx$$

Example : $\dfrac{dy}{dx} = \sec(x+y)$

Let $x + y = v$ $\qquad \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$

Then the differential equation becomes: $\dfrac{dv}{dx} = \sec v + 1 = \dfrac{1+\cos v}{\cos v} = \dfrac{2\cos^2\frac{v}{2}}{2\cos^2\frac{v}{2} - 1}$

$\displaystyle\int \left[1 - \frac{1}{2}\sec^2\left(\frac{v}{2}\right)\right] dv = \int dx$ implies $v - \tan\left(\dfrac{v}{2}\right) = x + c$

Substitute $v = x + y$ $\qquad y - \tan\left(\dfrac{x+y}{2}\right) = c$

Homogeneous Equations

A differential of first order and first degree is said to be homogenous, if It is of the form or can be put in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$\Downarrow$$

$$\frac{y}{x} = v \implies \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad v + x\frac{dv}{dx} = f(v)$$

(Separable form)

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + c$$

Example: $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0, \qquad x > 0$

$$\frac{dy}{dx} = -\frac{x^3 + 3xy^2}{y^3 + 3x^2y} = -\frac{1 + 3\left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^3 + 3\left(\frac{y}{x}\right)}$$

Substitute $\frac{y}{x} = v \quad \Longrightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = -\frac{1 + 3v^2}{v^3 + 3v} \qquad \Longrightarrow \qquad x\left(\frac{dv}{dx}\right) = \frac{-v^4 - 6v^2 - 1}{v^3 + 3v}$$

$$-\int \frac{4(v^3+3v)}{v^4+6v^2+1}\,dv = \int \frac{4}{x}\,dx$$

$$\Longrightarrow -\ln(v^4+6v^2+1) = 4\ln x + \ln c \qquad (x>0)$$

$$\Longrightarrow x^4 c(v^4+6v^2+1) = 1$$

$$\Longrightarrow cx^4\left(\frac{y^4}{x^4}+\frac{6y^2}{x^2}+1\right) = 1 \;\Longrightarrow\; c(y^4+6y^2x^2+x^4) = 1$$

Equation Reducible to Homogeneous Form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \qquad \text{where} \qquad \frac{a}{a'} \neq \frac{b}{b'}$$

Take $\quad x = X + h \quad y = Y + k$

where X & Y are new variables and h & k are constant to be so chosen that the resulting equation in X and Y becomes homogeneous.

$$y = Y + k \implies \frac{dy}{dx} = \frac{dY}{dX} \implies \frac{dY}{dX} = \frac{aX + bY + ah + bk + c}{a'X + b'Y + a'h + b'k + c'}$$

$$\frac{dY}{dX} = \frac{aX + bY + ah + bk + c}{a'X + b'Y + a'h + b'k + c'}$$

In order to make the above equation homogeneous, choose h and k such that

$$ah + bk + c = 0$$

$$a'h + b'k + c' = 0$$

(always possible because $ab' - a'b \neq 0$)

Getting h & k we have $X = x - h$ & $Y = y - k$

$$\Rightarrow \frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$

homogeneous in X & Y

Example: $$\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$$

Solution: Take $x = X + h$ & $y = Y + k$ so that $\dfrac{dy}{dx} = \dfrac{dY}{dX}$

$$\frac{dY}{dX} = \frac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)}$$

Choose h, k so that $h + 2k - 3 = 0$ & $2h + k - 3 = 0 \implies h = 1 \;\&\; k = 1$

$$X = x - 1 \qquad Y = y - 1$$

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y}$$ Take $Y = vX$ $\implies \dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} \quad \& \quad Y = vX, \qquad \frac{dY}{dX} = v + X\frac{dv}{dX} \qquad \Longrightarrow X\frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 - v^2}{2 + v}$$

$$\frac{dX}{X} = \left[\frac{1}{2}\left(\frac{1}{1 + v}\right) + \frac{3}{2}\left(\frac{1}{1 - v}\right)\right]dv$$

Integrating $\quad \ln X + \ln C = \frac{1}{2}[\ln(1 + v) - 3\ln(1 - v)]$

$$2\ln(XC) = \ln\left(\frac{1 + v}{(1 - v)^3}\right) \qquad \Longrightarrow X^2C^2 = \frac{1 + v}{(1 - v)^3}$$

Substitution $\quad v = \dfrac{y - 1}{x - 1} \qquad \Longrightarrow C^2(x - y)^3 = x + y - 2$

Conclusion

- Separation of Variables
- Equation Reducible to Separation of Variables
- Homogeneous Equations
- Equation Reducible to Homogeneous Form

Thank You