Indian Institute of Technology Kharagpur
Department of Mathematics
MA11004 - Linear Algebra, Numerical and Complex Analysis
Problem Sheet - 1
Spring 2021

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1. Determine which of the following sets form vector spaces under the given operations:
(i) The set of all triples of real numbers (x, y, z) with the operations $(x, y, z)+(x', y', z') = (x + x', y + y', z + z')$ and $k(x, y, z) = (kx, y, z)$, $k \in \mathbb{R}$, $\forall$ $(x, y, z),(x', y', z') \in \mathbb{R}^3$.
(ii) Let $V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$ with the operations $(x_1, x_2) + (y_1, y_2) = (\frac{1}{2}(x_1 + y_1), \frac{1}{2}(x_2 + y_2))$ and $r(x_1, x_2) = (rx_1, rx_2)$, $r \in \mathbb{R}$, $\forall$ $(x_1, x_2),(y_1, y_2) \in V$.
(iii) The set of all positive real numbers x with the operations $x + x' = xx'$ and $kx = x^k$, $k \in \mathbb{R}$.
(iv) The set of all 2×2 matrices of the form $\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$ with usual matrix addition and scalar multiplication.
(v) Let $V = \{f \in C(\mathbb{R}) : \exists p \in \mathbb{N}, f(x + p) = f(x), \forall x \in \mathbb{R}\}$. Does V form a vector space under the usual addition and scalar multiplication of $C(\mathbb{R})$, the set of all continuous functions over $\mathbb{R}$?

2. Determine which of the following subsets are the subspaces of the given vector spaces:
(i) All vectors of the form (a, b, c), with $b = a + c$ in $\mathbb{R}^3$.
(ii) All matrices with $A = A^T$ in $M_{n\times n}$, where $M_{n\times n}$ is the set of all $n \times n$ matrices.
(iii) All matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 0$ in $M_{2\times2}$.
(iv) All matrices with $\det(A) = 0$ in $M_{n\times n}$.
(v) All vectors of the form (a, b, c), where $ab = 0$ in $\mathbb{R}^3$.
(vi) Is the set $W = \{(a, b, c) : a^3 = b^3\}$ a subspace of both $\mathbb{R}^3$ and $\mathbb{C}^3$?

3. Let $S = \{f \in C[0, 1] : \int_0^1 f(x)dx = b, \text{for some fixed } b \in \mathbb{R}\}$. Then show that S is a subspace of $C[0, 1]$ if and only if $b = 0$.

4. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$, such that $f'(-1) = 3f(2)$ is a subspace of $C[-4, 4]$.

5. (a) Write $E = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ as a linear combination of $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.
(b) Write $p = 2 + 2x + 3x^2$ as a linear combination of $p_1 = 2 + x + 4x^2$, $p_2 = 1 - x + 3x^2$, $p_3 = 3 + 2x + 5x^2$.
(c) Which of the following are linear combinations of the vectors $u = (1, -1, 3)$, $v = (2, 4, 0)$:
(i) $(3, 3, 3)$, (ii) $(4, 2, 6)$, (iii) $(1, 5, 6)$, (iv) $(0, 0, 0)$.

6. In the vector space $\mathbb{R}^3$, let $u_1 = (1, 2, 1)$, $u_2 = (3, 1, 5)$, $u_3 = (3, -4, 7)$. Then show that $\text{span}\{u_1, u_2\} = \text{span}\{u_1, u_2, u_3\}$.

7. (a) Let $S = \{v_1, v_2, v_3, v_4\}$ spans a vector space V. Show that the set $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$ also spans V.
(b) Let $S = \{u_1, u_2, u_3\}$, $T = \{u_1, u_1+u_2, u_1+u_2+u_3\}$, and $U = \{u_1+u_2, u_2+u_3, u_3+u_1\}$ in $\mathbb{R}^4$. Show that span S=span T=span U.

8. Which of the following sets are linear independent:
(a) {(4,-4,8,0), (2,2,4,0), (6,0,0,2), (6,3,-3,0)} in $\mathbb{R}^4$.
(b) $\{2, 4\sin^2 x, \cos^2 x\}$ in $C[-\pi, \pi]$.
(c) $\{t^3 - 5t^2 - 2t + 3, t^3 - 4t^2 - 3t + 4, 2t^3 - 7t^2 - 7t + 9\}$ in $\mathbb{P}_3$, where $\mathbb{P}_3$ is the set of polynomials with degree at most 3.
(d) Let $f_1, f_2 \in C[-1, 1]$ be defined as $f_1(t) = t$, $t \in [-1, 1]$ and

$$f_2(t) = \begin{cases} -t, & t \in [-1, 0], \\ t, & t \in [0, 1]. \end{cases}$$

Show that the set $\{f_1, f_2\}$ is linearly dependent on $C[0, 1]$ and $C[-1, 0]$ and linearly independent on $C[-1, 1]$.
(e) Show that the set of vectors $\{1 + i, 1 - i\} \subset \mathbb{C}$ is linearly independent if $\mathbb{C}$ is taken as a vector space over $\mathbb{R}$. But it becomes linearly dependent when $\mathbb{C}$ is a vector space over $\mathbb{C}$.
(f) Let $S = \{p_0, p_1, \ldots, p_m\} \subset \mathbb{P}_m$, such that $p_j(2) = 0$ for $j = 0, 1, \ldots, m$. Prove that S is not linearly independent in $\mathbb{P}_m$.

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