Linear Algebra

Gauss elimination method to solve system of linear equations

We consider a system of m linear equations in the n unknown $x_1, x_2, \ldots x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \qquad \cdots (1) \\
\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

The system is said to be homogeneous if the constants $b_1, - b_m$ are all 0. A set of numbers $x_1, x_2, \ldots, x_n$ that satisfies all the m eqns. is called a solution of (1). A solution vector of (1) is a vector x whose components constitute a sol^n of (1).

The homogeneous system associated with (1) is

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \qquad \cdots (2) \\
\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

The above system always has a solution $x_1 = x_2 = \cdots = x_n = 0$ known as zero or trivial sol^n. Any other sol^n, if it exists is called a nonzero or nontrivial sol^n.

A set of real nos. $(x_1, x_2, \ldots x_n)$ is a sol^n. or a particular sol^n. if it satisfies each of the eqns.; the set of all such sol^{n}s. is termed as the sol^n. set or the general sol^n.

Theorem

Suppose u is a particular sol^n. of the non-homogeneous system (1) and suppose W is the general sol^n of the associated homogeneous system (2). Then $u + W = \{u + \omega, \omega \in W\}$ is the general sol^n. of the non-homogeneous system (1).

Matrix form of the linear system (1)

$$Ax = b$$

where the coefficient matrix $A = [a_{ij}]$ is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

are column vectors. The matrix

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ - & - & - & - & - \\ a_{m1} & & & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix of the system (1). We see that $\tilde{A}$ is obtained by augmenting A by the column b. $\tilde{A}$ is sometimes written

$$\tilde{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ - & - & - & - & - \\ a_{m1} & \cdots & \cdots & a_{mn} & b_m \end{array}\right]$$

where the vertical line is merely a reminder that the last column of $\tilde{A}$ is the right side of the system. The augmented matrix $\tilde{A}$ represents the system (1) completely because it contains all the given numbers appearing in (1).

Existence of solutions ; Geometrical interpretation

If $m = n = 2$, we have two eqns. in two unknowns x_1, x_2

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

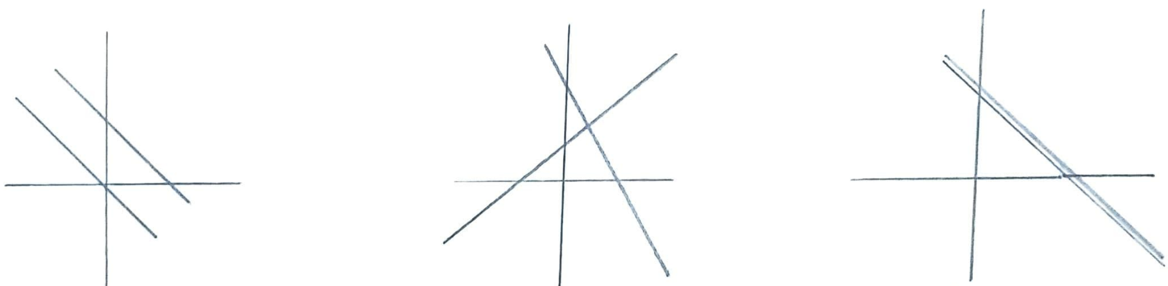
If we interpret x_1, x_2 as co-ordinates in the x_1x_2 plane, then each of the two eqns. represents a straight line and (x_1, x_2) is a solⁿ. iff the pt. $P(x_1, x_2)$ lies on both lines.

Hence there are three possible cases :

(a) No solⁿ. if the lines are parallel.

(b) Exactly one solⁿ. if they intersect

(c) Infinitely many solⁿ. if they coincide



If the system is homogeneous, case (a) cannot happen. Because then those two straight lines pass through the origin whose co-ord. (0,0) constitute the trivial solⁿ.

Gauss elimination with backward substitution

We consider
$$2x + y - 2z = 10$$
$$3x + 2y + 2z = 1$$
$$5x + 4y + 3z = 4$$

Pivot

Eqns.

Pivot eqn. $\rightarrow$ $2x + y - 2z = 10$

$3x + 2y + 2z = 1$

$5x + 4y + 3z = 4$

$$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 \end{array}\right]$$

First step: Elimination of x

$L_2 \rightarrow -3L_1 + 2L_2 \quad \left(L_2 - \frac{3}{2}L_1 \right.$

$L_3 \rightarrow -5L_1 + 2L_3 \quad \left. L_3 - \frac{5}{2}L_1 \right)$

$2x + y - 2z = 10$

Pivot 1 $\rightarrow$ $y + 10z = -28$

$3y + 16z = -42$

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 10 \\ 0 & 1 & 10 & -28 \\ 0 & 3 & 16 & -42 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow R_2 - \frac{3}{2} \\ R_3 \rightarrow R_3 - \frac{5}{2} \end{array}$$

Second step: Elimination of y

$L_3 \rightarrow -3L_2 + L_3$

$2x + y - 2z = 10$

$y + 10z = -28$ --- (3)

$-14z = 42$

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 10 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & -14 & 42 \end{array}\right] \quad R_3 \rightarrow R_3 - 3R$$

Third step: Back substitution

$-14z = 42$, $\left. \begin{array}{l} z = -3 \\ y = 2 \\ x = 1 \end{array} \right\}$ unique soln.

Elementary row operations: Row equivalent systems

(i) Interchange of two eqns.

(ii) Addition of a constant multiple of one eqn. to another eqn.

(iii) Multiplication of an eqn. by a non zero constant c

Similarly row operations for matrices

Theorem

Row-equivalent linear systems have the same set of solutions.

Def^n. The system (3) is said to be in echelon form;
the unknowns x_i which do not appear at the beginning of any eqn. are termed as free variables.

Theorem The sol^n. of a system in echelon form is as follows;

(i) If the no. of eqns. is equal to the no. of unknowns, then the system has a unique sol^n.

(ii) If the no. of eqns. is less than the no. of unknowns, then we can arbitrarily assign values to the $n-m$ free variables and obtain a sol^n. of the system.

A system(1) is called consistent if it has at least one sol^n. and inconsistent if it has no sol^n.

System of linear eqn

- Inconsistent
  - No sol^n.
- Consistent
  - Unique Sol^n
  - Many sol^n.

Note

(i) If an eqn. $0x_1 + \cdots + 0x_n = b$, $b \neq 0$ occurs, then the system is inconsistent and has no solution.

(ii) If an eqn. $0x_1 + \cdots + 0x_n = 0$ occurs, then the equation can be deleted without affecting the solution.

Theorem

A homogeneous system of linear equations with more unknowns than equations has a non-zero solution.

Gauss method if there is infinitely many solutions

$$
\begin{aligned}
2x - 3y + 6z + 2v - 5w &= 3 \\
y - 4z + v &= 1 \\
v - 3w &= 2
\end{aligned}
$$

The system is in echelon form. Since the equations begin with the unknowns x, y and v, the other unknowns z and w are the free variables.

Put $z = c_1$, $w = c_2$

$\therefore v = 2 + 3c_2$, $y = 4c_1 - 3c_2 - 1$, $x = 3c_1 - 5c_2 - 2$

$\therefore$ Solⁿ. is $(x, y, z, v, w) = (3c_1 - 5c_2 - 2, 4c_1 - 3c_2 - 1, c_1, 2 + 3c_2, c_2)$ where c_1, c_2 are arbitrary constants.

If the system would have been homogeneous, the solution would have been $(x, y, z, v, w) = (3c_1 - 5c_2, 4c_1 - 3c_2, c_1, 3c_2, c_2)$

* For a particular solⁿ. of non-homogeneous system put $c_1 = 2, c_2 = 1$. Then $(-1, 4, 2, 5, 1)$ is a part. solⁿ. Similary put $c_1 = 1, c_2 = 1$ in the homogeneous system. $(-2, 1, 1, 3, 1)$ is a part. solⁿ. Adding these two $(-3, 5, 3, 8, 2)$ is a solⁿ. of the non-homogeneous system.

Gauss method if no solution exists

$$x+2y-3z = -1$$
$$3x-y+2z = 7$$
$$5x+3y-4z = 2$$

$L_2 \to L_2-3L_1$
$L_3 \to L_3-5L_1$

$$x+2y-3z = -1$$
$$-7y+11z = 10$$
$$-7y+11z = 7$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array}\right]$$

$L_3 \to L_3-L_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

$\therefore$ The system is inconsistent.

Ex Determine the value of a so that the following system in unknowns x, y and z has (i) no solⁿ. (ii) more than one solⁿ. (iii) a unique solⁿ.

$$x+y-z=1$$
$$2x+3y+az=3$$
$$x+ay+3z=2$$

Solⁿ: $L_2 \to L_2-2L_1$
$L_3 \to L_3-L_1$

$$x+y-z=1$$
$$y+(a+2)z=1$$
$$(a-1)y+4z=1$$

$L_3 \to L_3-(a-1)L_2$

$$x+y-z=1$$
$$y+(a+2)z=1$$
$$(3+a)(2-a)z=2-a$$

(i) No solⁿ. if $a=-3$ (ii) many solⁿ. if $a=2$ (iii) unique solⁿ. if $a\neq 2$, $a\neq -3$

Rank of a matrix and its properties

Recap of Submatrix and minor

A matrix obtained by leaving some rows and columns from the original matrix is called a submatrix. If A be an $m \times n$ matrix then the determinant of every square sub-matrix of A is called a minor of the matrix A.

Rank of a matrix; Definition

A number r is said to be the rank of a matrix A if it possesses the following two properties

(i) There is at least one square submatrix of A of order r whose determinant is not equal to zero.

(ii) If the matrix A contains any square submatrix of order $r+1$, then the determinant of every square submatrix of A of order $r+1$ should be zero.

In short, the rank of a matrix is the order of any highest order non-vanishing minor of the matrix.

It is obvious that the rank r of an $m \times n$ matrix can at most be equal to the smaller of the numbers m and n, and it may be less. If there is a matrix A which has at least one non-zero minor of order n and there is no minor of A of order $n+1$, then the rank of A is n. Thus the rank of every non-singular matrix of order n is n. The rank of a square matrix A of order n can be less than n if and only if A is singular i.e. $|A|=0$

Note: Since the rank of every non-zero matrix is ≥ 1, we agree to assign the rank, zero, to every null matrix.

Echelon matrices

A matrix $A = a_{ij}$ is an echelon matrix or is said to be in echelon form, if the number of zeros preceding the first nonzero entry of a row increases row by row until only zero rows remain

i.e. if $\exists$ non-zero entries

$$a_{1j_1}, a_{2j_2}, \cdots, a_{rj_r} \text{ where } j_1 < j_2 < \cdots < j_r$$

with the property that $a_{ij} = 0$ for $i \leq r, j < j_i$ and for $i > r$.

We call $a_{1j_1}, \cdots, a_{rj_r}$ the distinguished elements of the matrix A.

Ex The following are echelon matrices where the distinguished elements have been circled

$$\begin{pmatrix} 2 & 3 & 2 & 0 & 4 & 5 & -6 \\ 0 & 0 & 7 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Important result The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

Theorem The rank of the transpose of a matrix is the same as that of the original matrix.

Ex Show that no skew-symmetric matrix can be of rank 1.

$$A = \begin{bmatrix} 0 & h & g & l \\ -h & 0 & f & m \\ -g & -f & 0 & n \\ -l & -m & -n & 0 \end{bmatrix}$$

Th The rank of a matrix is not changed by a finite chain of elementary transformations.

Defn: Equivalence of matrices

If B be an $m\times n$ matrix obtained from an $m\times n$ matrix A by finite number of elementary transformations of A, then A is called equivalent to B. Symbolically, we write $A\sim B$.

As we are doing only row operations, the matrices will be called row equivalent matrices.

Ex Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$$

Sol^n:

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$\therefore$ rank 1.

Ex Find the rank of the matrix

$$A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$$

Sol^n: $$A \sim \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{by} \quad \begin{array}{l} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_2 \end{array}$$

$\therefore$ rank $= 2$.