LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

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- **Continuity of Complex Functions**
- **Differentiability of Complex Functions**

RECALL: LIMIT OF FUNCTIONS OF A COMPLEX VARIABLE

We call $\lim_{z \to z_0} f(z) = w_0$ (w_0 the limit of $f(z)$ as z approaches z_0)

if the difference in absolute value between $f(z)$ and w_0 can be made arbitrarily small by choosing z close enough to z_0.

if and only if for given $\epsilon > 0$, there exists a positive number $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

CONTINUITY OF FUNCTIONS OF A COMPLEX VARIABLE

A function $f(z)$ is said to be continuous at $z = z_0$ if

- $\lim_{z \to z_0} f(z) = w_0$, i.e., the limit $\lim_{z \to z_0} f(z)$ exists
- $f(z)$ is defined at z_0, i.e., $f(z_0)$ exists
- $w_0 = f(z_0)$

Alternatively

A function $f(z) = u(x, y) + iv(x, y)$ is continuous at the point $z_0 = x_0 + i\, y_0$ if and only if the functions $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0).

If $\lim_{z \to z_0} f(z)$ exists but it is not equal to $f(z_0)$, we call z_0 **removable discontinuity** since by redefining $f(z_0)$ to be same as $\lim_{z \to z_0} f(z)$ the function becomes continuous.

Example: Discuss continuity of $f(z) = \begin{cases} \dfrac{iz^3 - 8}{z - 2i}, & z \neq 2i \\ -10\,i, & z = 2i \end{cases}.$

$$\lim_{z \to 2i} \frac{iz^3 - 8}{z - 2i} = \lim_{z \to 2i} \frac{iz^3 - 8\,i^4}{z - 2i}$$

$$= \lim_{z \to 2i} \frac{i(z^3 - 8\,i^3)}{z - 2i} = \lim_{z \to 2i} \frac{i(z - 2\hat{\imath})\,(z^2 - 4 + 2z\hat{\imath})}{z - 2i}$$

$$= \lim_{z \to 2i} i\,(z^2 - 4 + 2z\hat{\imath})$$

$$= \lim_{z \to 2i} i\,(-4 - 4 - 4) = -12\hat{\imath}$$

The given function is not continuous. (Removable Discontinuity)

DIFFERENTIABILITY OF FUNCTIONS OF A COMPLEX VARIABLE

A function $f(z)$ is said to be differentiable at a point $z_0 \in \mathbb{C}$ if

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists independent of the path in which $\Delta z \to 0$.

The limit, if exists, is called the derivative of f at z_0 and be denoted by $f'(z_0)$

Example: Find the derivative of $f(z) = z^2$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 + \Delta z^2 + 2z\Delta z - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Example: $f(z) = \bar{z}$ is not differentiable at any z.

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

If Δz approaches to 0 along the real axis:

$$\frac{\overline{\Delta z}}{\Delta z} = 1 \quad \text{as } \overline{\Delta z} = \Delta z$$

But if Δz approaches to 0 along the imaginary axis:

Let $\Delta z = ik$ for some real k $\quad \frac{\overline{\Delta z}}{\Delta z} = \frac{-ik}{ik} = -1$

$\Rightarrow \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist $\Rightarrow f$ has no derivative at any point.

DIFFERENTIABILITY IMPLIES CONTINUITY

Let f be differentiable at z_0, then f is continuous at z_0.

$$f(z_0+\Delta z)-f(z_0)=\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\times \Delta z, \qquad \Delta z\neq 0$$

$$\lim_{\Delta z\to 0}\big(f(z_0+\Delta z)-f(z_0)\big)=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\times \Delta z$$

$$=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\times \lim_{\Delta z\to 0}\Delta z$$

$$=f'(z_0)\times 0=0$$

$$\Longrightarrow \lim_{\Delta z\to 0} f(z_0+\Delta z)=f(z_0)$$

ANALYTIC FUNCTIONS

If the derivative $f'(z)$ exists at all points z of a domain D, then $f(z)$ is said to be analytic in D.

The terms regular, and holomorphic are also used for analytic.

A function $f(z)$ is said to be analytic at a point z_0 if there exists a neighborhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

CAUCHY-RIEMANN EQUATIONS (C-R EQUATIONS)

Let $f(z) = u(x,y) + i\, v(x,y)$ be a function of the complex variable z. The partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are called **Cauchy-Riemann** equations.

C-R EQUATIONS

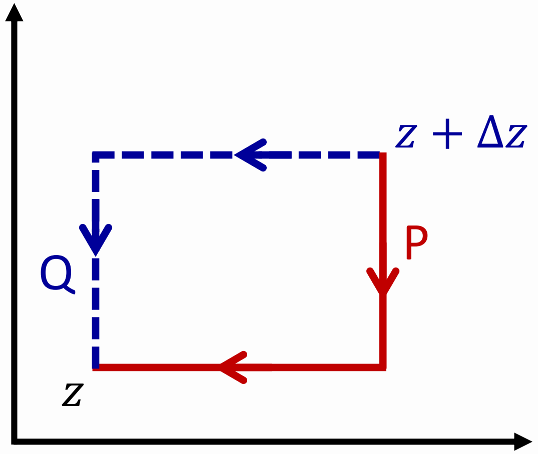
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \& \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

NECESSARY AND SUFFICIENT CONDITIONS OF ANALYTICITY

A necessary condition that $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D is that u & v satisfy C-R equations in D.

Moreover, if the partial derivatives appearing in CR equations are continuous then the C-R equations are sufficient for analyticity of $f(z)$ in D

Sketch of the Proof (Necessary Conditions):



Assume that $f(z)$ is analytic in $D \implies f'(z)$ exists at a point $z \in D$

$$\implies f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y}$$

Since, $f'(z)$ exists, the right hand limit should be the same along the all the paths $\Delta z \to 0$

Along path P : First $\Delta y \to 0$ and then $\Delta x \to 0$

Along path Q : First $\Delta x \to 0$ and then $\Delta y \to 0$

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

Along path P : First $\Delta y \to 0$ and then $\Delta x \to 0$

$$\Rightarrow f'(z) = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right] = u_x + iv_x$$

(existence of $f'(z) \Rightarrow$ existence of u_x, v_x)

Along path Q : First $\Delta x \to 0$ and then $\Delta y \to 0$

$$\Rightarrow f'(z) = \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y} \right] = v_y - iu_y$$

The above two limits implies $u_x = v_y$ & $v_x = -u_y$

CR equations in polar form

$u_x = v_y \;\; \& \;\; v_x = -u_y \quad \text{(cartesian)}$

Cartesian to polar: $x = r\cos\theta\,; \quad y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{1}{r}\left(\frac{\partial v}{\partial y}\, r\cos\theta - \frac{\partial v}{\partial x} r\sin\theta\right) = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Similarly by chain rule:

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

NOTE - 1 If the existence of the derivative is known then the following formula can be used for its evaluation:

$$f'(z) = u_x + iv_x = v_y - iu_y$$

NOTE - 2 The C-R equations are necessary condition for f to be differentiable at a point.

If they are not satisfied at a point, then $f'(z)$ does not exists at that point.

If the C-R equations hold at a point z_0, then f may or may not be differentiable at z_0

Summary

Differentiability at z_0: $\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

A function $f(z)$ is said to be analytic at a point z_0 if there exists a neighborhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

A necessary condition that $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D is that u & v satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \& \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- Analytic Functions
- Harmonic Functions
- Construction of Analytic Functions

CAUCHY-RIEMANN EQUATIONS (C-R EQUATIONS)

Let $f(z) = u(x, y) + i\, v(x, y)$ be a function of the complex variable z. The partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are called **Cauchy-Riemann** equations.

A function $f(z)$ is said to be **analytic** at a point z_0 if there exists a neighborhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

REMARK

The C-R equations are necessary conditions for f to be differentiable at a point.

If they are not satisfied at a point, then $f'(z)$ does not exists at that point.

If the C-R equations hold at a point z_0, then f may or may not be differentiable at z_0

Example: Consider: $f(z) = \bar{z}$

Note that $u(x,y) = x \;\; \& \;\; v(x,y) = -y$ $\qquad u_x = 1 \;\&\; u_y = 0 \qquad v_x = 0 \;\&\; v_y = -1$

CR- Equations: $u_x = v_y \;\; \& \;\; v_x = -u_y$

$\Rightarrow$ C-R equations do not hold at any point

$\Rightarrow$ The function f is not differentiable at any point

Example: Consider: $f(z) = z\,\mathrm{Re}(z) = (x+iy)x = x^2 + i\,xy$

$\Rightarrow\ u(x,y) = x^2\ \ \&\ \ v(x,y) = xy$ $\qquad u_x = 2x\ \&\ u_y = 0 \qquad v_x = y\ \&\ v_y = x$

CR - Equations: $u_x = v_y\ \&\ v_x = -u_y$

$\Rightarrow$ C-R equations do not hold at any point except $z = 0$

$\Rightarrow$ f is not differentiable at z if $z \neq 0$. (nowhere analytic!) It may have a derivative at 0.

$$\lim_{\Delta z\to 0}\frac{f(\Delta z)-f(0)}{\Delta z} = \lim_{\Delta z\to 0}\frac{\Delta z\,\mathrm{Re}(\Delta z)}{\Delta z} = \lim_{\Delta z\to 0}\mathrm{Re}(\Delta z) = 0$$

$\Rightarrow$ The function f is differentiable only at $z = 0$

Example: Consider $f(z) = |z|^2$

$$\Rightarrow \quad f(z) = |z|^2 = x^2 + y^2$$

$$\Rightarrow u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0$$

$$\Rightarrow u_x = 2x \quad \text{and} \quad v_y = 0$$

$$\Rightarrow u_y = 2y \quad \text{and} \quad v_x = 0$$

C-R equations are satisfied ONLY at $z = 0$

$\Rightarrow f$ is not differentiable at z if $z \neq 0$, but may have a derivative at $z = 0$

Consider $\displaystyle \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|\Delta z|^2}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0$

$\Rightarrow$ The function f is differentiable only at $z = 0$ (nowhere analytic!)

Example: Consider $f(z) = \sqrt{|xy|}$

$\Rightarrow$ $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$

$$\Rightarrow u_x = \frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\Rightarrow u_y = \frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = 0 \qquad \Rightarrow \quad v_x = 0 \text{ and } v_y = 0$$

$\Rightarrow$ C-R equations are satisfied.

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{\Delta z\to 0}\frac{\sqrt{|xy|}-0}{x+iy}$$

$f(z) = \sqrt{|xy|}$

Take $z \to 0$ along the path $y = mx$:

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{x\to 0}\frac{\sqrt{|mx^2|}}{(1+im)x} = \lim_{x\to 0}\frac{\sqrt{|m|}}{(1+im)}$$

$\Rightarrow f'(0)$ does not exist.

Hence, $f(z)$ is not differentiable at origin although C-R equations are satisfied.

Thank You