

ADVANCED CALCULUS

MA11003

SECTION 11, 12, & 15CD

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Concepts Covered

Differential Calculus

Functions of Several Variables

Differentiability – Two Variables (Cont.)

- ☐ Limit Test for Differentiability
- ☐ Worked Problems

Differentiability of Functions of Two Variables (Previous Lecture)

The function $z = f(x, y)$ is said to be differentiable at the point (x, y) , if at this point

$$\Delta z = a \Delta x + b \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

Necessary conditions

- Continuity of f
- Existence of partial derivatives f_x & f_y

Sufficient conditions

- Continuity of one/both partial derivatives

Testing Differentiability

$$\text{Differentiability} \iff \lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = 0, \quad \Delta\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

Let f be differentiable

$$\Delta z = \underbrace{a \Delta x + b \Delta y}_{dz} + \epsilon_1 \Delta x + \epsilon_2 \Delta y \implies \frac{\Delta z - dz}{\Delta\rho} = \epsilon_1 \frac{\Delta x}{\Delta\rho} + \epsilon_2 \frac{\Delta y}{\Delta\rho}$$

$$\lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho \rightarrow 0} \epsilon_1 \frac{\Delta x}{\Delta\rho} + \lim_{\Delta\rho \rightarrow 0} \epsilon_2 \frac{\Delta y}{\Delta\rho} = 0$$

Note that $\frac{\Delta x}{\Delta\rho} \leq 1$ & $\frac{\Delta y}{\Delta\rho} \leq 1$ and ϵ_1, ϵ_2 tend to zero as $\Delta\rho \rightarrow 0$

Testing Differentiability (cont.)

$$\text{Differentiability} \iff \lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = 0, \quad \Delta\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\text{Let } \lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = 0 \implies \frac{\Delta z - dz}{\Delta\rho} = \epsilon \quad \epsilon \rightarrow 0 \text{ as } \Delta\rho \rightarrow 0$$

$$\implies \Delta z - dz = \epsilon \Delta\rho = \epsilon \sqrt{\Delta x^2 + \Delta y^2} = \epsilon \frac{\Delta x^2 + \Delta y^2}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$= \left(\frac{\epsilon \Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} \right) \Delta x + \left(\frac{\epsilon \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \right) \Delta y$$

$$\implies \Delta z = dz + \epsilon_1 \Delta x + \epsilon_2 \Delta y \implies \text{Differentiability of } f$$

Problem – 1 (Continuous, partial derivatives exist but not differentiable)

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Existence of Partial Derivatives

$$f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = 0 \quad f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = 0$$

Continuity Changing to polar coordinates ($x = r \cos \theta$, $y = r \sin \theta$)

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) = \lim_{r \rightarrow 0} r \cos \theta \sin \theta = 0 = f(0, 0)$$

Differentiability

$$\lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = ?$$

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\Delta z = f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = \frac{\Delta x \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y = 0$$

$$\lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho \rightarrow 0} \left(\frac{\Delta x \Delta y}{\Delta x^2 + \Delta y^2} \right)$$

Along the path $\Delta y = m \Delta x$

$$\lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = \frac{m}{1 + m^2}$$

The given function is NOT differentiable.

Problem – 2 (Continuous, partial derivatives exist but not differentiable)

$$f(x, y) = \begin{cases} \frac{x^3 + 2y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Existence of Partial Derivatives

$$f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = 1 \quad f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = 2$$

Continuity Changing to polar coordinates ($x = r \cos \theta$, $y = r \sin \theta$)

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^3 + 2y^3}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta + 2r^3 \sin^3 \theta}{r^2} = 0 = f(0, 0)$$

Differentiability

$$\lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = ?$$

$$f(x, y) = \begin{cases} \frac{x^3 + 2y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\Delta z = f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = \frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2 + \Delta y^2}$$

$$dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y = \Delta x + 2 \Delta y$$

$$\lim_{\Delta\rho \rightarrow 0} \frac{1}{\sqrt{\Delta x^2 + \Delta y^2}} \left(\frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2 + \Delta y^2} - (\Delta x + 2 \Delta y) \right) = \lim_{\Delta\rho \rightarrow 0} \frac{-\Delta x \Delta y^2 - 2\Delta x^2 \Delta y}{(\Delta x^2 + \Delta y^2)^{3/2}}$$

$$\begin{aligned} \text{Along the path } \Delta y = m \Delta x & \\ &= \frac{-m^2 - 2m}{(1 + m^2)^{3/2}} \end{aligned}$$

The given function is NOT differentiable.

Problem – 3 (Differentiable but f_x & f_y are not continuous)

$$f(x, y) = \begin{cases} (x^2 + y^2) \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right), & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Existence of Partial Derivatives

$$f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = 0 \qquad f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = 0$$

Continuity Changing to polar coordinates ($x = r \cos \theta$, $y = r \sin \theta$)

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} (x^2 + y^2) \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right) = 0 = f(0, 0)$$

Differentiability

$$f(x, y) = \begin{cases} (x^2 + y^2) \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right), & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\Delta z = f(0 + \Delta x, 0 + \Delta y) - f(0, 0)$$

$$= (\Delta x^2 + \Delta y^2) \cos\left(\frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}\right)$$

$$dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y = 0$$

$$\begin{aligned} \lim_{\Delta \rho \rightarrow 0} \frac{\Delta z - dz}{\Delta \rho} &= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{(\Delta x^2 + \Delta y^2)}{\sqrt{\Delta x^2 + \Delta y^2}} \cos\left(\frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}\right) \\ &= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \sqrt{\Delta x^2 + \Delta y^2} \cos\left(\frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}\right) = 0 \end{aligned}$$

Hence, the given function is differentiable

Continuity of f_x & f_y

At $(x, y) \neq (0, 0)$

$$f(x, y) = \begin{cases} (x^2 + y^2) \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right), & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\begin{aligned} f_x(x, y) &= -(x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \left(-\frac{1}{2} \frac{2x}{(x^2 + y^2)^{\frac{3}{2}}}\right) + 2x \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \\ &= \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \left(\frac{x}{\sqrt{(x^2 + y^2)}}\right) + 2x \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \end{aligned}$$

Along x -axis

$$\lim_{x \rightarrow 0} f_x(x, y) = \lim_{x \rightarrow 0} \left(\frac{x}{|x|} \sin\left(\frac{1}{|x|}\right) + 2x \cos\left(\frac{1}{|x|}\right) \right) \neq 0$$

$\Rightarrow f_x$ is not continuous at $(0, 0)$. Similarly, f_y is not continuous at $(0, 0)$

Remark: The above example shows that continuity of partial derivatives is not a necessary condition for differentiability. A function can be differentiable without having continuous first order partial derivatives.

Example (Differentiable but f_x & f_y are not continuous)

$$f(x, y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + y^2 \cos\left(\frac{1}{y}\right), & x \neq 0, y \neq 0 \\ 0 & \text{elsewhere} \end{cases}$$

KEY TAKEAWAY

$$\text{Differentiability} \iff \lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = 0$$

- The function may not be differentiable at a point $P(x, y)$ even if the partial derivatives f_x and f_y exists at P .
(Existence of partial derivatives is a necessary condition)
- A function may be differentiable even if f_x and f_y are not continuous.
(Continuity of the f_x and/or f_y is a sufficient condition)

Concepts Covered

Differential Calculus

Functions of Several Variables

□ Differentiability – Worked Problems

Differentiability of Functions of Two Variables (Previous Lecture)

$$\Delta z = a \Delta x + b \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \iff \lim_{\Delta \rho \rightarrow 0} \frac{\Delta z - dz}{\Delta \rho} = 0, \quad \Delta \rho = \sqrt{\Delta x^2 + \Delta y^2}$$

Necessary Conditions

- Continuity of f
- Existence of partial derivatives f_x & f_y

Sufficient Conditions

- Existence of one partial derivative and continuity of the other

Problem – 1

Discuss the differentiability at origin of the function $f(x, y) = \begin{cases} \frac{xy^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

Necessary Conditions

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$$

$$f_x(0,0) = 0$$

$$f_y(0,0) = 0$$

Sufficient Conditions

$$f_x(x, y) = \begin{cases} \frac{-x^2 y^3 + y^5}{(x^2 + y^2)^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,0)} f_x(x, y) = 0 = f_x(0, 0)$$

Hence f_x is continuous.

Therefore the function f is differentiable at $(0,0)$.

Problem – 2

Discuss the differentiability at origin of the function $f(x, y) = \begin{cases} y^3 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Continuity of f and existence of partial derivatives can easily be shown.

$$f_x(x, y) = \begin{cases} -\frac{2y^3}{x^3} \cos\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \lim_{(x,y) \rightarrow (0,0)} f_x(x, y) = ?$$

Along the path $y = x$: $\lim_{(x,y) \rightarrow (0,0)} f_x(x, y) = -2 \lim_{x \rightarrow 0} \cos\left(\frac{1}{x^2}\right)$

Hence f_x is NOT continuous.

Differentiability at origin of the function $f(x, y) = \begin{cases} y^3 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$f_y(x, y) = \begin{cases} 3y^2 \cos\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\lim_{(x,y) \rightarrow (0,0)} f_y(x, y) = 0$ Hence f_y is continuous.

$\Rightarrow f_x$ exist at $(0,0)$ and f_y is continuous at $(0,0)$

$\Rightarrow f$ is differentiable at $(0,0)$

Problem – 3 Let $f(x, y) = \begin{cases} \sqrt{xy}, & xy \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$

Determine whether the function is differentiable at the origin.

Continuity: $\lim_{(x,y) \rightarrow (0,0)} \sqrt{xy} = 0$

Existence of Partial Derivatives:

$$f_x = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0 \quad f_y = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

Differentiability $f(x, y) = \begin{cases} \sqrt{xy}, & xy \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$

$$\lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = ?$$

$$dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y = 0 \qquad \Delta z = f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = \sqrt{\Delta x \Delta y}$$

(assuming $\Delta x \Delta y \geq 0$)

$$\lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho \rightarrow 0} \frac{\sqrt{\Delta x \Delta y}}{\sqrt{\Delta x^2 + \Delta y^2}}$$

Along the path $\Delta y = \Delta x$

$$\lim_{\Delta\rho \rightarrow 0} \frac{\Delta z - dz}{\Delta\rho} = \frac{1}{\sqrt{2}} \neq 0$$

The given function is NOT differentiable

Problem – 4 Discuss the differentiability at the origin of the function

$$f(x, y) = \begin{cases} x^{\frac{5}{2}} \sin\left(\frac{1}{\sqrt{x}}\right) + y^{\frac{5}{2}} \cos\left(\frac{1}{\sqrt{y}}\right), & x \neq 0, y \neq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Recall the definition of differentiability $\Delta z = a \Delta x + b \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$

$$\begin{aligned} f(\Delta x, \Delta y) - f(0, 0) &= \Delta x^{\frac{5}{2}} \sin\left(\frac{1}{\sqrt{\Delta x}}\right) + \Delta y^{\frac{5}{2}} \cos\left(\frac{1}{\sqrt{\Delta y}}\right) \\ &= 0 \cdot \Delta x + 0 \cdot \Delta y + \underbrace{\Delta x \left(\Delta x^{\frac{3}{2}} \sin\left(\frac{1}{\sqrt{\Delta x}}\right) \right)}_{\epsilon_1} + \underbrace{\Delta y \left(\Delta y^{\frac{3}{2}} \cos\left(\frac{1}{\sqrt{\Delta y}}\right) \right)}_{\epsilon_2} \end{aligned}$$

Problem – 5 Let $f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

Check the existence of f_x & f_y at origin. Is f differentiable at origin?

$$f_x = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = 0$$

$$f_y = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = 0$$

Continuity Check of f

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \text{ along } (x = y) = 0 \neq f(0, 0)$$

\Rightarrow the function f is not continuous at $(0, 0)$

\Rightarrow the function f is NOT differentiable at $(0, 0)$

Problem – 6 Let $f(x, y) = \begin{cases} \frac{x^2 + y^2}{|x| + |y|}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ Is f differentiable at origin?

Continuity Check

$$|f(x, y) - 0| = \frac{x^2 + y^2}{|x| + |y|} < \frac{(|x| + |y|)^2}{|x| + |y|} = |x| + |y| < \sqrt{2} \sqrt{x^2 + y^2} < \sqrt{2} \delta < \epsilon$$

Choose $\delta < \frac{\epsilon}{\sqrt{2}}$, then $|f(x, y) - f(0, 0)| < \epsilon$ whenever $0 < \sqrt{x^2 + y^2} < \delta$

This implies the function $f(x, y)$ is continuous.

Differentiability of $f(x, y) = \begin{cases} \frac{x^2 + y^2}{|x| + |y|}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

Existence of Partial Derivatives

$$\lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x^2}{|\Delta x| \Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{|\Delta x|} \quad \text{limit does not exist}$$

$\Rightarrow f_x(0, 0)$ does not exist.

Similarly, $f_y(0, 0)$ does not exist.

\Rightarrow The function f is NOT differentiable at $(0, 0)$

KEY TAKEAWAY

Investigation of Differentiability

Necessary Conditions:

Continuity & existence of partial derivatives

Sufficient Conditions:

Continuity of one of the partial derivatives

Final Check (Limit Test):

$$\lim_{\Delta \rho \rightarrow 0} \frac{\Delta z - dz}{\Delta \rho} = 0 \quad \text{or} \quad \Delta z = a \Delta x + b \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

Thank You