LINEAR ALGEBRA, NUMERICAL AND COMPLEX ANALYSIS

MA11004

SECTIONS 1 and 2

Dr. Jitendra Kumar

Professor
Department of Mathematics
Indian Institute of Technology Kharagpur
West Bengal 721302, India

Webpage: http://www.facweb.iitkgp.ac.in/~jkumar/

- **Complex Numbers**
  - **Modulus and Argument of a Complex Number**
  - **Some Properties of the Modulus & Argument**
- **Complex Functions**
  - **Definition**
  - **Limit**

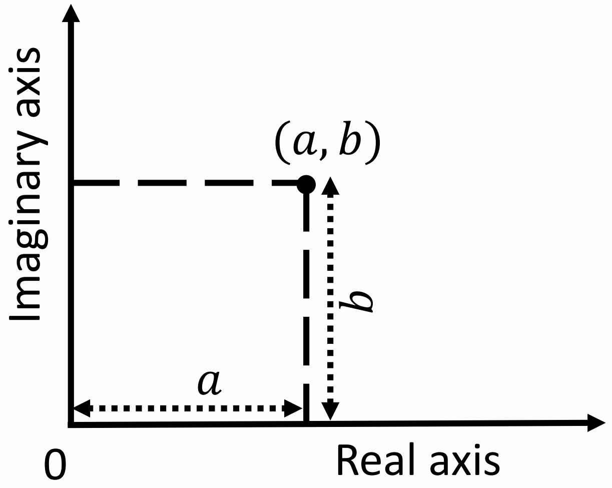
Complex Numbers

A complex number, say z, is written in the form $z = a + ib$, or equivalenty, $z = a + bi$

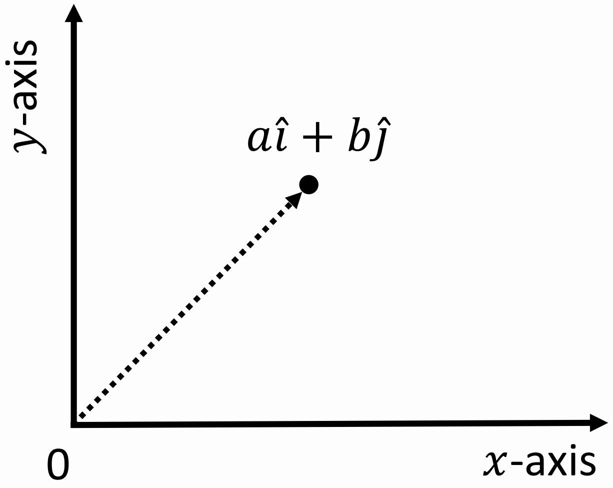
Here a and b are real numbers and i is an imaginary number that satisfy $i^2 = -1$.

The real numbers a and b are called **real** and **imaginary** part of z, respectively.

Set of all complex numbers is denoted by $\mathbb{C}$.



Point in a plane



Vector in a plane

ARITHMETIC ON COMPLEX NUMBERS

- Equality $a + ib = c + id$ exactly when $a = c$ & $b = d$
- Addition $(a + ib) + (c + id) = (a + c) + i(b + d)$
- Multiplication (first order polynomial & $i^2 = -1$)

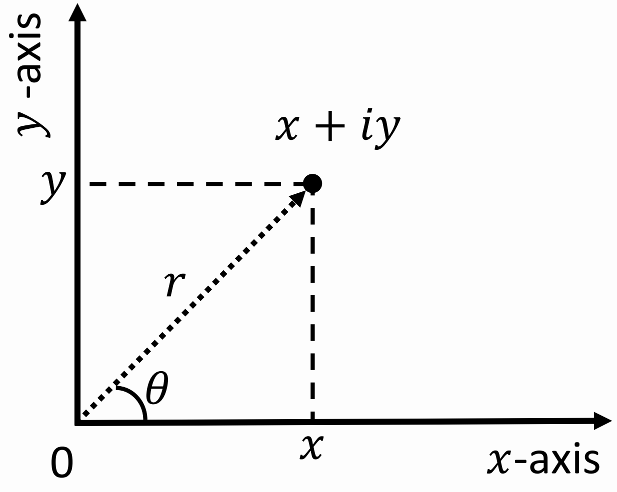
$$(a + ib)(c + id) = ac + adi + bci + bdi^2 = (ac - bd) + i(ad + bc)$$

COMPLEX CONJUGATE

The complex conjugate of $z = a + ib$ is defined as

$$\bar{z} = a - ib$$

MODULUS & ARGUMENT OF A COMPLEX NUMBER



The number r is called the **modulus** of the complex number $z = x + iy$

Modulus of $x + iy$ is denoted by $|x + iy|$ and is defined as

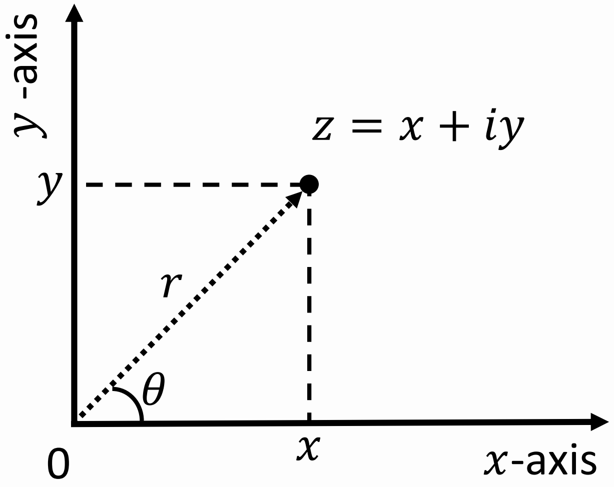
$$\sqrt{x^2 + y^2}$$

The angle θ is called the **argument** of z and is denoted by $\mathbf{arg}\, \boldsymbol{z}$ and is defined as

$$\theta = \arg z\,, \qquad \text{if} \quad \tan\theta = \frac{y}{x}$$

Among infinitely many values of θ, the one which lies in $(-\pi, \pi]$ is called the **principal value** $(\mathrm{Arg}\; z)$.

POLAR FORM OF A COMPLEX NUMBER



Note that $x = r\cos\theta$ and $y = r\sin\theta$

Then $z = x + iy$ may be written as

$$z = x + iy = r\cos\theta + i\,r\sin\theta$$

$$= r\,(\cos\theta + i\,\sin\theta) \quad \text{(trigonometric form)}$$

$$= r\,e^{i\theta} \quad \text{(polar form)}$$

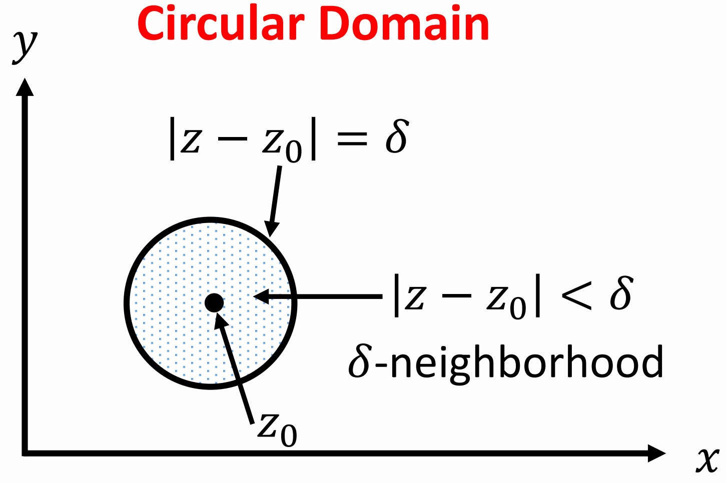
(or exponential form)

SOME PROPERTIES OF COMPLEX NUMBERS

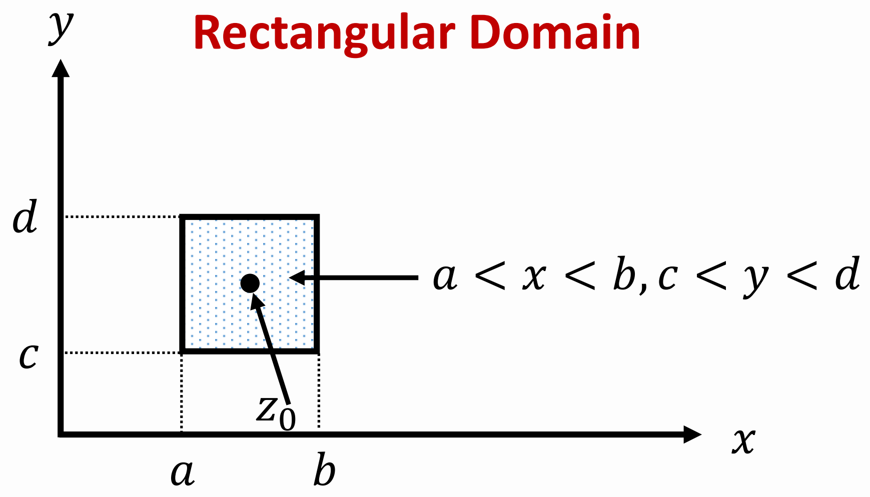
- $z\bar{z} = |z|^2$
- $|z| = 0 \iff z = 0$
- $|z| = |\bar{z}|$
- $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$
- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$
- $|z^n| = |z|^n$
- $\arg z^n = n \arg z$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$

NEIGHBOURHOOD OF A POINT

Circular Domain



Rectangular Domain



Any circular domain or rectangular domain around a point is called its **neighborhood**.

FUNCTION OF A COMPLEX VARIABLE

Let D be a set of complex numbers. A function f defined on D (domain of f) is a rule that assigns to each value of z in D a complex number w:

$$w = f(z) \quad \Leftrightarrow \quad u(x, y) + i\, v(x, y) = f(x + iy)$$

Example: $w = z^2$

$$\Longrightarrow w = z^2 = x^2 - y^2 + 2xy\, i$$

$$\Longrightarrow u(x, y) = x^2 - y^2 \;\; \& \;\; v(x, y) = 2xy$$

LIMIT OF A FUNCTION OF A COMPLEX VARIABLE

Let $f(z)$ be defined and single valued in a neighbored of $z = z_0$. Let w_0 be a complex number then,

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if for given $\epsilon > 0$, there exists a positive number $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

We call w_0 the limit of $f(x)$ as z approaches z_0.

OR, we call $\lim_{z \to z_0} f(z) = w_0$ if the difference in absolute value between $f(z)$ and w_0

can be made arbitrarily small by choosing z close enough to z_0.

LIMIT IN TERMS OF ITS REAL AND IMAGINERY PARTS OF A COMPLEX FUNCTION

Let $f(z) = u(x, y) + i\, v(x, y)$ and $z_0 = x_0 + iy_0$.

$$\lim_{z \to z_0} f(z) = u_0 + i\, v_0 \iff u_0 = \lim_{(x,y) \to (x_0, y_0)} u(x, y) \quad \& \quad v_0 = \lim_{(x,y) \to (x_0, y_0)} v(x, y)$$

Examples: $\displaystyle \lim_{z \to 2-3i} |z| = \lim_{(x,y) \to (2,-3)} \sqrt{x^2 + y^2} = \sqrt{13}$

$$\lim_{z \to 3} \frac{z^2 + 4z - 21}{z - 3} = \lim_{z \to 3} \frac{(z-3)(z+7)}{z-3} = \lim_{z \to 3}(z+7) = 10$$

SUMMARY

- **Representation of a Complex Number:** $z = x + iy$ $\quad z = r\left(\cos(\theta) + i\sin(\theta)\right)$ $\quad z = re^{i\theta}$

- **Complex Function:** $w = f(z) \quad \Leftrightarrow \quad u(x,y) + i\, v(x,y) = f(x+iy)$

- **Limit of Function of a Complex Variable:** $\lim\limits_{z \to z_0} f(z) = w_0$

Thank You