ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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Concepts Covered

Differential Calculus

Functions of Several Variables

- Partial Derivatives

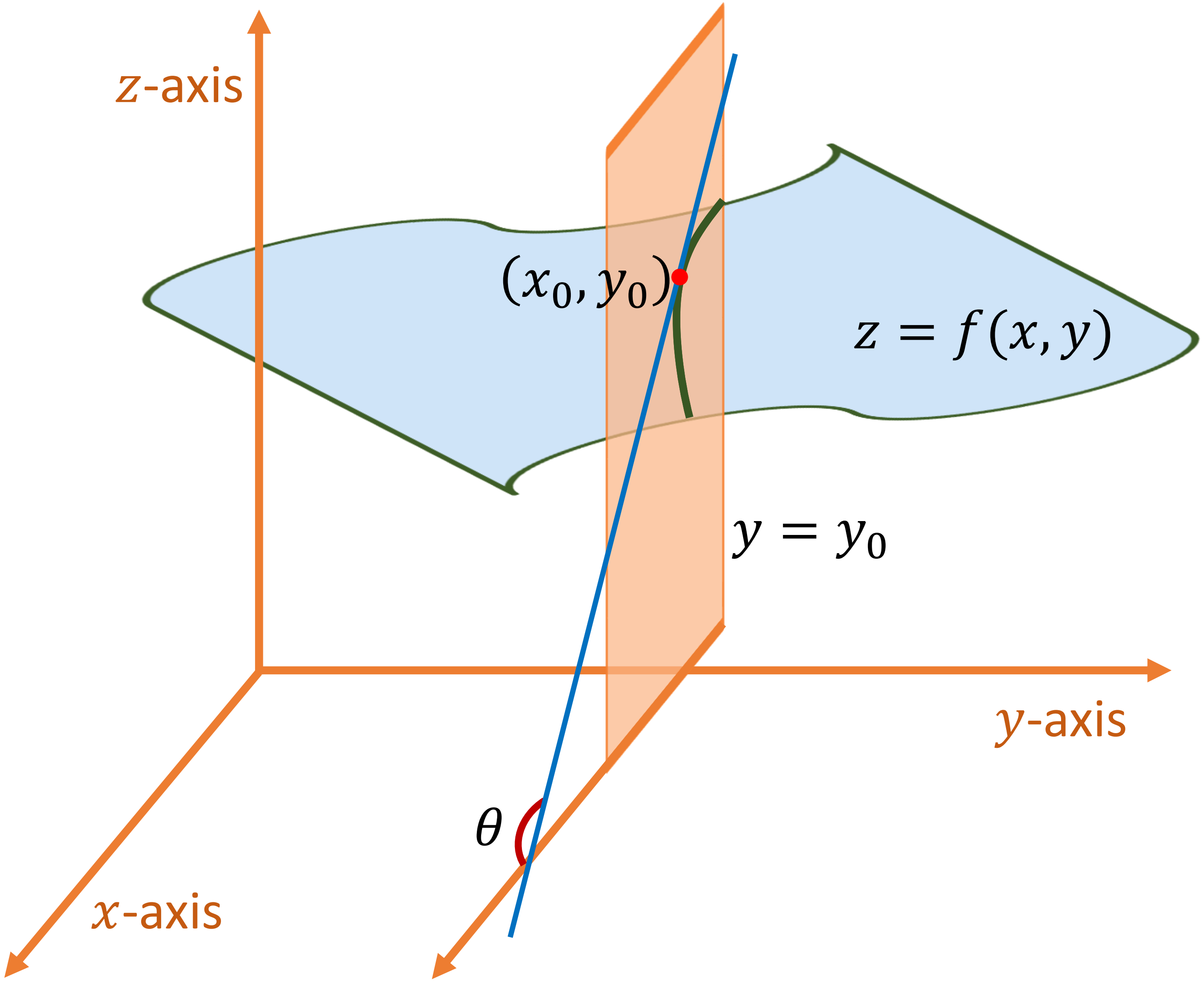
Partial Derivatives

The usual derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables as constant is called the partial derivatives of the function with respect to that variable.

Let $z = f(x, y); \quad (x, y) \in \mathbb{R}^2, \quad z \in \mathbb{R}$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \frac{d}{dx} f(x, y_0)\Big|_{x=x_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = \frac{d}{dy} f(x_0, y)\Big|_{y=y_0}$$



Geometrical Interpretation of Partial Derivatives

$$\tan\theta = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$$

Problem – 1: Find the value of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (x, y) of the function $f(x,y) = ye^{-x}$ from the first principal.

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{ye^{-(x+\Delta x)} - ye^{-x}}{\Delta x}$$

$$= ye^{-x} \lim_{\Delta x \to 0} \frac{e^{-\Delta x} - 1}{\Delta x} = -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(y+\Delta y)e^{-x} - ye^{-x}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} e^{-x} = e^{-x}$$

Relationship: Partial Derivatives & Continuity

A function can have partial derivatives with respect to both x and y at a point without being continuous there. On the other hand a continuous function may not have partial derivatives.

Problem – 2: Show that the function

$$f(x,y) = \begin{cases} (x+y)\sin\left(\dfrac{1}{x+y}\right), & (x+y) \neq 0 \\ 0, & \text{elsewhere} \end{cases}$$

is continuous at $(0,0)$ but its partial derivatives do not exist at $(0,0)$

Continuity at $(0,0)$ $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$,

Partial Derivatives at $(0,0)$ $f(x,y) = (x+y)\sin\left(\frac{1}{x+y}\right)$

$$\lim_{\Delta x\to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{1}{\Delta x}\Delta x \sin\left(\frac{1}{\Delta x}\right) = \lim_{\Delta x\to 0} \sin\left(\frac{1}{\Delta x}\right)$$

$\Longrightarrow$ The partial derivative w.r.t. x does not exist.

Similarly, the partial derivative w.r.t. y does not exist.

Problem – 3: Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 + 2y^2}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives exist at $(0,0)$

Choosing the path $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + 2y^2} = \frac{m}{(1 + 2m^2)}$$ Limit depends on the path

The limit does not exist. Hence the function is not continuous at $(0,0)$.

Partial Derivatives at $(\mathbf{0}, \mathbf{0})$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$$

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$\lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y} = \lim_{\Delta x \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

$\Longrightarrow$ The partial derivatives w.r.t. x & y exist at $(0,0)$.

Problem – 4: Let $f(x,y) = \begin{cases} \dfrac{2x^3 + 3y^3}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$

Compute $f_x(0,0)$ & $f_y(0,0)$.

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 2$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 3$$

The function $f(x,y)$ is continuous and also partial derivatives exist.

Problem – 5: Let $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & \text{elsewhere}\end{cases}$

Compute $\dfrac{\partial f}{\partial x}(x,y)$ & $\dfrac{\partial f}{\partial y}(x,y)$ and discuss the continuity of these partial derivatives

$$f_x(x,y)=\begin{cases}\dfrac{y^3}{(x^2+y^2)^{3/2}}, & (x,y)\neq(0,0)\\ 0, & \text{elsewhere}\end{cases}$$

$$f_y(x,y)=\begin{cases}\dfrac{x^3}{(x^2+y^2)^{3/2}}, & (x,y)\neq(0,0)\\ 0, & \text{elsewhere}\end{cases}$$

Continuity of partial derivatives

$$f_x(x,y) = \begin{cases} \dfrac{y^3}{(x^2+y^2)^{3/2}}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$$

$$f_y(x,y) = \begin{cases} \dfrac{x^3}{(x^2+y^2)^{3/2}}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}} = \lim_{r\to 0} \sin^3\theta \quad = \sin^3\theta$$

Limit does not exist

The same observation for f_y

Hence, both f_x & f_y are not continuous

Sufficient Condition for Continuity in a Region *R*

If a function $f(x, y)$ has partial derivatives f_x & f_y everywhere in a region *R* and these derivatives everywhere satisfy the inequalities

$$|f_x(x, y)| < M, \qquad |f_y(x, y)| < M$$

where M is independent of x & y, then $f(x, y)$ is continuous everywhere in *R*.

Sufficient Condition for Continuity at (x_0, y_0)

One of the first order partial derivatives exists and is bounded in the neighborhood of (x_0, y_0) and the other exists at (x_0, y_0)

KEY TAKEAWYAY

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \frac{d}{dx} f(x, y_0)\Big|_{x=x_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = \frac{d}{dy} f(x_0, y)\Big|_{y=y_0}$$

Concepts Covered

Differential Calculus

Functions of Several Variables

- Partial Derivatives of Higher Order

Partial Derivatives of f (Previous Lecture)

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Second Order Partial Derivatives of f

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$	$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$	$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$	$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$
f_{xx}	$f_{y\underline{x}}$	f_{yy}	$f_{x\underline{y}}$
$\frac{\partial^2 f}{\partial x^2}$	$\frac{\partial^2 f}{\partial y \partial x}$	$\frac{\partial^2 f}{\partial y^2}$	$\frac{\partial^2 f}{\partial x \partial y}$

The derivatives f_{xy} and f_{yx} are called mixed derivatives.

Problem – 1:

Compute $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ at the origin of $f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f_y}{\partial x} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

$$f_y(\Delta x, 0) = \Delta x$$

$$f_y(0,0) = 0$$

$\frac{\partial^2 f}{\partial y \partial x}$ at the origin of $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f_x}{\partial y} = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-\Delta y - 0}{\Delta y} = -1$$

$f_x(0,\Delta y) = -\Delta y$

Note that $\frac{\partial^2 f}{\partial x \partial y} = 1$

$f_x(0,0) = 0$

The equality of mixed partial derivatives

If (i) f_x, f_y, f_{yx} all exist in the neighborhood of the point (x_0, y_0)

& (ii) f_{yx} is continuous at (x_0, y_0), then

a) f_{xy} also exists at (x_0, y_0), and

b) $f_{xy} = f_{yx}$

OR

If the mixed derivatives f_{yx} & f_{xy} are continuous in an open domain D, then at any point $(x, y) \in D$

$$f_{xy} = f_{yx}$$

Problem – 2:

Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function $f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & x \neq -y^2 \\ 0, & \text{elsewhere} \end{cases}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f_y}{\partial x} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(\Delta x, 0) = 0$$

$$f_y(0,0) = 0$$

$f_{yx}(0,0)$ for the function $f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & x \neq -y^2 \\ 0, & \text{elsewhere} \end{cases}$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f_x}{\partial y} = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y - 0}{\Delta y} = 1$$

$f_x(0,\Delta y) = \Delta y$ $\qquad$ $f_x(0,0) = 0$

Since $f_{xy}(0,0) \neq f_{yx}(0,0)$, f_{xy} and f_{yx} are not continuous at $(0,0)$.

Continuity Check of $\boldsymbol{f_{xy}}$ & $\boldsymbol{f_{yx}}$

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & x \neq -y^2 \\ 0, & \text{elsewhere} \end{cases}$$

For $x \neq -y^2$

$$f_x(x,y) = \frac{y^5}{(x+y^2)^2}$$

$$f_{yx}(x,y) = \frac{y^6 + 5xy^4}{(x+y^2)^3} = f_{xy}(x,y)$$

Along the path $x = my^2$ the limit $\lim_{(x,y)\to(0,0)} f_{yx}(x,y) = \dfrac{1+5m}{(m+1)^3}$

Limit depends on the path

The limit does not exist. Hence f_{yx} or f_{xy} is not continuous at $(0,0)$

Problem 3: Showing existence of second order partial derivative though the function is not continuous

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y \\ 0, & \text{elsewhere} \end{cases}$$

Take a path $y = x\cos x$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^3 + x^3\cos^3 x}{x - x\cos x} = \ 4$$

The function is not continuous at $(0,0)$.

Evaluation of $f_{xx}(0,0)$:

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y \\ 0, & \text{elsewhere} \end{cases}$$

$$f_{xx}(0,0) = \lim_{\Delta x \to 0} \frac{f_x(\Delta x, 0) - f_x(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x - 0}{\Delta x} = 2$$

$$f_x(x,0) = \quad 2x$$

$$f_x(0,0) = \; 0$$

Evaluation of $f_{yy}(0,0)$:

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y \\ 0, & \text{elsewhere} \end{cases}$$

$$f_{yy}(0,0) = \lim_{\Delta y \to 0} \frac{f_y(0,\Delta y) - f_y(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-2\Delta y - 0}{\Delta y} = -2$$

$$f_y(0,\mathrm{y}) = \quad -2y$$

$$f_y(0,0) = \ 0$$

KEY TAKEAWAY

Partial Derivatives of Higher Order

Continuity of f_{yx} & $f_{xy} \Longrightarrow f_{yx} = f_{xy}$

Thank You