Indian Institute of Technology Kharagpur
Department of Mathematics
MA11003 - Advanced Calculus (Autumn 2020)
Answer Hints Tutorial Sheet - 7

---

1. Discuss the convergence of improper integrals using definition:

   (i) convergent
   (ii) divergent
   (iii) divergent
   (iv) divergent
   (v) convergent
   (vi) divergent
   (vii) divergent
   (viii) divergent
   (ix) convergent
   (x) divergent

2. Discuss the convergence of the following integrals :

   (i) convergent, apply μ test.
   (ii) convergent, apply μ test.
   (iii) divergent, as $0 \le x \le 1$ so $e^x \le e$ and $x(x+e^x) \le x(e+1)$,then apply comparison test.
   (iv) convergent, apply comparison test.
   (v) convergent, apply comparison test.
   (vi) convergent, apply μ test.
   (vii) convergent, no point of infinite discontinuity.
   (viii) convergent, $\frac{\cos x}{e^x} < \frac{1}{x^2}$ when $x > 1$, then apply comparison test.
   (ix) convergent, for $x \ge 1 \qquad e^{-(x+x^{-1})} \le e^{-x}$, apply comparison test.
   (x) convergent, apply comparison test.

3. Examine the convergence of the following integrals :

   (i) Convergent, 0 and 1 are points of infinite discontinuity. Examine the convergence of $\int_0^{\frac{1}{2}} \frac{1}{(x+2)\sqrt{x(1-x)}}dx$ at $x=0$ and convergence of $\int_{\frac{1}{2}}^1 \frac{1}{(x+2)\sqrt{x(1-x)}}dx$ at $x=1$; In both cases apply μ test.
   (ii) Convergent, 0 and ∞ are the point of infinite discontinuity. Examine the convergence of $\int_0^1 x^{-\frac{1}{2}}e^{-x}dx$ at $x=0$ and convergence of $\int_1^\infty x^{-\frac{1}{2}}e^{-x}dx$ at $x=\infty$ For first integral use comparison test and for second integral use $\frac{1}{e^x} < \frac{1}{x}$ for all $x \ge 1$ for comparison test.

(iii) Divergent, ∞ is the only point of discontinuity. Apply comparison test taking $g(x) = \frac{1}{x^{\frac{3}{4}}}$.

(iv) Convergent, modulus of integrand is $\leq \frac{1}{\sqrt{x^3+x}}$.First check that $\int\limits_0^\infty \frac{1}{\sqrt{x^3+x}}dx$ is convergent by applying comparison test and use every absolutely convergent integral is convergent.

(v) Divergent, 1 is a point of infinite discontinuity. If $p < 1$ then 0 is also a point of infinite discontinuity. Examine the convergence of $\int\limits_0^{\frac{1}{2}} \frac{x^{p-1}}{1-x}dx$ at $x = 0$ when $p < 1$ and convergence of $\int\limits_{\frac{1}{2}}^1 \frac{x^{p-1}}{1-x}dx$ at $x = 1$. The second integral will be divergent.

4. Only point of infinite discontinuity is at $x = 0$. Apply comparison test by taking $g(x) = \frac{1}{x^{n-m}}$.

5. Here 0 is the point of infinite discontinuity. As $|\frac{\sin(\frac{1}{x})}{\sqrt{x}}| \leq \frac{1}{\sqrt{x}}$ for all $x \in (0, 1]$,apply comparision test and use every absolutely convergent integral is convergent .

6. The only point of infinite discontinuity is at $x = \infty$. Examine the convergence of the integral with taking $g(x) = \frac{1}{x^2}$.

7. Convergent, Apply comparison test using $e^{-x^2} \leq e^{-x}$ for all $x \in [1, \infty)$.

8. Convergent, 0 is the point of infinite discontinuity of the integrand. Check that $\int\limits_0^1 \ln x x^{n-1} dx$ is convergent if $n > 0$.For this case $n = \frac{1}{2}$.

9. Here 0 is point of infinite discontinuity if $m < 1$ and 1 is the point of infinite discontinuity if $n < 1$. Examine the convergence of $\int\limits_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ when $m < 1$ and convergence of $\int\limits_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}dx$ when $n < 1$. In both cases apply comparison test.

10. Apply $\int\limits_0^\infty \frac{\phi(ax) - \phi(bx)}{x}dx = (\lim\limits_{x\to 0} \phi(x) - \lim\limits_{x\to\infty} \phi(x)) \log(\frac{a}{b})$. Here $\phi(x) = \tan^{-1}(x)$ for $x \geq 0$

11. Similarly to previous problem, take $\phi(x) = \frac{\sin(x)}{x}$.
