INTEGRAL CALCULUS

CONVERGENCE OF IMPROPER INTEGRALS:

PROPER INTEGRAL:

$\int_a^b f(x)\,dx$ Range of integration is finite and integrand is bounded.

IMPROPER INTEGRAL:

Integral $\int_a^b f(x)\,dx$ is called improper if

(i) $a = -\infty$ and/or $b = \infty$ and $f(x)$ is bounded — first kind

(ii) $f(x)$ is unbounded at one or more points of $a \le x \le b$ — Second kind

(iii) Both (i) & (ii) type - Third kind or mixed kind.

Example: $\int_0^\infty \cos x\,dx$ - first kind

$\int_0^1 \frac{dx}{x-1}$ — Second kind

$\int_0^\infty \frac{dx}{(1-x)^2}$ — third kind

Evaluation of integrals of first kind:

(i) $\int_a^\infty f(x)\,dx = \lim\limits_{R\to\infty} \int_a^R f(x)\,dx$

(ii) $\int_{-\infty}^b f(x)\,dx = \lim\limits_{R\to\infty} \int_{-R}^b f(x)\,dx$

(iii) $\int_{-\infty}^\infty f(x)\,dx = \lim\limits_{R_1\to\infty} \int_{-R_1}^c f(x)\,dx + \lim\limits_{R_2\to\infty} \int_c^{R_2} f(x)\,dx$

OR

$$= \lim_{\substack{R_1\to\infty \\ R_2\to\infty}} \int_{-R_1}^{R_2} f(x)\,dx$$

Example -

(i) $\int_0^\infty \sin x\,dx = \lim\limits_{R\to\infty} \int_0^R \sin x\,dx$

$= \lim\limits_{R\to\infty} (1-\cos R)$ does not exist!

(ii) $\int_2^\infty \frac{2x^2}{x^4-1}\,dx = \lim\limits_{R\to\infty} \int_2^R \frac{2x^2}{x^4-1}\,dx = \lim\limits_{R\to\infty} \int_2^R \left(\frac{1}{x^2+1} + \frac{1}{x^2-1}\right) dx$

$= \lim\limits_{R\to\infty} \left[\int_2^R \frac{1}{x^2+1}\,dx + \frac{1}{2}\int_2^R \frac{1}{x-1}\,dx - \frac{1}{2}\int_2^R \frac{1}{x+1}\,dx\right]$

$= \lim\limits_{R\to\infty} \left[\tan^{-1} R - \tan^{-1}(2) + \frac{1}{2}\ln\left(\frac{R-1}{R+1}\right) + \frac{1}{2}\ln 3\right]$

$= \frac{\pi}{2} - \tan^{-1}(2) + \frac{1}{2}\ln(3).$

Evaluation of improper integrals of the second kind

(i) If $f(x) \to \infty$ as $x \to b$ then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)\,dx$$

(ii) If $f(x) \to \infty$ as $x \to a$ then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} f(x)\,dx$$

(iii) If $f(x) \to \infty$ as $x \to c$ only. Here

$a < c < b$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon \to 0+} \int_{c+\varepsilon}^{b} f(x)\,dx$$

(iv) If $f(x) \to \infty$ as $x \to a$ and $x \to b$

$$\int_a^b f(x)\,dx = \lim_{\substack{\varepsilon_1 \to 0+ \\ \varepsilon_2 \to 0+}} \int_{a+\varepsilon_1}^{b-\varepsilon_2} f(x)\,dx$$

Example - 1: $\int_0^1 \frac{dx}{\sqrt{1-x}}$

$$= \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{\varepsilon \to 0+} \left[-2\sqrt{1-x}\right]_0^{1-\varepsilon}$$

$$= -\lim_{\varepsilon \to 0+} 2\left(\sqrt{\varepsilon}-1\right)$$

$$= 2$$

Example - 2: $\int_0^2 \frac{dx}{2x-x^2}$

$$= \lim_{\varepsilon_1 \to 0+} \int_{\varepsilon_1}^{1} \frac{dx}{2x-x^2} + \lim_{\varepsilon_2 \to 0+} \int_1^{2-\varepsilon_2} \frac{dx}{2x-x^2}$$

$$= \lim_{\varepsilon_1 \to 0+} \frac{1}{2}\left[\ln \frac{x}{2-x}\right]_{\varepsilon_1}^{1} + \lim_{\varepsilon_2 \to 0+} \frac{1}{2}\left[\ln \frac{x}{2-x}\right]_1^{2-\varepsilon_2}$$

$$= -\frac{1}{2} \lim_{\varepsilon_1 \to 0+} \ln\left(\frac{\varepsilon_1}{2-\varepsilon_1}\right) + \frac{1}{2} \lim_{\varepsilon_2 \to 0+} \ln\left(\frac{2-\varepsilon_2}{\varepsilon_2}\right)$$

$$= \infty$$

$\Rightarrow$ Integral diverges

CONVERGENCE TEST FOR IMPROPER INTEGRALS

– TYPE-I INTEGRALS

COMPARISON TEST – I:

If f and g are positive or non-negative, $f \geq 0$, $g \geq 0$ and $f(x) \leq g(x)$, $\forall x$ in $[a, \infty]$, then

(i) $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges

(ii) $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges

COMPARISON TEST – II:

Suppose $f(x) \geq 0$ & $g(x) > 0$ $\forall x > a$.

If $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = K \ (\neq 0)$. Then both the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge together.

In case: $K = 0$ and $\int_a^\infty g(x)\,dx$ converges then

$$\int_a^\infty f(x)\,dx \text{ converges}$$

In case $K = \infty$ and $\int_a^\infty g(x)\,dx$ diverges then

$$\int_a^\infty f(x)\,dx \text{ diverges.}$$

A useful comparison test:

Consider $a>0$ and

$$\int_a^R \frac{c}{x^n}\,dx = \begin{cases} c\ln\left(\frac{R}{a}\right), & n=1 \\ \frac{c}{1-n}\left[\frac{1}{R^{n-1}} - \frac{1}{a^{n-1}}\right], & n\neq 1 \end{cases}$$

$$\Rightarrow \int_a^\infty \frac{c}{x^n}\,dx = \lim_{R\to\infty}\int_a^R \frac{c}{x^n}\,dx = \begin{cases} +\infty, & n\leq 1. \\ \frac{c}{(n-1)\,a^{n-1}}, & n>1. \end{cases}$$

μ-test (comparison test + above result)

Let $f(x)\geq 0$ in the interval $[a,\infty)$, $a>0$. (OR $f(x)\leq 0$)

a] If $\exists\ \mu>1$ such that $\lim\limits_{x\to\infty} x^\mu f(x)$ exists then

$$\int_a^\infty f(x)\,dx \text{ is convergent.}$$

b] If $\exists\ \mu\leq 1$ such that $\lim\limits_{x\to\infty} x^\mu f(x)$ exists and $\neq 0$, then the integral $\int_a^\infty f(x)\,dx$ is divergent and the same is true if $\lim\limits_{x\to\infty} x^\mu f(x)$ is $+\infty$ (or $-\infty$)

OR, in short:

If $\lim\limits_{x\to\infty} x f(x) = A \neq 0$ (or $=\pm\infty$), then

$$\Rightarrow \int_a^\infty f(x)\,dx \text{ diverges}$$

Test fails if $A=0$.

Examples:

i) $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$

Sol: Note that $\frac{1}{x\sqrt{x^2+1}} \sim \frac{1}{x^2}$ so, let $f(x) = \frac{1}{x\sqrt{x^2+1}}$ and $g(x) = \frac{1}{x^2}$. Further $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1 \; (\neq 0)$

$\Rightarrow \int_1^\infty f(x)\,dx$ & $\int_1^\infty g(x)\,dx$ converge or diverge together.

As $\int_1^\infty \frac{dx}{x^2}$ converges $\Rightarrow \int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ converges.

OR apply μ-test as $\mu = 2$.

ii) $\int_1^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$

Let $f(x) = \frac{x^2}{\sqrt{x^5+1}} \quad \left(\sim \frac{1}{\sqrt{x}}\right)$

and $g(x) = \frac{1}{\sqrt{x}}$.

Note that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{\sqrt{x^5+1}} \cdot \sqrt{x} = 1$

As $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, by comparison test

$\int_0^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$ diverges.

OR apply μ-test as $\mu = \frac{1}{2}$ in this case.

iii) $\int_0^\infty e^{-x^2}\,dx = \underbrace{\int_0^1 e^{-x^2}\,dx}_{\text{PROPER}} + \int_1^\infty e^{-x^2}\,dx$

We know that: $e^{x^2} = 1 + x^2 + \frac{x^4}{\lfloor 2} + \cdots > x^2 \quad \forall \begin{smallmatrix} x>0 \\ x<0 \end{smallmatrix}$

$$\Rightarrow e^{-x^2} < \frac{1}{x^2}$$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, the integral $\int_1^\infty e^{-x^2}\,dx$ converges.

OR $\mu = 2$ & $\lim_{x\to\infty} x^2 e^{-x^2} = 0 \Rightarrow \int_0^\infty e^{-x^2}\,dx$ converges

(iv) $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \underbrace{\int_0^1 \frac{\sin^2 x}{x^2}\,dx}_{\text{PROPER}} + \int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

Also $\frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}$ converges

$\Rightarrow \int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ converges.

(v) $\int_1^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}}\,dx$

$f(x) = \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} = \frac{\tan^{-1} x}{x^{1/3}(1+x^{-4})^{1/3}} \quad (\sim x^{-1/3} \text{ at } \infty)$

$g(x) = \frac{1}{x^{1/3}} \qquad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{1/3}\cdot \tan^{-1} x}{x^{1/3}\,(1+x^{-4})^{1/3}} = \pi/2$

$\Rightarrow \int_1^\infty f(x)\,dx$ diverges

OR Apply μ-test for $\mu = 1/3$ (<1) $\rightarrow$ divergence of $\int_1^\infty f(x)\,dx$.

ABSOLUTE CONVERGENCE:

Def: The integral $\int_a^\infty f(x)\,dx$ converges absolutely

$$\Leftrightarrow \int_a^\infty |f(x)|\,dx \text{ converges.}$$

Def: The integral $\int_a^\infty f(x)\,dx$ converges conditionally $\Leftrightarrow$ it converges but not absolutely.

Example: $\int_1^\infty \frac{\sin x}{x^2}\,dx$ converges absolutely $\left(\text{OR } \int_1^\infty \frac{\sin x}{x^p}\,dx,\ p>1\right)$

Note that $\frac{|\sin x|}{x^2} \leq \frac{1}{x^2}$

By comparison test $\int_1^\infty \frac{|\sin x|}{x^2}\,dx$ converges.

Theorem: $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty |f(x)|\,dx$ converges but the converse is not true.

Example: $\int_0^\infty \frac{\sin x}{x}\,dx$ converges conditionally (to be discussed later)

DIRICHLET TEST: Let $f, g: [a,\infty) \to \mathbb{R}$ be such that

i) f is integrable on each interval $[a,b]$, $b>a$ and the integrals $\int_a^b f(x)\,dx$ are uniformly bounded, i.e., $\exists\, C>0$, s.t.

$$\left|\int_a^b f(x)\,dx\right| \leq C \quad \text{for all } b>a \quad (b<\infty)$$

ii) g is monotone and bounded on $[a,\infty)$ and $\lim_{x\to\infty} g(x) = 0$

Then the improper integral $\int_a^\infty f(x)\,g(x)\,dx$ converges.

Example: $\int_1^\infty \frac{\sin x}{x^p}\,dx$ is convergent for $p>0$.

Let $f(x) = \sin x$ & $g(x) = \frac{1}{x^p}$

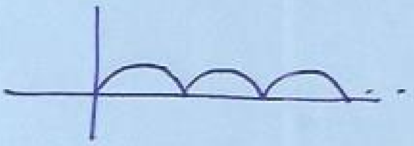
Note that $\left|\int_1^b \sin x\, dx\right| = |\cos(1)-\cos(b)| \leq |\cos(1)|+|\cos(b)|$
$\leq 2.$ for $1 \leq b < \infty$

Also $g(x) = \frac{1}{x^p}$ is monotone decreasing function tending to 0 as $x \to \infty$, for $p>0$.

Using Dirichlet test $\int_1^\infty f(x)\,g(x)\,dx$ converges for $p>0$.

Example: Show that $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ does not converge.

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx$$



Subst. $x = n\pi + y$, then

$$\sum_{n=0}^{\infty} \int_0^\pi \frac{|\sin(n\pi+y)|}{n\pi+y}\,dy = \sum_{n=0}^{\infty} \int_0^\pi \frac{|(-1)^n \sin y|}{n\pi+y}$$

$$= \sum_{n=0}^{\infty} \int_0^\pi \frac{\sin y}{n\pi+y}\,dy \geq \sum_{n=0}^{\infty} \int_0^\pi \frac{\sin y}{n\pi+\pi}\,dy$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)} \cdot \frac{2}{\pi} \rightarrow \text{divergent series.}$$

$$\Rightarrow \sum_{n=0}^{\infty} \int_0^\pi \frac{\sin y}{n\pi+y}\,dy \text{ diverges}$$

and hence the improper integral

$$\int_0^\infty \frac{|\sin x|}{x}\,dx \text{ diverges.}$$

Example: Test the convergence of $\int_0^\infty \frac{\sin x}{x} \cdot e^{-x}\, dx$

$$\int_0^\infty \frac{\sin x}{x} e^{-x}\, dx = \underbrace{\int_0^1 \frac{\sin x}{x} \cdot e^{-x}\, dx}_{\text{PROPER}} + \int_1^\infty \frac{\sin x}{x} e^{-x}\, dx$$

Note that $\int_1^b \frac{\sin x}{x}\, dx \leq \int_1^b \sin x\, dx \leq 2$

Further, e^{-x} is monotone and bounded, and $\lim_{x\to\infty} e^{-x} = 0$

Hence by Dirichlet's test $\int_0^\infty \frac{\sin x}{x} e^{-x}\, dx$ converges.

Example: $\int_a^\infty (1-e^{-x}) \frac{\cos x}{x^2}\, dx \qquad a>0$

$$\int_a^\infty (1-e^{-x}) \frac{\cos x}{x^2}\, dx = \underbrace{-\int_a^\infty e^{-x} \frac{\cos x}{x^2}}_{\substack{\text{Converges} \\ \text{(Similar as above)}}} + \underbrace{\int_a^\infty \frac{\cos x}{x^2}\, dx}_{\substack{\text{converges} \\ \text{(conv. absolulety)}}}$$

$$\Rightarrow \int_a^\infty (1-e^{-x}) \frac{\cos x}{x^2}\, dx \text{ converges.}$$

INTEGRAL OF THE TYPE:

$$\int_{-\infty}^b f(x)\, dx$$

Subst. $x = -t$: $\int_{-b}^\infty f(-t)\, dt$

Review of convergence test for $\int_a^\infty f(x)dx$

Comparison Tests: Let $0 \leq f(x) \leq g(x)$.

(I)

$(a)\quad \int_a^\infty g(x)dx$ conveges $\Rightarrow \int_a^\infty f(x)dx$ conveges

$(b)\quad \int_a^\infty f(x)dx$ diverges $\Rightarrow \int_a^\infty g(x)dx$ diverges

(II)

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \mathrm{k}$$

(a) if $k \neq 0$ then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ behave the same

(b) if $k = 0$ and $\int_a^\infty g(x)dx$ conveges then $\int_a^\infty f(x)dx$ conveges

(c) if $k = \infty$ and $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)dx$ diverges

Test Integral:

$$\int_a^\infty \frac{1}{x^p}\ dx \quad \text{converges for } p > 1 \ \text{\& diverges if } \ p \leq 1$$

$\boldsymbol{\mu}$ **− test**: Comparison test (II) with $g(x) = \dfrac{1}{x^\mu}$

Dirichlet's Test:

If (1) $\left|\int_a^b f(x)dx\right| < C$ for all $b > a$, (2) g is monotone, bounded and $\lim_{x\to\infty} g(x) = 0$, then

$$\int_a^b f(x)g(x)\ dx \text{ conveges}$$

CONVERGENCE OF IMPROPER INTEGRALS OF SECOND TYPE:

TEST INTEGRAL:

$$\int_a^b \frac{dx}{(x-a)^n}$$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon\to 0+} \frac{1}{1-n}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right]$$

if $n \neq 1$.

$$= \begin{cases} \frac{1}{(1-n)(b-a)^{n-1}} & \text{if } n<1 \\ \infty & \text{if } n>1. \end{cases}$$

For $n=1$:

$$\int_a^b \frac{dx}{(x-a)} = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b \frac{dx}{x-a} = \lim_{\varepsilon\to 0+} \Big[\ln|x-a|\Big]\Big|_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon\to 0+} \left[\ln(b-a) - \ln\varepsilon\right] = \infty$$

Hence: $\int_a^b \frac{dx}{(x-a)^n}$ converges if $n<1$ and diverges if $n \geq 1$.

Note: Notation: $\int_{a+}^b f(x)\,dx$ ($f(x)$ becomes unbounded at $x=a$)

For the case $\int_a^{b-} f(x)\,dx$ we can set

$x = b-t$ and get $\int_{0+}^{b-a} f(b-t)\,dt$.

Example: Test the convergence of $\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$.

Note that the integrand is unbounded at upper end.

Set $3-x = t \Rightarrow dx = -dt$

$$\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}} = \int_0^3 \frac{dt}{t\sqrt{(3-t)^2+1}}$$

Let $g(t) = \frac{1}{t}$ $\left(\underbrace{\frac{1}{t\sqrt{(3-t)^2+1}}}_{=:f(t)} \times t = \frac{1}{\sqrt{(3-t)^2+1}} \right)$

Note that $\lim\limits_{t\to 0} \frac{f(t)}{g(t)} = \lim\limits_{t\to 0} \frac{1}{\sqrt{(3-t)^2+1}} = \frac{1}{\sqrt{10}}$

$\Rightarrow \int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$ diverges since $\int_0^3 g(t)\,dt$ diverges.

Example: $\int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}\,dx$

Notice: $\left| \frac{\sin x}{\sqrt[3]{x-\pi}} \right| \le \frac{1}{\sqrt[3]{x-\pi}}$ and

$\int_{\pi}^{4\pi} \frac{1}{\sqrt[3]{x-\pi}}\,dx$ converges

$\Rightarrow \int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}$ converges absolutely.

Note: Improper integrals of the third kind can be expressed in terms of improper integrals of the first and second kind.

Review of convergence test for $\displaystyle\int_{a+}^{b} f(x)dx$

Comparison Tests: Let $0 \le f(x) \le g(x)$.

(I)

(a) $\quad \displaystyle\int_{a+}^{b} g(x)dx$ conveges $\Rightarrow$ $\displaystyle\int_{a+}^{b} f(x)dx$ conveges

(b) $\quad \displaystyle\int_{a+}^{b} f(x)dx$ diverges $\Rightarrow$ $\displaystyle\int_{a+}^{b} g(x)dx$ diverges

(II)

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \mathrm{k}$$

(a) if $k \neq 0$ then $\displaystyle\int_{a+}^{b} f(x)dx$ and $\displaystyle\int_{a+}^{b} g(x)dx$ behave the same

(b) if $k = 0$ and $\displaystyle\int_{a+}^{b} g(x)dx$ conveges then $\displaystyle\int_{a+}^{b} f(x)dx$ conveges

(c) if $k = \infty$ and $\displaystyle\int_{a+}^{b} g(x)dx$ diverges then $\displaystyle\int_{a+}^{b} f(x)dx$ diverges

Test Integral:

$$\int_{a}^{b} \frac{1}{(x-a)^p}\, dx \quad \text{converges for } p<1 \text{ \& diverges if } \ p \ge 1$$

$\boldsymbol{\mu}$ **− test**:

if $\exists\ 0 < \mu < 1$ such that $\lim\limits_{x\to a+} (x-a)^{\mu} f(x)$ exsits then $\displaystyle\int_{a+}^{b} f(x)\,dx$ conveges absolutely

if $\exists\ \mu \ge 1$ such that $\lim\limits_{x\to a+} (x-a)^{\mu} f(x)$ exsits $(\neq 0,$ it may be $\pm\infty)$ then $\displaystyle\int_{a+}^{b} f(x)\,dx$ diverges

Dirichlet's Test:

If $\quad (1) \left|\displaystyle\int_{a+\epsilon}^{b} f(x)dx\right| < C, \ \forall \quad b > a, \quad (2)\ g$ is monotone, bounded and $\lim\limits_{x\to a} g(x) = 0,$ then

$$\int_{a+}^{b} f(x)g(x)\,dx \ \text{ conveges}$$

Beta & Gamma Function

Definition:

Beta function:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad , \quad m>0,\ n>0$$

Gamma function:

$$\overline{|n} = \int_0^\infty e^{-x} x^{n-1}\,dx \quad , \quad n>0.$$

Convergence of Beta function:

Case-I: $m, n \geq 1$, the integral is proper. Hence it is convergent.

Case-II: $m, n < 1$.

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \underbrace{\int_0^c x^{m-1}(1-x)^{n-1}\,dx}_{I_1} + \underbrace{\int_c^1 x^{m-1}(1-x)^{n-1}\,dx}_{I_2}$$

where $0<c<1$.

Consider

$$I_1 = \int_0^c x^{m-1}(1-x)^{n-1}\,dx$$

Then.
$$\lim_{x\to 0+} x^{\mu}\, x^{m-1}(1-x)^{n-1} = \lim_{x\to 0} x^{\mu+m-1}(1-x)^{n-1}$$
$$= 1 \quad \text{if} \quad \mu+m-1=0$$
$$\Rightarrow \mu = -m+1.$$

If $0<m<1$, then $0<\mu<1$ and hence the integral converges.

If $m\leq 0$ then $\mu \geq 1$ and hence the integral diverges.

Similarly: consider $I_2 = \int_c^1 x^{m-1} (1-x)^{n-1} dx$

$$\lim_{x \to 1} (1-x)^{\mu} \cdot x^{m-1} (1-x)^{n-1} = \lim_{x \to 1} x^{m-1} (1-x)^{\mu+n-1}$$

If $0 < n < 1$, the integral converges

If $n \leq 0$, the integral diverges.

Therefore

$\int_0^1 x^{m-1} (1-x)^{n-1} dx$ converges if both $m \,\&\, n > 0$.

otherwise it is divergent.

Convergence of Gamma function:

$$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx$$

Case I: $n \geq 1$

The integrand is bounded in $0 < x \leq a$, where a is arbitrary.

We check convergence of $\int_a^{\infty} x^{n-1} e^{-x} dx$

Consider $\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} \frac{x^{\mu} \cdot x^{n-1}}{e^x}$

$= 0$ for all values of μ and n.

Using μ test ($\mu > 1$), the integral $\int_a^{\infty} x^{n-1} e^{-x} dx$ is convergent for all values of n.

Case II: Let $0<n<1$: Then

$$\int_0^\infty e^{-x} x^{n-1}\,dx = \int_0^a e^{-x}\cdot x^{n-1}\,dx + \underbrace{\int_a^\infty e^{-x} x^{n-1}\,dx}_{\text{converges (see above)}}$$

Note that $\lim_{x\to 0} x^{\mu} x^{n-1} e^{-x} = 1$ if $\mu+n-1=0$, i.e.,

if $\mu = 1-n$

Since n lies between 0 & 1, μ also lies between 0 & 1.

Hence $\int_0^a e^{-x} x^{n-1}\,dx$ is convergent.

Therefore the integral converges for $0<n<1$.

Case III Let $n\leq 0$.

$$\lim_{x\to 0} x^{\mu} x^{n-1} e^{-x}$$

Take $\mu = 1$:

$$\lim_{x\to 0} x^{n} e^{-x} = \begin{cases} 1, & n=0 \\ \infty & n<0 \end{cases}$$

$$\Rightarrow \int_0^a e^{-x} x^{n-1}\,dx \text{ diverges.}$$

PROPERTIES OF BETA and GAMMA function:

a) $B(m,n) = B(n,m)$

Subst. $1-x=y$ ..

b) Evaluation of $B(m,n)$

Suppose n is a positive integer.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Integrating by parts keeping $(1-x)^{n-1}$ as first function.

$$B(m,n) = \left[\frac{x^m}{m}(1-x)^{n-1}\right]_0^1 + \int_0^1 \frac{x^m}{m}(n-1)(1-x)^{n-2}\,dx$$

$$= \frac{(n-1)}{m}\int_0^1 x^m (1-x)^{n-2}\,dx$$

$$\vdots$$

$$= \frac{(n-1)(n-2)\cdots 1^{=(n-(n-1))}}{m(m+1)\cdots(m+n-2)}\int_0^1 x^{m+n-2}\,dx$$

$$= \frac{\lfloor(n-1)}{m(m+1)\cdots(m+n-2)(m+n-1)}$$

If m is a positive integer

$$B(m,n) = \frac{\lfloor(m-1)}{n(n+1)\cdots(n+m-1)}$$

If both m and n are integer

$$B(m,n) = \frac{\lfloor(n-1)\,\lfloor(m-1)}{\lfloor m+n-1}$$

c) Evaluation of Gamma function:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$$

integrating by parts:

$$= -x^n e^{-x}\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\, dx$$

$$\boxed{\Gamma(n+1) = n\,\Gamma(n)}$$

Note that if n is an integer

$$\Gamma(n) = (n-1)(n-2)\cdots 3\cdot 2\cdot \Gamma(1)$$

where $\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1.$

$\Rightarrow$ $\boxed{\Gamma(n) = \lfloor(n-1)\text{, if } n \text{ is a positive int.}}$

d) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\qquad \Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$

Subst. $x = y^2 \Rightarrow dx = 2y\, dy$

$$\Rightarrow \Gamma(n) = \int_0^\infty y^{2n-1} e^{-y^2}\, 2\, dy$$

Set $n = \frac{1}{2} \Rightarrow \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty y^0 e^{-y^2}\, dy$

$$= 2\int_0^\infty e^{-y^2}\, dy = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

$\Rightarrow$ $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

Different forms of Γn :

a) $\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$

Subst. $x = \lambda y \Rightarrow dx = \lambda dy$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-\lambda y} \cdot \lambda^{n-1} y^{n-1} \lambda \, dy$$

$$\Rightarrow \boxed{\int_0^\infty e^{-\lambda y} y^{n-1} \, dy = \frac{\Gamma n}{\lambda^n}}$$

b) Subst. $x^n = z \Rightarrow n x^{n-1} dx = dz$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-z^{1/n}} \frac{1}{n} dz \Rightarrow \boxed{\int_0^\infty e^{-z^{1/n}} dz = n \Gamma n = \Gamma n+1}$$

c) Subst $e^{-x} = t \Rightarrow -e^{-x} dx = dt$

$$\Rightarrow \Gamma n = -\int_1^0 \left[\ln\left(\frac{1}{t}\right)\right]^{n-1} dt$$

$$\Rightarrow \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{n-1} dt = \Gamma n$$

Different forms of Beta function:

a) $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$

Subst. $x = \frac{1}{1+y} \Rightarrow dx = -\frac{1}{(1+y)^2} dy$

$$\boxed{B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy}$$

b) $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta\, d\theta$$

Relation between Gamma & Beta function:

We know for m and n being integers

$$B(m,n) = \frac{\lfloor m-1 \,\lfloor n-1}{\lfloor m+n-1}$$

$$\Rightarrow \boxed{B(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}}$$

This result also holds for $m,n>0$ (not necessary only for integers)

Some other deductions:

1. $\boxed{\Gamma n\, \Gamma 1-n = \frac{\pi}{\sin n\pi}}$ $0<n<1$.

We know $B(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}$, putting $m = 1-n$

$\Rightarrow B(1-n,n) = \Gamma 1-n\, \Gamma n = \int_0^\infty \frac{y^{n-1}}{(1+y)} dy = \frac{\pi}{\sin n\pi}$ if $0<n<1$

(Complicated, Residue th. comp. ana.)

2. $\boxed{\Gamma n+1\, \Gamma 1-n = \frac{n\pi}{\sin n\pi}}$

3. Put $n = 1/2$ in 1. $\Rightarrow \Gamma\frac{1}{2}\,\Gamma\frac{1}{2} = \pi$

$$\Rightarrow \boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$

4. We know

$$B(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Let $2m-1 = p$ & $2n-1 = q$

$$\Rightarrow \boxed{\int_0^{\pi/2} \cos^p\theta \, \sin^q\theta \, d\theta = \frac{\Gamma\frac{(p+1)}{2} \, \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}}$$

Let $p=0$ then

$$\boxed{\int_0^{\pi/2} \sin^q\theta \, d\theta = \frac{\Gamma\frac{q+1}{2}}{\Gamma\frac{q+2}{2}} \cdot \frac{\sqrt{\pi}}{2}}$$

Let $q=0$ then

$$\boxed{\int_0^{\pi/2} \cos^p\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}}{\Gamma\frac{p+2}{2}} \cdot \frac{\sqrt{\pi}}{2}}$$

Let $p=0$, $q=0$ then

$$\frac{\pi}{2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2 \Rightarrow \boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$

5. $B(m,n) = B(m+1,n) + B(m,n+1)$

$$\text{R.H.S} = \frac{\Gamma m+1 \, \Gamma n}{\Gamma m+n+1} + \frac{\Gamma m \, \Gamma n+1}{\Gamma m+n+1}$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n+1}(m+n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} = B(m,n).$$

Example - 1: Evaluate

$$\int_0^1 x^4 (1-\sqrt{x})^5 \, dx$$

Let $\sqrt{x} = t$ or $x = t^2 \Rightarrow dx = 2t\,dt$

$$\int_0^1 t^8 (1-t)^5 \, 2t \, dt$$

$$= 2\int_0^1 t^9 (1-t)^5 dt$$

$$= 2 \cdot B(10,6) = 2 \cdot \frac{\Gamma 10 \, \Gamma 6}{\Gamma 16} = 2 \cdot \frac{\lfloor 9 \, \lfloor 5}{\lfloor 15} = \frac{1}{15015}$$

Example 2. Show that $\int_0^{\pi/2} (\cot\theta)^{1/2} d\theta = \frac{\pi}{\sqrt{2}}$

$$I = \int_0^{\pi/2} (\cot\theta)^{1/2} d\theta = \int_0^{\pi/2} \cos^{1/2}\theta \, \sin^{-1/2}\theta \, d\theta$$

$$= \frac{\Gamma\frac{-\frac{1}{2}+1}{2} \, \Gamma\frac{\frac{1}{2}+1}{2}}{2\,\Gamma\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}} = \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{2}$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin(\pi/4)}$$

$$= \frac{\pi}{\sqrt{2}}.$$

Differentiation under integral sign (Leibnitz Rule)

Let $$\Phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\, dx$$

$$\Delta\Phi = \Phi(\alpha+\Delta\alpha) - \Phi(\alpha)$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\, dx - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\, dx$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_1(\alpha)} f(x,\alpha+\Delta\alpha)\, dx + \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha+\Delta\alpha)\, dx$$

$$+ \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\, dx - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\, dx$$

$$= \int_{u_1(\alpha)}^{u_2(\alpha)} \left[f(x,\alpha+\Delta\alpha) - f(x,\alpha)\right] dx + \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\, dx$$

$$- \int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\, dx$$

Using mean value theorem:

$$\int_{u_1(\alpha)}^{u_2(\alpha)} \left[f(x,\alpha+\Delta\alpha) - f(x,\alpha)\right] dx = \Delta\alpha \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\xi_1)\, dx$$

$$\int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\, dx = f(\xi_2,\alpha+\Delta\alpha)\left[u_2(\alpha+\Delta\alpha) - u_2(\alpha)\right]$$

$$\int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\, dx = f(\xi_3,\alpha+\Delta\alpha)\left[u_1(\alpha+\Delta\alpha) - u_1(\alpha)\right]$$

where $\xi_1 \in (\alpha, \alpha+\Delta\alpha)$, $\xi_2 \in (u_2(\alpha), u_2(\alpha+\Delta\alpha))$, $\xi_3 \in (u_1(\alpha), u_1(\alpha+\Delta\alpha))$

Dividing by $\Delta\alpha$:

$$\frac{\Delta\phi}{\Delta\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x, \xi_1)\, dx + f(\xi_2, \alpha+\Delta\alpha)\frac{\Delta u_2}{\Delta\alpha} - f(\xi_3, \alpha+\Delta\alpha)\frac{\Delta u_1}{\Delta\alpha}$$

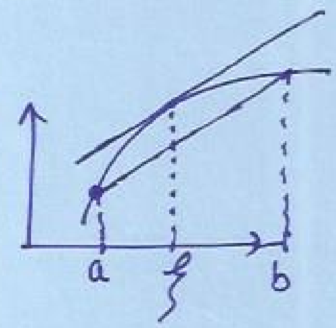
Taking the limit as $\Delta\alpha \to 0$,

$$\frac{d\phi}{d\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\alpha)\, dx + f(u_2(\alpha),\alpha)\frac{du_2}{d\alpha} - f(u_1(\alpha),\alpha)\frac{du_1}{d\alpha}$$

Note: We have used the following mean value theorems in the above proof.

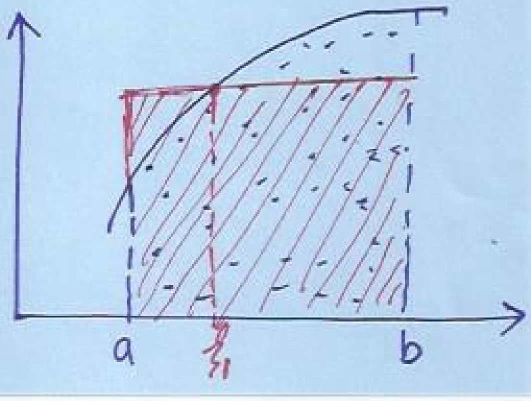
I. Lagrange mean value theorem:

$$\frac{f(b)-f(a)}{b-a} = f'(\xi) \;; \quad \xi \in (a,b)$$



II. Mean value theorem of the integral calculus:

$$\int_a^b f(x)\, dx = (b-a)\, f(\xi_1) \;; \quad \xi_1 \in (a,b)$$



A Particular Case: Assume that $u_1(\alpha)$ and $u_2(\alpha)$ are some constants. Then,

$$\frac{d\Phi(\alpha)}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx$$

OR

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\, dx = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx.$$

<u>Note</u>: Leibnitz rule is not applicable, in general, in the case of improper integrals. In all examples given in this lecture we assume that differentiation under integral sign is valid.

<u>Example</u>: Show that

$$\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\, dx = \frac{\pi}{2}\ln(1+a) \text{ if } a \geq 0.$$

Let $$\varphi(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\, dx$$

$$\Rightarrow \varphi'(a) = \int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)}\, dx$$

$$= \int_0^\infty \frac{1}{(1-a^2)}\left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right] dx$$

$$= \frac{1}{(1-a^2)}\left[\tan^{-1}x - a\tan^{-1}ax\right]_0^\infty = \frac{1}{(1-a^2)}\frac{\pi}{2}(1-a)$$

$$\Rightarrow \varphi'(a) = \frac{\pi}{2(1+a)}$$

Integrading

$$\varphi(a) = \frac{\pi}{2} \ln(1+a) + c$$

Note that $\varphi(0) = 0$

$$\Rightarrow 0 = \frac{\pi}{2} \ln(1) + c \Rightarrow c = 0$$

$$\Rightarrow \varphi(a) = \frac{\pi}{2} \ln(1+a)$$

Example: Prove $\int_0^\infty e^{-x^2} \cos \alpha x \, dx = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}$

$$\varphi(\alpha) = \int_0^\infty e^{-x^2} \cos \alpha x \, dx$$

$$\varphi'(\alpha) = -\int_0^\infty e^{-x^2} \sin \alpha x \cdot x \, dx$$

Integrating right hand side by parts

$$\varphi'(\alpha) = \frac{e^{-x^2}}{2} \sin \alpha x \Big|_0^\infty + \int_0^\infty \left(-\frac{e^{x^2}}{2}\right) \cos \alpha x \cdot \alpha \cdot dx$$

$$= -\frac{\alpha}{2} \varphi(\alpha)$$

$$\Rightarrow \frac{\varphi'(\alpha)}{\varphi(\alpha)} = -\frac{\alpha}{2} \Rightarrow \ln \varphi(\alpha) = -\frac{\alpha^2}{4} + c$$

$$\Rightarrow \varphi(\alpha) = c_1 e^{-\alpha^2/4}$$

Note that $\varphi(0) = \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

$$\Rightarrow \sqrt{\pi}/2 = c_1$$

$$\Rightarrow \int_0^\infty e^{-x^2} \cos \alpha x \, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}.$$

Example: Starting with a suitable integral, show that

$$\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)}$$

Solution: Consider $\varphi(a,x) = \int_0^x \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_0^x$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Diff. w.r.t a:

$$\frac{\partial\varphi}{\partial a} = \int_0^x -\frac{1}{(x^2+a^2)^2}\cdot 2a\, dx = \frac{1}{a}\frac{1}{\left(1+\frac{x^2}{a^2}\right)}\left(-\frac{x}{a^2}\right) - \frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \int_0^x \frac{1}{(x^2+a^2)^2}\, dx = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2}\frac{1}{(x^2+a^2)}$$

Example: Let $\varphi(\alpha) = \int_\alpha^{\alpha^2} \frac{\sin \alpha x}{x}\, dx$. Find $\varphi'(\alpha)$ where $\alpha \neq 0$.

$$\varphi'(\alpha) = \int_\alpha^{\alpha^2} \frac{\cos \alpha x}{x}\cdot x\, dx + 2\alpha\cdot\frac{\sin\alpha^3}{\alpha^2} - \frac{\sin\alpha^2}{\alpha}$$

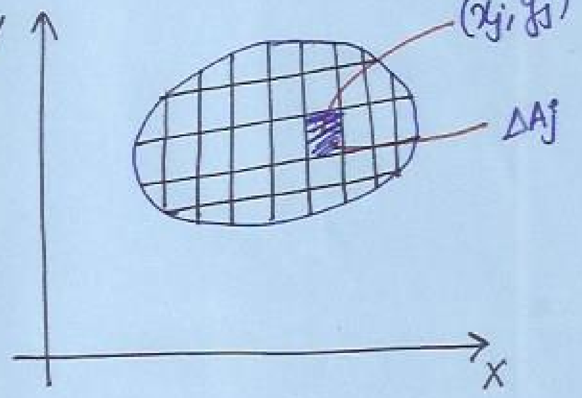
$$= \frac{\sin\alpha x}{\alpha}\Big|_\alpha^{\alpha^2} + \frac{2\sin\alpha^3}{\alpha} - \frac{\sin\alpha^2}{\alpha}$$

$$= \frac{3\sin\alpha^3 - 2\sin\alpha^2}{\alpha}.$$

Multiple Integrals

Double Integrals: Let $f(x,y)$ be defined in a closed region D of the xy plane. Divide D into n subregions of area ΔA_j, $j=1,2,\ldots,n$. Let (x_j, y_j) be some point of ΔA_j. Then Consider

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j, y_j)\,\Delta A_j$$

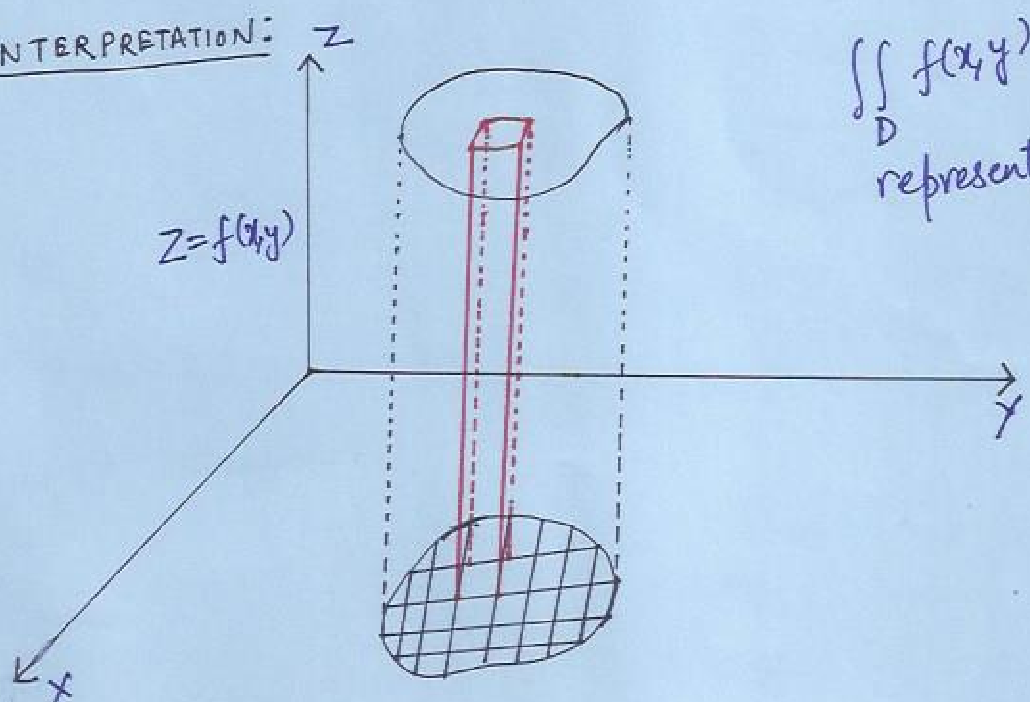


If this limit exists, then it is denoted by

$$\iint_D f(x,y)\,dA \quad \text{or} \quad \iint_D f(x,y)\,dx\,dy$$

Note: It can be proved that the above limit exists if $f(x,y)$ is continuous or piecewise continuous in D.

Physical Interpretation:

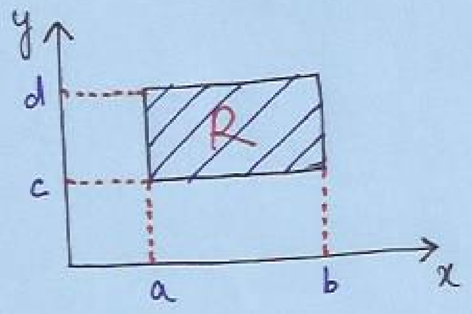


$\iint_D f(x,y)\,dA$ represents Volume

Evaluation of Double Integrals

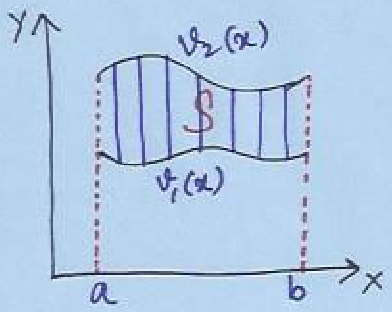
a) If $f(x,y)$ is continuous* on rectangular region $R: a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \underbrace{\left\{\int_a^b f(x,y)\,dx\right\}}_{\psi(y)} dy = \int_a^b \underbrace{\left\{\int_c^d f(x,y)\,dy\right\}}_{\psi(x)} dx$$



* or $f(x,y)$ is defined and bounded on R.

b)

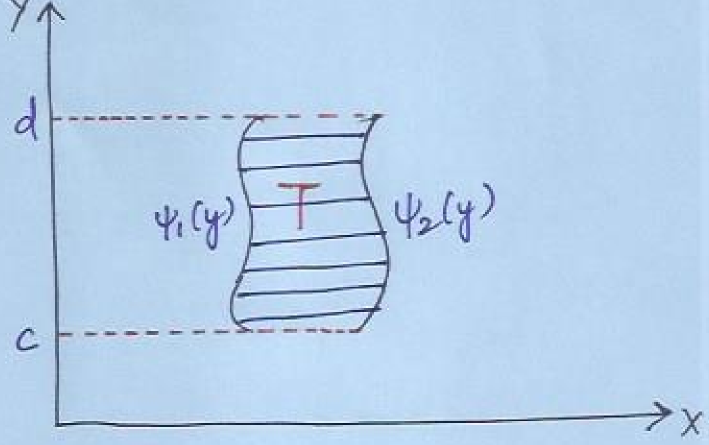


- $v_1(x)$ and $v_2(x)$ are continuous between 'a' and 'b'.
- $f(x,y)$ be defined and bounded on S.

Then,

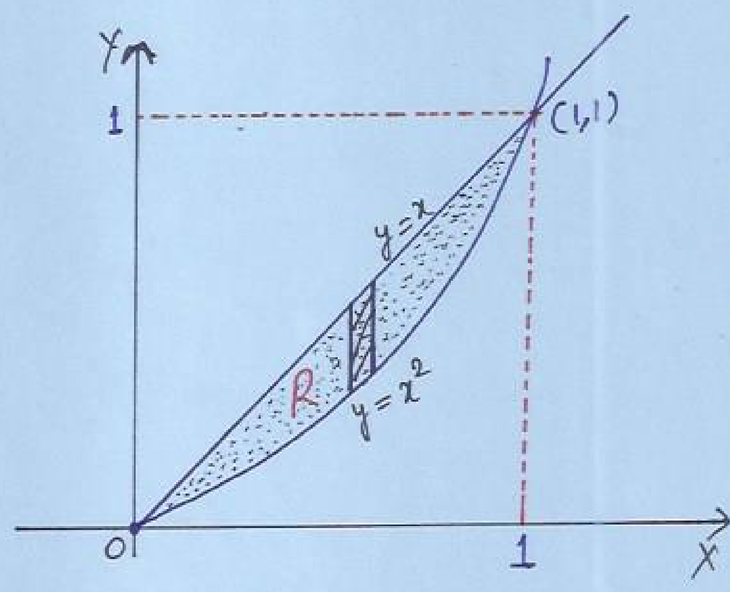
$$\iint_S f(x,y)\,dA = \int_a^b \int_{v_1(x)}^{v_2(x)} f(x,y)\,dy\,dx$$

c)



$$\iint_T f(x,y)\,dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$

Example-1: Evaluate $\iint_R xy(x+y)\,dA$ where R is the region bounded by the line $y=x$ and the curve $y=x^2$.



Solution: $$I = \int_{x=0}^{1} \int_{x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\cdot x^2 + x\cdot\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

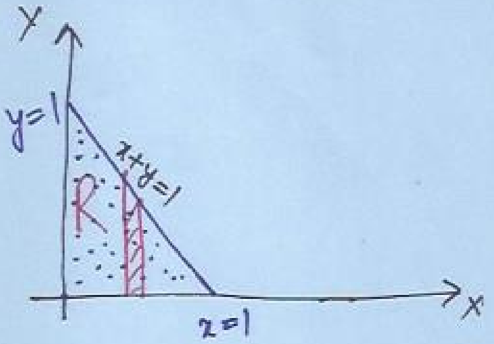
$$= \frac{5}{6}\cdot\frac{1}{5} - \frac{1}{2}\cdot\frac{1}{7} - \frac{1}{3}\cdot\frac{1}{8} = \frac{3}{56}$$

or

$$I = \int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} xy(x+y)\,dx\,dy = \dots = \frac{3}{56}$$

Example-2: Evaluate $\iint_R e^{2x+3y}\,dx\,dy$, R is a triangle bounded by $x=0$, $y=0$ and $x+y=1$.

$$I=\int_{x=0}^{1}\int_{y=0}^{1-x} e^{2x+3y}\,dy\,dx$$



$$=\int_0^1 e^{2x}\left[\frac{e^{3y}}{3}\right]_0^{1-x} dx$$

$$=\int_0^1 e^{2x}\,\frac{1}{3}\cdot\left\{e^{3-3x}-1\right\}dx$$

$$=\frac{1}{3}\int_0^1\left(e^{3-x}-e^{2x}\right)dx=\frac{1}{3}\left[-e^{3-x}-\frac{e^{2x}}{2}\right]_0^1$$

$$=-\frac{1}{3}\left[e^2+\frac{e^2}{2}-e^3-\frac{1}{2}\right]=-\frac{1}{3}\left[\frac{3e^2}{2}-e^3-\frac{1}{2}\right]$$

Ans...

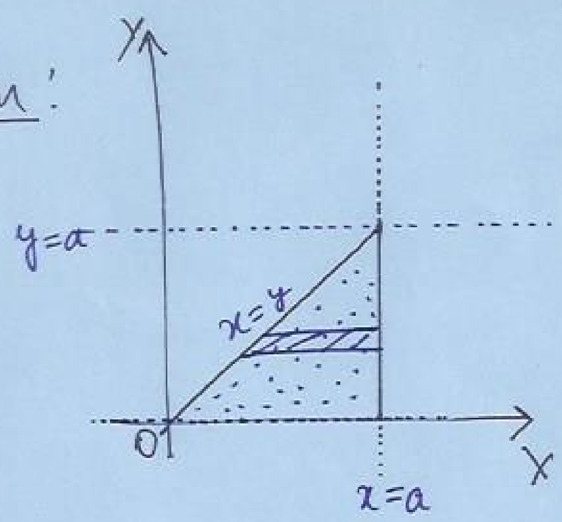
Change of Order of Integration:

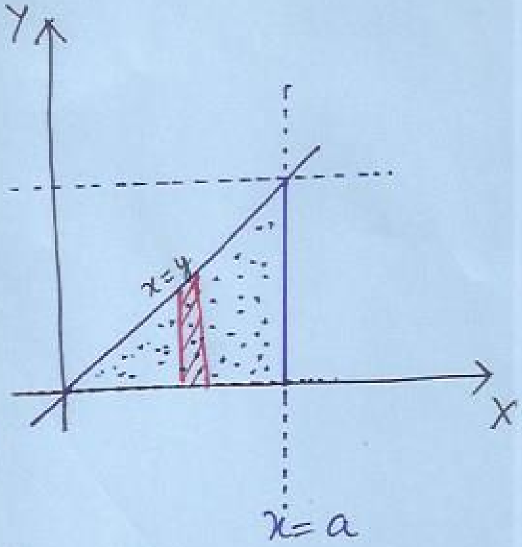
Why? To make the integration easier.

Example: Change the order of integration

$$\int_{y=0}^{a}\int_{x=y}^{a}\frac{x}{x^2+y^2}\,dx\,dy$$ and evaluate.

<u>Solution:</u>





$$\int_{y=0}^{a}\int_{x=y}^{a}\frac{x}{x^2+y^2}\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{x}\frac{x}{x^2+y^2}\,dy\,dx$$

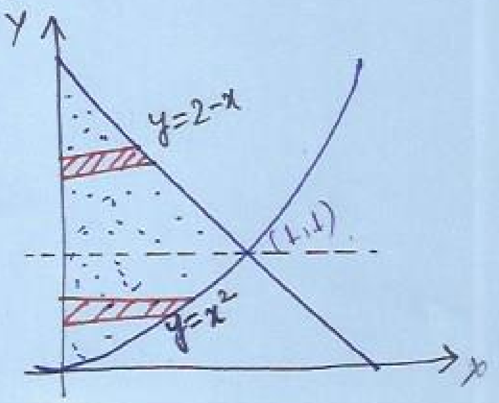
$$= \int_{x=0}^{a}\left[x\cdot\frac{1}{x}\cdot\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_0^a\left[\tan^{-1}(1)-\tan^{-1}(0)\right]dx$$

$$= \frac{\pi}{4}(x)_0^a = \frac{\pi a}{4}$$

<u>Example:</u> Change the order of integration

$\int_0^1\int_{y=x^2}^{2-x} xy\,dy\,dx$ and evaluate.



<u>Solution:</u> $\int_0^1\int_{y=x^2}^{2-x} xy\,dy\,dx$

$$= \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\, dx\, dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\, dx\, dy$$

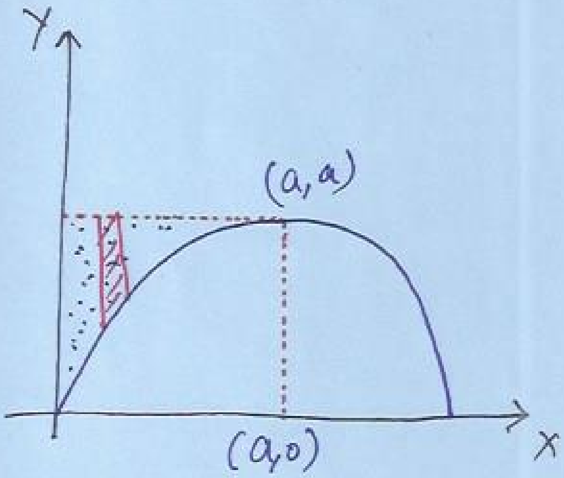
$$= \cdots$$

$$= \frac{3}{8}. \quad \text{Ans...}$$

Example: change the order of integration

$$\int_{y=0}^{a}\int_{x=0}^{a-\sqrt{a^2-y^2}} \frac{xy \cdot \log(x+a)}{(x-a)^2}\, dx\, dy \text{ and evaluate.}$$

Solution:



$$\int_{y=0}^{a}\int_{x=0}^{a-\sqrt{a^2-y^2}} \cdots\, dx\, dy = \int_{x=0}^{a}\int_{y=\sqrt{a^2-(x-a)^2}}^{a} \frac{xy\log(x+a)}{(x-a)^2}\, dy\, dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2} \cdot \frac{1}{2} \cdot \left[a^2 - \{a^2-(x-a)^2\}\right] dx$$

$$= \frac{1}{2}\int_0^a x\log(x+a)\, dx$$

$$= \frac{1}{2}\left[\left\{\frac{x^2}{2}\log(x+a)\right\}_0^a - \int_0^a \frac{x^2}{2}\cdot\frac{1}{(x+a)}\, dx\right]$$

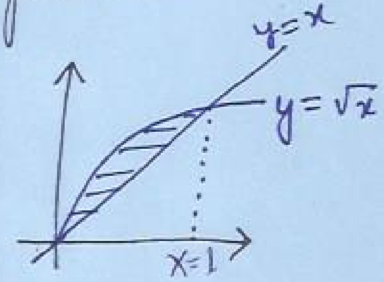
$$= \frac{1}{2}\left[\left\{\frac{a^2}{2}\log(2a)\right\} - \frac{1}{2}\int_0^a \left[(x-a) + \frac{a^2}{x+a}\right] dx\right]$$

$$= \frac{1}{2}\left[\frac{a^2\log(2a)}{2} - \frac{1}{2}\left\{\frac{a^2}{2} - a^2 + a^2\log(2a) - a^2\log a\right\}\right]$$

$$= \frac{a^2}{8}\left[1 + 2\log a\right]$$

Example → Change the order of integration

$$\int_0^1 \int_x^{\sqrt{x}} f(x,y)\, dy\, dx$$

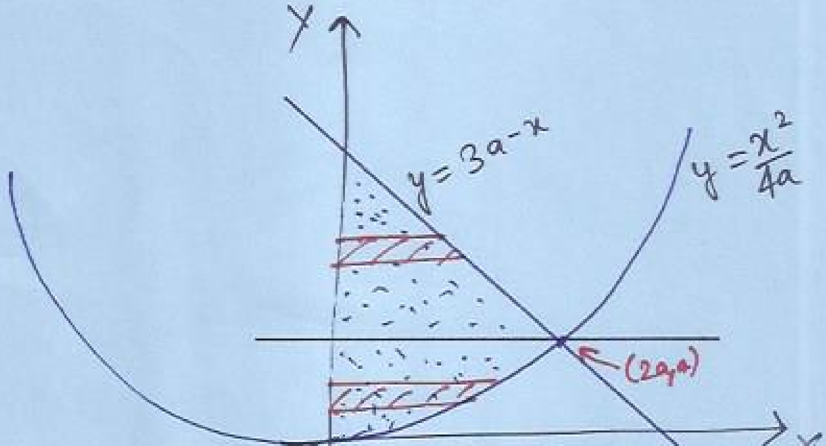


Ans: $\int_0^1 \int_{y^2}^{y} f(x,y)\, dx\, dy$

Q: Change the order of integration in

$$I = \int_0^{2a} \int_{\frac{x^2}{4a}}^{3a-\cancel{x}} F(x,y)\, dy\, dx$$

Solution:

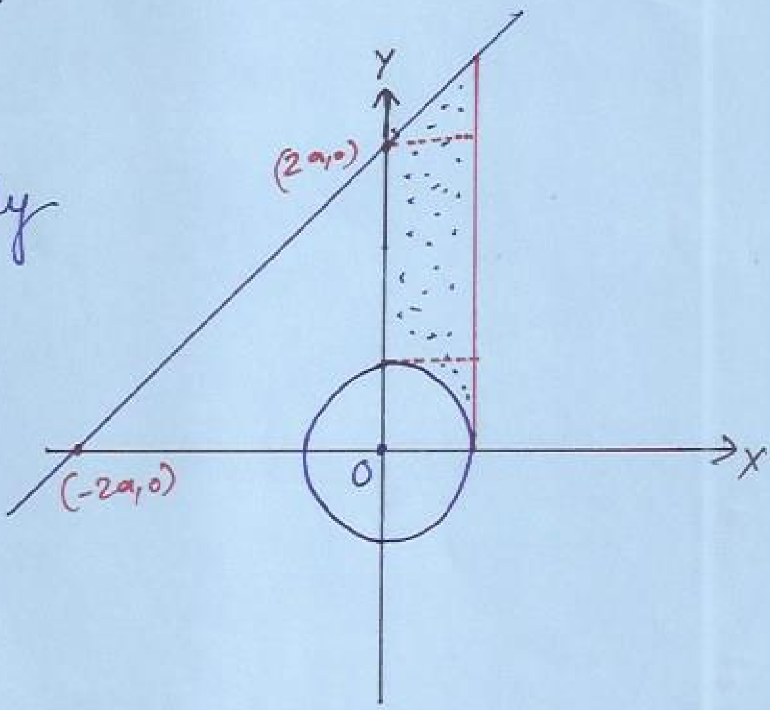


$$I = \int_0^a \int_0^{2\sqrt{ay}} F(x,y)\, dx\, dy$$

$$+ \int_a^{3a} \int_0^{3a-y} F(x,y)\, dx\, dy$$

Example: Change the order of integration in

$$I=\int_0^a \int_{\sqrt{a^2-x^2}}^{x+2a} F(x,y)\,dy\,dx$$

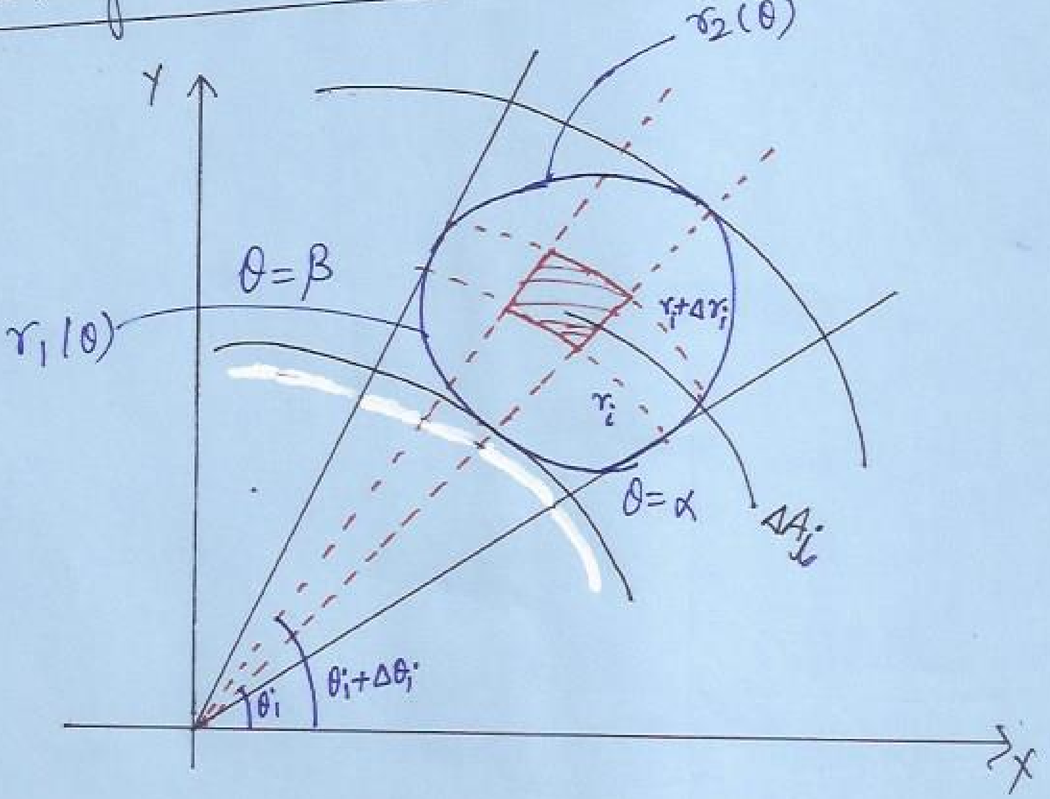


Solution: $I=\int_0^a \int_{\sqrt{a^2-y^2}}^{a} f(x,y)\,dx\,dy$

$$+\int_a^{2a} \int_0^{a} F(x,y)\,dx\,dy$$

$$+\int_{2a}^{3a} \int_{y=2a}^{a} F(x,y)\,dx\,dy$$

Double Integral in Polar Co-ordinate



$$\Delta A_i = (r_i + \Delta r_i)^2 \frac{\Delta\theta_i}{2} - r_i^2 \frac{\Delta\theta_i}{2}$$

$$= \cancel{2r_i\Delta\Delta}\cdot\left(2 r_i \Delta r_i + \Delta r_i^2\right)\frac{\Delta\theta_i}{2}$$

$$= \left(\frac{2r_i + \Delta r_i}{2}\right)\cdot \Delta r_i \, \Delta\theta_i$$

$$= \left(r_i + \frac{\Delta r_i}{2}\right)\Delta r_i \, \Delta\theta_i$$

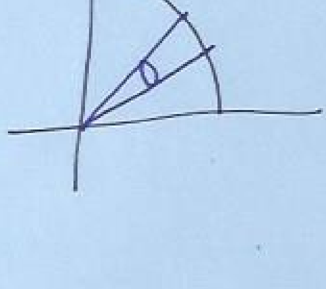
$$= r_i^{*} \, \Delta r_i \, \Delta\theta_i$$

$$I = \lim_{n\to\infty} \sum_{j=1}^{n} f(r_j, \theta_j)\, \Delta A_j$$

$$I = \int_{\theta=\alpha}^{\beta} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

<u>Example</u> → Compute area of first quadrant of a circle.

$$A = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r\, dr\, d\theta = \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}$$



Change of Variables:

Analogous to the method of substitution in Single Variable.

$$x = g(t) \qquad = \frac{dx}{dt}$$

$$\int_a^b f(x)\,dx = \int_c^d f(g(t))\, g'(t)\, dt$$

where $a = g(c)$ and $b = g(d)$

We can change variables in two dimensional case.

Let the variables x, y in the double integral.

$$\iint_R f(x,y)\,dx\,dy$$

be changed to new-variables u, v by means of relations.

$$x = \phi(u,v) \quad , \; y = \psi(u,v)$$

then double integral is transformed into

$$\iint_{R'} f\{\phi(u,v), \psi(u,v)\} \underbrace{|J|\, du\, dv}$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ Jacobian

R' is the region in uv plane which corresponds to the region R in the xy-plane.

Special Case: Cartesian to Polar Co-ordinate

$x = r\cos\theta$, $y = r\sin\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

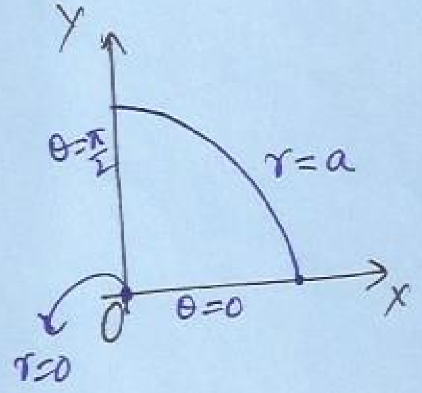
$$\Rightarrow \iint_S f(x,y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example 1: Volume of one Octant of a sphere of radius a,

$$\iint_S \sqrt{a^2-x^2-y^2}\,dx\,dy$$

where S is the first quadrant of the circular disk

$$x^2+y^2 \le a^2$$



Change of variables

$x = r\cos\theta$, $y = r\sin\theta$

$|J| = r$

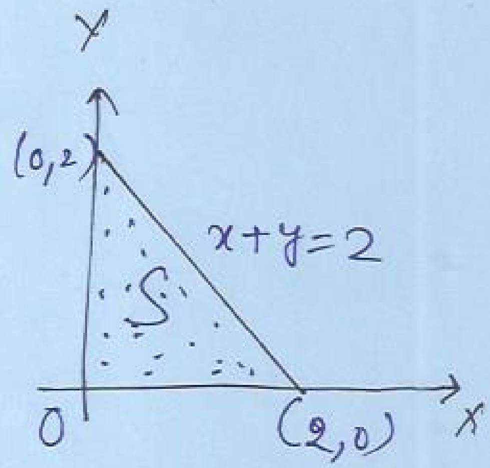
$$\iint_S \sqrt{a^2-x^2-y^2}\,dx\,dy = \iint_R \sqrt{a^2-r^2}\; r\,dr\,d\theta$$

$$\iint_R \sqrt{a^2-r^2}\; r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} \sqrt{a^2-r^2}\; r\,dr\,d\theta$$

$$= \frac{\pi}{2} \cdot \left(-\frac{1}{2}\right) \frac{(a^2-r^2)^{3/2}}{3/2} \Bigg|_0^a$$

$$= \frac{\pi}{2} \cdot \left(-\frac{1}{2}\right) (-a^3) \cdot \frac{2}{3}$$

$$= \frac{\pi}{6} a^3 .$$



Example – 2: $\displaystyle\iint_S e^{(y-x)/(y+x)}\, dx\, dy$

Change of variables

$$\begin{aligned} y-x&=u \\ y+x&=v \end{aligned} \Rightarrow \quad x = \frac{v-u}{2}, \quad y = \frac{v+u}{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

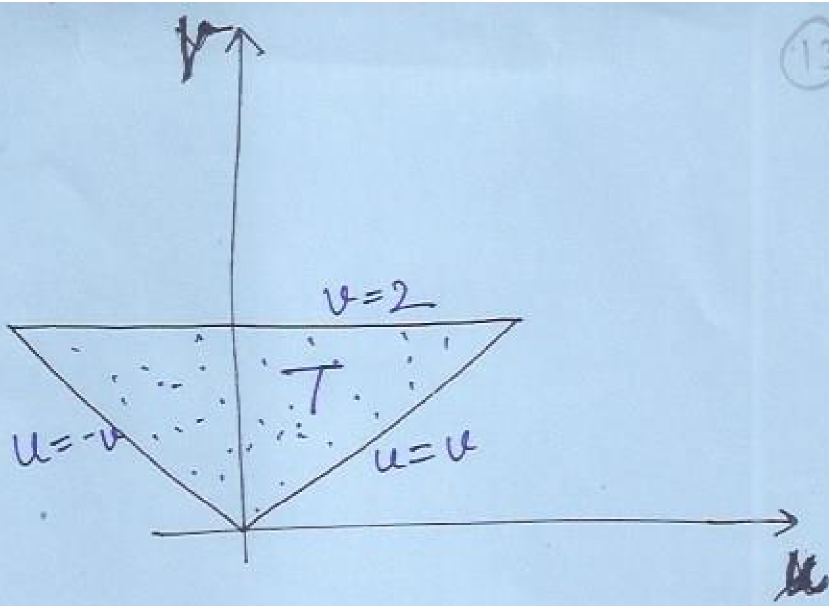
Domain in the uv-plane

line $x=0$ maps to $v=u$

line $y=0$ maps to $v=-u$

line $x+y=2$ maps to $v=2$

$$\iint_S e^{(y-x)/(y+x)}\,dx\,dy$$



$$= \iint_T e^{u/v}\,\frac{1}{2}\,du\,dv$$

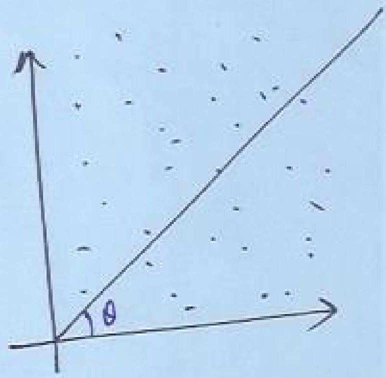
$$= \frac{1}{2}\int_{v=0}^{2}\int_{u=-v}^{v} e^{u/v}\,du\,dv$$

$$= \frac{1}{2}\int_0^2 v\left(e-\frac{1}{e}\right)dv$$

$$= e-\frac{1}{e}.$$

Example: Change into polar coordinates and evaluates

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$$



$x=r\cos\theta$, $y=r\sin\theta$

$$\Rightarrow \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta = \int_0^{\pi/2}\frac{1}{2}\,d\theta = \frac{\pi}{4}.$$

Note: Let $I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-y^2} dy$

$$\Rightarrow I^2 = \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \frac{\pi}{4}$$

$$\boxed{I = \frac{\sqrt{\pi}}{2}}$$

$$\Rightarrow \boxed{\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$$

Example: Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$ by changing to polar coordinates.

Solution: The region of integration is bounded by

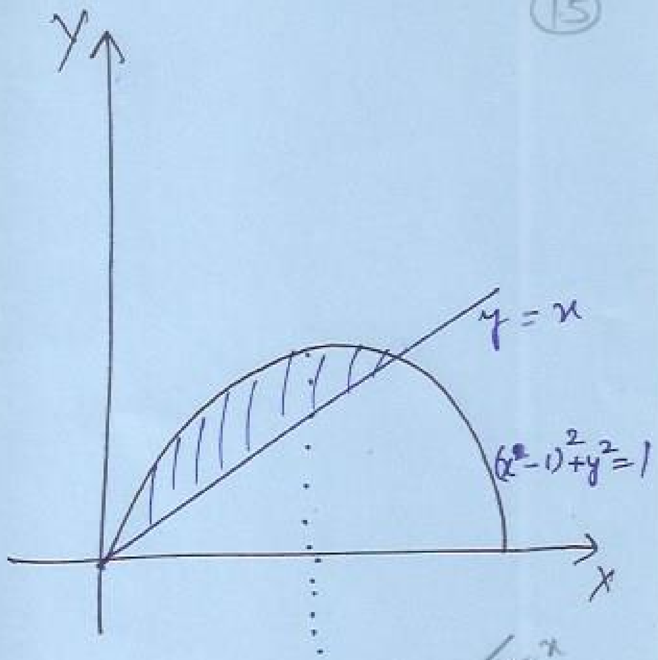
$y = x$, $y = \sqrt{2x - x^2}$, $x = 0$ & $x = 1$

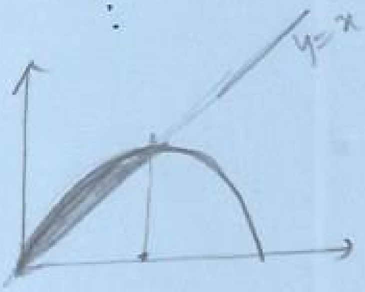
Polar Equation of the circle

$$(r\cos\theta-1)^2+r^2\sin^2\theta=1$$

$$\Rightarrow r^2-2r\cos\theta=0$$

$$\Rightarrow r=2\cos\theta$$





$$\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$$

$$=\int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2\cdot r\,dr\,d\theta$$

$$=\int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta=\int_{\pi/4}^{\pi/2} 4\cos^4\theta\cdot d\theta$$

$$=\int_{\pi/4}^{\pi/2} (2\cos^2\theta)^2\,d\theta=\int_{\pi/4}^{\pi/2} (1+\cos 2\theta)^2\,d\theta$$

$$=\int_{\pi/4}^{\pi/2} [1+\cos^2 2\theta+2\cos 2\theta]\,d\theta$$

$$=\int_{\pi/4}^{\pi/2} \left[1+\frac{1}{2}(1+\cos 4\theta)+2\cos 2\theta\right] d\theta$$

$$= \cdots = \frac{1}{8}(3\pi - 8)$$

Example: Evaluate the integral

$\iint_R \sqrt{x^2+y^2}\,dx\,dy$ by changing to polar co-ordinates, where R is the region in the x-y plane bounded by the circles $x^2+y^2=4$ and $x^2+y^2=9$.

Solution: $x = r\cos\theta$, $y = r\sin\theta$

$$|J| = r$$

$$I = \int_0^{2\pi}\int_2^3 r\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^3}{3}\right]_2^3 d\theta$$

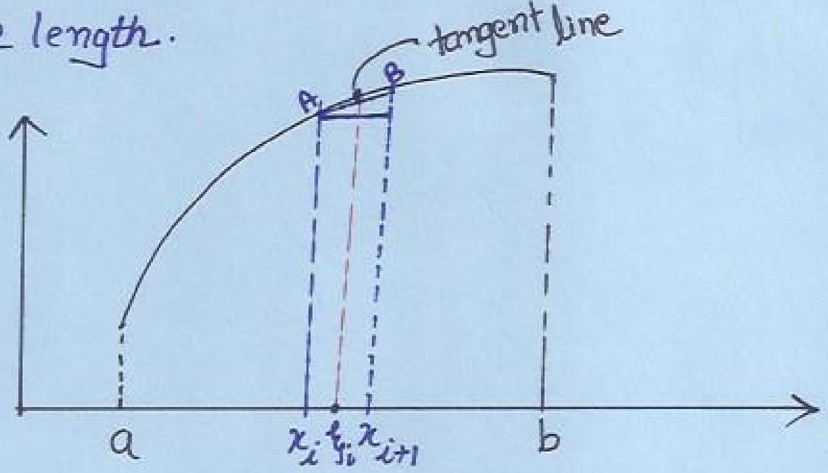
$$= \left(\frac{27-8}{3}\right)2\pi$$

$$= \frac{19}{3}\cdot 2\pi$$

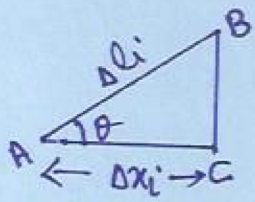
$$= \frac{38}{3}\cdot\pi$$

COMPUTING THE AREA OF A SURFACE:

Let us consider the case of 1-dimension, i.e. computation of the curve length.



length of the curve



$$L = \lim_{n\to\infty} \sum_{i=1}^{n-1} \Delta l_i$$

$$\frac{\Delta x_i}{\Delta l_i} = \cos\theta$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n-1} \sqrt{1+f'(\xi_i)^2} \cdot \Delta x_i$$

$$\Rightarrow \Delta l_i = \Delta x_i \frac{1}{\cos\theta}.$$

$$= \int_a^b \sqrt{1+(f'(x))^2} \cdot dx$$

Also $\cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}}$

$$\frac{1}{\cos\theta} = \sqrt{1+\tan^2\theta}$$

$$= \sqrt{1+f'(\xi_i)^2}$$

In two dimension case we consider tangent plane instead of tangent line and similar to one dimensional case we get surface area.

$$\Rightarrow \Delta l_i = \Delta x_i \sqrt{1+f'(\xi_i)^2}$$

$$\boxed{S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy}$$

wher D is the projection of the surface in xy-plane.

Similarly if the equation is given in the form:

$$x = \mu(y,z) \text{ or in the form } y = \psi(x,z)$$

then

$$S = \iint_{\tilde{D}} \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz$$

$$S = \iint_{\tilde{\tilde{D}}} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\, dx\, dz.$$

where $\tilde{D}$ and $\tilde{\tilde{D}}$ are the domains in the yz and xz planes in which the given surface is projected.

<u>Example</u>: Compute the surface area of the sphere

$$x^2+y^2+z^2 = R^2$$

<u>Solution</u>: Equation of the surface

$$z = \sqrt{R^2-x^2-y^2} \quad \text{(upper half)}$$

In this case: $$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2-x^2-y^2}}$$

Domain of integration: $x^2+y^2 \leq R^2$

$$S = 2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} \underbrace{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}_{\hookrightarrow \frac{R}{\sqrt{R^2-x^2-y^2}}}\, dy\, dx$$

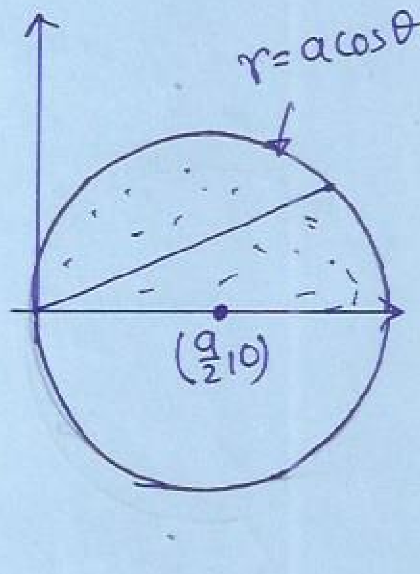
transformation to polar coordinate gives:

$$S = 2\int_0^{2\pi}\int_0^{R} \frac{R}{\sqrt{R^2-r^2}}\, r.\, dr\, d\theta$$

$$= 2\pi \cdot 2R\left(-\sqrt{R^2-r^2}\right)_0^R$$

$$= 4\pi R^2$$

Question: Find the area of that part of the sphere $x^2+y^2+z^2=a^2$ which is cut off by the cylinder $x^2+y^2=ax$.



$$x^2+y^2-ax=0 \Rightarrow \left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$$

$$S = 2.2\int_0^{\pi/2}\int_{r=0}^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}}\, r.\, dr\, d\theta$$

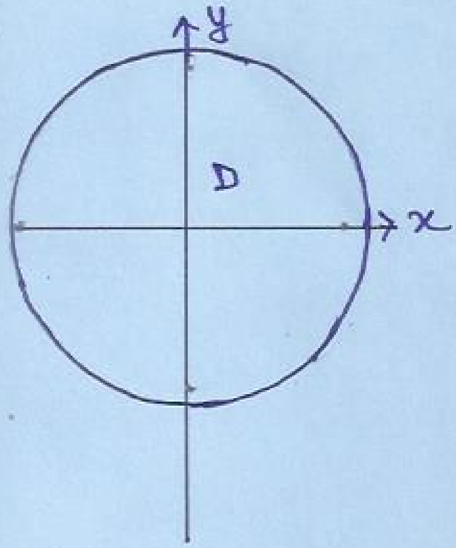
$$= 4 \cdot a\int_0^{\pi/2}\left(-\sqrt{a^2-r^2}\right)_0^{a\cos\theta} d\theta$$

$$= 4a \cdot \int_0^{\pi/2}\left[-a\sin\theta + a\right] d\theta$$

$$= 4a \cdot \left[\left[a\cos\theta\right]_0^{\pi/2} + a\{\theta\}_0^{\pi/2}\right] = 4a \cdot \left[-a + a \cdot \frac{\pi}{2}\right]$$

$$= 2a^2(\pi - 2).$$

Ex. Determine the surface area of the part of $z = xy$ that lies in the cyl. $x^2+y^2=1$.



Solution:

$$z = f(x,y) = xy$$

$$z_x = y \quad \& \quad z_y = x$$

$$S = \iint_D \sqrt{1+x^2+y^2}\, dA$$

In polar coordinate

$$S = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\sqrt{1+r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\cdot\frac{2}{3}\left[(1+r^2)^{3/2}\right]_0^1 d\theta = \frac{2\pi}{3}\left(2^{3/2}-1\right) \quad \text{Ans.}$$

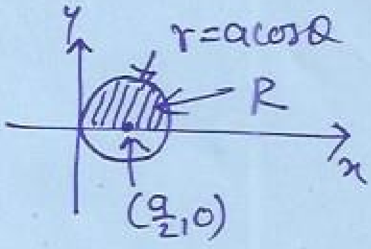
Evaluation of volume:

$$V = \iint_D \underset{f(x,y)}{\underset{\parallel}{z}}\, dx\, dy \quad \text{OR} \quad \iint_{\bar{D}} \mu(y,z)\, dy\, dz$$

$$\text{OR} \quad \iint_{\bar{\bar{D}}} \psi(x,z)\, dx\, dz.$$

Example: Find the volume common to sphere $x^2+y^2+z^2=a^2$ and a circular cylinder $x^2+y^2=ax$.

Required volume :



$$V = 4\iint_R z \, dx\,dy$$

$$= 4\iint_R \sqrt{a^2-x^2-y^2}\, dx\,dy$$

Subst. $x = r\cos\theta \quad y = r\sin\theta$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} \sqrt{a^2-r^2}\; r\,dr\,d\theta$$

$$= \frac{4}{-2}\int_0^{\pi/2} \frac{2}{3}\left(a^2-r^2\right)^{3/2}\Big|_0^{a\cos\theta} d\theta$$

$$= -2\cdot\frac{2}{3}\cdot\int_0^{\pi/2}\left(a^3\sin^3\theta - a^3\right)d\theta$$

$$= \frac{-4}{3}a^3\cdot\left[2/3 - \pi/2\right] = \frac{2}{9}a^3\left(3\pi-4\right)$$

Ans.

TRIPLE INTEGRALS

Divide the region V into n sub-regions of respective volumes $\delta V_1, \delta V_2, \cdots \delta V_n$. Let (x_r, y_r, z_r) be an arbitrary point in the rth sub-region.

Consider the sum

$$\sum_{j=1}^{n} f(x_j, y_j, z_j)\,\delta V_j$$

If the limit of this sum exists as $n \to \infty$ and $\delta V_j \to 0$, then

$$\iiint_V f(x,y,z)\,dV = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j, y_j, z_j)\,\delta V_j$$

Evaluation:

$$\iiint_V f(x,y,z)\,dV = \int_{z=a}^{b} \left[\int_{y=\varphi_1(z)}^{\varphi_2(z)} \left\{ \int_{x=f_1(y,z)}^{f_2(y,z)} f(x,y,z)\,dx \right\} dy \right] dz$$

Note: Similar to double integrals, the order of integration is immaterial if the limits of integration are constants.

$$\int_a^b \int_c^d \int_e^f F(x,y,z)\,dx\,dy\,dz = \int_e^f \int_c^d \int_a^b F(x,y,z)\,dz\,dy\,dx$$

$$= \int_c^d \int_e^f \int_a^b F(x,y,z)\,dz\,dx\,dy$$

Example: Evaluate $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx$

$$I = \int_0^a \int_0^x e^{x+y+z} \Big|_0^{x+y} \cancel{dz} \, dy \, dx$$

$$= \int_0^a \int_0^x \left(e^{2(x+y)} - e^{x+y} \right) dy \, dx$$

$$= \int_0^a \frac{e^{2(x+y)}}{2} \Big|_0^x dx - \int_0^a e^{x+y} \Big|_0^x dx$$

$$= \frac{1}{2} \left(\int_0^a \left\{ \left(e^{4x} - e^{2x} \right) - 2 \left(e^{2x} - e^x \right) \right\} dx \right.$$

$$= \frac{1}{2} \int_0^a \left(e^{4x} - 3e^{2x} + 2e^x \right) dx$$

$$= \frac{1}{2} \left[\frac{e^{4x}}{4} \Big|_0^a - \frac{3}{2} e^{2x} \Big|_0^a + 2 e^x \Big|_0^a \right]$$

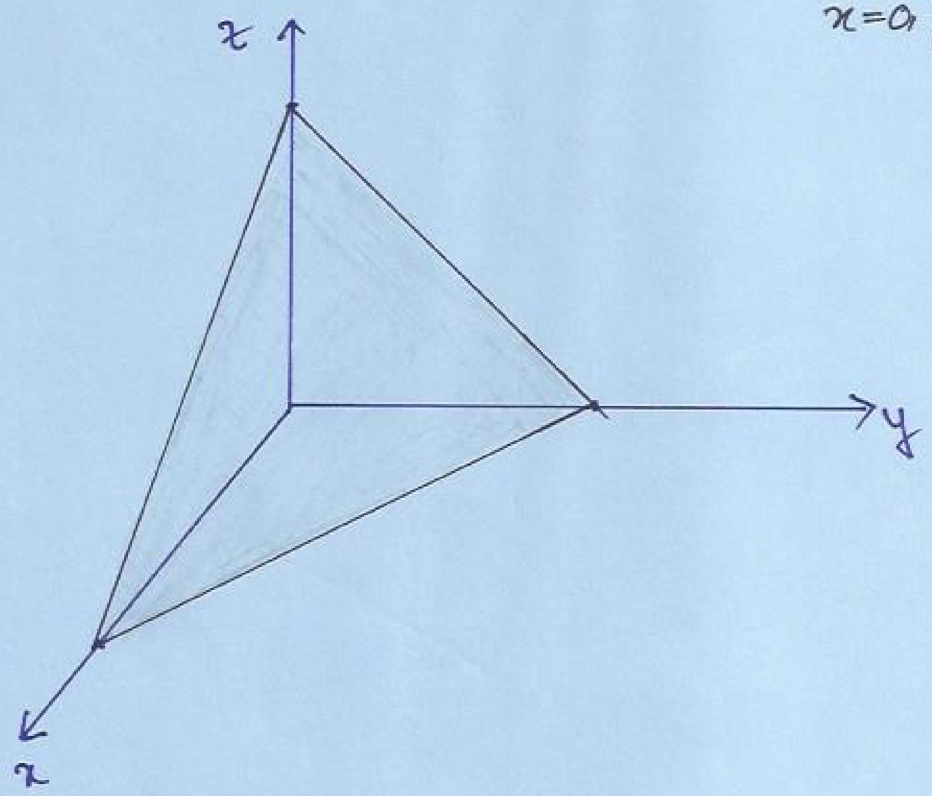
$$= \frac{1}{2} \left[\frac{e^{4a}}{4} - \frac{3}{2} e^{2a} + 2e^a - \frac{1}{4} + \frac{3}{2} - 2 \right]$$

$$= \frac{1}{2} \left[\frac{e^{4a}}{4} - \frac{3}{2} e^{2a} + 2e^a - \frac{3}{4} \right]$$

$$= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^a - \frac{3}{8}$$

Ans

Example: Evaluate $\iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$; R is the region bounded by $x=0, y=0, z=0$ & $x+y+z=1$



$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3} \cdot dz\, dy\, dx$$

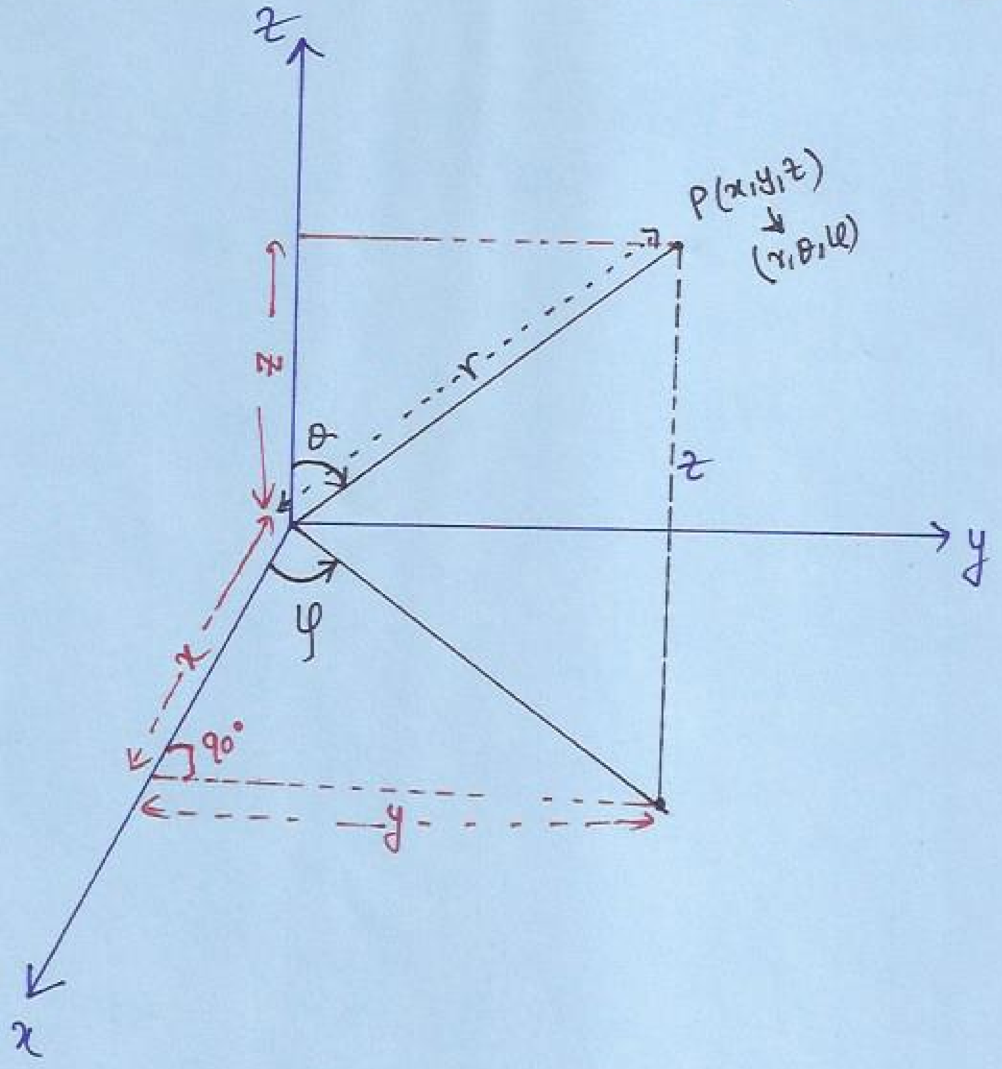
$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2}(x+y+z+1)^{-2}\right]_0^{1-x-y} dy\, dx$$

$\vdots$

$$= \frac{1}{2}\left[\ln 2 - \frac{5}{8}\right]$$ Ans

Change of Variables in TRIPLE integrals:

- Cartesian co-ordinate (x,y,z) to spherical polar coordinates



$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

Note that $x^2+y^2+z^2=r^2$

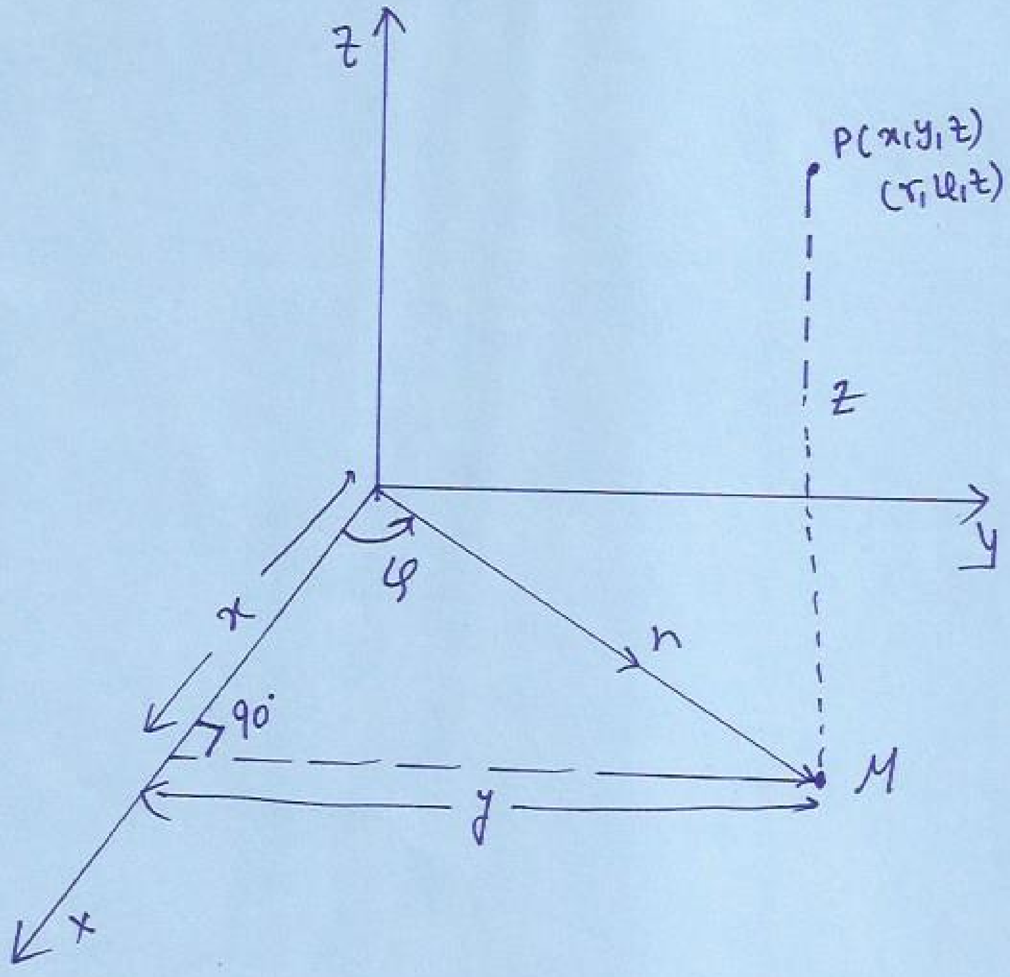
$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{\tilde{D}} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\, r^2\sin\theta\, dr\,d\theta\,d\varphi.$$

ii) Cartesian Coordinates (x, y, z) to Cylindrical coordinates (r, φ, z)

$$(x, y, z) \rightarrow (r, \varphi, z)$$



$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = z$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = r$$

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iiint_{\tilde{D}} f(r\cos\varphi, r\sin\varphi, z)\, r\, dr\, d\varphi\, dz$$

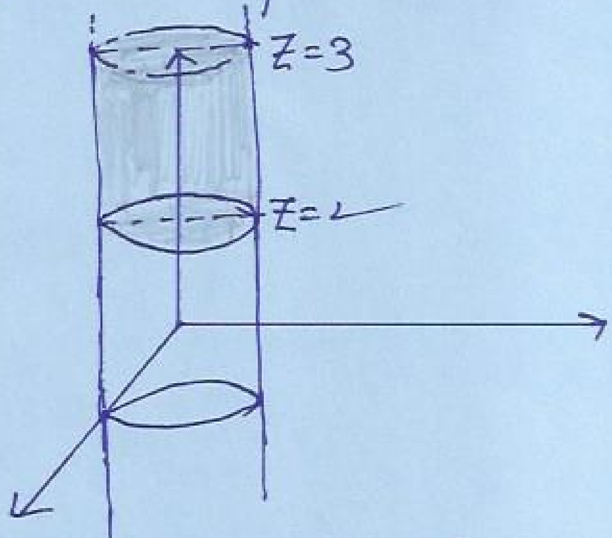
Ex: Changing to cylindrical coordinate, evaluate

$$\iiint z(x^2+y^2)\,dx\,dy\,dz\ ; \qquad x^2+y^2 \le 1$$
$$2 \le z \le 3$$

Solution: $x = r\cos\theta \quad y = r\sin\varphi \quad z = z$

Note that $x^2+y^2 = r^2$

& $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \vdots & & \\ \vdots & & \end{vmatrix} = r$



$$\iiint z(x^2+y^2)\,dx\,dy\,dz = \int_{z=2}^{3}\int_{\varphi=0}^{2\pi}\int_{r=0}^{1} z\cdot r^2\, r\,dr\,d\varphi\,dz$$

$$= \int_2^3\int_0^{2\pi} \frac{1}{4} z\,d\varphi\,dz$$

$$= \frac{1}{4}\,2\pi\,\frac{1}{2}(9-4) = \frac{5\pi}{4}.$$

Example: Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} \frac{1}{\sqrt{x^2+y^2+z^2}}\,dz\,dy\,dx$ by changing into spherical polar coordinate.

Solution:

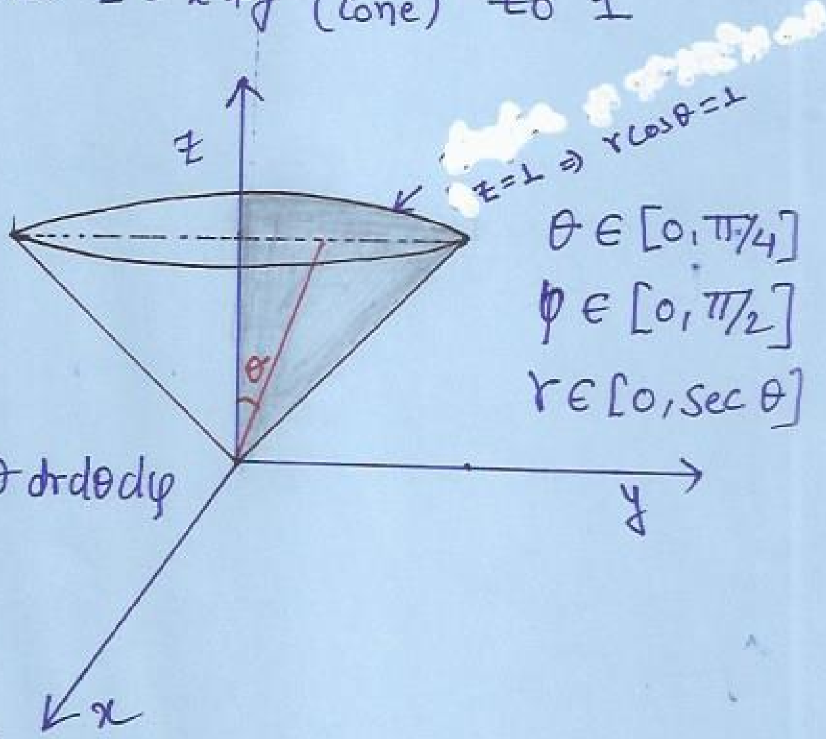
$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$J = r^2\sin\theta \qquad x^2+y^2+z^2 = r^2$$

x varies from 0 to 1

y varies from 0 to $y=\sqrt{1-x^2}$ i.e., $y^2+x^2=1$

z varies from $\sqrt{x^2+y^2}$, i.e. $z^2=x^2+y^2$ (cone) to 1



$\theta \in [0, \pi/4]$

$\phi \in [0, \pi/2]$

$r \in [0, \sec\theta]$

$$I = \int_0^{\pi/2}\int_{\theta}^{\pi/4}\int_{r=0}^{\sec\theta} \frac{1}{r}\cdot r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/4} \frac{1}{2}\sec^2\theta\sin\theta\,d\theta\,d\varphi$$

$$= \frac{\pi}{4}\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta$$

$$= \frac{\pi}{4}\sec\theta\Big|_0^{\pi/4} = \frac{(\sqrt{2}-1)\pi}{4}$$

Ex. Evaluate $$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$$

by changing to spherical polar coordinates.

Sol: $x = r\sin\theta\cos\varphi \quad y = r\sin\theta\sin\varphi \quad z = r\cos\theta$

$J = r^2\sin\theta.$

$$I = \int_{\theta=0}^{\pi/2}\int_{\varphi=0}^{\pi/2}\int_{r=0}^{1} \frac{r^2\sin\theta}{\sqrt{1-r^2}}\, dr\, d\varphi\, d\theta$$

First evaluate: $\displaystyle\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr$ subst $r = \sin t$, $dr = \cos t\, dt$

$$= \int_0^{\pi/2} \frac{\sin^2 t}{\cos t} \cos t\, dt = \frac{\pi}{4}$$

$$I = \frac{\pi}{4}\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\, d\varphi\, d\theta$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{2}\cdot\left[-\cos\theta\right]\Big|_0^{\pi/2}$$

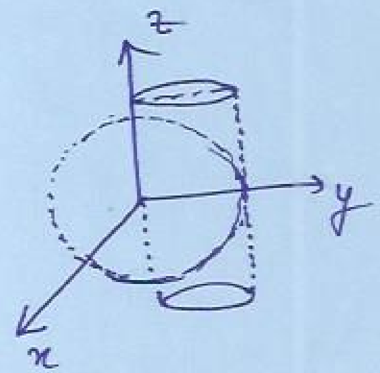
$$= \frac{\pi^2}{8}\cdot 1.$$

$$= \frac{\pi^2}{8}$$ Ans

Q. Using triple integral find the volume common to a sphere $x^2+y^2+z^2=a^2$ and a circular cylinder $x^2+y^2=ax$.

$$V=\iiint_V dx\,dy\,dz = \iiint_V dz\,dy\,dx$$

$$= 4\int_0^a \int_{y=0}^{\sqrt{ax-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$



$$= 4\int_0^a \int_0^{\sqrt{ax-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$\vdots$

proceed as in double integral

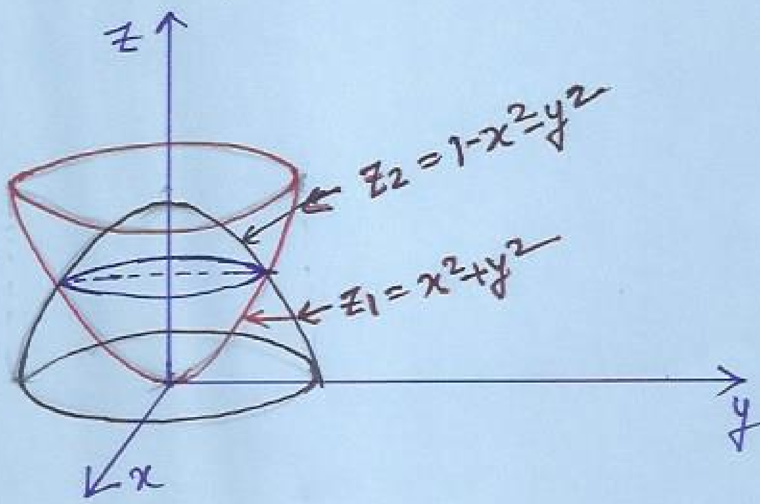
$$= \frac{2}{9}a^3\left(\pi-\frac{4}{3}\right).$$

Q. Find the volume of the solid formed by two paraboloids: $z_1=x^2+y^2$ & $z_2=1-x^2-y^2$

Intersecting curve:

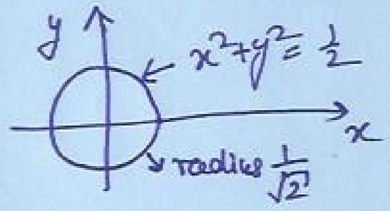
$x^2+y^2=1-x^2-y^2$

$\Rightarrow x^2+y^2=\frac{1}{2}$



$$V = \iiint_V dx\,dy\,dz = \int_{x=-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{y=-\sqrt{\frac{1}{2}-x^2}}^{+\sqrt{\frac{1}{2}-x^2}} \int_{x^2+y^2}^{1-x^2-y^2} dz\,dy\,dx$$

Projection on xy plane:



Changing to cylindrical coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$V = 4\int_0^{\pi/2} \int_{r=0}^{\frac{1}{\sqrt{2}}} \int_{z=r^2}^{1-r^2} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \int_{r=0}^{\frac{1}{\sqrt{2}}} r\cdot(1-r^2-r^2)\,dr\,d\theta$$

$$= 2\pi \int_0^{\frac{1}{\sqrt{2}}} r(1-2r^2)\,dr$$

$$= 2\pi\left[\frac{1}{2}\left(\frac{1}{2}-0\right) - \frac{2}{4}\cdot\left(\frac{1}{4}-0\right)\right]$$

$$= 2\pi\cdot\left[\frac{1}{4}-\frac{1}{8}\right]$$

$$= 2\pi\cdot\frac{1}{8}$$

$$= \frac{\pi}{4} \quad \text{Ans.}$$