Indian Institute of Technology Kharagpur
Department of Mathematics
MA11004 - Linear Algebra, Numerical and Complex Analysis
Problem Sheet - 2
Spring 2021

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1. Determine which of the following form a basis of the respective vector spaces:

   (a) $\{4t^2 - 2t + 3, 6t^2 - t + 4, 8t^2 - 8t + 7\}$ of $\mathbb{P}_2(\mathbb{R})$.

   (b) Let V be a real vector space with $\{\alpha, \beta, \gamma\}$ as a basis. Check whether $\{\alpha+\beta+\gamma, \beta+\gamma, \gamma\}$ is also a basis of V.

   (c)
   $$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$
   for V, where V is the vector space of all 2×2 real matrices.

2. Determine a basis and the dimension of the following subspaces:

   (a) The subspace V of all 2×2 real symmetric matrices.

   (b) $U = \{(x, y, z, w) \in \mathbb{R}^4 : x + 2y - z = 0, 2x + y + w = 0\}$ of $\mathbb{R}^4$.

   (c) Let $U = \{p \in \mathbb{P}_4(\mathbb{R}) : \displaystyle\int_{-1}^{1} p(t)dt = 0\}$.

3. If $U = \text{span}(\{(1, 2, 1), (2, 1, 3)\}), W = \text{span}(\{(1, 0, 0), (0, 0, 1)\})$, show that U and W are subspaces of $\mathbb{R}^3$. Find the dimensions of $U, W, U \cap W$.

4. Check the following mappings are linear transformation or not:

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x^2, |y| + z)\ \forall (x, y, z) \in \mathbb{R}^3$.

   (b) $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_4(\mathbb{R})$ defined by $T(p(x)) = (1 - x)p'(0) - xp(x)$.

5. Consider the vector space $\mathbb{C}$ over the field $\mathbb{C}$. Give an example of a function $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi(w + z) = \phi(w) + \phi(z)\ \forall\ w, z \in \mathbb{C}$, but ϕ is not a linear transformation .

6. Find the null space and range space of the following linear transformations. Also find their respective dimensions and verify the rank-nullity theorem.

   (a) $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ defined by $T(f(x)) = 2f'(x) + \displaystyle\int_0^x 3f(t)dt$.

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$, defined by $T(x, y, z) = (\frac{x-y-z}{2}, \frac{z}{2})$.

   (c) $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T(A) = \dfrac{A - A^T}{2}, \forall A \in M_{2\times2}(\mathbb{R})$.

7. (a) Determine the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1, 1) = (1, 0, 2),\ T(2, 3) = (1, -1, 4)$.

   (b) Determine the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors $\{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$ of $\mathbb{R}^3$ to the vectors $\{(1, 1),\ (2, 3),\ (3, 2)\}$ respectively.

   (i) Find $T(1, 1, 0),\ T(6, 0, -1)$,

(ii) Find $N(T)$ and $R(T)$,

(iii) Prove that T is not one-to-one but onto.

8. Find the matrix of the linear transformations with respect to the given ordered bases:

   (a) $D : \mathbb{P}_4(\mathbb{R}) \to \mathbb{P}_4(\mathbb{R})$ defined by $D(p(x)) = 3\frac{d^3}{dx^3}(p(x))$, with respect to the ordered basis $\{1, x, x^2, x^3, x^4\}$ for both $\mathbb{P}_4(\mathbb{R})$.

   (b) $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{bmatrix} 2f''(0) & f(3) \\ 0 & f'(2) \end{bmatrix}$$

   with respect to the ordered basis $\{1, x, x^2, x^3\}$ and $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

9. Prove that there does not exist a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that $R(T) = N(T)$.

10. Solve the following system of equations by Gauss-elimination method:

$$(a) \quad \begin{aligned} 9x + 3y + 4z &= 7 \\ 4x + 3y + 4z &= 8 \\ x + y + z &= 3 \end{aligned} \qquad\qquad (b) \quad \begin{aligned} x + 2y + 3z + 2w &= -1 \\ -x - 2y - 2z + w &= 2 \\ 2x + 4y + 8z + 12w &= 4 \end{aligned}$$

11. Find the rank of the matrix A using definition where

$$(i)\ A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix}. \qquad (ii) \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{bmatrix}$$

12. Determine the rank of the following matrices by reducing to row echelon form.

$$(a) \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 3 & 6 & 6 & 3 \end{bmatrix} \qquad (b) \begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$$

13. Find all x such that the rank of the matrix $\begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$ is less than 3.

14. Find the value of k for which the system of equations has non-trivial solution.

$$\begin{aligned} x + 2y + z &= 0 \\ 2x + y + 3z &= 0 \\ x + ky + 3z &= 0 \end{aligned}$$

15. Solve the system of equations in integers

$$\begin{aligned} x + 2y + z &= 1 \\ 3x + y + 2z &= 3 \\ x + 7y + 2z &= 1 \end{aligned}$$

16. Solve if possible

$$\begin{aligned} x + 2y + z - 3w &= 1 \\ 2x + 4y + 3z + w &= 3 \\ 3x + 6y + 4z - 2w &= 5 \end{aligned}$$

17. Determine the condition for which the system

$$\begin{aligned} x + y + z &= b \\ 2x + y + 3z &= b + 1 \\ 5x + 2y + az &= b^2 \end{aligned}$$

admits of (i) only one solution, (ii) no solution, (iii) infinitely many solutions.

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