ADVANCED CALCULUS
MA11003

SECTION 11 & 12

Dr. Jitendra Kumar

Professor
Department of Mathematics
Indian Institute of Technology Kharagpur
West Bengal 721302, India

Webpage: http://www.facweb.iitkgp.ac.in/~jkumar/

Concepts Covered

Differential Equations of Higher Order

- Higher Order Linear Differential Equations
- Introduction

Linear Differential Equations of Higher Order with Constant Coefficients

The general form:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X$$

where $a_1, a_2, \ldots, a_n$ are constants and X is a function of x

free from arbitrary constants

General Solution = Complementary Function (C.F.) + Particular Integral (P.I.)

contains n arbitrary constants

Complementary Function (C.F.)

It is the general solution of the homogeneous equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$$

Particular Integral (P.I.)

If v is any particular solution, then

$$\frac{d^n v}{dx^n} + a_1 \frac{d^{n-1} v}{dx^{n-1}} + \cdots + a_n v = X$$

Linear Independence of Solution

Two functions y_1 & y_2 are **linearly independent** if one is not the constant multiple of other.

or $\dfrac{y_1}{y_2} \neq \text{constant}$

In other words, the two functions $y_1(x)$ & $y_2(x)$ are linearly dependent if for some c_1 & $c_2 \neq 0$

$$c_1 y_1(x) + c_2 y_2(x) = 0 \text{ for all } x \text{ in some interval } x \in [a, b]$$

Examples:

$y_1 = \sin x\,,\qquad y_2 = \cos x$ Linearly Independent

$y_1 = \sin 2x\,,\qquad y_2 = \sin x$ Linearly Independent

$y_1 = e^{\alpha_1 x},\qquad y_2 = e^{\alpha_2 x}$ Linearly Independent

$y_1 = 2\sin^2 x\,,\qquad y_2 = (1 - \cos^2 x)$ Linearly Dependent

Linear Independence of Solution

For n functions $y_1, y_2, \dots, y_n$ are said to be **linearly dependent,**

if for some $c_1, c_2, \dots, c_n$ (not all zero), $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$

Usually, it is difficult to verify linear independence using this definition.

For n functions (differentiable) $y_1, y_2, \dots, y_n$, if the Wronskian $W(y_1, y_2, \dots, y_n) \neq 0$,

for some $x \in [a, b]$ then they are linearly independent on $[a, b]$.

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ {y_1}' & {y_2}' & \cdots & {y_n}' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Consider $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$

Let $y_{1,} y_2$ be any two linearly independent solution, then $c_1 y_1 + c_2 y_2$ is also a solution of the above equation, where c_1, c_2 are arbitrary constants:

$$\frac{d^n}{dx^n}(c_1 y_1 + c_2 y_2) + a_1 \frac{d^{n-1}}{dx^{n-1}}(c_1 y_1 + c_2 y_2) + \cdots + a_n(c_1 y_1 + c_2 y_2)$$

$$= c_1 \left(\frac{d^n}{dx^n} y_1 + a_1 \frac{d^{n-1}}{dx^{n-1}} y_1 + \cdots + a_n y_1 \right)$$

$$+ c_2 \left(\frac{d^n}{dx^n} y_2 + a_1 \frac{d^{n-1}}{dx^{n-1}} y_2 + \cdots + a_n y_2 \right) = 0$$

Generalization:

If $y_1, y_2 \dots y_n$ be any n linearly independent solutions of homogeneous differential equation, then

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

is the general solution of the homogeneous differential equation.

Here $c_1, c_2, \cdots, c_n$ are arbitrary constants

If u be the general solution of the associated **homogeneous equation** and v be any **particular solution** of given differential equation, then $(u+v)$ is the general solution of the given nonhomogeneous differential equation.

$$\frac{d^n}{dx^n}(u+v)+a_1\frac{d^{n-1}}{dx^{n-1}}(u+v)+\cdots+a_n(u+v)$$

$$=\left(\frac{d^n}{dx^n}u+a_1\frac{d^{n-1}}{dx^{n-1}}u+\cdots+a_nu\right)+\left(\frac{d^n}{dx^n}v+a_1\frac{d^{n-1}}{dx^{n-1}}v+\cdots+a_nv\right)$$

$$=0+X=X$$

OPERATORS: $\frac{d}{dx}, \frac{d^2}{dx^2} \cdots$

For the sake of convenience , the operators

$\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \cdots$ are denoted by $D, D^2, D^3, \cdots$

Product of operators

$(D-\alpha)(D-\beta)\, y = (D-\beta)(D-\alpha)\, y,$ $\quad \alpha, \beta$ being any constant

$$(D-\alpha)(D-\beta)y = (D-\alpha)\left(\frac{dy}{dx}-\beta y\right) = \frac{d}{dx}\left(\frac{dy}{dx}-\beta y\right)-\alpha\left(\frac{dy}{dx}-\beta y\right)$$

$$= \frac{d^2y}{dx^2}-\beta\frac{dy}{dx}-\alpha\frac{dy}{dx}+\alpha\beta y = \frac{d^2y}{dx^2}-(\alpha+\beta)\frac{dy}{dx}+\alpha\beta y = [D^2-(\alpha+\beta)D+\alpha\beta]y$$

Similarly, one can show that $(D-\beta)(D-\alpha)y = [D^2-(\alpha+\beta)D+\alpha\beta]y$

$$(D-\alpha)(D-\beta) \equiv (D-\beta)(D-\alpha)$$

So, the order of operational factors is immaterial.

Also note that $\underbrace{(D-\beta)(D-\alpha)}y = \underbrace{[D^2-(\alpha+\beta)D+\alpha\beta]}\,y$

Same

In General: $[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]y = \mathrm{X}$

$$\Longrightarrow\ [(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)]y = \mathrm{X}$$

Conclusion

Linear Differential Equations of Higher Order with Constant Coefficients

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X$$

General solution =

Complementary Function (C.F.) + Particular Integral (P.I.)

$$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]y = \mathrm{X}$$

$$\Longrightarrow \; [(D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n)]y = \mathrm{X}$$

Thank You