Solⁿ: The difference table is

	x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
x_0	0.20	1.6596			
x_1	0.22	1.6698	0.0102		
x_2	0.24	1.6804	0.0106	0.0004	
x_3	0.26	1.6912	0.0108	0.0002	−0.0002
x_4	0.28	1.7024	0.0112	0.0004	0.0002
x_5	0.30	1.7139	0.0115	0.0003	−0.0001

Here $x_n = 0.30$, $x = 0.29$, $h = 0.02$, $u = \frac{x - x_n}{h} = \frac{0.29 - 0.30}{0.02} = -0.5$

Then

$$f(0.29) = f(x_n) + u \nabla f(x_n) + \frac{u(u+1)}{2!} \nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!} \nabla^3 f(x_n)$$

$$= 1.7139 - 0.5 \times 0.0115 + \frac{-0.5(-0.5+1)}{2} \times 0.0003$$

$$+ \frac{-(0.5)(-0.5+1)(-0.5+2)}{6} \times (-0.0001)$$

$$= 1.7139 - 0.00575 - 0.0000375 + 0.00000625$$

$$= 1.70811875 \simeq 1.7081$$

Error in Newton's backward interpolation formula

The error is

$$E(x) = (x - x_n)(x - x_{n-1}) \cdots (x - x_0) \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$= u(u+1)(u+2) \cdots (u+n) h^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

where $u = \frac{x - x_n}{h}$ and ξ lies between $\min\{x_0, x_1, \ldots x_n, x\}$ and $\max\{x_0, x_1, \ldots, x_n, x\}$.

Ex If $y(75) = 246$, $y(80) = 202$, $y(85) = 118$, $y(90) = 40$, find $y(79)$.

Sol^n: Here $x_0 = 75$, $h = 5$.

From Newton's forward interpolation formula

$$\varphi(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
75	246			
		-44		
80	202		-40	
		-84		46
85	118		6	
		-78		
90	40			

Here $x = 79$; $u = \frac{79-75}{5} = 0.8$

$$y_{0.8} = 246 + 0.8(-44) + \frac{0.8 \times (0.8-1)}{2!}(-40) + \frac{0.8(0.8-1)(0.8-2)}{3!}\,46$$

$$= 215.472$$

Ex Find a cubic polynomial which takes the following values

x	0	1	2	3
$f(x)$	1	2	1	10

Sol^n:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f(x_0)}{1!\,h} + (x-x_0)(x-x_0-h)\frac{\Delta^2 f(x_0)}{2!\,h^2} + \cdots$$

Here $x_0 = 0$, $h = 1$

$$\therefore f(x) = 1 + (x-0)\frac{1}{1!} + (x-0)(x-1)\frac{-2}{2!} + (x-0)(x-1)(x-2)\frac{12}{3!}$$

$$= 1 + x - x(x-1) + 2x(x-1)(x-2)$$

$$= 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x$$

$$= 2x^3 - 7x^2 + 6x + 1$$

Ex Find the value of y from the following table at $x=2.65$

x	-1	0	1	2	3
y	-21	6	15	12	3

Solⁿ:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
-1	-21				
0	6	27			
1	15	9	-18		
2	12	-3	-12	6	
3	3	-9	-6	6	0

$$\phi(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \cdots$$

$$v = \frac{x - x_n}{h} = -0.35$$

$$y_{2.65} = 3 + (-0.35)(-9) + \frac{(-0.35)(-0.35+1)}{2!}(-6)$$

$$+ \frac{(-0.35)(-0.35+1)(-0.35+2)}{3!} \times 6$$

$$= 6.4571$$

Lagranges interpolation

Given $(x_0, y_0) \ldots (x_n, y_n)$ with arbitrarily spaced x_i. Lagrange had the idea of multiplying each y_i by a polynomial that is 1 at x_i and 0 at the other n nodes and then to take sum of these $n+1$ polynomials to get the interpolating polynomial of degree n or less.

$$\varphi(x) = \sum_{i=0}^{n} L_i(x)\, y_i$$

$$L_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Let us begin with the simplest case.

Linear interpolation

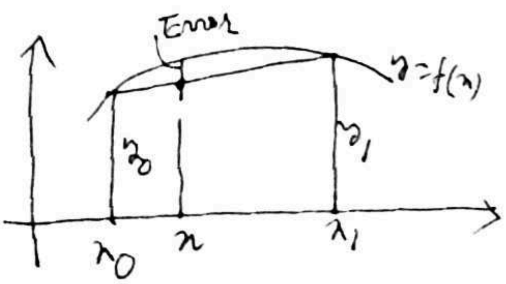
Linear Lagrange polynomial $\varphi_1(x) = L_0 y_0 + L_1 y_1$

$L_0(x_0)=1 \quad L_0(x_1)=0 \quad L_1(x_1)=1, L_1(x_0)=0$

with L_0 is 1 at x_0 and 0 at x_1; similarly L_1 is 0 at x_0 and 1 at x_1.

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$\varphi_1(x) = L_0(x)\, y_0 + L_1(x)\, y_1 = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} \cdot y_1$$



Quadratic interpolation

$$\varphi_2(x) = L_0(x)\, y_0 + L_1(x)\, y_1 + L_2(x)\, y_2$$

$L_0(x_0) = 1 \quad L_1(x_1) = 1 \quad L_2(x_2) = 1$

$L_0(x_1) = L_0(x_2) = 0$ etc

$$L_0(x) = \frac{l_0(x)}{l_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \qquad L_1(x) = \frac{l_1(x)}{l_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{l_2(x)}{l_2(x_2)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

In general $\varphi(x) = \sum_{k=0}^{n} L_k(x)\, y_k = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f_k$

Lagrange's interpolation polynomial

$$\phi(x) = \sum_{i=0}^{n} L_i(x) y_i$$

where each $L_i(x)$ is polynomial in x, of degree less than or equal to n, called the Lagrangian function.

The polynomial $\phi(x)$ satisfies $\phi(x_i) = y_i$, $i = 0$ to n

if $L_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

$\therefore$ $L_i(x)$ vanishes at $x_0, x_1, \ldots, x_n$. So it can be written in the form

$$L_i(x) = a_i (x - x_0)(x - x_1) \cdots (x - x_n)$$

where a_i is determined by $L_i(x_i) = 1$.

$$\therefore a_i (x_i - x_0)(x_i - x_1) \cdots (x_i - x_n) = 1$$

$$\therefore a_i = \frac{1}{(x_i - x_0) \cdots (x_i - x_n)}$$

$$\therefore L_i(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_n)}$$

$$= \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right)$$

$$\phi(x) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) y_i$$

Ex Obtain Lagrange's interpolating polynomial for $f(x)$ and find an approximate value of the f^n. $f(x)$ at $x=0$ given that $f(-2)=-5$, $f(-1)=-1$ and $f(1)=1$

Sol^n:

$$x_0=-2, \quad x_1=-1, \quad x_2=1$$

$$f(x_0)=-5 \quad f(x_1)=-1 \quad f(x_2)=1$$

$$\therefore f(x) \simeq \phi(x) = \sum_{i=0}^{2} L_i(x)\, f(x_i)$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x+1)(x-1)}{(-2+1)(-2-1)} = \frac{x^2-1}{3}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+2)(x-1)}{(-1+2)(-1-1)} = \frac{x^2+x-2}{-2}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+2)(x+1)}{(1+2)(1+1)} = \frac{x^2+3x+2}{6}$$

$$f(x) \simeq \frac{x^2-1}{3}\times(-5) + \frac{x^2+x-2}{-2}\times(-1) + \frac{x^2+3x+2}{6}\times 1$$

$$= 1+x-x^2$$

Uniqueness of the interpolating polynomial

We assume that we are given an interval $[a,b]$ and a function $f(x)$ which is continuous on $[a,b]$. Further, we assume that we have $n+1$ distinct points $a \le x_0 < x_1 < x_2 \cdots < x_n \le b$ of $[a,b]$ and that the values of a f^n. $f(x)$ are known at these points. We seek to find the polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

and $P(x_i) = f(x_i) \qquad i = 0, 1, 2, \ldots n$

Substituting the conditions, we obtain the system of eqns.

$$a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n = f(x_0)$$
$$a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n = f(x_1)$$
$$\cdots \quad \cdots \quad \cdots$$
$$a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^n = f(x_n)$$

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \cdots & \cdots & \cdots & & \cdots \\ \cdots & \cdots & \cdots & & \cdots \\ 1 & x_n & x_n^2 & & x_n^n \end{vmatrix} \neq 0$$

$\therefore$ Unique polynomial.

General quadrature formula based on Newton's forward interpolation

The Newton's forward interpolation formula for the equispaced points $x_i, i=0,1,\dots,n$, $x_i = x_0 + ih$ is

$$\phi(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $u = \frac{x-x_0}{h}$, h is the spacing.

Let the interval $[a,b]$ be divided into n equal subintervals such that $a = x_0 < x_1 < x_2 \cdots < x_n = b$. Then

$$I = \int_a^b f(x)\,dx \simeq \int_{x_0}^{x_n} \phi(x)\,dx$$

$$= \int_{x_0}^{x_n} \left[y_0 + u\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \cdots \right] dx$$

Since $x = x_0 + uh$, $dx = h\,du$, when $x = x_0$ then $u=0$ and $x = x_n$, then $u = n$. Thus

$$I = \int_0^n \left[y_0 + u\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \cdots \right] h\,du$$

$$= h\left[y_0 (u)_0^n + \Delta y_0 \left(\frac{u^2}{2}\right)_0^n + \frac{\Delta^2 y_0}{2!}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)_0^n + \cdots \right]$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{2n^2-3n}{12}\Delta^2 y_0 + \frac{n^3-4n^2+4n}{24}\Delta^3 y_0 + \cdots \right]$$

From this formula, one can generate different integration formulae by substituting $n = 1,2,3,\cdots$.

Trapezoidal rule

Substituting $n=1$, all differences higher than the first difference become zero

$$\int_{x_0}^{x_1} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1)$$

This formula is known as trapezoidal rule.

In this formula, the interval $[a,b]$ is considered as a single interval and it gives a very rough answer. But if the interval $[a,b]$ is divided into several subintervals and this formula is applied to each of these subintervals then a better approximate result may be obtained. This formula is known as composite formula, deduced below.

Composite trapezoidal rule

Let the interval $[a,b]$ be divided into n equal subintervals by the pts $a = x_0, x_1, \ldots x_n = b$, where $x_i = x_0 + ih$, $i = 1, 2, \ldots, n$. Applying the trapezoidal rule to each of the subintervals, one can find the composite formula as

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

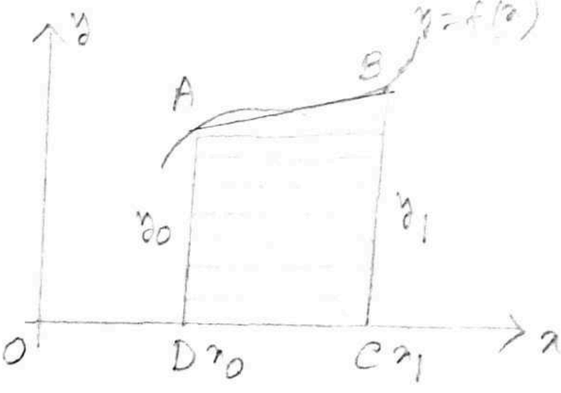
$$\simeq \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + \cdots + \frac{h}{2}[y_{n-1} + y_n]$$

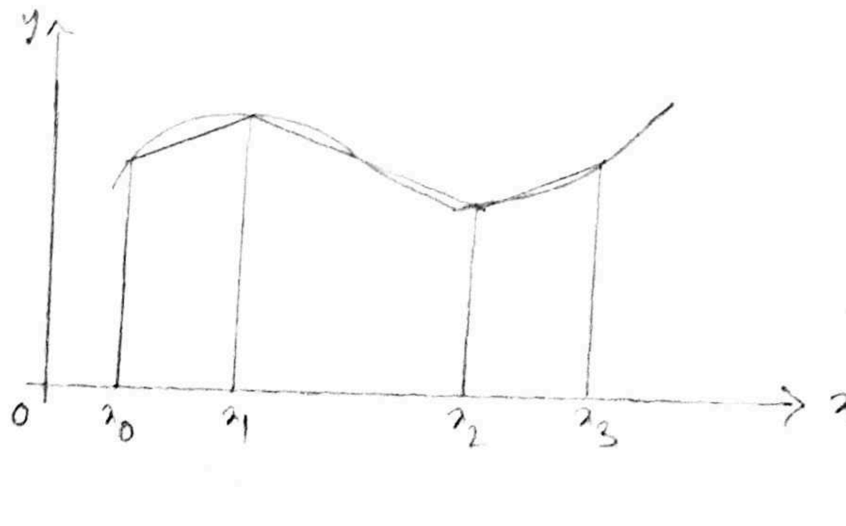
$$= \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

Geometrical interpretation of trapezoidal rule

In this rule, the curve $y=f(x)$ is replaced by the line joining the points $A(x_0,y_0)$ and $B(x_1,y_1)$. Thus the area bounded by the curve $y=f(x)$, the ordinates $x=x_0$, $x=x_1$ and the x axis is then approximately equivalent to the area of the trapezium (ABCD) bounded by the line AB, $x=x_0$, $x=x_1$ and x axis.

The geometrical significance of composite trapezoidal rule is that the curve $y=f(x)$ is replaced by n straight lines joining the points (x_0,y_0) and (x_1,y_1); (x_1,y_1) and (x_2,y_2) .... (x_{n-1},y_{n-1}) and (x_n,y_n). Then the area bounded by the curve $y=f(x)$, the lines $x=x_0$, $x=x_n$ and the x axis is then approximately equal to the sum of the area of n trapeziums.





Simpson's $\frac{1}{3}$rd rule

Here the interval $[a,b]$ is divided into two equal sub-intervals by the points x_0, x_1, x_2 where $h = \frac{b-a}{2}$, $x_1 = x_0 + h$ and $x_2 = x_1 + h$.

The rule is obtained by putting $n=2$ in . In this case, the third and higher order differences do not exist. The eqn. is simplified as

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\int_{x_0}^{x_n} f(x)\,dx \simeq 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Composite Simpson's $\frac{1}{3}$rd rule

Let the interval $[a,b]$ be divided into n, an even no. equal subintervals by the points $x_0, x_1, \dots, x_n$, where $x_i = x_0 + ih$, $i = 1, 2 \dots n$. Then

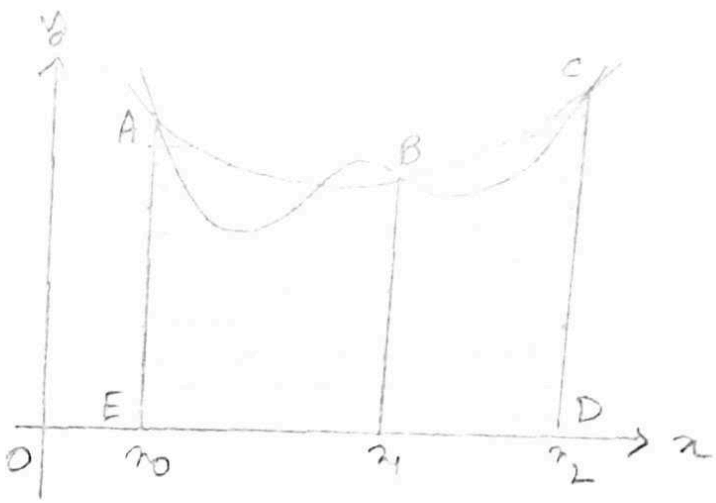
$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] + \frac{h}{3}\left[y_2 + 4y_3 + y_4\right] + \dots + \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

$$= \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2}) + y_n\right]$$

Geometrical interpretation of Simpson's $\frac{1}{3}$rd rule

In Simpson's $\frac{1}{3}$rd rule, the curve $y=f(x)$ is replaced by the 2nd degree parabola passing through the points $A(x_0, y_0)$, $B(x_1, y_1)$ and $C(x_2, y_2)$. Therefore the area bounded by the curve $y=f(x)$, the ordinates $x=x_0$, $x=x_2$ and the x axis is approximated to the area bounded by the parabola ABC, the straight lines $x=x_0$, $x=x_2$ and x axis i.e. the area of the shaded region $ABCDEA$.



Ex Evaluate $\int_0^1 e^{-x^2}\,dx$ by dividing the range into 4 equal parts using trapezoidal rule.

Sol^n: Let $y = e^{-x^2}$. $h = 0.25$

x	0	0.25	0.5	0.75	1.0
y	1	0.9394	0.7788	0.5698	0.3679

$$\therefore \int_0^1 e^{-x^2}\,dx = \frac{h}{2}\left[(y_0+y_4) + 2(y_1+y_2+y_3)\right]$$

$$= \frac{0.25}{2}\left[1.3679 + 2(2.288)\right] = 0.7430$$

Ex Find the value of $\log 2^{\frac{1}{3}}$ from $\int_0^1 \frac{x^2\,dx}{1+x^3}$ using Simpson's $\frac{1}{3}$rd rule with $h = 0.25$.

Sol^n: $y = \frac{x^2}{1+x^3}$

x	0	0.25	0.5	0.75	1
y	0	0.0615	0.2222	0.3956	0.5

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{h}{3}\left[(y_0+y_4) + 2y_2 + 4(y_1+y_3)\right]$$

$$= \frac{0.25}{3}\left[0.5 + 2\times 0.2222 + 4(0.0615+0.3956)\right]$$

$$= 0.2311$$

By actual integration,

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^3}\,dx$$

$$= \frac{1}{3}\left[\log(1+x^3)\right]_0^1$$

$$= \frac{1}{3}\log 2 = \log 2^{\frac{1}{3}}$$

$$\therefore \log 2^{\frac{1}{3}} = 0.2311$$

Ex Evaluate $\int_0^{10} \frac{dx}{1+x^2}$ by using

(i) Trapezoidal rule

(ii) Simpson's $\frac{1}{3}$rd rule (10 intervals)

Sol^n:

x	0	1	2	3	4	5	6	7	8	9	10
y	1	0.5	0.2	0.1	0.0588	0.0385	.0270	.02	.0154	.0122	.0099

(i) Trapezoidal rule

$$\int_0^{10} \frac{dx}{1+x^2} = \frac{1}{2}\left[(y_0+y_{10}) + 2(y_1+y_2 \cdots + y_9)\right]$$

$$= \frac{1}{2}\left[(1+0.0099 + 2(0.5 + \cdots + 0.0122)\right] = 1.4769$$

(ii) Simpson's $\frac{1}{3}$rd rule

$$\int_0^{10} \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_{10}) + 2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= \frac{1}{3}\left[(1+.0099 + 2(0.2+0.0588+0.027+.0154) + 4(.5+ \cdots + .0122)\right]$$

$$= \frac{1}{3}(4.2951)$$

$$= 1.4317$$