ADVANCED CALCULUS
MA11003

SECTION 11, 12, & 15CD

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Beta & Gamma Functions

- Beta & Gamma Functions
- Convergence

Recall (Previous Lectures) $0 \le f(x)\ \&\ 0 < g(x), \quad \forall a < x \qquad \lim\limits_{\substack{x \to a^+ \\ (x\to\infty)}} \frac{f(x)}{g(x)} = \mathrm{k}$

if $k \ne 0$ then $\int_a^{b(\infty)} f(x)\,dx$ and $\int_a^{b(\infty)} g(x)\,dx$ behave the same

if $k = 0$ & $\int_a^{b(\infty)} g(x)\,dx$ conveges $\Longrightarrow \int_a^{b(\infty)} f(x)\,dx$ conveges

if $k = \infty$ & $\int_a^{b(\infty)} g(x)\,dx$ diverges $\Longrightarrow \int_a^{b(\infty)} f(x)\,dx$ diverges

Recall (Previous Lectures)

Test Integrals

$$\int_a^b \frac{1}{(x-a)^p}\,dx \text{ converges for } p < 1 \quad \& \quad \text{diverges if } p \geq 1$$

$$\int_a^\infty \frac{1}{x^p}\,dx \text{ converges for } p > 1 \quad \& \quad \text{diverges if } p \leq 1$$

Beta & Gamma Functions

Beta function:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\,, m > 0, n > 0$$

Gamma function:

$$\Gamma(n) = \int_0^\infty e^{-x}\ x^{n-1} dx\,, n > 0$$

Convergence of Beta function: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\,, m > 0, n > 0$

Case 1: $m, n \geq 1$ The integral is proper. Hence it is convergent.

Case 2: $m, n < 1$

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \underbrace{\int_0^c x^{m-1}(1-x)^{n-1}\,dx}_{I_1} + \underbrace{\int_c^1 x^{m-1}(1-x)^{n-1}\,dx}_{I_2}$$

Consider $I_1 = \int_0^c x^{m-1}(1-x)^{n-1}\,dx$

$f(x) = x^{m-1}(1-x)^{n-1}$

$g(x) = \dfrac{1}{x^{1-m}}$

$$\lim_{x\to 0^+} x^{1-m} \times x^{m-1}(1-x)^{n-1} = 1$$

$$\int_0^c \frac{1}{x^{1-m}}\,dx \begin{cases} \text{converges for } 1-m<1 & \Rightarrow m>0 \\ \text{diverges for } 1-m\geq 1 & \Rightarrow m\leq 0 \end{cases}$$

If $0<m<1$, the integral converges

If $m\leq 0$, the integral diverges

Consider $$I_2 = \int_c^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$f(x) = x^{m-1}(1-x)^{n-1}$$

$$g(x) = \frac{1}{(1-x)^{1-n}}$$

$$\lim_{x\to 1^-} (1-x)^{1-n} \times x^{m-1}(1-x)^{n-1} = 1$$

$$\int_c^1 \frac{1}{(1-x)^{1-n}}\,dx \begin{cases} \text{converges for } 1-n<1 & \Rightarrow n>0 \\ \text{diverges for } 1-n\geq 1 & \Rightarrow n\leq 0 \end{cases}$$

If $0<n<1$, the integral converges

If $n\leq 0$, the integral diverges

Beta function:

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

converges if $m\ \&\ n > 0$

diverges if $m\ \&\ n \leq 0$

Convergence of Gamma function: $\Gamma(n) = \int_0^\infty e^{-x}\ x^{n-1} dx\,, \qquad n > 0$

Case 1: $n \geq 1$

The integrand is bounded in $0 < x \leq a$, where a is arbitrary

We check the convergence of $\int_a^\infty e^{-x}\ x^{n-1}\, dx$

Consider $f(x) = e^{-x}\ x^{n-1} \qquad g(x) = \frac{1}{x^2}$ or $\frac{1}{x^p}, \qquad p > 1$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{n+1}}{e^x} = 0$$ $\Longrightarrow$ Note that $\int_a^\infty e^{-x}\ x^{n-1}\, dx$ converges for any value of n

$\Longrightarrow \Gamma(n)$ converges for $n \geq 1$

Convergence of Gamma function: $\Gamma(n) = \int_0^\infty e^{-x}\ x^{n-1} dx\,, \qquad n > 0$

Case 2: $0 < n < 1$

$$\int_0^\infty e^{-x} x^{n-1} dx = \int_0^a e^{-x} x^{n-1} dx + \underbrace{\int_a^\infty e^{-x} x^{n-1} dx}_{\text{converges}}$$

$f(x) = e^{-x} x^{n-1} \quad g(x) = x^{n-1}$

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} e^{-x} = 1 \neq 0$$

$\int_0^a \frac{1}{x^{1-n}} dx$ conveges for $0 < n < 1 \Longrightarrow \Gamma(n)$ converges for $0 < n < 1$

Convergence of Gamma function: $\Gamma(n)=\int_0^\infty e^{-x}\ x^{n-1}dx\,, \qquad n>0$

Case 3: $n \le 0$

$$\int_0^\infty e^{-x}x^{n-1}dx = \int_0^a e^{-x}x^{n-1}dx + \underbrace{\int_a^\infty e^{-x}x^{n-1}dx}_{\text{converges}}$$

$f(x)=e^{-x}x^{n-1} \quad g(x)=x^{n-1}$

$$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+} e^{-x} = 1 \neq 0$$

$\int_0^a \frac{1}{x^{1-n}}dx$ diverges for $n\le 0$ $\quad\Longrightarrow \Gamma(n)$ diverges for $n\le 0$

Conclusion:

Beta function:

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

converges if $m \,\&\, n > 0$

diverges if $m \,\&\, n \leq 0$

Gamma function:

$$\int_0^\infty e^{-x}\ x^{n-1}\,dx$$

converges if $n > 0$

diverges if $n \leq 0$

Beta & Gamma Functions

- Beta & Gamma Functions
- Properties & Evaluation

Symmetry Property of Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

- $B(m,n) = B(n,m)$

Subst. $1 - x = y$

Evaluation of Beta function

Suppose n is a positive integer. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$B(m,n) = \left[\frac{x^m}{m}(1-x)^{n-1}\right]_0^1 + \int_0^1 \frac{x^m}{m}(n-1)(1-x)^{n-2}dx$$

Integrating by parts keeping

$$= \frac{(n-1)}{m}\int_0^1 x^m(1-x)^{n-2}dx$$

$$\vdots$$

$$= \frac{(n-1)(n-2)\cdots(n-(n-1))}{m(m+1)\cdots(m+n-2)}\int_0^1 x^{m+n-2}dx$$

$$= \frac{(n-1)!}{m(m+1)\cdots(m+n-1)}$$

Evaluation of Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Suppose n is a positive integer.

$$B(m,n) = \frac{(n-1)!}{m(m+1)\cdots(m+n-1)}$$

Suppose m is a positive integer.

$$B(m,n) = \frac{(m-1)!}{n(n+1)\cdots(m+n-1)}$$

Suppose both m and n are integer

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

Evaluation of Gamma function

$$\Gamma(n+1) = \int_0^\infty e^{-x}\ x^n\ dx$$

Integrating by parts gives

$$\Gamma(n+1) = -x^n e^{-x}\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-x} dx \quad \Longrightarrow \ \Gamma(n+1) = n\,\Gamma n.$$

Note that if n is a positive integer

$$\Gamma(n) = (n-1)(n-2)\cdots(2)(1)\Gamma(1) \qquad \Gamma(1) = \int_0^\infty e^{-x}\, dx = 1$$

$$\Longrightarrow \ \Gamma(n) = (n-1)!$$

- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma(n) = \int_0^\infty e^{-x}\ x^{n-1}\, dx$$

Subst. $x = y^2$

$$\Gamma(n) = 2\int_0^\infty y^{2n-1}\, e^{-y^2} dy$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2} dy\ = 2\ \frac{\sqrt{\pi}}{2}$$

Different forms of Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Substitute $x = \dfrac{1}{1+y}$

$$\boxed{B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy}$$

Different forms of Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Substitute $x = \sin^2\theta$

$$\boxed{B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\;d\theta = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\,\cos^{2m-1}\theta\,d\theta}$$

Different forms of Gamma function $\Gamma(n) = \int_0^\infty e^{-x}\ x^{n-1}\, dx$

Substitute $x = \lambda y$

$$\Gamma n = \int_0^\infty e^{-\lambda y} \lambda^{n-1} y^{n-1} \lambda\, dy$$

$$\boxed{\int_0^\infty e^{-\lambda y} y^{n-1} dy = \frac{\Gamma n}{\lambda^n}}$$

Different forms of Gamma function $\Gamma(n) = \int_0^\infty e^{-x}\ x^{n-1}\, dx$

Substitute $e^{-x} = t$

$$\Gamma(n) = -\int_1^0 \left[\ln\left(\frac{1}{t}\right)\right]^{n-1} dt$$

$$\boxed{\int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{n-1} dt = \Gamma(n)}$$

Relation between Gamma and Beta functions:

We know that m and n being integers

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

$$\Rightarrow B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

(This result also hold for $m, n > 0$)

Example: $\int_0^1 x^4(1-\sqrt{x})^5\,dx$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $\sqrt{x} = t$ or $x = t^2$

$$\int_0^1 t^8(1-t)^5\,2t\,dt = 2\int_0^1 t^9(1-t)^5\,dt$$

$$= 2\,B(10,6) = 2\,\frac{\Gamma 10\,\Gamma 6}{\Gamma 16} = 2\,\frac{9!\,5!}{15!} = \frac{1}{15015}$$

Conclusion:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x}\ x^n\,dx$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$B(m,n) = \frac{\Gamma \mathrm{m}\,\Gamma n}{\Gamma(m+n)}$$

Thank You