Exploring the Blossom Algorithm

MATH 6404: Graph Theory, Spring 2025

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May 7, 2025

Section Outline

What is the Blossom Algorithm?

The Blossom Algorithm: A Quick Intro

In class we discussed the augmenting path algorithm for finding a maximum matching on an X, Y-bigraph.
While even cycles occur and are handled easily by the augmenting path algorithm, odd cycles present a problem and require additional steps.
The blossom algorithm tackles these odd cycles and extends this algorithm to general graphs using a structure called a blossom.

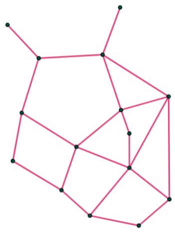


Figure: A general graph G

Theorem (Berge's Theorem [Wes21])

A matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path.

The blossom algorithm finds blossoms, contracts them down to a single point, continues iteratively, and then finds augmenting paths in the graph G' formed by contracting the blossom's.

What is a Blossom?

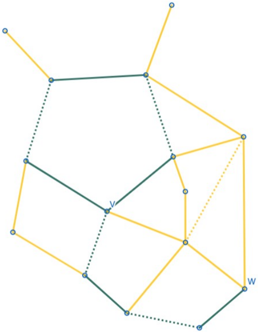


Figure: A graph G with a flower

Definition

A vertex is called exposed if it meets no edge of M. [Edm65b]

Definition

A blossom $B = B(M)$ in (G, M) is an odd circuit in G for which $M \cap B$ is a maximum matching in B with an exposed vertex v. [Edm65b]

Definition

A stem in (G, M) is either an exposed vertex or an alternating path with an exposed vertex at one end and a matching edge at the other end. [Edm65b]

A blossom and stem together are called a flower.

Contracting a Blossom

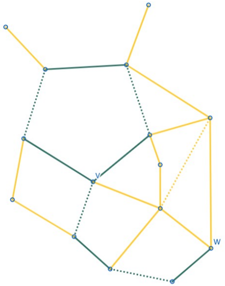


Figure: A graph G with a flower

$\rightarrow$

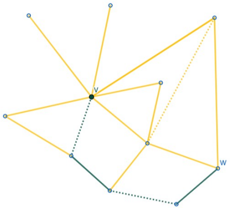


Figure: The graph G' with a contracted blossom

As we saw briefly in class, we contract a blossom by treating the entire odd cycle as a single point (illustrated above) with every edge incident to any vertex in the cycle being incident to the vertex v' in the contracted graph G'.

The Role of an Augmenting Path

Definition (Augmenting Path as Defined by Edmunds in [Edm65b])

An augmenting path is an alternating path A in (G, M) joining two exposed vertices containing one more edge of $\overline{M}$ than of M. $M + A$ is a matching of G larger than M by one.

Once an augmenting path is found, the blossom algorithm performs a *matching augmentation* along the augmenting path P, replacing M with a new matching, as we saw in class for the an X, Y-bipartite graph.

The Role of an Augmenting Path (cont.)

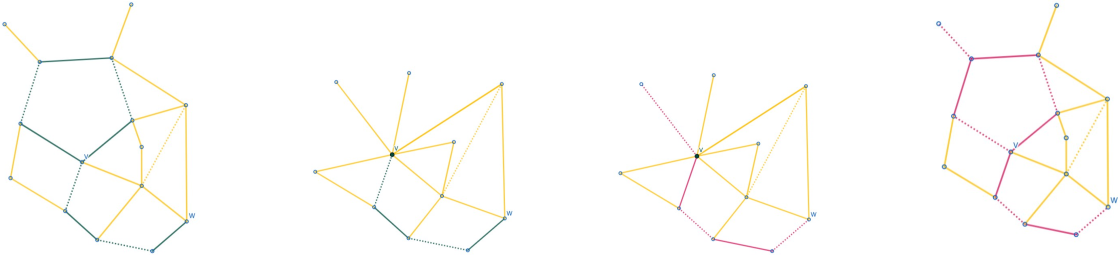
Results of an Iteration [Alg]

At each iteration, the algorithm does one of 3 things:

1. finds an augmenting path
2. finds a blossom and recurses onto the corresponding contracted graph
3. concludes there are no augmenting paths

This is clearly more possible outcomes than given by Berge's theorem, however outcome 2 uses the blossom contraction then begins again, until all blossoms have been contracted, all augmenting paths of the original matching M have been found, and we have concluded that as there are no more augmenting paths, Berge's theorem tells us that we must have a maximum matching.
The efficiency of the algorithm comes from not having to search, and back up, search, and back up over and over due to the odd cycles.

Visual Example



(a) The graph G

(b) The graph G' with a contracted blossom

(c) The graph G' with a contracted blossom and $M + A$

(d) The graph G with it's new matching and the blossom re-expanded

The Blossom Algorithm in Context

Running Time

The blossom algorithm was the first proof of a polynomial time algorithm for finding a maximum-size matching on general graphs [Alg], running in

$$\mathcal{O}(|E||V|^2)$$

Faster running times have been acheived with more complex algorithms such as that of Micali and Vazirani's 1980 algorithm running in $\mathcal{O}(|E|\sqrt{|V|})$

Trivia

Edmonds includes a 2 page long digression in the 19 page paper "Paths, Trees, and Flowers" discussing the use of the of the words "efficient algorithm". He seems to have had opinions on this!

Weighted Matchings

In the same year as ''Paths, Trees, and Flowers" was published, Jack Edmonds published another paper titled ''Maximum Matching and a Polyhedron With 0,1-Vertices" [Edm65a].
This work gave a linear programming polyhedral description of the matching polytope:

Definition (Matching Polytope [MC14])

The matching polytope of G is the convex hull of all characteristic vectors of matchings of G.

Description of the Matching Polytope [Edm65a]

The matching polytope is the convex hull of the vectors associated with the matchings in a graph.

This yielded an algorithm for minimum-*weight* matching and had significance in terms of it's proof of integrality which ''does not simply follow just from total unimodularity, and it's description was a breakthrough in polyhedral combinatorics". [Alg], quoting from a 2003 book by Alexander Schrijver titled ''Combinatorial Optimization: Polyhedra and Efficiencey".

Bibliography

[Edm65a] Jack Edmonds. “Maximum matching and a polyhedron with 0, 1-vertices”. In: *Journal of research of the National Bureau of Standards B* 69.125-130 (1965), pp. 55–56.

[Edm65b] Jack Edmonds. “Paths, Trees, and Flowers”. In: *Canadian Journal of Mathematics* 17 (1965), pp. 449–467.

[MC14] Giacomo Zambell Michele Conforti Gérard Cornuejols. *Integer Programmin*. Graduate Texts in Mathematics'. Springer International Publishing, 2014. ISBN: 9783319110073'.

[Wes21] Douglas B. West. *Combinatorial Mathematics*. Cambridge University Press, 2021. ISBN: 9781107058583.

[Alg] *Blossom Algorithm*. URL: https://en.wikipedia.org/wiki/Blossom_algorithm. (accessed: 04.30.2025).