# Math 152 Final Exam Practice Problems

Note: The following problems cover a broad selection of topics from Calculus 2. However, this problem set is not intended to be "just like the real exam with different numbers." This review problem set is longer than an actual final exam.

**1.** Evaluate $\displaystyle\int \tan^5(2x)\sec(2x)\,dx$.

**Solution:**

$$\begin{aligned}\int \tan^5(2x)\sec(2x)\,dx &= \int \tan^4(2x)\sec(2x)\tan(2x)\,dx\\ &= \int \left(\tan^2(2x)\right)^2\sec(2x)\tan(2x)\,dx\\ &= \int \left(\sec^2(2x)-1\right)^2\sec(2x)\tan(2x)\,dx\end{aligned}$$

Let $u=\sec(2x)$ and $du = 2\sec(2x)\tan(2x)\,dx$, so that $\sec(2x)\tan(2x) = \frac{1}{2}du$. Then the integral on the preceding line becomes

$$\begin{aligned}\int (u^2-1)^2(\tfrac{1}{2}du) &= \frac{1}{2}\int u^4 - 2u^2 + 1\,du\\ &= \frac{1}{2}\left(\frac{u^5}{5} - \frac{2u^3}{3} + u\right)\\ &= \frac{u^5}{10} - \frac{u^3}{3} + \frac{u}{2}\\ &= \boxed{\frac{\sec^5(2x)}{10} - \frac{\sec^3(2x)}{3} + \frac{\sec(2x)}{2} + C}\end{aligned}$$

**2.** Let $\mathcal{R}$ be the region in the $xy$-plane bounded between $y = 4 - x^2$ and $y = 1 + x^2$. Set up an integral that is equal to the volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = -5$ using (a) the cylindrical shell method, and (b) the washer method. Do not evaluate the integral(s).

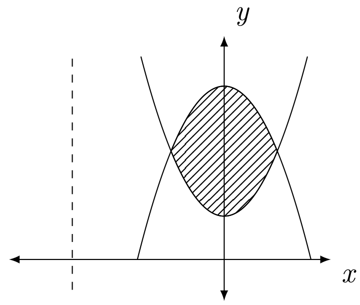
**Solution: Part (a):** First, we solve for the points of intersection of the two curves by equating the two functions:

$$\begin{aligned}1 + x^2 &= 4 - x^2\\ 2x^2 &= 3\\ x^2 &= \tfrac{3}{2}\\ x &= \pm\sqrt{\tfrac{3}{2}}\end{aligned}$$

The formula for a volume of revolution about a vertical axis using shells is

$$2\pi \int (\text{radius of shell})(\text{height of shell})\, dx$$

To find the height of the shell, we need to substract the bottom curve from the top curve. If we graph two curves, it should be apparent which one is on top. The following drawing shows both curves, together with the axis of revolution $x = -5$:




Alternatively, we can identify which curve is on top by substituting a value of $x$ that lies between the points of intersection. Zero is a convenient value between $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$. Since $4 - 0^2 > 1 + 0^2$, the curve $y = 4 - x^2$ is the top curve in the interval between the points of intersection. Therefore, the height of each shell is (top curve) − (bottom curve) = $(4 - x^2) - (1 + x^2) = 3 - 2x^2$. Since the axis is $x = -5$ and the axis lies to the left of region $\mathcal{R}$, the radius of each shell is $x - (-5) = x + 5$. Therefore, the integral that gives the volume of revolution is:

$$\boxed{2\pi \int_{-\sqrt{3/2}}^{\sqrt{3/2}} (x+5)(3-2x^2)\, dx}$$

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**Part (b):** Recall that the shell and washer solutions to the same problem always use opposite variables. When a solid is formed by revolving a region in the $xy$-plane about a vertical axis, the shell integral has the variable $x$, so the washer integral has the variable $y$. Therefore, we first need to invert the functions. $y = 4 - x^2$ corresponds to $x = \pm\sqrt{4-y}$ (with the positive and negative square roots corresponding to the right and left sides of the upper parabola), and the equation $y = x^2 + 1$ corresponds to $x = \pm\sqrt{y-1}$ (again, corresponding to the right and left sides of the lower parabola). The curves intersect at $x = \pm\sqrt{\frac{3}{2}}$, and substituting these values into either of the curves shows that they intersect at $y = \frac{5}{2}$. (E.g., $y = 4 - \left(\pm\sqrt{\frac{3}{2}}\right)^2 = 4 - \frac{3}{2} = \frac{5}{2}$.) Note that the right and left curves are defined by $x = \pm\sqrt{4-y}$ *above* the point of intersection and by $x = \pm\sqrt{y-1}$ *below* the point of intersection, so we have to break the volume calculation into two integrals. Since the axis of revolution lies to the left of the region, the outer curves are the branches of the parabolas on the right, and the inner curves are the ones on the left.

Putting this all together, the washer solution is:

$$\boxed{\pi \int_1^{5/2} \left[(\sqrt{y-1}+5)^2 - (5-\sqrt{y-1})^2\right] dy + \pi \int_{5/2}^{4} \left[(\sqrt{4-y}+5)^2 - (5-\sqrt{4-y})^2\right] dy}$$

It is not necessary to simplify the answer (and it is actually preferable not to, since it is easier for us to see that you constructed it correctly if you leave all of its different components visible). However, the above answer does simplify to

$$\boxed{20\pi \int_1^{5/2} \sqrt{y-1}\,dy + 20\pi \int_{5/2}^{4} \sqrt{4-y}\,dy}$$

(which also would have been accepted, *if* you include the work showing all the algegra needed to get from the previous solution to this simplified one).

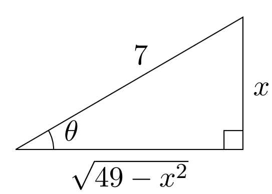
**3.** Evaluate each of the following integrals:

**(a)** $\displaystyle\int \frac{dx}{(49-x^2)^{3/2}}$

**Solution:** Since there is no apparent way to use $u$-substitution and the integrand contains an expression of the form $\sqrt{a^2-x^2}$, use the substitution $x = 7\sin\theta$ and $dx = 7\cos\theta\,d\theta$. Then:

$$\begin{aligned}
\int \frac{dx}{(49-x^2)^{3/2}} &= \int \frac{7\cos\theta\,d\theta}{\left(49-49\sin^2\theta\right)^{3/2}} \\
&= \int \frac{7\cos\theta\,d\theta}{\left(49-49(1-\cos^2\theta)\right)^{3/2}} \\
&= \int \frac{7\cos\theta\,d\theta}{\left(49\cos^2\theta\right)^{3/2}} = \int \frac{7\cos\theta\,d\theta}{7^3\cos^3\theta} \\
&= \int \frac{d\theta}{49\cos^2\theta} = \int \frac{\sec^2\theta\,d\theta}{49} \\
&= \tfrac{1}{49}\tan\theta + C
\end{aligned}$$

The substitution $x = 7\sin\theta$ implies that $\sin\theta = \frac{x}{7}$, which is true if $\theta$ is part of the right triangle shown. It can be seen from the triangle that $\tan\theta = \dfrac{x}{\sqrt{49-x^2}}$. Therefore, the answer is




$$\boxed{\int \frac{dx}{(49-x^2)^{3/2}} = \frac{x}{49\sqrt{49-x^2}} + C}$$

It should be noted that the answer should be in terms of $x$, and any trigonometric and inverse trignometric functions that can be removed by simplification must be so-removed. So, for example, a final answer of $\frac{1}{49}\tan\theta + C$ is incorrect because it is in $\theta$ rather than $x$. Moreover, the answer $\frac{1}{49}\tan\left(\sin^{-1}\left(\frac{x}{7}\right)\right) + C$ would also be incorrect, since it is possible to simplify the composition of tan and $\sin^{-1}$ into an expression that has no trig or inverse trig functions.

**(b)** $\displaystyle\int \frac{dx}{(x^2+4x+13)^2}$

**Solution:** Observe that

$$x^2+4x+13 = (x+2)^2 - 4 + 13 = (x+2)^2 + 9$$

so the integral in the problem can be rewritten as $\displaystyle\int \frac{dx}{\left((x+2)^2+9\right)^2}$. We then let $x+2 = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, so

$$\begin{aligned}
\int \frac{dx}{\left((x+2)^2+9\right)^2} &= \int \frac{3\sec^2\theta\,d\theta}{\left((3\tan\theta)^2+9\right)^2} \\
&= \int \frac{3\sec^2\theta\,d\theta}{\left(9\tan^2\theta+9\right)^2} = \int \frac{3\sec^2\theta\,d\theta}{\left(9(\tan^2\theta+1)\right)^2} \\
&= \int \frac{3\sec^2\theta\,d\theta}{(9\sec^2\theta)^2} = \int \frac{3\sec^2\theta\,d\theta}{81\sec^4\theta}
\end{aligned}$$

$$= \frac{1}{27}\int \cos^2\theta\,d\theta$$

Simplify the fraction $\frac{3}{81}$ and the various factors of sec. Then note that cos is the reciprocal of sec.

$$= \frac{1}{27}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

Half-angle formula

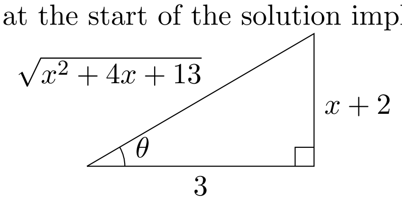
$$= \frac{1}{54}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C$$

Integrate

$$= \frac{1}{54}\left(\theta + \sin\theta\cos\theta\right) + C$$

Double-angle formula $\sin 2\theta = 2\sin\theta\cos\theta$. Why use it here? Because the triangle we're about to draw doesn't have the angle $2\theta$ on it, so we're going to need trig function of the angle $\theta$, not the angle $2\theta$.

The substitution we chose at the start of the solution implies this triangle:




Since $x+2 = 3\tan\theta$, it follows that $\theta = \tan^{-1}\left(\frac{x+2}{3}\right)$. And the triangle above shows that $\sin\theta = \frac{x+2}{\sqrt{x^2+4x+13}}$ and $\cos\theta = \frac{3}{\sqrt{x^2+4x+13}}$. Therefore:

$$\begin{aligned}
\frac{1}{54}\left(\theta + \sin\theta\cos\theta\right) + C &= \frac{1}{54}\left(\tan^{-1}\left(\frac{x+2}{3}\right) + \frac{3(x+2)}{x^2+4x+13}\right) + C \\
&= \boxed{\frac{1}{54}\tan^{-1}\left(\frac{x+2}{3}\right) + \frac{x+2}{18(x^2+4x+13)} + C}
\end{aligned}$$

**4.** Evaluate $\displaystyle\int x^2 e^{-4x}\,dx$

**Solution:** Use integration by parts with

$$\begin{aligned} u &= x^2 & dv &= e^{-4x}\,dx \\ du &= 2x\,dx & v &= -\tfrac{1}{4}e^{-4x} \end{aligned}$$

Then,

$$\begin{aligned} \int x^2 e^{-4x}\,dx &= -\frac{x^2 e^{-4x}}{4} - \int 2x \cdot \left(-\tfrac{1}{4}\right) e^{-4x}\,dx \\ &= -\frac{x^2 e^{-4x}}{4} + \frac{1}{2}\int x e^{-4x}\,dx \qquad (1) \end{aligned}$$

We use integration by parts again on the last integral, with

$$\begin{aligned} u &= x & dv &= e^{-4x}\,dx \\ du &= dx & v &= -\tfrac{1}{4}e^{-4x} \end{aligned}$$

The expression on line (1) therefore equals

$$\begin{aligned} \cdots &= -\frac{x^2 e^{-4x}}{4} + \frac{1}{2}\left(-\frac{x e^{-4x}}{4} - \int \left(-\tfrac{1}{4}\right) e^{-4x}\,dx\right) \\ &= -\frac{x^2 e^{-4x}}{4} - \frac{x e^{-4x}}{8} + \frac{1}{8}\int e^{-4x}\,dx \\ &= -\frac{x^2 e^{-4x}}{4} - \frac{x e^{-4x}}{8} - \frac{1}{32}e^{-4x} + C \\ &= \boxed{-e^{-4x}\left(\frac{x^2}{4} + \frac{x}{8} + \frac{1}{32}\right) + C} \end{aligned}$$

**5.** Evaluate $\int e^{3x}\sin(2x)\,dx$.

**Solution:** Using integration by parts with

$$\begin{aligned} u &= e^{3x} & dv &= \sin(2x)\,dx \\ du &= 3e^{3x}\,dx & v &= -\tfrac{1}{2}\cos(2x) \end{aligned}$$

We have

$$\int e^{3x}\sin(2x)\,dx = -\tfrac{1}{2}e^{3x}\cos(2x) + \tfrac{3}{2}\int e^{3x}\cos(2x)\,dx$$

Using integration by parts again with

$$\begin{aligned} u &= e^{3x} & dv &= \cos(2x)\,dx \\ du &= 3e^{3x}\,dx & v &= \tfrac{1}{2}\sin(2x) \end{aligned}$$

we now have

$$\begin{aligned}
\int e^{3x}\sin(2x)\,dx &= -\tfrac{1}{2}e^{3x}\cos(2x) + \tfrac{3}{2}\left(\tfrac{1}{2}e^{3x}\sin(2x) - \tfrac{3}{2}\int e^{3x}\sin(2x)\right)\\
\int e^{3x}\sin(2x)\,dx &= -\tfrac{1}{2}e^{3x}\cos(2x) + \tfrac{3}{4}e^{3x}\sin(2x) - \tfrac{9}{4}\int e^{3x}\sin(2x)\\
\left(\tfrac{9}{4}+1\right)\int e^{3x}\sin(2x)\,dx &= -\tfrac{1}{2}e^{3x}\cos(2x) + \tfrac{3}{4}e^{3x}\sin(2x)\\
\tfrac{13}{4}\int e^{3x}\sin(2x)\,dx &= -\tfrac{1}{2}e^{3x}\cos(2x) + \tfrac{3}{4}e^{3x}\sin(2x)\\
\int e^{3x}\sin(2x)\,dx &= \tfrac{4}{13}\left(-\tfrac{1}{2}e^{3x}\cos(2x) + \tfrac{3}{4}e^{3x}\sin(2x)\right)\\
\int e^{3x}\sin(2x)\,dx &= \boxed{-\tfrac{2}{13}e^{3x}\cos(2x) + \tfrac{3}{13}e^{3x}\sin(2x) + C}
\end{aligned}$$

**6.** Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt[3]{5n^{12}+1}}{\sqrt{3n^{12}-1}}$ converges or diverges.

**Solution:** Use the Limit Comparison Test (LCT) to compare the given series with an appropriately chosen series $\sum b_n$. In order to choose $b_n$, since the general term of the given series has only roots and powers of $n$ (without any exponential, logarithmic, or trig functions, and without any factorials), a good way to choose $b_n$ is to look only at the radicals and the highest powers of $n$ in the numerator and denominator. Therefore, we choose $b_n = \frac{\sqrt[3]{n^{12}}}{\sqrt{n^{12}}} = \frac{n^{12/3}}{n^{12/2}} = \frac{n^4}{n^6} = \frac{1}{n^2}$.

We now calculate the LCT limit:

$$\begin{aligned}
\lim_{n\to\infty}\frac{\left[\frac{\sqrt[3]{5n^{12}+1}}{\sqrt{3n^{12}-1}}\right]}{1/n^2} &= \lim_{n\to\infty}\frac{\sqrt[3]{5n^{12}+1}}{\sqrt{3n^{12}-1}}\cdot\frac{n^2}{1}\\
&= \lim_{n\to\infty}\frac{\sqrt[3]{5n^{12}+1}}{\sqrt{3n^{12}-1}}\cdot\frac{\sqrt[3]{n^6}}{1}\\
&= \lim_{n\to\infty}\frac{\sqrt[3]{5n^{18}+n^6}}{\sqrt{3n^{12}-1}}\cdot\frac{1/n^6}{1/n^6}\\
&= \lim_{n\to\infty}\frac{\sqrt[3]{5n^{18}+n^6}}{\sqrt{3n^{12}-1}}\cdot\frac{\sqrt[3]{1/n^{18}}}{\sqrt{1/n^{12}}} \qquad \text{Note that the fraction on the right is equal to 1}\\
&= \lim_{n\to\infty}\frac{\sqrt[3]{5+1/n^{12}}}{\sqrt{3-1/n^{12}}}\\
&= \frac{\sqrt[3]{5+0}}{\sqrt{3-0}} = \frac{\sqrt[3]{5}}{\sqrt{3}}
\end{aligned}$$

Since $0 < \frac{\sqrt[3]{5}}{\sqrt{3}} < \infty$, the given series behaves the same as $\sum\frac{1}{n^2}$, which converges because it is a $p$-series with $p = 2 > 1$. Therefore, the given series converges.

**7.** Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos^4(2^n+n!)}{n^{3/2}+1}$ converges or diverges.

**Solution:** Since cos(anything) is between $-1$ and 1, and $(\cos(\text{anything}))^4$ cannot be negative, we know that $0 \le \cos^4(2^n+n!) \le 1$. Therefore,

$$0 \le \frac{1+\cos^4(2^n+n!)}{n^{3/2}+1} \le \frac{2}{n^{3/2}+1} \le \frac{2}{n^{3/2}}$$

for all $n \ge 1$. $\sum \frac{2}{n^{3/2}}$ converges because it is a $p$-series with $p = 3/2 > 1$. Therefore, by the DCT, $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos^4(2^n+n!)}{n^{3/2}+1}$ $\boxed{\text{converges}}$.

**8.** Determine whether $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$ converges or diverges.

**Solution:** Using the Root Test,

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n+1}{n}\right)^{n^2}\right|} &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2}} \\
&= \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)^{n^2}\right]^{1/n} \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n} \\
&= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} \\
&= e
\end{aligned}$$

The expression inside the absolute value is never negative for $n \ge 1$

You're allowed to use the limits from Theorem 5 in section 10.1. This is limit #5 from that theorem.

Since $e > 1$, the series given in the question $\boxed{\text{diverges}}$.

**9.** Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(3n-2)!}{(2n)!(n-1)!\,8^n}$ converges or diverges.

**Solution:**

Using the Ratio Test, we calculate:

$$\begin{aligned}
&\lim_{n\to\infty}\left|\frac{(3(n+1)-2)!}{(2(n+1))!((n+1)-1)!\cdot 8^{n+1}}\cdot\frac{(2n)!(n-1)!\,8^n}{(3n-2)!}\right| \\
&=\lim_{n\to\infty}\left|\frac{(3n+1)!}{(2n+2)!n!\cdot 8^{n+1}}\cdot\frac{(2n)!(n-1)!\,8^n}{(3n-2)!}\right| \\
&=\lim_{n\to\infty}\left|\frac{(3n+1)!}{(3n-2)!}\cdot\frac{(2n)!}{(2n+2)!}\cdot\frac{(n-1)!}{n!}\cdot\frac{8^n}{8^{n+1}}\right| \\
&=\lim_{n\to\infty}\left|\frac{(3n+1)(3n)(3n-1)(3n-2)!}{(3n-2)!}\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\cdot\frac{(n-1)!}{n(n-1)!}\cdot\frac{1}{8}\right| \\
&=\lim_{n\to\infty}\left|\frac{(3n+1)(3n)(3n-1)}{(2n+2)(2n+1)\cdot n\cdot 8}\right| \\
&=\lim_{n\to\infty}\left|\frac{27n^3-3n}{32n^3+48n^2+16n}\right|=\frac{27}{32}<1
\end{aligned}$$

Therefore, the series converges.

**10.** Evaluate $\displaystyle\int_0^\infty \frac{4e^{3x}\,dx}{e^{6x}+1}$.

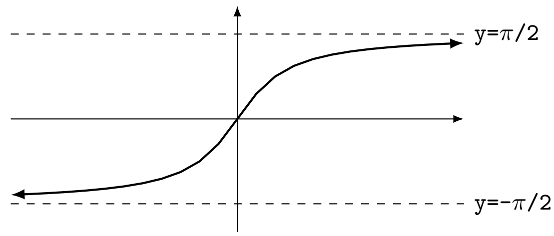
**Solution:**

$$\int_0^\infty \frac{4e^{3x}\,dx}{e^{6x}+1}=\lim_{b\to\infty}\int_0^b \frac{4e^{3x}\,dx}{e^{6x}+1}$$

We need $u$-substitution to evaluate this integral, but note that the common mistake is assuming that the $u$ should equal the denominator. That choice works *sometimes*. This isn't one of those times. Instead, use the substitution $u=e^{3x}$, $du=3e^{3x}\,dx$, so $4e^{3x}\,dx=\frac{4}{3}du$, and continue:

$$\begin{aligned}
\cdots&=\lim_{b\to\infty}\int_1^{e^{3b}}\frac{\frac{4}{3}du}{u^2+1} && e^{6x}=(e^{3x})^2=u^2.\text{ And note that the limits of integration change from } x \text{ to } u. \\
&=\lim_{b\to\infty}\frac{4}{3}\int_1^{e^{3b}}\frac{du}{u^2+1} && \text{Simplify} \\
&=\frac{4}{3}\lim_{b\to\infty}\arctan u\Big|_1^{e^{3b}} && \text{Integrate} \\
&=\frac{4}{3}\lim_{b\to\infty}\left(\arctan e^{3b}-\arctan 1\right) && (1)
\end{aligned}$$

$\arctan 1=\pi/4$. As to $\arctan e^{3b}$, note that the graph of $y=\arctan x$ is:




Since $e^{3b} \to \infty$ as $b \to \infty$, from the graph above we can conclude that $\lim_{b\to\infty} \arctan e^{3b} = \pi/2$. Therefore, the expression on line (1) above evaluates to:

$$\frac{4}{3}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{4}{3} \cdot \frac{\pi}{4} = \boxed{\frac{\pi}{3}}$$

**11.** For each series: if the series converges, find its sum; if it diverges, show that it diverges.

**(a)** $\displaystyle\sum_{n=-1}^{\infty} \frac{1+3^{-n}}{2^n}$

**Solution:**

$$\begin{aligned}
\sum_{n=-1}^{\infty} \frac{1+3^{-n}}{2^n} &= \sum_{n=-1}^{\infty} \left(\frac{1}{2^n} + \frac{3^{-n}}{2^n}\right) \\
&= \sum_{n=-1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{2^n \cdot 3^n}\right) \\
&= \sum_{n=-1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{6^n}\right) \\
&= \sum_{n=-1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=-1}^{\infty} \left(\frac{1}{6}\right)^n
\end{aligned}$$

Since these are two geometric series with common ratios of $1/2$ and $1/6$ (and both ratios have an absolute value less than 1), we can use the formula $\displaystyle\sum_{n=M}^{\infty} r^n = \frac{r^M}{1-r}$ for $|r| < 1$. For the series on the left, $r = 1/2, M = -1$, and for the series on the right, $r = 1/6, M = -1$:

$$\begin{aligned}
\sum_{n=-1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=-1}^{\infty} \left(\frac{1}{6}\right)^n &= \frac{(1/2)^{-1}}{1-(1/2)} + \frac{(1/6)^{-1}}{1-(1/6)} \\
&= \frac{2}{1/2} + \frac{6}{5/6} \\
&= 4 + \frac{36}{5} = \boxed{\frac{56}{5}}
\end{aligned}$$

**(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1} - 7^{n+1}}{3^{2n}}$

**Solution:**

$$\sum_{n=1}^{\infty} \frac{3^{n-1} - 7^{n+1}}{3^{2n}} = \sum_{n=1}^{\infty} \frac{3^n \cdot \frac{1}{3} - 7^n \cdot 7}{9^n}$$

$3^{-1} = \frac{1}{3}$, $7^1 = 7$, and $3^{2n} = (3^2)^n = 9^n$

$$= \frac{1}{3}\sum_{n=1}^{\infty} \frac{3^n}{9^n} - 7\sum_{n=1}^{\infty} \frac{7^n}{9^n}$$

$$= \frac{1}{3}\sum_{n=1}^{\infty} \left(\frac{3}{9}\right)^n - 7\sum_{n=1}^{\infty} \left(\frac{7}{9}\right)^n$$

$$= \frac{1}{3}\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n - 7\sum_{n=1}^{\infty} \left(\frac{7}{9}\right)^n$$

Technically not necessary to reduce $\frac{3}{9}$ as $\frac{1}{3}$, but makes the next calculation a bit easier

These are two geometric series, with $r = \frac{1}{3}$ and $r = \frac{7}{9}$, both $|r| < 1$. Therefore, use the formula $\sum_{n=M}^{\infty} r^n = \frac{r^M}{1-r}$ on both series.

$$= \frac{1}{3} \cdot \frac{(1/3)^1}{1 - \frac{1}{3}} - 7 \cdot \frac{(7/9)^1}{1 - \frac{7}{9}}$$

$$= \frac{1}{9} \cdot \frac{1}{2/3} - \frac{49}{9} \cdot \frac{1}{2/9}$$

$$= \frac{1}{9} \cdot \frac{3}{2} - \frac{49}{9} \cdot \frac{9}{2}$$

$$= \frac{1}{6} - \frac{49}{2} = \frac{1}{6} - \frac{147}{6} = -\frac{146}{6} = \boxed{-\frac{73}{3}}$$

**12.** Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n/5} + 1}$. Use the Alternating Series Estimation Theorem to find the minimum value of $N$ such that $s_N$ (the N-th partial sum of the $S$) approximates $S$ with an error whose absolute value is less than 1/9.

**Solution:** According to the Alternating Series Estimation Theorem, if an alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the hypotheses of the Alternating Series Test, then the N-th partial sum approximates the sum of the series with an error whose absolute value less than $u_{n+1}$. Therefore, to find the minimum number of terms $N$, we set $u_{N+1} \leq 1/9$ and solve for $N$.

Observe that, based on the definition of the series, $u_n = \dfrac{1}{2^{n/5}+1}$. Then:

$$\begin{aligned} u_{N+1} &\leq \frac{1}{9} \\ \frac{1}{2^{(N+1)/5}+1} &\leq \frac{1}{9} \\ 2^{(N+1)/5}+1 &\geq 9 \\ 2^{(N+1)/5} &\geq 8 \\ \log_2 2^{(N+1)/5} &\geq \log_2 8 \\ (N+1)/5 &\geq 3 \\ N+1 &\geq 15 \\ N &\geq 14 \end{aligned}$$

Since the question asks for the minimum value value of $N$, we choose the smallest integer that satisfies the inequality $N \geq 14$, which is $N = 14$.

**13.** Find the points on the parametric curve $x = t^4 - 8t^2$, $y = t^2 - 6t$ where the tangent line to the curve is **(a)** horizontal, **(b)** vertical.

**Solution:** $x'(t) = 4t^3 - 16t$ and $y'(t) = 2t - 6$. Therefore

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t-6}{4t^3-16t} = \frac{t-3}{2t(t^2-4)}$$

To get the last expression, multiply by $\frac{1/2}{1/2}$ and then factor the denominator

**Part(a):** The tangent line is horizontal where $\frac{dy}{dx} = 0$, which occurs when the numerator is zero and the denominator is nonzero. This occurs at $t = 3$, which corresponds to the point $\boxed{(9, -9)}$.

**Part (b):** The tangent line is vertical where $\frac{dy}{dx}$ is infinite, which occurs where the denominator is zero and the numerator is nonzero. This occurs at $t = 0$ and $t = \pm 2$, which corresponds to the points $\boxed{(0,0)}$, $\boxed{(-16,-8)}$, and $\boxed{(-16,16)}$.

**14.** Find the area of the surface generated when the curve $y = 8\sqrt{x}$ is revolved about the $x$-axis for $9 \leq x \leq 20$.

**Solution:**

$$A = \int_9^{20} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Substituting $y = 8\sqrt{x}$ and $\frac{dy}{dx} = 4x^{-1/2}$, we obtain

$$\begin{aligned} A = \int_9^{20} 2\pi(8\sqrt{x})\sqrt{1 + \frac{16}{x}}\, dx &= 16\pi \int_9^{20} \sqrt{x}\frac{\sqrt{x+16}}{\sqrt{x}} dx \\ &= 16\pi \int_9^{20} \sqrt{x+16}\, dx \\ &= 16\pi \left[ \frac{(x+16)^{3/2}}{3/2} \right]_9^{20} \\ &= \frac{32\pi}{3}(216 - 125) \\ &= \frac{2912\pi}{3} \end{aligned}$$

**15.** Consider the parametric curve $x = \sin^3 t$, $y = \cos^3 t$ for the interval $\pi/6 \le t \le \pi/3$.

**(a)** Find an equation (in the variables $x$ and $y$) of the line tangent to the curve at $t = \pi/4$.

**(b)** Find the arc length of the curve.

**(c)** Find the area of the surface obtained by revolving the curve about the $x$-axis.

**Solution:**

**Part (a):** Since $x(t) = \sin^3 t$ and $y(t) = \cos^3 t$, we have $x'(t) = 3\sin^2 t \cos t$ and $y'(t) = -3\cos^2 t \sin t$. Therefore,

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-3\cos^2 t \sin t}{3\sin^2 t \cos t} = -\frac{\cos t}{\sin t} = -\cot t$$

$-\cot\frac{\pi}{4} = -1$, so the slope at $t = \pi/4$ is $-1$. The point of tangency at $t = \pi/4$ is $(x(\pi/4), y(\pi/4)) = \left(\left(\frac{1}{\sqrt{2}}\right)^3, \left(\frac{1}{\sqrt{2}}\right)^3\right) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$. Therefore, an equation for the line tangent to the curve at $t = \pi/4$ (in point-slope form) is:

$$\boxed{y - \frac{1}{2\sqrt{2}} = -\left(x - \frac{1}{2\sqrt{2}}\right)}$$

Note that point-slope for is usually the most convenient way to answer questions that ask for an equation of a line tangent to ⟨*something*⟩. The question doesn't specifically ask for point-slope form, so we would accept other $xy$ forms like slope-intercept or standard form. But the question *does* ask for an equation in the variable $x$ and $y$, so describing the tangent line, say, parametrically in the variable $t$ would not be acceptable and would be marked wrong.

**Part (b):** The formula for the arc length of a parametric curve is:

$$s = \int_{t=a}^{t=b} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Using the expressions for $x'(t)$ and $y'(t)$ from part (a), we have

$$\begin{aligned}
s &= \int_{\pi/6}^{\pi/3} \sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t}\,dt \\
&= \int_{\pi/6}^{\pi/3} \sqrt{9\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)}\,dt \\
&= \int_{\pi/6}^{\pi/3} 3|\sin t\cos t|\sqrt{\sin^2 t + \cos^2 t}\,dt \\
&= \int_{\pi/6}^{\pi/3} 3\sin t\cos t\sqrt{1}\,dt && \text{$\sin t$ and $\cos t$ are both positive in the interval $\frac{\pi}{6} \le t \le \frac{\pi}{3}$, so we can remove the absolute value signs} \\
&= \int_{\pi/6}^{\pi/3} \frac{3}{2}\sin 2t\,dt && \text{Double-angle formula: $2\sin t\cos t = \sin 2t$} \\
&= -\frac{3}{4}\cos 2t\Big|_{\pi/6}^{\pi/3} = -\frac{3}{4}\left(-\frac{1}{2}-\frac{1}{2}\right) = -\frac{3}{4}(-1) = \boxed{\frac{3}{4}}
\end{aligned}$$

**Part(c):** The formula for the area of the surface obtained by revolving a parametric curve about the $x$-axis is

$$S = 2\pi\int_{t=a}^{t=b} y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

Using the expressions for $x'(t)$ and $y'(t)$ from part (a), we have

$$\begin{aligned}
S &= 2\pi\int_{\pi/6}^{\pi/3} \cos^3 t\sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t}\,dt \\
&= 2\pi\int_{\pi/6}^{\pi/3} \cos^3 t\sqrt{9\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)}\,dt \\
&= 2\pi\int_{\pi/6}^{\pi/3} 3\cos^3 t\,|\sin t\cos t|\sqrt{\sin^2 t + \cos^2 t}\,dt \\
&= 2\pi\int_{\pi/6}^{\pi/3} 3\cos^4 t\sin t\sqrt{1}\,dt && \text{$\sin t$ and $\cos t$ are both positive in the interval $\frac{\pi}{6} \le t \le \frac{\pi}{3}$, so we can remove the absolute value signs} \\
&= -6\pi\int_{\sqrt{3}/2}^{1/2} u^4\,du && \text{$u$-substitution with $u = \cos t$ and $du = -\sin t\,dt$} \\
&= -6\pi\cdot\frac{u^5}{5}\Big|_{\sqrt{3}/2}^{1/2} \\
&= \frac{-6\pi}{5}\left((1/2)^5 - (\sqrt{3}/2)^5\right) = \frac{6\pi}{5}\left((\sqrt{3}/2)^5 - (1/2)^5\right) = \boxed{\frac{3\pi}{80}\left(9\sqrt{3}-1\right)}
\end{aligned}$$

**16.** Determine whether $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)}$ converges absolutely, converges conditionally, or diverges.

**Solution:** We begin by investigating whether the positive term version of the given series converges — i.e., whether $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n(\ln n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ converges.

To investigate the positive series, we use the Integral Test. Let $f(x) = \frac{1}{x(\ln x)}$, and observe that $f(n) = \frac{1}{n(\ln n)}$ for all $n = 2, 3, 4, \ldots$. To apply the Integral Test, $f$ must be positive, continuous, and decreasing. The function $f(x)$ is clearly positive for all $x \geq 2$. It is also continuous for all $x \geq 2$, since it is a product and quotient of functions that are continuous for all $x \geq 2$ (see continuity theorems from Calculus 1). To verify that $f$ is decreasing for $x \geq 2$, calculate $f'(x) = \frac{x(\ln x) \cdot 0 - (x \cdot \frac{1}{x} + \ln x)}{[x(\ln x)]^2} = \frac{-(1 + \ln x)}{[x(\ln x)]^2}$ (using the quotient rule). The latter expression for $f'(x)$ is clearly negative for all $x \geq 2$, which shows that $f$ is decreasing for all $x \geq 2$.

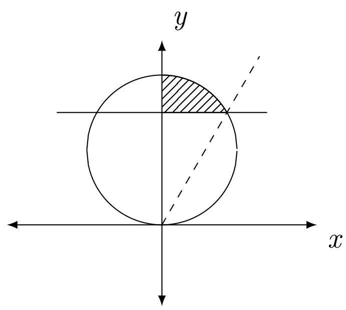
We now evaluate the integral $\int_2^{\infty} f(x)\, dx$ using the substitution $u = \ln x$ and $du = \frac{dx}{x}$: $\int_2^{\infty} \frac{dx}{x(\ln x)} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)} = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u} = \int_{\ln 2}^{\infty} \frac{du}{u}$. The latter integral is a $p$-integral with $p = 1$. Since this integral diverges (because the interval of integration is of the form $[\text{constant}, \infty]$ and $p \leq 1$), so does the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$.

We now test the original series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)}$. Let $b_n = \frac{1}{n(\ln n)}$, so that the given series is $\sum_{n=2}^{\infty} (-1)^n b_n$. The analysis above shows that $b_n$ is positive and decreasing. (Note: "continuous" is not a relevant observation about $b_n$; continuity was only relevant to the function $f$ that was used in the Integral Test.) Using the Alternating Series Test (also called the A.S.T., or the Leibniz Test), we observe that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n(\ln n)} = 0$. (Zero is the limit since the denominator is a product of two terms that both approach infinity.) Therefore, by the Alternating Series Test, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)}$ converges.

Since the given series converges and its positive-term counterpart diverges, we say that the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)}$ **converges conditionally**.

**17.** Consider the region inside the circle $x^2 + (y-2)^2 = 4$, above the line $y = 3$, and to the right of the line $x = 0$. Set up an integral in polar coordinates that is equal to the area of this region. Do not evaluate the integral. (It may help to sketch the region.)

**Solution:** The objects described by the equation above are: (1) a circle of radius 2 and center $(0, 2)$, and (2) a horizontal line passing through the point $(0, 3)$. Therefore, a sketch of the region is:




To find the equivalent equations in polar coordinates, substitute $x = r\cos\theta$ and $y = r\sin\theta$ into the original equations. Therefore, the equation of the horizontal line is $r\sin\theta = 3$, or $r = 3\csc\theta$.

The equation of the circle is found the same way; only the algebra is more complicated:

$$\begin{aligned}
x^2 + (y-2)^2 &= 4 \\
r^2\cos^2\theta + (r\sin\theta - 2)^2 &= 4 \\
r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta + 4 &= 4 \\
r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta &= 0 \\
r^2(\cos^2\theta + \sin^2\theta) &= 4r\sin\theta \\
r &= 4\sin\theta
\end{aligned}$$

We now need to find the angles where the circle and the line intersect, which we do by setting their expresions equal and solving for $\theta$:

$$\begin{aligned}
4\sin\theta &= 3\csc\theta \\
\frac{\sin\theta}{\csc\theta} &= \frac{3}{4} \\
\sin^2\theta &= \frac{3}{4}
\end{aligned}$$

Since the relevant point of intersection lies in Quadrant I (see the angle indicated by the dashed line), we want an angle in Quadrant I that satisfies that equation. Such an angle is $\theta = \pi/3$, which is the polar equation of the dashed line. The shaded region lies between the dashed line and the positive $y$-axis, whose polar equation is $\theta = \pi/2$.

Putting this all together, we are finding the area between the outer polar curve ($r = 4\sin\theta$) and the inner polar curve ($r = 3\csc\theta$), for the interval $\pi/3 \le \theta \le \pi/2$. The formula for area between polar curves is $\frac{1}{2}\int_\alpha^\beta \left[f(\theta)^2 - g(\theta)^2\right] d\theta$ (where $r = f(\theta)$ and $r = g(\theta)$ are the outer and inner curves, and $\alpha$ and $\beta$ are the lower and upper angles of the region). Therefore, the answer is:

$$\text{Area} = \boxed{\frac{1}{2}\int_{\pi/3}^{\pi/2} \left((4\sin\theta)^2 - (3\csc\theta)^2\right) d\theta}$$

**18.** Find all complex-number solutions to the equation $z^5 = 4\sqrt{2} + \left(4\sqrt{2}\right)i$. Write your answers using the exponential form of a complex number.

**Solution:** The given complex number is in the form $a + bi$ with $a = 4\sqrt{2}$ and $b = 4\sqrt{2}$. Therefore, the radius (modulus) of the complex number is $\sqrt{a^2+b^2} = \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{32+32} = \sqrt{64} = 8$. The angle (argument) of the complex number is an angle such that $\theta = \tan(b/a) = \tan\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right) = \tan 1$. Since the complex number appears in Quadrant I of the complex plane, the angle whose tangent is 1 is $\pi/4$. Therefore, the original equation can be written as $z^5 = 8e^{i\frac{\pi}{4}}$.

To solve this equation, we raise both sides to the power 1/5:

$$\begin{aligned}(z^5)^{1/5} &= \left(8e^{i(\pi/4)}\right)^{1/5} \\ z &= 8^{1/5}e^{i(\pi/20)}\end{aligned}$$

This gives one solution to the equation. There are five solutions to the equation, and the other solutions have the same radius as the first solution ($8^{1/5}$), but lie 1/5 of the way around a circle from each other. Therefore, we keep adding 1/5 of a circle (or an angle of $2\pi/5$) to the first solution until we have all the solutions. Those angles are: $9\pi/20$, $17\pi/20$, $25\pi/20$, and $33\pi/20$.

Therefore, the five solutions are: $\boxed{8^{1/5}e^{i(\pi/20)}}$, $\boxed{8^{1/5}e^{i(9\pi/20)}}$, $\boxed{8^{1/5}e^{i(17\pi/20)}}$, $\boxed{8^{1/5}e^{i(5\pi/4)}}$, and $\boxed{8^{1/5}e^{i(33\pi/20)}}$.

**19.** Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n(x+2)^n}{\sqrt{n}}$.

**Solution:** First, we find the radius of convergence of the given power series using the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-3)^{n+1}(x+2)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-3)^n(x+2)^n}\right| &= \lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{(-3)^n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{(x+2)^{n+2}}{(x+2)^n}\right| \\ &= \lim_{n\to\infty}|(-3)\cdot 1 \cdot (x+2)| \\ &= (|-3| \cdot |x+2|) \\ &= 3|x+2|\end{aligned}$$

We set the latter expression $< 1$ giving

$$\begin{aligned} 3|x+2| &< 1 \\ |x+2| &< \tfrac{1}{3} \\ -\tfrac{1}{3} < x+2 &< \tfrac{1}{3} \\ -\tfrac{1}{3} - 2 < x &< \tfrac{1}{3} - 2 \\ -\tfrac{7}{3} < x &< -\tfrac{5}{3} \end{aligned}$$

The last inequality gives the bounds of the interval of convergence. However, to complete the problem we need to test the behavior of the series at the endpoints, since the the series might converge at either, both, or neither of its endpoints.

We cannot use the Ratio Test to test convergence at the endpoints, since the endpoints — by definition — are points where the Ratio Test limit is 1, so the Ratio Test would be inconclusive at those points.

First, test $x = -\frac{7}{3}$ by substituting $x = -\frac{7}{3}$ into the original series: $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n(-\frac{7}{3}+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-3)^n(-\frac{1}{3})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. It is clear from the latter expression that the series diverges, because it is a $p$-series with $p = \frac{1}{2} \leq 1$.

Next, test $x = -\frac{5}{3}$ in a similar way: $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n(-\frac{5}{3}+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-3)^n(\frac{1}{3})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. The latter expression is an **alternating** $p$-series with $p = \frac{1}{2} > 0$, and therefore it converges. (Caution: Only **alternating** $p$-series converge for $p > 0$. For positive $p$-series, the criterion for convergence is $p > 1$.)

Since the series diverges at the endpoint $-\frac{7}{3}$ and converges at the endpoint $-\frac{5}{3}$, the interval of convergence is $\boxed{(-\frac{7}{3}, -\frac{5}{3}]}$.

**20.** **(a)** Find the terms up to the $x^6$ term of the Maclaurin series for $(x^2+1)\cos(2x)$.

**(b)** Find the Maclaurin series (in $\Sigma$ notation) for $x^3\cos(2x)$.

**(c)** Use the first two nonzero terms of your answer to part (b) to find the approximate value of $\displaystyle\int_0^{1/2} x^3 \cos(2x)\,dx$.

**Solution:**

**Part (a):** Since $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, it follows that $\cos(2x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!} = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \cdots$. Therefore $x^2\cos(2x) = x^2 - \frac{4x^4}{2!} + \frac{16x^6}{4!} + \cdots$. Adding these terms together, we conclude that the terms up to $x^6$ of $x^2+1$ are:

$$1+\left(1-\frac{4}{2!}\right)x^2+\left(\frac{16}{4!}-\frac{4}{2!}\right)x^4+\left(\frac{16}{4!}-\frac{64}{6!}\right)x^6$$

**Part (b):** In part (a), we found the Sigma form of the Maclaurin series for $\cos(2x)$. Therefore,

$$\begin{aligned} x^3\cos(2x) &= x^3\sum_{n=0}^{\infty}\frac{(-1)^n4^nx^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n4^nx^{2n+3}}{(2n)!} \end{aligned}$$

**Part (c):** The first two nonzero terms of the series from part (b) are $x^3-\frac{4x^5}{2!}=x^3-2x^5$. Therefore,

$$\begin{aligned} \int_0^{1/2} x^3\cos(2x)\,x &\approx \int_0^{1/2} x^3-2x^5\,dx \\ &= \left[\frac{x^4}{4}-\frac{2x^6}{6}\right]_0^{1/2} \\ &= \left[\frac{x^4}{4}-\frac{x^6}{3}\right]_0^{1/2} \\ &= \frac{(1/2)^4}{4}-\frac{(1/2)^6}{3} \\ &= \frac{1}{64}-\frac{1}{3\cdot 64} \\ &= \frac{1}{64}(1-\tfrac{1}{3}) \\ &= \frac{1}{64}\cdot\frac{2}{3}=\boxed{\frac{1}{96}} \end{aligned}$$

**21.** Evaluate $\displaystyle\int_0^{1/10}\arcsin(10x)\,dx$.

**Solution:** Using integration by parts with

$$\begin{aligned} u &= \arcsin(10x) & dv &= dx \\ du &= \frac{10\,dx}{\sqrt{1-100x^2}} & v &= x \end{aligned}$$

We find

$$\int \arcsin(10x)\,dx = x\arcsin(10x) - \int \frac{10x\,dx}{\sqrt{1-100x^2}} \tag{1}$$

Using $u$-substiution on the last question (where we'll use the variable $t$ to avoid reusing $u$), we have $t = 1-100x^2$ and $dt = -200x\,dx$, so $10x\,dx = -\frac{1}{20}dt$. Then the expression on line (1) equals:

$$\begin{aligned}
\cdots &= x\arcsin(10x) - \int \frac{-\frac{1}{20}dt}{\sqrt{t}} \\
&= x\arcsin(10x) + \frac{1}{20}\int t^{-1/2}\,dt \\
&= x\arcsin(10x) + \frac{1}{20}\cdot\frac{t^{1/2}}{1/2} \\
&= x\arcsin(10x) + \frac{\sqrt{1-100x^2}}{10} \qquad (2)
\end{aligned}$$

We now substitute 0 and 1/10 into the antiderivative on line (2), and subtract:

$$\left(\tfrac{1}{10}\arcsin\left(10\cdot\tfrac{1}{10}\right) + \frac{\sqrt{1-100\left(\frac{1}{10}\right)^2}}{10}\right) - \left(0\arcsin(10\cdot 0) + \frac{\sqrt{1-100\cdot 0^2}}{10}\right)$$
$$= \left(\frac{1}{10}\cdot\frac{\pi}{2}+0\right) - \left(0+\frac{1}{10}\right) = \boxed{\frac{\pi-2}{20}}$$

**22.** Use Maclaurin series to evaluate $\displaystyle\lim_{x\to 0}\frac{1-\cos(5x^2)}{x\sin(4x)-4x^2}$.

**Solution:** We only need the first few terms of each Maclaurin series to calculate the limit. The Maclaurin series for the numerator and denominator are:

$$1-\cos(5x^2) = 1-\underbrace{\left(1-\frac{(5x^2)^2}{2!}+\frac{(5x^2)^4}{4!}-\cdots\right)}_{=\cos(5x^2)} = \frac{5^2x^4}{2!}-\frac{5^4x^8}{4!}+\cdots$$

$$\begin{aligned}
x\sin(4x)-4x^2 &= x\cdot\underbrace{\left(4x-\frac{(4x)^3}{3!}+\frac{(4x)^5}{5!}-\cdots\right)}_{=\sin(4x)}-4x^2 \\
&= \left(4x^2-\frac{4^3x^4}{3!}+\frac{4^5x^6}{5!}-\cdots\right)-4x^2 = -\frac{4^3x^4}{3!}+\frac{4^5x^6}{5!}-\cdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos(5x^2)}{x\sin(4x)-4x^2} &= \lim_{x\to 0}\frac{\frac{5^2x^4}{2!}-\frac{5^4x^8}{4!}+\cdots}{-\frac{4^3x^4}{3!}+\frac{4^5x^6}{5!}-\cdots}\cdot\frac{\frac{1}{x^4}}{\frac{1}{x^4}} \\
&= \lim_{x\to 0}\frac{\frac{5^2}{2!}-\frac{5^4x^4}{4!}+\cdots}{-\frac{4^3}{3!}+\frac{4^5x^2}{5!}-\cdots} \\
&= \frac{\frac{5^2}{2!}-0+\cdots}{-\frac{4^3}{3!}+0-\cdots} \\
&= \frac{25}{2}\cdot\frac{-6}{64} = -\frac{150}{128} = \boxed{-\frac{75}{64}}
\end{aligned}$$

All terms except the leftmost terms have positive powers of $x$, so substituting $x = 0$ makes all of those terms 0