The Syntax of Sudoku (Grids) – Roadmap

This is a preliminary draft
and will be updated step by step;
be patient!

Alex Pfaff
nonintersective@gmail.com

August 13, 2025

Abstract

1 Basics & Basic Terminology

Sudoku Puzzle

3	1			4		5		8
5	8	6	7	3	9	2		
4		2		8				
9	3	4		2		1		
7		8		9	1	4	3	5
6		1		7	3	9		
	6	5		1			2	9
	7	3		5	2		4	6
			8		7			1

Sudoku Grid

3	1	7	2	4	6	5	9	8
5	8	6	7	3	9	2	1	4
4	9	2	1	8	5	6	7	3
9	3	4	5	2	8	1	6	7
7	2	8	6	9	1	4	3	5
6	5	1	4	7	3	9	8	2
8	6	5	3	1	4	7	2	9
1	7	3	9	5	2	8	4	6
2	4	9	8	6	7	3	5	1

(3, 1, 7, 2, 4, 6, 5, 9, 8, 5, 8, 6 . . . . 2, 8, 6, 9, 1, 4, 3 . . . 8, 4, 6, 2, 4, 9, 8, 6, 7, 3, 5, 1)
Grid Sequence
(abbreviated)

2 Some Basic Numbers

2.1 Top-Down: Demarcations

Given a 9 X 9 Grid $\rightarrow$ 81 slots and values $v = v_1, v_2, ... v_{81}$ to fill into the slots, where $v \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$; suppose there are no further constraints / rules (= arbitrary distribution of values in v and arbitrary distribution of v across the grid). In this case, there are **1.9 x 10^{77}** possible grid assignments.

If we impose a cardinality constraint: $v \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}^9$ to the effect that every digit 1-9 occurs exactly nine times, there are **5.3 x 10^{70}** possible grid assignments.

In contrast, the number of valid grid assignments, where the values are distributed according to the rules of Sudoku, is 6,670,903,752,021,072,936,960 $\sim$ **6.7 x 10^{21}** (see Felgenhauer and Jarvis 2005).[1] Based on this number, in turn, we can approximate the maximum number of hypothetical Sudoku puzzles as $\sum_{k=1}^{81}$ x **6.7 x 10^{21}** x $\binom{81}{k}$ where k is the number of blanks.[2]

[1]See also:
https://en.wikipedia.org/wiki/Mathematics_of_Sudoku;
http://sudopedia.enjoysudoku.com/Mathematics_of_Sudoku.html.

[2]There are some caveats heres.

(i) (McGuire et al., 2014) proved that, in a classical Sudoku (with no further constraints), the minimum of given digits for a puzzle to have a unique solution is 17 ($\leftrightarrow$ 64 blanks). Thus the range of the summation index should be reduced: $\sum_{k=1}^{64}$.

(ii) Even with this restriction, not every blanked grid is a valid Sudoku puzzle, i.e. has a unique solution. For instance, if all all nine instances of two digits are missing (e.g. no 3s and no 7s), there are two possible solutions. In other words, for $18 \leq k \leq 64$, there are actually less than 6.7 x 10^{21} x $\binom{81}{k}$ valid puzzles with a unique solution – hence the label "hypothetical Sudoku puzzle".

1.9 x 10^{77}	9^{81}	**No rules** – $v \in$ {1, 2, 3, 4, 5, 6, 7, 8, 9} – 9 x 9 = 81 slots
5.3 x 10^{70}	$\frac{81!}{9!^9}$	**Cardinality Constraint** – $v \in$ {1, 2, 3, 4, 5, 6, 7, 8, 9}^{9} – 9 x 9 = 81 slots
6.7 x 10^{21}	6670903752021072936960	**Valid Sudoku Grids** – $v \in$ {1, 2, 3, 4, 5, 6, 7, 8, 9}^{9} – 9 x 9 = 81 slots distribution of v according to Sudoku rules
6.7 x 10^{21} x $\binom{81}{k}$		**Hypothetical(!) Sudoku Puzzles with k blanks**

Table 1: Number of possible and valid Grids

2.2 Bottom-up: Expansions

Given a valid grid, if we swap two columns within a *boxcolumn*, we get another valid grid (see the illustration below, left and middle); obviously the same applies within all three boxcolumns. Likewise, if we swap two boxcolumns themselves, we get another valid grid (see the illustration below, middle and right).

3	1	7	2	4	6	5	9	8
5	8	6	7	3	9	2	1	4
4	9	2	1	8	5	6	7	3
9	3	4	5	2	8	1	6	7
7	2	8	6	9	1	4	3	5
6	5	1	4	7	3	9	8	2
8	6	5	3	1	4	7	2	9
1	7	3	9	5	2	8	4	6
2	4	9	8	6	7	3	5	1

3	7	1	2	4	6	5	9	8
5	6	8	7	3	9	2	1	4
4	2	9	1	8	5	6	7	3
9	4	3	5	2	8	1	6	7
7	8	2	6	9	1	4	3	5
6	1	5	4	7	3	9	8	2
8	5	6	3	1	4	7	2	9
1	3	7	9	5	2	8	4	6
2	9	4	8	6	7	3	5	1

3	7	1	5	9	8	2	4	6
5	6	8	2	1	4	7	3	9
4	2	9	6	7	3	1	8	5
9	4	3	1	6	7	5	2	8
7	8	2	4	3	5	6	9	1
6	1	5	9	8	2	4	7	3
8	5	6	7	2	9	3	1	4
1	3	7	8	4	6	9	5	2
2	9	4	3	5	1	8	6	7

These permutations can be combined, and an exhaustive application of all combination yields $3!^4 =$ **1,296** column permutations. Exactly the same goes for row permutations, thus altogether, we have $3!^4$ x $3!^4 = 3!^8 =$ 1,296 x 1,296 $=$ **1,679,616** valid grid permutations.

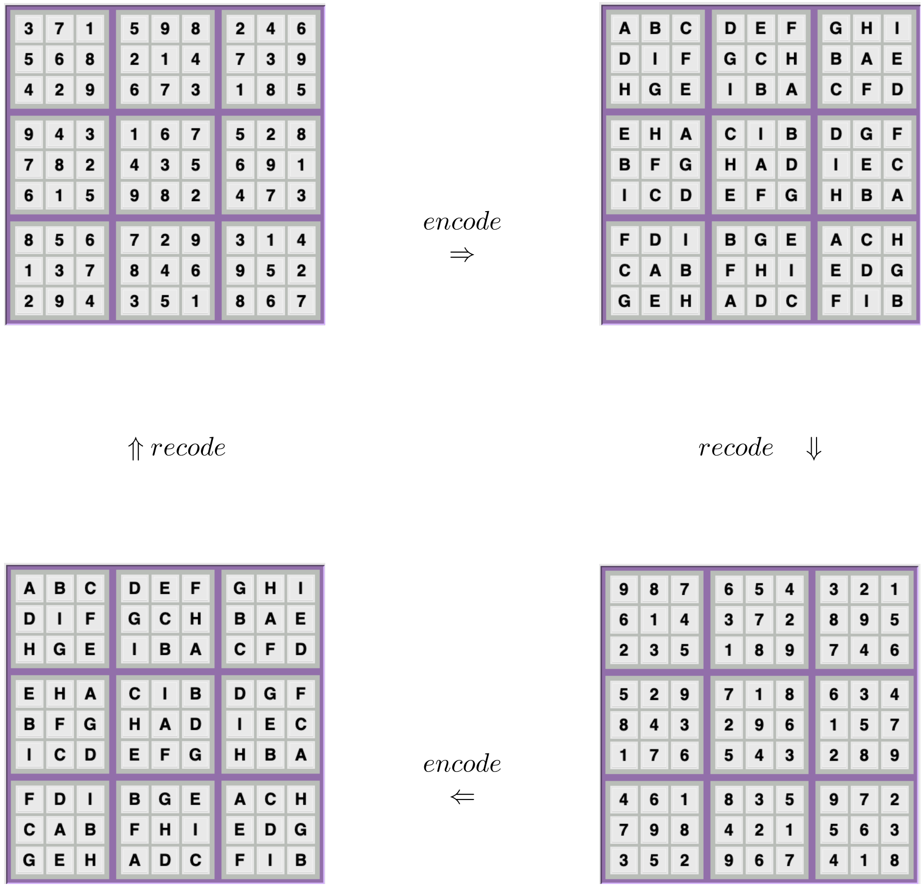
Interestingly, rotation does not lead to new grids, but reflection does.[3] So every valid grid in the permutation series can be mapped to its reflection, which is a new grid not already contained in the permutations generated by column/row permutation. In short, one valid grid gives rise to 2 x 1,679,616 $=$ **3,359,232** valid grids through a series of simple geometric transformations.

Next, consider substitution; we will make use of an intermediate alphabetic representation (= *abc-grid*). To this end, we will make use of a key which is a simple, but rigid mapping:

top_row_of_grid $\quad\Leftrightarrow\quad$ (A, B, C, D, E, F, G, H, I)

i.e. the values in the top row in a numeric grid (*int grid*) are systematically mapped to the letters in a fixed sequence. Next, all numeric values in the grid are systematically replaced (encoded) by the corresponding letter in the key. Notice that the key works both ways in that the sequence (A, B, C, D, E, F, G, H, I) may be mapped to a sequence of digits, and subsequently, the abc grid can be recoded as an integer grid. The letter sequence in the key is konstant (immutable), thus encoding is deterministic. Recoding, on the other hand, allows some degrees of freedom in that any valid Sudoku row can be substitued for the letter sequence.

[3] I offer no formal proof, but merely a systematic observation: generating all grid permutations, then generating four rotations for each grid, and finally adding the reflection for each grid outputs a list `grids` with `len(grids)` = 13436928 (= 1,679,616 x 4 x 2), but for the corresponding set, we get: `len(set(grids))` = 3359232 (= $\frac{13,436,928}{4}$ = 1,679,616 x 2). In the course of generating all permutations, all rotations are also generated, but not the reflections.
$\Rightarrow$ Permutation X Rotation X Reflection = Permutation X Reflection.



Obviously, an abc-grid is distributionally equivalent to the int-grid from which it was created, and any further recodings are distributionally equivalent to the abc grid – the distribution of the value in cell 1 is identical across the grid no matter whether it is labelled 3, A or 9 (see above illustration), or any other value; same goes for the values in other cells. In other words, all int grids that can be obtained from an abc-grid by recoding are distributionally equivalent, the only difference being the labeling. Since the number of equivalent int-grids corresponds to the number of possible keys, there are $9! =$ **362,880** possible grids that are distributionally equivalent with only the values being permuted. We can refer to the abc-grid as the deep structure (or prototype) for one particular distribution of digits, and the corresponding int-grids as surface structure (or substitutions). At face value, int-grids top left and bottom right (illustration above) are distinct because different values occur in the same position, respectively, but at a more abstract level, they have something in common, viz. the distributional pattern of those values: both translate to the same abc-grid.

As a matter of (descriptive) terminology, I will refer to a collection of grids that are obtained via grid permutations (+ reflection) as the **Vertical Permutation Series**, whereas grids that stem from value substitution, i.e. grids that are obtained via recoding from one given abc-grid, will be referred to as the **Horizontal Permutation Series**.

	123456789	....	**987654321**
abc_grid_1	int_grid$_{1,1}$	...	int_grid$_{1,362880}$
abc_grid_2	int_grid$_{2,1}$	...	int_grid$_{2,362880}$
...	...	...	...
abc_grid_1679616	int_grid$_{1679616,1}$	...	int_grid$_{1679616,362880}$

Table 2: Motivating the terminology *Vertical* and *Horizontal (Permutation) Series* by using abc-grids of grid permutations as row labels, and int keys for recoding as column labels.

Grids in the vertical series are distinct as such, but moreover, they are distributionally distinct, insofar as positions (rows, columns) are permuted, not simply the values at those positions. Permutation plus reflection thus gives rise to 3.359.232 distinct abc-grids via encoding. Every single abc-grid in this collection, in turn, can be recoded in 362.880 ways. With all that taken into consideration, it transpires that one single valid grid gives rise to 3.359.232 x 362.880 = 1,218,998,108,160 = **1.2 x 10^{12}** (superficially) distinct valid grids.

3 Syntax of Sudoku

4 PseudoQ – Data Generation

5 Sudoku & Deep Learning I: Classification

for the moment and illustration, see:
- psq_binary_clf.ipynb,
- psq_multi_clf.ipynb,
- psq_solver_player.ipynb

`https://github.com/A-Lex-McLee/PseudoQ-2.1`

5.1 Binary Classification

5.2 Multiclass Classification

6 Sudoku & Deep Learning II: Predict Several Values

7 Sudoku & Deep Learning III: Solve the Puzzle

References

Felgenhauer, Bertram, and Frazer Jarvis. 2005. Enumerating possible sudoku grids.

McGuire, Gary, Bastian Tugemann, and Gilles Civario. 2014. There is no 16-clue Sudoku: Solving the Sudoku minimum number of clues problem via hitting set enumeration. *Experimental Mathematics* 23 (2): 190–217.