

# Preface

The **Encyclopedia of Proof Systems** aims at providing a reliable, technically accurate, historically informative, concise and convenient central repository of proof systems for various logics. The goal is to facilitate the exchange of information among logicians with mathematical, computational or philosophical backgrounds; in order to foster and accelerate the development of new proof systems and automated deduction tools that rely on them.

Preparatory work for the creation of the Encyclopedia, such as the implementation of the LaTeX template and the setup of the Github repository, started in October 2014, triggered by the call for workshop proposals for the 25th Conference on Automated Deduction (CADE). Christoph Benzmüller, CADE’s conference chair, and Jasmin Blanchette, CADE’s workshop co-chair, encouraged me to submit a workshop proposal and supported my alternative idea to organize instead a special poster session based on encyclopedia entries. I am thankful for their encouragement and support.

In December 2014, Björn Lellmann, Giselle Reis and Martin Riener kindly accepted my request to beta-test the template and the instructions I had created. They submitted the first few example entries to the encyclopedia and provided valuable feedback, for which I am grateful. Their comments were essential for improving the templates and instructions before the public announcement of the encyclopedia.

Discussions with Lev Beklemishev, Björn Lellmann, Roman Kuznets, Sergei Soloviev and Anna Zamansky brainstormed many ideas for improving the organization and structure of the encyclopedia. Many of these ideas still need to be fully implemented.

May 2015

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**Part I**  
*Introductions*

**ToDo:**

We plan to add a few short introductory chapters addressing various aspects that are orthogonal to several entries, such as:

- basic technical notions for each type of proof system (e.g. tableaux, natural deduction systems, sequent calculi, resolution calculi ...),
- logical languages
- logics (classical, intuitionistic, modal, substructural, linear, paraconsistent, ...)
- application domains

The goals of these introductory chapters will be to:

- provide a global technical and historical view of the entries,
- reduce repetition of basic notions in the entries,
- increase the understandability of the entries,
- make the encyclopedia more self-contained.

The exact structure and content of these chapters will be decided later, after the collection of sufficiently many entries.

For now, entries are sorted in chronological order only, and various indexes are provided in the backmatter. If the need arises, we might consider grouping entries according to various criteria.



**Part II**  
***Proof Systems***



## Intuitionistic Natural Deduction NJ (1935)

$\frac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A} \& \mathfrak{B}} UE$	$\frac{\mathfrak{A} \& \mathfrak{B}}{\mathfrak{A}} UB$	$\frac{\mathfrak{A} \& \mathfrak{B}}{\mathfrak{B}} UB$	
$\frac{\mathfrak{A}}{\mathfrak{A} \vee \mathfrak{B}} OE$	$\frac{\mathfrak{B}}{\mathfrak{A} \vee \mathfrak{B}} OE$	$\frac{\mathfrak{A} \vee \mathfrak{B} \quad \frac{[\mathfrak{A}] \quad \vdots \quad \mathfrak{C}}{\mathfrak{C}} \quad \frac{[\mathfrak{B}] \quad \vdots \quad \mathfrak{C}}{\mathfrak{C}}}{\mathfrak{C}} OB$	
$\frac{\mathfrak{F}\alpha}{\forall x \mathfrak{F}x} AE$	$\frac{\forall x \mathfrak{F}x}{\mathfrak{F}\alpha} AB$	$\frac{\mathfrak{F}\alpha}{\exists x \mathfrak{F}x} EE$	$\frac{\exists x \mathfrak{F}x \quad \frac{[\mathfrak{F}\alpha] \quad \vdots \quad \mathfrak{C}}{\mathfrak{C}}}{\mathfrak{C}} EB$
$\frac{[\mathfrak{A}] \quad \vdots \quad B}{\mathfrak{A} \supset \mathfrak{B}} FE$	$\frac{\mathfrak{A} \quad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}} FB$	$\frac{[\mathfrak{A}] \quad \vdots \quad \neg \mathfrak{A}}{\neg \mathfrak{A}} NE$	$\frac{\mathfrak{A} \quad \neg \mathfrak{A}}{\wedge} NB \quad \frac{\wedge}{\mathfrak{D}}$

The eigenvariable  $\alpha$  of an  $AE$  must not occur in the formula designated in the schema by  $\forall x \mathfrak{F}x$ ; nor in any assumption formula upon which that formula depends. The eigenvariable  $\alpha$  of an  $EB$  must not occur in the formula designated in the schema by  $\exists x \mathfrak{F}x$ ; nor in any assumption formula upon which that formula depends, with the exception of the assumption formulae designated by  $\mathfrak{F}\alpha$ .

**Clarifications:** The names of the rules are those originally given by Gentzen [1]:  $U$  = und (and),  $O$  = oder (or),  $A$  = all,  $E$  = es-gibt (exists),  $F$  = folgt (follows),  $N$  = nicht (not),  $E$  = Einführung (introduction),  $B$  = Beseitigung (elimination).

**History:** The main novelty introduced by Gentzen in this proof system is its *assumption* handling mechanism, which allows formal proofs to reflect more naturally the logical reasoning involved in mathematical proofs.

**Remarks:** In [1], completeness of **NJ** is proven by showing how to translate proofs in the Hilbert-style calculus **LHJ** to **NJ**-proofs, and soundness is proven by showing how to translate **NJ**-proofs to **LJ**-proofs [3].

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[1] Gerhard Gentzen. “Untersuchungen über das logische Schließen I”. In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.

## Classical Sequent Calculus LK (1935)

$\overline{A \vdash A}$		$\frac{\Gamma \vdash \Lambda, A \quad A, \Delta \vdash \Theta}{\Gamma, \Delta \vdash \Lambda, \Theta} \text{ cut}$	
$\frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} w_l$		$\frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, A} w_r$	
$\frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta} e_l$	$\frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} c_l$	$\frac{\Gamma \vdash \Theta, B, A, \Delta}{\Gamma \vdash \Theta, A, B, \Delta} e_r$	$\frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A} c_r$
$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta} \neg_l$		$\frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A} \neg_r$	
$\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta} \wedge_l$		$\frac{\Gamma \vdash \Theta, A \quad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \wedge B} \wedge_r$	
$\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta} \vee_l$		$\frac{\Gamma \vdash \Theta, A_i}{\Gamma \vdash \Theta, A_1 \vee A_2} \vee_r$	
$\frac{\Gamma \vdash \Lambda, A \quad B, \Delta \vdash \Theta}{A \rightarrow B, \Gamma, \Delta \vdash \Lambda, \Theta} \rightarrow_l$		$\frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \rightarrow B} \rightarrow_r$	
$\frac{A[\alpha], \Gamma \vdash \Theta}{\exists x.A[x], \Gamma \vdash \Theta} \exists_l$	$\frac{A[t], \Gamma \vdash \Theta}{\forall x.A[x], \Gamma \vdash \Theta} \forall_l$	$\frac{\Gamma \vdash \Theta, A[\alpha]}{\Gamma \vdash \Theta, \forall x.A[x]} \forall_r$	$\frac{\Gamma \vdash \Theta, A[t]}{\Gamma \vdash \Theta, \exists x.A[x]} \exists_r$
<p>The eigenvariable <math>\alpha</math> should not occur in <math>\Gamma</math>, <math>\Theta</math> or <math>A[x]</math>.  The term <math>t</math> should not contain variables bound in <math>A[t]</math>.</p>			

**History:** This is a modern presentation of Gentzen's original **LK** calculus[1], using modern notations and rule names.

**Remarks:** **LK** is complete relative to **NK** (i.e. **NJ** {1} with the axiom of excluded middle) and sound relative to a Hilbert-style calculus **LHK** [2]. Cut is eliminable (*Hauptsatz* [1]), and hence classical predicate logic is consistent. Any *prenex* cut-free proof may be further transformed into a shape with only propositional inferences above and only quantifier and structural inferences below a *midsequent* [2].

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- [1] Gerhard Gentzen. "Untersuchungen über das logische Schließen I". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.
  - [2] Gerhard Gentzen. "Untersuchungen über das logische Schließen II". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 405–431.

## Intuitionistic Sequent Calculus LJ (1935)

$\overline{A \vdash A}$	$\frac{\Gamma \vdash A \quad A, \Delta \vdash \Theta}{\Gamma, \Delta \vdash \Theta} \text{ cut}$
$\frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} w_l$	$\frac{\Gamma \vdash}{\Gamma \vdash A} w_r$
$\frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta} e_l$	$\frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} c_l$
$\frac{\Gamma \vdash A}{\neg A, \Gamma \vdash} \neg_l$	$\frac{A, \Gamma \vdash}{\Gamma \vdash \neg A} \neg_r$
$\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta} \wedge_l$	$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_r$
$\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta} \vee_l$	$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2} \vee_r$
$\frac{\Gamma \vdash A \quad B, \Delta \vdash \Theta}{A \rightarrow B, \Gamma, \Delta \vdash \Theta} \rightarrow_l$	$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_r$
$\frac{A[\alpha], \Gamma \vdash \Theta}{\exists x.A[x], \Gamma \vdash \Theta} \exists_l$	$\frac{\Gamma \vdash A[t]}{\Gamma \vdash \exists x.A[x]} \exists_r$
$\frac{A[t], \Gamma \vdash \Theta}{\forall x.A[x], \Gamma \vdash \Theta} \forall_l$	$\frac{\Gamma \vdash A[\alpha]}{\Gamma \vdash \forall x.A[x]} \forall_r$

The eigenvariable  $\alpha$  should not occur in  $\Gamma$ ,  $\Theta$  or  $A[x]$ .  
The term  $t$  should not contain variables bound in  $A[t]$ .

**Clarifications:** **LJ** and **LK** {2} have exactly the same rules, but in **LJ** the succedent of every sequent may have at most one formula. This restriction is equivalent to forbidding the axiom of excluded middle in natural deduction.

**Remarks:** The cut rule is eliminable (*Hauptsatz* [Gentzen1935]), and hence intuitionistic predicate logic is consistent and its propositional fragment is decidable [Gentzen1935a]. **LJ** is complete relative to **NJ** {1} and sound relative to the Hilbert-style calculus **LHJ** [Gentzen1935a].

## Multi-Conclusion Sequent Calculus LJ' (1954)

$$\begin{array}{c}
\frac{}{A \vdash A} \quad \frac{\Gamma \vdash \Theta, A \quad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Theta, \Lambda} \text{ cut} \\
\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta} \wedge_l \quad \frac{\Gamma \vdash \Theta, A \quad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \wedge B} \wedge_r \\
\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta} \vee_l \quad \frac{\Gamma \vdash \Theta, A_i}{\Gamma \vdash \Theta, A_1 \vee A_2} \vee_r \\
\frac{\Gamma \vdash \Theta, A \quad B, \Delta \vdash \Lambda}{A \rightarrow B, \Gamma, \Delta \vdash \Theta, \Lambda} \rightarrow_l \quad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_r \\
\\
\frac{A\alpha, \Gamma \vdash \Theta}{\exists x.Ax, \Gamma \vdash \Theta} \exists_l \quad \frac{\Gamma \vdash \Theta, At}{\Gamma \vdash \Theta, \exists x.Ax} \exists_r \quad \frac{At, \Gamma \vdash \Theta}{\forall x.Ax, \Gamma \vdash \Theta} \forall_l \quad \frac{\Gamma \vdash A\alpha}{\Gamma \vdash \forall x.Ax} \forall_r \\
\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta} \neg_l \quad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A} \neg_r \quad \frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta} e_l \quad \frac{\Gamma \Delta \vdash \Theta, B, A, \Lambda}{\Gamma \vdash \Theta, A, B, \Lambda} e_r \\
\frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} c_l \quad \frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A} c_r \quad \frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta} w_l \quad \frac{\Gamma \vdash}{\Gamma \vdash A} w_r
\end{array}$$

The eigenvariable  $\alpha$  should not occur in  $\Gamma, \Theta$  or  $A[x]$ .  
 The term  $t$  should not contain variables bound in  $A[t]$ .

**Clarifications:** While **LJ** {3} is defined by restricting **LK** {2} to single conclusion, in **LJ'** only the rules  $\neg_r$ ,  $\rightarrow_r$  and  $\forall_r$  have this restriction.

**History:** **LJ'** was proposed in [6] and used to prove the completeness of **LJ** {3} in [4]. It also appears in [3] (as GHPC) and [5] (as L').

**Remarks:** **LJ'** is equivalent to **LJ**, and this is established by translating sequents of the form  $\Gamma \vdash A_1, \dots, A_n$  into sequents of the form  $\Gamma \vdash A_1 \vee \dots \vee A_n$ . Cut can be eliminated by using a combination of the rewriting rules for cut-elimination in **LJ** and **LK** and permutation of inferences, as shown by Schellinx [2] and Reis [1].

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- [1] Giselle Reis. “Cut-elimination by resolution in intuitionistic logic”. PhD thesis. Vienna University of Technology, July 2014.
  - [2] Harold Schellinx. “Some Syntactical Observations on Linear Logic”. In: *J. Log. Comput.* 1.4 (1991), pp. 537–559.
  - [3] A. G. Dragalin. *Mathematical Intuitionism: Introduction to Proof Theory*. American Mathematical Society, 1988.
  - [4] Gaisi Takeuti. *Proof Theory*. 2nd Edition. North Holland, 1987.
  - [5] Michael Dummett. *Elements of Intuitionism*. Oxford: Clarendon Press, 1977.
  - [6] Shôji Maehara. “Eine Darstellung der intuitionistischen Logik in der klassischen”. In: *Nagoya Math. J.* 7 (1954), pp. 45–64.

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Entry 4 by: Giselle Reis and Valeria de Paiva

## Second Order $\lambda$ -Calculus (System F) (1971)

$\frac{(x : T) \in E}{\Gamma; E \vdash x : T} \text{ assumption}$	
$\frac{\Gamma; E, (x : T) \vdash e : S}{\Gamma; E \vdash \lambda(x : T.e) : (T \rightarrow S)} \rightarrow I$	$\frac{\Gamma; E \vdash f : (T \rightarrow S) \quad \Gamma; E \vdash e : T}{\Gamma; E \vdash (fe) : \tau} \rightarrow E$
$\frac{\Gamma X; E \vdash e : T}{\Gamma; E \vdash (\lambda X : Tp.e) : (\forall X : Tp.T)} \forall I^*$	$\frac{\Gamma; E \vdash f : (\forall X : Tp.T) \quad \Gamma \vdash S : Tp}{\Gamma; E \vdash fS : [S/X]T} \forall E$
<p>* <math>X</math> must be not free in the type of any free term variable in <math>E</math>.</p>	

**Clarifications:** The presentation from [3] with minor corrections is used. Below  $X, Y, Z, \dots$  are type-variables and  $x, y, \dots$  term variables.

Type expressions:  $T := X | (T \rightarrow S) | (\forall X : Tp.T)$ .

Term expressions:  $e := x | (ee) | (\lambda x : T.e) | (\lambda X : Tp.e)$ .

$\forall, \lambda$  and  $\lambda$  are variable binders. All expressions are considered up to renaming of bound variables ( $\alpha$ -conversion). An unbound variable is free.  $FV(R)$  is the set of free variables for any (type or term) expression;  $[e/x]$ ,  $[S/X]$  mean capture-avoiding substitution in term- and type-expressions respectively (defined by induction). A context is a finite set  $\Gamma$  of type variables;  $\Gamma X$  stands for  $\Gamma \cup X$ . A type  $T$  is legal in  $\Gamma$  iff  $FV(T) \subseteq FV(\Gamma)$ . A type assignment in  $\Gamma$  is a finite list  $E = (x_1 : T_1), \dots, (x_n : T_n)$  where any  $T_i$  is legal in  $\Gamma$ . The typing relation  $\Gamma; E \vdash e : T$ , where  $E$  is a type assignment legal in  $\Gamma$ ,  $e$  is a term expression and  $T$  is a type expression, is defined by the rules above.

The *conversion relation* between well-typed terms is very important. It is defined by the following axioms:  $(\beta) (\lambda x : T.f)e = [e/x]f$ ;  $(\beta_2) (\lambda X : Tp.e)S = [S/X]e$ ;  $(\eta) \lambda x : T.(ex) = e$  if  $x \notin FV(e)$ ;  $(\eta_2) \lambda X : Tp.(eX) = e$  if  $X \notin FV(e)$ , and by usual rules that turn “=” into congruence. The system  $\mathbf{F}_c$  is obtained if one more equality axiom is added:  $(C) \quad eT = eT'$  for  $\Gamma; E \vdash e : \forall X.S$  and  $X \notin FV(S)$ .

**History:** Introduced by Girard [8] and Reynolds [6]. Inspired works on higher order type systems. Included by Barendregt in his  $\lambda$ -cube [4]. Various extensions were considered, for example,  $\mathbf{F}_c$  [2],  $\mathbf{F}$  with subtyping [5, 1]. Important for functional programming languages.

**Remarks:** A strong normalization theorem for  $\mathbf{F}$  was proved by Girard [7]. It implies a normalization theorem and consistency for second order arithmetic  $PA_2$ . For  $\mathbf{F}_c$ , a *genericity theorem* holds [2].

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- [1] G. Longo, K. Milsted, and S. Soloviev. “Coherence and Transitivity of Subtyping as Entailment”. In: *Journal of Logic and Computation* 10 (2000).

- [2] G. Longo, K. Milsted, and S. Soloviev. “The Genericity Theorem and the Notion of Parametricity in the Polymorphic  $\lambda$ -calculus”. In: *Theoretical Computer Science* 121 (1993).
- [3] Andrea Asperti and Giuseppe Longo. *Categories, Types and Structures*. Cambridge, Mass., London, England: The MIT Press, 1991.
- [4] H.P. Barendregt. “Introduction to generalized type systems”. In: *Journal of Functional Programming* 2 (1991).
- [5] L. Cardelli, S. Martini, J.C. Mitchell, and A. Scedrov. “An Extension of System F with Subtyping”. In: *Lecture Notes in Computer Science* 526 (1991).
- [6] J.C. Reynolds. “Towards a Theory of Type Structure”. In: *Lecture Notes in Computer Science* 19 (1974).
- [7] J.-Y. Girard. “Interprétation fonctionnelle et élimination des coupures de l’arithmétique d’ordre supérieur”. PhD thesis. Université Paris VII, 1972.
- [8] J.-Y. Girard. “Une extension de l’interprétation fonctionnelle de Gödel à l’analyse et son application à l’élimination des coupures dans et la théorie des types”. In: *Proc. 2nd Scandinavian Logic Symposium*. North-Holland (1971).



## Expansion Proofs (1983)

*Expansion trees, eigenvariables, and the function  $\text{Sh}(-)$  (read *shallow formula of*), that maps an expansion tree to a formula, are defined as follows:*

1. If  $A$  is  $\top$  (true),  $\perp$  (false), or a literal, then  $A$  is an expansion tree with top node  $A$ , and  $\text{Sh}(A) = A$ .
2. If  $E$  is an expansion tree with  $\text{Sh}(E) = [y/x]A$  and  $y$  is not an eigenvariable of any node in  $E$ , then  $E' = \forall x.A +^y E$  is an expansion tree with top node  $\forall x.A$  and  $\text{Sh}(E') = \forall x.A$ . The variable  $y$  is called an *eigenvariable* of (the top node of)  $E'$ . The set of eigenvariables of all nodes in an expansion tree is called the *eigenvariables of the tree*.
3. If  $\{t_1, \dots, t_n\}$  (with  $n \geq 0$ ) is a set of terms and  $E_1, \dots, E_n$  are expansion trees with pairwise disjoint eigenvariable sets and with  $\text{Sh}(E_i) = [t_i/x]A$  for  $i \in \{1, \dots, n\}$ , then  $E' = \exists x.A +^{t_1} E_1 \dots +^{t_n} E_n$  is an expansion tree with top node  $\exists x.A$  and  $\text{Sh}(E') = \exists x.A$ . The terms  $t_1, \dots, t_n$  are known as the *expansion terms* of (the top node of)  $E'$ .
4. If  $E_1$  and  $E_2$  are expansion trees that share no eigenvariables and  $\circ \in \{\wedge, \vee\}$ , then  $E_1 \circ E_2$  is an expansion tree with top node  $\circ$  and  $\text{Sh}(E_1 \circ E_2) = \text{Sh}(E_1) \circ \text{Sh}(E_2)$ .

In the expansion tree  $\forall x.A +^x E$  (resp. in  $\exists x.A +^{t_1} E_1 \dots +^{t_n} E_n$ ), we say that  $x$  (resp.  $t_i$ ) *labels* the top node of  $E$  (resp.  $E_i$ , for any  $i \in \{1, \dots, n\}$ ). A term  $t$  *dominates* a node in an expansion tree if it labels a parent node of that node in the tree.

For an expansion tree  $E$ , the quantifier-free formula  $\text{Dp}(E)$ , called the *deep formula of  $E$* , is defined as:

- $\text{Dp}(E) = E$  if  $E$  is  $\top$ ,  $\perp$ , or a literal;
- $\text{Dp}(E_1 \circ E_2) = \text{Dp}(E_1) \circ \text{Dp}(E_2)$  for  $\circ \in \{\wedge, \vee\}$ ;
- $\text{Dp}(\forall x.A +^y E) = \text{Dp}(E)$ ; and
- $\text{Dp}(\exists x.A +^{t_1} E_1 \dots +^{t_n} E_n) = \text{Dp}(E_1) \vee \dots \vee \text{Dp}(E_n)$  if  $n > 0$ , and  $\text{Dp}(\exists x.A) = \perp$ .

Let  $\mathcal{E}$  be an expansion tree and let  $<_{\mathcal{E}}^0$  be the binary relation on the occurrences of expansion terms in  $\mathcal{E}$  defined by  $t <_{\mathcal{E}}^0 s$  if there is an  $x$  which is free in  $s$  and which is the eigenvariable of a node dominated by  $t$ . Then  $<_{\mathcal{E}}$ , the transitive closure of  $<_{\mathcal{E}}^0$ , is called the *dependency relation* of  $\mathcal{E}$ .

An expansion tree  $\mathcal{E}$  is said to be an *expansion proof* if  $<_{\mathcal{E}}$  is acyclic and  $\text{Dp}(\mathcal{E})$  is a tautology; in particular,  $\mathcal{E}$  is an *expansion proof of  $\text{Sh}(\mathcal{E})$* .

**Clarifications:** The soundness and completeness theorem for expansion trees is the following. A formula  $B$  is a theorem of first-order logic if and only if there is an expansion proof  $Q$  such that  $\text{Sh}(Q) = B$ .

**History:** Expansion trees and expansion proofs [3, 4] provide a simple generalization of both Herbrand's disjunctions and Gentzen's mid-sequent theorem to formulas that are not necessarily in prenex-normal form. These proof structures were

originally defined for higher-order classical logic and used to provide a generalization of Herbrand's theorem for higher-order logic as well as a soundness proof for skolemization in the presence of higher-order quantification. Expansion trees are an early example of a matrix-based proof system that emphasizes parallelism within proof structures in a manner similar to that found in linear logic proof nets [2]. That parallelism is explicitly analyzed in [1] using a multi-focused version of LKF [31].

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- [1] Kaustuv Chaudhuri, Stefan Hetzl, and Dale Miller. "A Multi-Focused Proof System Isomorphic to Expansion Proofs". In: *J. of Logic and Computation* (June 2014).
  - [2] Jean-Yves Girard. "Linear Logic". In: *Theoretical Computer Science* 50 (1987), pp. 1–102.
  - [3] Dale Miller. "A Compact Representation of Proofs". In: *Studia Logica* 46.4 (1987), pp. 347–370.
  - [4] Dale Miller. "Proofs in Higher-order Logic". PhD thesis. Carnegie-Mellon University, Aug. 1983.

## Bledsoe's Natural Deduction - Prover (1973-78)

**SPLIT: basic rules of Natural Deduction**(see {1}), for example

To prove  $A \wedge B$ , prove  $A$  and prove  $B$

To prove  $p \rightarrow A \wedge B$ , prove  $(p \rightarrow A) \wedge (p \rightarrow B)$

To prove  $p \vee q \rightarrow A$ , prove  $(p \rightarrow A) \wedge (q \rightarrow A)$

To prove  $\exists x P(x) \rightarrow D$ , prove  $P(y) \rightarrow D$ , where  $y$  is a new variable

**REDUCE: conversion rules**, for example

To prove  $x \in A \cap B$ , prove  $x \in A \wedge x \in B$

To prove  $S \in \mathcal{P}(A)$ , prove  $S \subset A \wedge S \in \mathcal{U}$

To prove  $x \in \sigma F$ , prove  $\exists y(y \in F \wedge x \in y)$

**DEFINITIONS**, example

$A \subset B$  is defined by  $\forall x(x \in A \rightarrow x \in B)$  and is replaced by  $x \in A \rightarrow x \in B$  or by  $x_o \in A \rightarrow x_o \in B$ , depending on the position of the formula in the theorem.

**IMPLY:** in addition to SPLIT and REDUCE rules,

- search for substitutions which unify some hypotheses and a conclusion and compose them until obtaining the empty substitution (theorem proved) or failing
- forward chaining : if  $P$  and  $P'$  are unified by  $\theta$  ( $P\theta = P'\theta$ ), then a hypothesis  $P' \wedge (P \rightarrow Q)$  is converted into  $P' \wedge (P \rightarrow Q) \wedge Q\theta$
- PEEK forward chaining : if  $P\theta = P'\theta$  and  $A$  has the definition  $(P \rightarrow Q)$ , then a hypothesis  $P' \wedge A$  is converted into  $P' \wedge A \wedge Q\theta$
- backward chaining : if  $A \rightarrow D$  and  $D\theta = C\theta$ , replace the conclusion  $C$  by  $A\theta$

**Clarifications:** Bledsoe's natural deduction may be seen as both an extension and a restriction of formal natural deduction {1}. In SPLIT and REDUCE, there is reduction but not expansion. Some subroutines convert expressions into forms convenient for applying the rules. The notions of hypothesis and conclusion are privileged.

**History:** After having applied the rules of IMPLY and REDUCE, the first version of **Prover** [3] called a resolution program if necessary. Then, in [2], these calls to resolution are completely replaced by IMPLY. **Prover** has been working in set theory, limit theorems, topology and program verification.

**Remarks:** The system is sound but not complete. Bledsoe emphasizes the fact that, with these methods, provers may succeed because they proceed in a natural human-like way [1].

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- [1] W. W. Bledsoe. "Non-resolution theorem proving". In: *Artificial Intelligence* 9 (1977), pp. 1–35.
  - [2] R. S. Boyer W. W. Bledsoe and W. H. Henneman. "Computer proofs of Limit theorems". In: 3 (1972).
  - [3] W. W. Bledsoe. "Splitting and reduction heuristics in automatic theorem proving". In: *Artificial Intelligence* 2 (1971), pp. 55–77.

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Entry 7 by: Dominique Pastre

## Natural Knowledge Bases - Muscadet (1984)

### Some of the rules given to the system :

Basic rules of Natural Deduction (similar to Bledsoe's SPLIT rules {7}).

Flatten : Replace  $P(f(x))$  by  $\exists y(y : f(x) \wedge P(y))$  or by  $\forall y(y : f(x) \Rightarrow P(y))$  depending on the position (positive or negative) of the formula in the theorem to be proved and in the definitions and lemmas.

### Rules automatically built by metarules from definitions :

If  $A \subset B$  and  $x \in A$  then  $x \in B$       If  $x \in \sigma E$ , then  $\exists y(y \in E \wedge x \in y)$

If  $C : A \cap B$  and  $x \in C$ , then  $x \in A$       If  $C : A \cap B$ ,  $x \in A$  and  $x \in B$ , then  $x \in C$

in place of (and more general than) given REDUCE conversion rules of {7}.

### and from universal hypotheses :

Universal hypotheses are removed and replaced by local rules (for a sub-theorem).

This replaces and generalizes PEEK forward-chaining of {7}.

**Clarifications:** “If  $C : A \cap B$ ” expresses that  $C$  is  $A \cap B$  which has already been introduced. Flattening is used to recursively create and name objects such as  $f(x)$ , and in a certain manner to “eliminate” functional symbols since the expression  $y : f(x)$  will be handled as if it was a predicate expression  $F(x)$ .

Rules are conditional actions. Actions may be defined by packs of rules. Metarules build rules from definitions, lemmas and universal hypotheses.

**History:** **Muscadet** [3, 2] is a knowledge-based system. Facts are hypotheses and the conclusion of a theorem or a sub-theorem to be proved, and all sorts of facts which give relevant information during the proof search process. Universal hypotheses are handled as local definitions (no skolemization). **Muscadet** worked in set theory, mappings and relations, topology and topological linear spaces, elementary geometry, discrete geometry, cellular automata, and TPTP problems. It attended CASC competitions. It is open software, freely available.

**Muscadet** is efficient for everyday mathematical problems which are expressed in a natural manner, and problems which involve many axioms, definitions or lemmas, but not for problems with only one large conjecture and few definitions.

**Remarks:** The system is sound but not complete (because of the use of many selective rules and heuristics). It displays proofs easily readable by a human reader.

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- [1] D. Pastre. *Muscadet version 4.1 : user's manual*. 2011, pp. 1–22. URL: <http://www.normalesup.org/~pastre/muscadet/manual-en.pdf>.
  - [2] D. Pastre. “Automated Theorem Proving in Mathematics”. In: *Annals on Artificial Intelligence and Mathematics* 8.3-4 (1993), pp. 425–447.
  - [3] D. Pastre. “MUSCADET : An Automatic Theorem Proving System using Knowledge and Metaknowledge in Mathematics”. In: *Artificial Intelligence* 38.3 (1989), pp. 257–318.

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Entry 8 by: Dominique Pastre

## Linear Logic (1986)

$\frac{}{\vdash A^\perp, A}$	$\frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}$	$\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}$
$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$	$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}$	$\frac{}{\vdash 1} \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}$
$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}$	$\frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$	$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \quad \frac{}{\vdash \Gamma, \top}$
$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$	$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}$	$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} \quad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}$
$(A \otimes B)^\perp = A^\perp \wp B^\perp \quad 1^\perp = \perp$ $(X^\perp)^\perp = X \quad (A \wp B)^\perp = A^\perp \otimes B^\perp \quad \perp^\perp = 1$ $(!A)^\perp = ?(A^\perp) \quad (A \oplus B)^\perp = A^\perp \& B^\perp \quad 0^\perp = \top$ $(?A)^\perp = !(A^\perp) \quad (A \& B)^\perp = A^\perp \oplus B^\perp \quad \top^\perp = 0$		
$\Gamma$ and $\Delta$ are lists of formulas. $\sigma$ is a permutation.		

**Clarifications:** If  $\Gamma = A_1, \dots, A_n$  then  $?\Gamma = ?A_1, \dots, ?A_n$ . Negation is not a connective. It is defined using De Morgan's laws so that  $(A^\perp)^\perp = A$ . The linear implication can be defined as  $A \multimap B = A^\perp \wp B$ .

**History:** **LL** [1] comes from the analysis of intuitionistic logic through Girard's decomposition of the intuitionistic implication into the linear implication:  $A \rightarrow B = !A \multimap B$ .

**Remarks:** Cut elimination holds. **LL** is sound and complete with respect to phase semantics [1]. **LL** is not decidable. Sequent calculi **LK** [2] and **LJ** [3] can be translated into **LL**.

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[1] Jean-Yves Girard. "Linear logic". In: *Theoretical Computer Science* 50 (1987), pp. 1–102.

## Intuitionistic Linear Logic (ILL) (1987)

STRUCTURAL			
$\frac{}{A \vdash A}$	$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} (cut)$	$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$	
MULTIPLICATIVE			
$\frac{}{\vdash 1}$	$\frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}$	$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$	$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$
	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$	$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}$	
ADDITIVE			
$\frac{}{\Gamma \vdash \top}$	$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$	$\frac{\Gamma, A_i \vdash B}{\Gamma, A_1 \& A_2 \vdash B}$	
$\frac{}{\Gamma, 0 \vdash A}$	$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \oplus A_2}$	$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}$	
EXPONENTIAL			
$\frac{! \Gamma \vdash A}{! \Gamma \vdash !A}$	$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}$	$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$	$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$

**Clarifications:** Succedents are sole formulas. Antecedents are ordered list of formulas. If  $\Gamma$  is the list  $A_1, \dots, A_n$  of formulas,  $! \Gamma$  denotes the list  $!A_1, \dots, !A_n$ . First order quantifier can be added with rules similar to **LJ** {3}. Conversely, removing the exponential rules leads to the intuitionistic multiplicative additive linear logic (IMALL). And by further removing the additive rules, the intuitionistic multiplicative linear logic (IMLL) [2] is obtained.

**History:** Introduced by Girard and Lafont in [1] as intuitionistic variant of LL. **ILL** has multiple applications in categorical logic. By contrast with the full intuitionistic linear logic {12}, there is no multiplicative disjunction in **ILL**.

**Remarks:** Enjoys cut elimination [1].

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- [1] Jean-Yves Girard and Yves Lafont. “Linear Logic and Lazy Computation”. In: *Theory and Practice of Software Development*. 1987, pp. 52–66.
  - [2] Grigori Efremovich Mints. “Closed categories and the theory of proofs”. In: *Zapiski Nauchnykh Seminarov POMI* 68 (1977), pp. 83–114.

## Pure Type Systems (1989)

$$\begin{array}{c}
 \frac{}{\vdash c : s} \text{ axiom } (c : s) \in \mathcal{A} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{ start } (x \notin \Gamma) \\
 \\
 \frac{\Gamma \vdash M : B \quad \Gamma \vdash A : s}{\Gamma, x : A \vdash M : B} \text{ weakening } (x \notin \Gamma) \\
 \\
 \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A. B : s_3} \text{ product } (s_1, s_2, s_3) \in \mathcal{R} \\
 \\
 \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \text{ abstraction} \\
 \\
 \frac{\Gamma \vdash M : \Pi x : A. B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B[x := N]} \text{ application} \\
 \\
 \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s \quad A \equiv_\beta B}{\Gamma \vdash M : B} \text{ conversion}
 \end{array}$$

**Clarifications:** *Pure type systems* (PTS) are a general class of typed  $\lambda$  calculus. They represent logical systems through the Curry-Howard correspondence and the “propositions as types” interpretation. The syntax is given by the grammar:

$$\mathcal{T} ::= \mathcal{V} \mid C \mid \Pi \mathcal{V} : \mathcal{T}. \mathcal{T} \mid \lambda \mathcal{V} : \mathcal{T}. \mathcal{T} \mid \mathcal{T} \mathcal{T}$$

where  $\mathcal{V}$  is a set of variables and  $C$  is a set of constants. A PTS is parameterized by a *specification*  $(\mathcal{S}, \mathcal{A}, \mathcal{R})$  where  $\mathcal{S} \subseteq C$  is the set of *sorts*,  $\mathcal{A} \subseteq C \times \mathcal{S}$  is the set of *axioms*, and  $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S} \times \mathcal{S}$  is the set of *rules*.

**History:** Pure type systems were independently introduced by Berardi and Terlouw as a generalization of systems of the  $\lambda$  cube, and further developed and popularized by Barendregt, Geuvers, Nederhof [4, 3, 2, 1]. Many important systems can be expressed as PTSs, including simply typed  $\lambda$  calculus ( $\lambda \rightarrow$ ),  $\lambda II$  calculus [15] ( $\lambda P$ ), system F [5] ( $\lambda 2$ ), and the calculus of constructions ( $\lambda C$ ):

$$\begin{array}{lll}
 \mathcal{S} = \{*, \square\} & \mathcal{A} = \{(*, \square)\} & \mathcal{R}_{\rightarrow} = \{(*, *, *)\} \\
 \mathcal{R}_P = \mathcal{R}_{\rightarrow} \cup \{(*, \square, \square)\} & \mathcal{R}_2 = \mathcal{R}_{\rightarrow} \cup \{(\square, *, *)\} & \mathcal{R}_C = \mathcal{R}_P \cup \mathcal{R}_2 \cup \{(\square, \square, \square)\}
 \end{array}$$

as well as intuitionistic higher-order logic ( $\lambda HOL$ ). Pure type systems form the basis of many proof assistants such as Automath, Lego, Coq, Agda, and Matita.

**Remarks:** Soundness and decidability of type checking in PTSs are closely related to *strong normalization* (SN), i.e. the property that all well-typed terms terminate. Not all pure type systems are SN. Examples of PTSs that are *not* SN (and are therefore inconsistent) are the universal PTS  $\lambda *$ :

$$\mathcal{S} = \{*\} \quad \mathcal{A} = \{(*, *)\} \quad \mathcal{R} = \{(*, *, *)\}$$

and Girard’s system U.

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- [1] Herman Geuvers. “Logics and type systems”. PhD thesis. University of Nijmegen, 1993.
  - [2] Henk Barendregt. “Lambda Calculi with Types”. In: *Handbook of Logic in Computer Science*. Ed. by Samson Abramsky, Dov M. Gabbay, and Thomas S. E. Maibaum. Vol. 2. Oxford University Press, 1992, pp. 117–309.
  - [3] Henk Barendregt. “Introduction to generalized type systems”. In: *Journal of Functional Programming* 1.2 (1991), pp. 125–154.
  - [4] Herman Geuvers and Mark-Jan Nederhof. “Modular proof of strong normalization for the calculus of constructions”. In: *Journal of Functional Programming* 1.2 (1991), pp. 155–189.



## Full Intuitionistic Linear Logic (FILL) (1990)

$$\begin{array}{c}
\frac{}{x:A \vdash x:A} Ax \qquad \frac{\Gamma \vdash t:A \mid \Delta \quad \Gamma', y:A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta \mid [t/y]\Delta'} Cut \\
\\
\frac{\Gamma \vdash \Delta}{\Gamma, x:\top \vdash \text{let } x \text{ be } * \text{ in } \Delta} \top_L \qquad \frac{}{\cdot \vdash * : \top} \top_R \\
\\
\frac{}{x:\perp \vdash \cdot} \perp_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \circ : \perp \mid \Delta} \perp_R \\
\\
\frac{\Gamma, x:A, y:B \vdash \Delta}{\Gamma, z:A \otimes B \vdash \text{let } z \text{ be } x \otimes y \text{ in } \Delta} \otimes_L \qquad \frac{\Gamma \vdash t_1:A \mid \Delta \quad \Gamma' \vdash t_2:B \mid \Delta'}{\Gamma, \Gamma' \vdash t_1 \otimes t_2 : A \otimes B \mid \Delta \mid \Delta'} \otimes_R \\
\\
\frac{\Gamma \vdash t:A \mid \Delta \quad \Gamma', x:B \vdash t_i:C_i}{\Gamma, y:A \multimap B, \Gamma' \vdash [y t/x]t_i : C_i \mid \Delta} \multimap_L \qquad \frac{\Gamma, x:A \vdash t:B \quad x \notin \text{FV}(\Delta)}{\Gamma \vdash \lambda x. t : A \multimap B \mid \Delta} \multimap_R \\
\\
\frac{\Gamma, x:A \vdash t_i:C_i \quad \Gamma', y:B \vdash t_j:D_j}{\Gamma, \Gamma', z:A \wp B \vdash \text{let-pat } z(x \wp -)t_i : C_i \mid \text{let-pat } z(- \wp y)t_j : D_j} \wp_L \\
\\
\frac{\Gamma \vdash \Delta \mid t_1 : A \mid t_2 : B \mid \Delta'}{\Gamma \vdash \Delta \mid t_1 \wp t_2 : A \wp B \mid \Delta'} \wp_R
\end{array}$$

**Clarifications:** Both the left-hand and right-hand sides of sequents above are multisets of formulas, denoted  $\Gamma$  and  $\Delta$ . The terms annotating formulas are standard terms used in the simply typed  $\lambda$ -calculus. Capture avoiding substitution is denoted by  $[t/x]t'$ , and uniformly replaces every occurrence of  $x$  in  $t'$  with  $t$ . The definition of the let-pattern function used in the rule  $\wp_L$  is defined as follows:

$$\begin{array}{l}
\text{let-pat } z(x \wp -)t = t \quad \text{let-pat } z(- \wp y)t = t \quad \text{let-pat } z p t = \text{let } z \text{ be } p \text{ in } t \\
\text{where } x \notin \text{FV}(t) \qquad \text{where } y \notin \text{FV}(t)
\end{array}$$

We denote vectors of terms (resp. types) by  $t_i$  (resp.  $A_j$ ). The function  $\text{FV}(\Delta)$  constructs the set of all free variables in each term found in  $\Delta$ .

**History:** The original formulation of FILL by Valeria de Paiva in her thesis [7] did not satisfy cut-elimination, as shown by Schellinx. Martin Hyland and Valeria de Paiva [6] added a term assignment system to cope with the notion of dependency in the right implication rule and obtain cut-elimination. However, there was still a mistake in the par rule in [6], which was corrected independently, with different proof methods, by Bierman [5], Bellin [4], Brauner/dePaiva [3], dePaiva/Ritter [2]. The version here is the minimal modification suggested by Bellin, (who used proofnets), but using a traditional term assignment, as described in Eades/dePaiva [1].

- [1] Harley Eades III and Valeria de Paiva. “Multiple Conclusion Intuitionistic Linear Logic and Cut Elimination”. <http://metatheorem.org/papers/FILL-cut-report.pdf>.
- [2] Valeria de Paiva and Eike Ritter. “A Parigot-style Linear Lambda-calculus for Full Intuitionistic Linear Logic”. In: *Theory and Applications of Categories* 17.3 (2006), pp. 127–152.
- [3] Torben Braüner and Valeria de Paiva. “A formulation of linear logic based on dependency-relations”. In: *Computer Science Logic*. Ed. by Mogens Nielsen and Wolfgang Thomas. Vol. 1414. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 1998, pp. 129–148.
- [4] Gianluigi Bellin. “Subnets of proof-nets in multiplicative linear logic with MIX”. In: *Mathematical Structures in Computer Science* 7 (Dec. 1997), pp. 663–669.
- [5] Gavin Bierman. “A note on full intuitionistic linear logic”. In: *Annals of Pure and Applied Logic* 79.3 (1996), pp. 281–287.
- [6] Martin Hyland and Valeria de Paiva. “Full intuitionistic linear logic (extended abstract)”. In: *Annals of Pure and Applied Logic* 64.3 (1993), pp. 273–291.
- [7] Valeria de Paiva. “The Dialectica Categories”. PhD thesis. University of Cambridge, 1990.

## Constructive Classical Logic LC (1991)

$\frac{}{\vdash \neg P; P}$	$\frac{\vdash \Gamma; P \quad \vdash \Delta, \neg P; \Pi}{\vdash \Gamma, \Delta; \Pi}$	$\frac{\vdash \Gamma, N; \quad \vdash \Delta, \neg N; \Pi}{\vdash \Gamma, \Delta; \Pi}$	
$\frac{\vdash \Gamma; \Pi}{\vdash \sigma(\Gamma); \Pi}$	$\frac{\vdash \Gamma; P}{\vdash \Gamma, P;}$	$\frac{\vdash \Gamma, A, A; \Pi}{\vdash \Gamma, A; \Pi}$	$\frac{\vdash \Gamma; \Pi}{\vdash \Gamma, A; \Pi}$
$\frac{}{\vdash ; V}$	$\frac{}{\vdash \Gamma, \neg F; \Pi}$		
$\frac{\vdash \Gamma; P \quad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \wedge Q}$	$\frac{\vdash \Gamma, M; \Pi \quad \vdash \Delta, N; \Pi}{\vdash \Gamma, \Delta, M \wedge N; \Pi}$		
$\frac{\vdash \Gamma; P \quad \vdash \Delta, N;}{\vdash \Gamma, \Delta; P \wedge N}$	$\frac{\vdash \Gamma, M; \quad \vdash \Delta; Q}{\vdash \Gamma, \Delta; M \wedge Q}$		
$\frac{\vdash \Gamma, A, B; \Pi}{\vdash \Gamma, A \vee B; \Pi}$	$A \vee B \text{ negative}$	$\frac{\vdash \Gamma; P}{\vdash \Gamma; P \vee Q}$	$\frac{\vdash \Gamma; Q}{\vdash \Gamma; P \vee Q}$
$\neg \neg X = X$	$\neg(A \wedge B) = \neg A \vee \neg B$	$\neg(A \vee B) = \neg A \wedge \neg B$	
Formulas:	$A, B ::= P \mid N$		
Positive formulas:	$P, Q ::= X \mid V \mid F \mid P \wedge Q \mid P \wedge N \mid M \wedge Q \mid P \vee Q$		
Negative formulas:	$M, N ::= \neg X \mid \neg V \mid \neg F \mid M \vee N \mid M \vee Q \mid P \vee N \mid M \wedge N$		
$\Gamma$ and $\Delta$ are lists of formulas, and $\Pi$ consists of 0 or 1 positive formula. $\sigma$ is a permutation.			

**Clarifications:** Negation is not a connective. It is defined using De Morgan's laws so that  $\neg \neg A = A$ . There are two atomic formulas for truth (a positive one  $V$  and a negative one  $\neg F$ ) and two atomic formulas for falsity (a positive one  $F$  and a negative one  $\neg V$ ). Sequents have the shape  $\vdash \Gamma; \Pi$  where  $\Pi$  is called the stoup.

**History:** **LC** [2] comes from the analysis of classical logic inside the coherent semantics of linear logic [1] together with the use of the focusing property [1].

**Remarks:** Cut elimination holds. **LK** [2] can be translated into **LC**, but not in a canonical manner. **LC** satisfies constructive properties such as the disjunction property: if  $\vdash ; P \vee Q$  is provable then  $\vdash ; P$  or  $\vdash ; Q$  as well. **LC** admits a denotational semantics through correlation spaces [2] (a variant of coherence spaces [1]).

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- [1] Jean-Marc Andreoli. "Logic Programming with Focusing Proofs in Linear Logic". In: *Journal of Logic and Computation* 2.3 (1992), pp. 297–347.
  - [2] Jean-Yves Girard. "A new constructive logic: classical logic". In: *Mathematical Structures in Computer Science* 1.3 (1991), pp. 255–296.

## Classical Natural Deduction ( $\lambda\mu$ -calculus) (1992)

STRUCTURAL SUBSYSTEM	
$\frac{c : \Gamma \vdash A^\alpha, \Delta}{\mu\alpha.c : \Gamma \vdash A \mid \Delta} \text{Focus}$	$\frac{A^a \in \Gamma}{a : \Gamma \vdash A \mid \Delta} Ax$
	$\frac{p : \Gamma \vdash A \mid \Delta \quad A^\alpha \in \Delta}{[\alpha]p : \Gamma \vdash \Delta} \text{Unfocus}$
INTRODUCTION RULES	
$\frac{p : \Gamma \vdash A_1 \wedge A_2 \mid \Delta}{\pi_1(p) : \Gamma \vdash A_i \mid \Delta} \wedge_E^i$	$\frac{p_1 : \Gamma \vdash A_1 \mid \Delta \quad p_2 : \Gamma \vdash A_2 \mid \Delta}{(p_1, p_2) : \Gamma \vdash A_1 \wedge A_2 \mid \Delta} \wedge_I$
$\frac{p : \Gamma \vdash A_1 \vee A_2 \mid \Delta \quad p_1 : \Gamma, A_1^{a_1} \vdash C \mid \Delta \quad p_2 : \Gamma, A_2^{a_2} \vdash C \mid \Delta}{\text{case } p \text{ of } [a_1 \Rightarrow p_1 \mid a_2 \Rightarrow p_2] : \Gamma \vdash C \mid \Delta} \vee_E$	
$\frac{q : \Gamma \vdash A_i \mid \Delta}{\iota_i(q) : \Gamma \vdash A_1 \vee A_2 \mid \Delta} \vee_I^i$	
$\frac{p : \Gamma \vdash A \rightarrow B \mid \Delta \quad q : \Gamma \vdash A \mid \Delta}{pq : \Gamma \vdash B \mid \Delta} \rightarrow_E$	$\frac{p : \Gamma, A^a \vdash B \mid \Delta}{\lambda a.p : \Gamma \vdash A \rightarrow B \mid \Delta} \rightarrow_I$
$\frac{p : \Gamma \vdash \exists x A \mid \Delta \quad q : \Gamma, A[y/x]^a \vdash C \mid \Delta}{\text{dest } p \text{ as } (y, a) \text{ in } q : \Gamma \vdash C \mid \Delta} \exists_E$	$\frac{p : \Gamma \vdash A[t/x] \mid \Delta}{(t, p) : \Gamma \vdash \exists x A \mid \Delta} \exists_I$
$\frac{p : \Gamma \vdash \forall x A \mid \Delta}{pt : \Gamma \vdash A[t/x] \mid \Delta} \forall_E$	$\frac{p : \Gamma \vdash A[y/x] \mid \Delta}{\lambda y.p : \Gamma \vdash \forall x A \mid \Delta} \forall_I$
$\frac{p : \Gamma \vdash \perp \mid \Delta}{\text{efq } p : \Gamma \vdash C \mid \Delta} \perp_E$	$\frac{}{\Gamma \vdash () : \top \mid \Delta} \top_I$

**Clarifications:** There are two kinds of sequents: first  $p : \Gamma \vdash A \mid \Delta$  with a distinguished formula on the right for typing the so-called *unnamed* term  $p$ , second  $c : \Gamma \vdash \Delta$  with no distinguished formula for typing the so-called *named* term  $c$ . The syntax of the underlying  $\lambda\mu$ -calculus is:

$$\begin{aligned}
 c &::= [\alpha]p \\
 p, q &::= a \mid \mu\alpha.c \mid (p, p) \mid \pi_i(p) \mid \iota_i(p) \mid \text{case } p \text{ of } [a_1 \Rightarrow p_1 \mid a_2 \Rightarrow p_2] \\
 &\quad \mid \lambda a.p \mid pq \mid \lambda x.p \mid pt \mid (t, p) \mid \text{dest } p \text{ as } (x, a) \text{ in } q \mid () \mid \text{efq } p
 \end{aligned}$$

The variables used for referring to assumptions in  $\Gamma$  and to conclusions in  $\Delta$  range over distinct classes (denoted by Latin and Greek letters respectively). In the rules  $\exists_E$  (resp.  $\forall_I$ ),  $y$  is assumed fresh in  $\Gamma, \Delta$  and  $\exists x A$  (resp.  $\forall x A$ ).

**History:** This system, defined in Parigot [6], highlights that classical logic in natural deduction can be obtained from allowing several conclusions with contraction and

weakening on the right of the sequent, as in Gentzen’s LK. Additionally, the system assigns to this form of classical reasoning a computational content, based on the  $\mu$  and bracket operator which provides with a fine-grained decomposition of the operators `call-cc` (from Scheme/ML) or  $C$  (from [8]) that were known at this time to provide computational content to classical logic [7], as well as a decomposition of Prawitz’s classical elimination rule of negation [9].

The original presentation [6] only contains implication as well as first-order and second-order universal quantification à la Curry (i.e. without leaving trace of the quantification in the proof-term, what corresponds to computationally interpreting quantification as an intersection type). The presentation above has quantification à la Church (i.e. with an explicit trace in the proof term) what makes the calculus compatible with several reduction strategies such as both call-by-name or call-by-value (see e.g. [2]). Variants with multiplicative disjunctions can be found in [4] or [3], or multiplicative conjunctions in [2].

A standard variant originating in [5] uses only one kind of sequents, interpreting  $c : \Gamma \vdash \Delta$  as  $c : \Gamma \vdash \perp \mid \Delta$  (and hence removing  $\perp_E$  and merging the syntactic categories  $c$  and  $p$  into one). This variant is logically equivalent to the original presentation (in the presence of  $\perp$ ), but not computationally equivalent [1].

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## Z. Luo's LF (1994)

$$\begin{array}{c}
\frac{}{\langle \rangle \vdash \mathbf{valid}} \quad \frac{\Gamma \vdash K \mathbf{kind} \quad x \notin FV(\Gamma)}{\Gamma, x : K \vdash \mathbf{valid}} \quad \frac{\Gamma, x : K, \Gamma' \vdash \mathbf{valid}}{\Gamma, x : K, \Gamma' \vdash x : K} \quad (1) \\
\\
\frac{\Gamma \vdash k : K \quad \Gamma \vdash K = K'}{\Gamma \vdash k : K'} \quad \frac{\Gamma \vdash k = k' : K \quad \Gamma \vdash K = K'}{\Gamma \vdash k = k' : K'} \quad (2)^* \\
\\
\frac{\Gamma, x : K, \Gamma' \vdash J \quad \Gamma \vdash k : K}{\Gamma, [k/x]\Gamma' \vdash [k/x]J} \quad (3)^{**} \\
\\
\frac{\Gamma \vdash K \mathbf{kind} \quad \Gamma, x : K \vdash K' \mathbf{kind}}{\Gamma \vdash (x : K)K' \mathbf{kind}} \quad \frac{\Gamma \vdash K_1 = K_2 \quad \Gamma, x : K_1 \vdash K'_1 = K'_2}{\Gamma \vdash (x : K_1)K'_1 = (x : K_2)K'_2} \\
\frac{\Gamma \vdash K \mathbf{kind} \quad \Gamma, x : K \vdash K' \mathbf{kind}}{\Gamma \vdash (x : K)K' \mathbf{kind}} \quad \frac{\Gamma \vdash K_1 = K_2 \quad \Gamma, x : K_1 \vdash K'_1 = K'_2}{\Gamma \vdash (x : K_1)K'_1 = (x : K_2)K'_2} \\
\frac{\Gamma \vdash K \mathbf{kind} \quad \Gamma, x : K \vdash K' \mathbf{kind}}{\Gamma \vdash (x : K)K' \mathbf{kind}} \quad \frac{\Gamma \vdash K_1 = K_2 \quad \Gamma, x : K_1 \vdash K'_1 = K'_2}{\Gamma \vdash (x : K_1)K'_1 = (x : K_2)K'_2} \\
\frac{\Gamma \vdash K \mathbf{kind} \quad \Gamma, x : K \vdash K' \mathbf{kind}}{\Gamma \vdash (x : K)K' \mathbf{kind}} \quad \frac{\Gamma \vdash K_1 = K_2 \quad \Gamma, x : K_1 \vdash K'_1 = K'_2}{\Gamma \vdash (x : K_1)K'_1 = (x : K_2)K'_2} \quad (4) \\
\\
\frac{\Gamma \vdash \mathbf{valid}}{\Gamma \vdash \mathbf{Typekind}} \quad \frac{\Gamma \vdash A : \mathbf{Type}}{\Gamma \vdash El(A) \mathbf{kind}} \quad (5)
\end{array}$$

**Clarifications:** We follow [3]. Terms of **LF** are of the forms **Type**,  $El(A)$ ,  $(x : K)K'$  (dependent product),  $[x : K]K'$  (abstraction),  $f(k)$ , and judgements of the forms  $\Gamma \vdash \mathbf{valid}$  (validity of context),  $\Gamma \vdash K \mathbf{kind}$ ,  $\Gamma \vdash k : K$ ,  $\Gamma \vdash k = k' : K$ ,  $\Gamma \vdash K = K'$ . Rule groups: (1) rules for contexts and assumptions; (2)\* equality rules (reflexivity, symmetry and transitivity rules are omitted); (3)\*\* substitution rules ( $J$  denotes the right side of any of the five forms of judgement); (4) rules for dependent product kinds; (5) and the kind **Type**.

**History:** The calculus was defined in [3], ch. 9. LF is a typed version of Martin-Löf's logical framework [4]. Type theories specified in **LF** were used as basis of proof-assistants Lego and Plastic. Later the system was extended to include coercive subtyping [LuoSolXue:14, 1].

**Remarks:** The proof-theoretical analysis of LF above was used in meta-theoretical studies of larger theories defined on its basis, *e.g.*, UTT (Universal Type Theory) that includes inductive schemata, second order logic SOL with impredicative type *Prop* and a hierarchy of predicative universes [3]. H. Goguen defined a typed operational semantics for UTT and proved strong normalization theorem [2]. For **LF** with coercive subtyping conservativity results were obtained [LuoSolXue:14, 1].

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## $\bar{\lambda}$ -calculus (1994)

CUT-FREE SYSTEM	
$\frac{}{\Gamma; \cdot : A \vdash \cdot () : A} Ax$	$\frac{\Gamma; \cdot : A \vdash \cdot (l) : C \quad (a : A) \in \Gamma}{\Gamma \vdash a(l) : C} Cont$
$\frac{\Gamma \vdash p : A \quad \Gamma; \cdot : B \vdash \cdot (l) : C}{\Gamma \mid (p, l) : A \rightarrow B \vdash C} \rightarrow_L$	$\frac{\Gamma, a : A \vdash p : B}{\Gamma \vdash \lambda a. p : A \rightarrow B} \rightarrow_R$
CUT RULES	
$\frac{\Gamma \vdash p : A \quad \Gamma; \cdot : A \vdash \cdot (l) : C}{\Gamma \vdash p(l) : C} Cut_H^l$	$\frac{\Gamma; \cdot : A \vdash \cdot (l) : B \quad \Gamma; \cdot : B \vdash \cdot (l') : C}{\Gamma; \cdot : A \vdash \cdot (l@l') : C} Cut_H^2$
$\frac{\Gamma \vdash p : A \quad \Gamma, a : A, \Gamma' \vdash q : C}{\Gamma, \Gamma' \vdash q[p/a] : C} Cut_M^l$	$\frac{\Gamma \vdash p : A \quad \Gamma, a : A, \Gamma'; \cdot : B \vdash \cdot (l) : C}{\Gamma, \Gamma'; \cdot : B \vdash \cdot (l[p/a]) : C} Cut_M^2$

**Clarifications:** This calculus can be seen as an organization of the rules of Gentzen's intuitionistic sequent calculus in a way such that: there is computational interpretation of proofs as  $\lambda$ -calculus-like terms; there is a simple one-to-one correspondence between cut-free proofs and normal proofs of natural deduction.

The definition of the calculus is based on two kinds of sequents: the sequents  $\Gamma \vdash p : A$  have a focus on the right and are annotated by a program  $p$ ; the sequents  $\Gamma; \cdot : A \vdash \cdot (l) : B$  have an extra focussed formula on the left annotated by a placeholder name  $\cdot$  while the formula on the right is annotated by a program referring to this placeholder. The syntax of the underlying calculus is:

$$\begin{aligned} (l), (l') &::= () \mid (p, l) \mid (l@l') \mid (l[p/a]) \\ p, q &::= a(l) \mid \lambda a. p \mid p(l) \mid q[p/a] \end{aligned}$$

with  $()$  and  $(p, l)$  denoting lists of arguments,  $l@l'$  denoting concatenation of lists,  $l[p/a]$  and  $p[q/a]$  denoting explicit substitution,  $x(l)$  and  $p(l)$  denoting cut-free and non cut-free application, respectively. The first two items of each entry characterize the syntax of cut-free proofs.

**History:** The  $\bar{\lambda}$ -calculus has been designed in [2, 3]. It can be seen as the direct counterpart for sequent calculus of what  $\lambda$ -calculus is for natural deduction, along the lines of the Curry-Howard correspondence between proofs and programs. The idea of focussing a specific formula of the sequent comes from Girard [5] which himself credits it to Andreoli [4] (see also [31]). With proof annotations removed, the calculus can be seen as the intuitionistic fragment LJT of the subsystem LKT of LK [1], with LKT and LKQ representing two dual ways to add asymmetric focus to LK.



Extensions to other connectives than implication can be given. Extensions to classical logic, namely a computational presentation of LKT, can be obtained by adding the  $\mu$  and bracket operators of  $\lambda\mu$ -calculus [14] and by considering instead three kinds of sequents,  $\Gamma \vdash p : A \mid \Delta$ , or  $\Gamma; \cdot : A \vdash \cdot (I) : B$ , or  $c : (\Gamma \vdash \Delta)$  (see [3]). A variant with implicit substitution is possible.

The symmetrization of  $\bar{\lambda}$ -calculus led to  $\mathbf{LK}_{\mu\bar{\mu}}$  [21].

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## Sequent Calculus G3c (1996)

$\frac{}{P, \Gamma \vdash \Delta, P} \text{Ax}$	$\frac{}{\perp, \Gamma \vdash \Delta} \text{L}\perp$
$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \text{L}\wedge$	$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \text{R}\wedge$
$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \text{L}\vee$	$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \text{R}\vee$
$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \rightarrow B, \Gamma \vdash \Delta} \text{L}\rightarrow$	$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B} \text{R}\rightarrow$
$\frac{\forall x A, A[x/t], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \text{L}\forall$	$\frac{\Gamma \vdash \Delta, A[x/y]}{\Gamma \vdash \Delta, \forall x A} \text{R}\forall$
$\frac{A[x/y], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \text{L}\exists$	$\frac{\Gamma \vdash \Delta, A[x/t], \exists x A}{\Gamma \vdash \Delta, \exists x A} \text{R}\exists$

$P$  should be atomic in Ax and  $y$  should not be free in the conclusion of R $\forall$  and L $\exists$

**Clarifications:** Sequents are based on multisets. A formula  $A[x/t]$  is the result of uniformly substituting the term  $t$  for the variable  $x$  in  $A$ , renaming bound variables to prevent clashes with the variables in  $t$ .

**Remarks:** G3c is sound and complete w.r.t. classical first-order logic. Weakening and contraction are depth-preserving admissible and all rules are depth-preserving invertible.

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## Graph-based tableaux for modal logics (1997)

<b>BOOLEAN RULES</b>		
$\frac{\Gamma, A, \neg A \bullet}{\Gamma, A, \neg A, \perp \bullet} (\perp)$	$\frac{\Gamma, A \wedge B \bullet}{\Gamma, A \wedge B, A, B \bullet} (\wedge)$	$\frac{\Gamma, A_1 \vee A_2 \bullet}{\Gamma, A_1 \vee A_2, A_i \bullet} (\vee)$
<b>DIAMOND RULE</b>		
$\frac{\Gamma, \Diamond A \bullet}{\Gamma, \Diamond A \bullet \rightarrow \bullet A} (\Diamond)$		
<b>PROPAGATION RULES</b>		
$\frac{\Gamma, \Box A \bullet \rightarrow \bullet \Delta}{\Gamma, \Box A \bullet \rightarrow \bullet \Delta, A} (K)$		
$\frac{\Gamma, \Box A \bullet}{\Gamma, \Box A, A \bullet} (T)$	$\frac{\Gamma, \Box A \bullet \rightarrow \bullet \Delta}{\Gamma, \Box A \bullet \rightarrow \bullet \Delta, \Box A} (4)$	$\frac{\Gamma \bullet \rightarrow \bullet \Delta, \Box A}{\Gamma, A \bullet \rightarrow \bullet \Delta, \Box A} (B)$
$\frac{\Gamma \bullet \rightarrow \bullet \Delta_1, \Box A}{\Gamma \bullet \rightarrow \bullet \Delta_1, \Box A} (5_{\rightarrow})$	$\frac{\Gamma \bullet \rightarrow \bullet \Delta, \Box A}{\Gamma, \Box A \bullet \rightarrow \bullet \Delta, \Box A} (5_{\uparrow})$	$\frac{\Gamma \bullet \rightarrow \bullet \Delta_1, \Box A}{\Gamma \bullet \rightarrow \bullet \rightarrow \bullet \Delta_2, \Box A} (5_{\downarrow})$
<b>STRUCTURAL RULES</b>		
$\frac{\Gamma \bullet}{\Gamma \bullet \rightarrow \bullet \emptyset} (D)$	$\frac{\Gamma \bullet \rightarrow \bullet \Delta}{\Gamma \bullet \rightarrow \bullet \rightarrow \bullet \Delta} (De)$	
$\frac{\Gamma \bullet \rightarrow \bullet \Delta_1}{\Gamma \bullet \rightarrow \bullet \rightarrow \bullet \Delta_2, \Delta_1} (C_0)$	$\frac{\Gamma \bullet \rightarrow \bullet \Delta}{\Gamma \bullet \rightarrow \bullet \rightarrow \bullet \emptyset} (C_1)$	

**Clarifications:** The method constructs a collection of rooted directed acyclic graphs with vertices labeled with sets of formulas.  $\Gamma \bullet$  denotes a vertex labeled with  $\Gamma$ . To each branch of the tableau corresponds a graph. The only branching rule is  $(\vee)$ , for

which  $i$  must be chosen among  $\{1, 2\}$ . A branch is closed if the corresponding graph contains a vertex of the form  $\Gamma, \perp \bullet$ . For modal logic K, only rules  $(\perp)$ ,  $(\wedge)$ ,  $(\vee)$ ,  $(\diamond)$  and  $(K)$  are used. To each additional axiom T, 4, B, 5, D, De and C corresponds a set of rules to add, as detailed in Table 18.1 below.

Axiom	Model's property	Rules
$T = \Box A \rightarrow A$	reflexivity	$(T)$
$4 = \Box A \rightarrow \Box \Box A$	transitivity	$(4)$
$B = \Diamond \Box A \rightarrow A$	symmetry	$(B)$
$5 = \Diamond \Box A \rightarrow \Box A$	euclideanity	$(5_{\rightarrow}), (5_{\uparrow}), (5_{\downarrow})$
$D = \Box A \rightarrow \Diamond A$	seriality	$(D)$
$De = \Diamond A \rightarrow \Diamond \Diamond A$	density	$(De)$
$C = \Diamond \Box A \rightarrow \Box \Diamond A$	confluence	$(C_0), (C_1)$

**Table 18.1** Correspondences between modal axioms and graph-based tableaux rules.

**History:** Tableaux methods for modal logics have a long history started by M. Fitting [5]. The present method, introduced in [4] and extended in [3], distinguish itself by its ability to deal with properties like confluence or density. Moreover, it can be easily adapted to multimodal logics. The method has been enhanced and implemented in the LoTREC prover [1].

**Remarks:** The method is sound and complete for any combination of axioms. Termination is more problematic and has been investigated in [3, 2, 1].

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## Synthetic Tableaux (2000)

The synthesizing rules are:

$$\begin{array}{ccc}
 \frac{\neg A}{A \rightarrow B} \mathbf{r}_{\rightarrow}^1 & \frac{B}{A \rightarrow B} \mathbf{r}_{\rightarrow}^2 & \frac{A}{\neg(A \rightarrow B)} \mathbf{r}_{\rightarrow}^3 \\
 \\
 \frac{A}{A \vee B} \mathbf{r}_{\vee}^1 & \frac{B}{A \vee B} \mathbf{r}_{\vee}^2 & \frac{\neg A}{\neg(A \vee B)} \mathbf{r}_{\vee}^3 \\
 \\
 \frac{\neg A}{\neg(A \wedge B)} \mathbf{r}_{\wedge}^1 & \frac{\neg B}{\neg(A \wedge B)} \mathbf{r}_{\wedge}^2 & \frac{A}{A \wedge B} \mathbf{r}_{\wedge}^3 & \frac{A}{\neg\neg A} \mathbf{r}_{\neg}
 \end{array}$$

The premises of rules  $\mathbf{r}_{\rightarrow}^3$ ,  $\mathbf{r}_{\vee}^3$ ,  $\mathbf{r}_{\wedge}^3$  may occur in any order.

The branching rule:

$$\begin{array}{c}
 \wedge \\
 p_i \quad \neg p_i
 \end{array}$$

**Clarifications:** A Synthetic Tableau for a formula  $A$  is a finite tree with the following properties: the tree is generated by the above rules (the root is empty), each formula labelling a node of the tree is a subformula of  $A$  or the negation of a subformula of  $A$ , each leaf is labelled with  $A$  or  $\neg A$ . The tableau is a proof of  $A$  if each leaf is labelled with  $A$ .

**History:** The method has been first presented in [4], [5], [3]. In [3], [2] and [1] it is also presented for some extensional many-valued logics and for some paraconsistent logics.

**Remarks:** The method is sound and complete with respect to Classical Propositional Logic and constitutes a decision procedure for CPL. The same holds with respect to the non-classical logics for which the method has been described, see [3], [2], [1].

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## Polarized Linear Logic (2000)

$\frac{}{\vdash P^\perp, P}$	$\frac{\vdash \Gamma, P \quad \vdash \Delta, P^\perp, \Pi}{\vdash \Gamma, \Delta, \Pi}$	$\frac{\vdash \Gamma, \Pi}{\vdash \sigma(\Gamma), \Pi}$
$\frac{\vdash \Gamma, P \quad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}$	$\frac{\vdash \Gamma, N, M, \Pi}{\vdash \Gamma, N \wp M, \Pi}$	$\frac{}{\vdash 1} \quad \frac{\vdash \Gamma, \Pi}{\vdash \Gamma, \perp, \Pi}$
$\frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q}$	$\frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}$	$\frac{\vdash \Gamma, M, \Pi \quad \vdash \Gamma, N, \Pi}{\vdash \Gamma, M \& N, \Pi} \quad \frac{}{\vdash \Gamma, \top, \Pi}$
$\frac{\vdash \Gamma, P}{\vdash \Gamma, ?P}$	$\frac{\vdash \Gamma, N}{\vdash \Gamma, !N}$	$\frac{\vdash \Gamma, N, N, \Pi}{\vdash \Gamma, N, \Pi} \quad \frac{\vdash \Gamma, \Pi}{\vdash \Gamma, N, \Pi}$
$  \begin{array}{lll}  (P \otimes Q)^\perp = P^\perp \wp Q^\perp & 1^\perp = \perp & \\  (!N)^\perp = ?(N^\perp) & (P \oplus Q)^\perp = P^\perp \& Q^\perp & 0^\perp = \top \\  (X^\perp)^\perp = X & (N \wp M)^\perp = N^\perp \otimes M^\perp & \perp^\perp = 1 \\  (?P)^\perp = !(P^\perp) & (N \& M)^\perp = N^\perp \oplus M^\perp & \top^\perp = 0  \end{array}  $		
Positive formulas: $P, Q ::= X \mid P \otimes Q \mid 1 \mid P \oplus Q \mid 0 \mid !N$ Negative formulas: $N, M ::= X^\perp \mid N \wp M \mid \perp \mid N \& M \mid \top \mid ?P$ $\Gamma$ and $\Delta$ are lists of negative formulas. $\Pi$ consists of 0 or 1 positive formula. $\sigma$ is a permutation.		

**Clarifications:** Negation is not a connective. It is defined using De Morgan's laws so that  $(A^\perp)^\perp = A$ . Negative connectives which turn negative formulas into negative formulas ( $\wp$ ,  $\perp$ ,  $\&$  and  $\top$ ) are the reversible connectives of **LL** {9}. Their dual, the positive connectives ( $\otimes$ ,  $1$ ,  $\oplus$ ,  $0$ ) have the focusing property **[focal]**, related here with the “at most one positive formula” property of sequents.

**History:** **LLP** [1] comes from the natural embedding of Girard's **LC** {13} into linear logic {9}. It is obtained by restricting **LL** to polarized formulas and then by generalizing the structural rules (contraction, weakening and context of promotion) to arbitrary negative formulas, not only those starting with a  $?$ -connective.

**Remarks:** Cut elimination holds. In the categorical models of **LL**, positive formulas are interpreted as  $\otimes$ -comonoids while negative formulas are interpreted as  $\wp$ -monoids.

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- [1] Olivier Laurent. “Étude de la polarisation en logique”. Thèse de Doctorat. Université Aix-Marseille II, Mar. 2002.

## LK<sub>μ $\tilde{\mu}$</sub> (2000)

### STRUCTURAL SUBSYSTEM

$$\frac{(a : A) \in \Gamma}{\Gamma \vdash a : A \mid \Delta} Ax_R \quad \frac{\Gamma \vdash v : A \mid \Delta \quad \Gamma \mid e : A \vdash \Delta}{\langle v \mid e \rangle : (\Gamma \vdash \Delta)} Cut \quad \frac{(\alpha : A) \in \Delta}{\Gamma \mid \alpha : A \vdash \Delta} Ax_L$$

$$\frac{c : (\Gamma, a : A \vdash \Delta)}{\Gamma \mid \tilde{\mu}a.c : A \vdash \Delta} Focus_L \quad \frac{c : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha.c : A \mid \Delta} Focus_R$$

### INTRODUCTION RULES

$$\frac{\Gamma \mid e : A_i \vdash \Delta}{\Gamma \mid \pi_i \cdot e : A_1 \wedge A_2 \vdash \Delta} \wedge_L^i \quad \frac{\Gamma \vdash v_1 : A_1 \mid \Delta \quad \Gamma \vdash v_2 : A_2 \mid \Delta}{\Gamma \vdash (v_1, v_2) : A_1 \wedge A_2 \mid \Delta} \wedge_R$$

$$\frac{\Gamma \mid e_1 : A_1 \vdash \Delta \quad \Gamma \mid e_2 : A_2 \vdash \Delta}{\Gamma \mid [e_1, e_2] : A_1 \vee A_2 \vdash \Delta} \vee_L \quad \frac{\Gamma \vdash v : A_i \mid \Delta}{\Gamma \vdash u_i(v) : A_1 \vee A_2 \mid \Delta} \vee_R^i$$

$$\frac{\Gamma \vdash v : A \mid \Delta \quad \Gamma \mid e : B \vdash \Delta}{\Gamma \mid v \cdot e : A \rightarrow B \vdash \Delta} \rightarrow_L \quad \frac{\Gamma, a : A \vdash v : B \mid \Delta}{\Gamma \vdash \lambda a.v : A \rightarrow B \mid \Delta} \rightarrow_R$$

$$\frac{\Gamma \mid e : A[y] \vdash \Delta}{\Gamma \mid \tilde{\lambda}x.e : \exists x A[x] \vdash \Delta} \exists_L \quad \frac{\Gamma \vdash v : A[t] \mid \Delta}{\Gamma \vdash t \cdot v : \exists x A[x] \mid \Delta} \exists_R$$

$$\frac{\Gamma \mid e : A[t] \vdash \Delta}{\Gamma \mid t \cdot e : \forall x A[x] \vdash \Delta} \forall_L \quad \frac{\Gamma \vdash v : A[y] \mid \Delta}{\Gamma \vdash \lambda x.v : \forall x A[x] \mid \Delta} \forall_R$$

$$\frac{}{\Gamma \mid [] : \perp \vdash \Delta} \perp_L \quad \frac{}{\Gamma \vdash () : \top \mid \Delta} \top_R$$

**Clarifications:** There are three kinds of sequents: first  $\Gamma \vdash v : A \mid \Delta$  with a distinguished formula on the right for typing the program  $v$ , second  $\Gamma \mid e : A \vdash \Delta$  with a distinguished formula on the left for typing the evaluation context  $e$ , and finally  $c : (\Gamma \vdash \Delta)$  with no distinguished formula for typing command  $c$ , i.e. the interaction of a program within an evaluation context. The typing contexts  $\Gamma$  and  $\Delta$  are lists of named formulas so that a non-ambiguous correspondence with  $\lambda$ -calculus is possible (if it were sets or multisets, there were e.g. no way to distinguish the two distinct proofs of  $x : A, x : A \vdash x : A \mid$ ). Weakening rules are implemented implicitly at the level of axioms. Contraction rules are derived, using a cut against an axiom. No exchange rule is needed. Not all cuts are eliminable: only those not involving an axiom rule are. Negation  $\neg A$  can be defined as  $A \rightarrow \perp$ . In the rules  $\exists_E$  and  $\forall_R$ ,  $y$  is assumed fresh in  $\Gamma, \Delta$  and  $A[x]$ . The syntax of the underlying  $\lambda$ -calculus is:

$$c ::= \langle v \mid e \rangle$$

$$e ::= \alpha \mid \tilde{\mu}a.c \mid \pi_i \cdot e \mid [e, e] \mid v \cdot e \mid (t, e) \mid \tilde{\lambda}x.e \mid []$$

$$v ::= a \mid \mu\alpha.c \mid (v, v) \mid u_i(v) \mid \lambda a.v \mid \lambda x.v \mid (t, v) \mid ()$$



**History:** The purpose of this system is to provide with a  $\lambda$ -calculus-style computational meaning to Gentzen’s LK [2] and to highlight how the symmetries of sequent calculus show computationally. Seeing the rules as typing rules, the left/right symmetry is a symmetry between programs and their evaluation contexts. At the level of cut elimination, giving priority to the left-hand side relates to call-by-name evaluation while giving priority to the right-hand side relates to call-by-value evaluation [3]. Thanks to the presence of two dual axiom rules and implicit contraction rules, the system supports a tree-like sequent-free presentation like originally presented by Gentzen for natural deduction [**HerbelinHdR**] (see [25]). The system can be seen as a symmetric variant of  $\bar{\lambda}$ -calculus [16].

The structural subsystem can be adapted to various sequent calculi. Restriction to intuitionistic logic can be obtained by demanding that the right-hand side has exactly one formula.

The presentation of this calculus with conjunctive and disjunctive additive connectives has been studied in [**HerbelinHdR**, 2]. A variant with only commands, called  $X$ , has been studied in [1], based on previous work in [4]. Various extensions of the system emphasizing different symmetries can be found in the literature.

**Remarks:** The system is obviously logically equivalent to Gentzen’s **LK** when equipped with the corresponding connectives and observed through the sequents of the form  $\Gamma \vdash A$ .

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- [1] Steffen van Bakel, Stephane Lengrand, and Pierre Lescanne. “The Language X: Circuits, Computations and Classical Logic”. In: *Theoretical Computer Science, 9th Italian Conference, ICTCS 2005, Siena, Italy, October 12-14, 2005, Proceedings*. Ed. by Mario Coppo, Elena Lodi, and G. Michele Pinna. Vol. 3701. Springer, 2005, pp. 81–96. ISBN: 3-540-29106-7.
  - [2] Philip Wadler. “Call-by-value is dual to call-by-name”. In: *Proceedings of ICFP 2003, Uppsala, Sweden, August 25-29, 2003*. Ed. by Colin Runciman and Olin Shivers. Vol. 38(9). SIGPLAN Notices. ACM, 2003, pp. 189–201. ISBN: 1-58113-756-7.
  - [3] Pierre-Louis Curien and Hugo Herbelin. “The duality of computation”. In: *Proceedings of the Fifth ACM SIGPLAN International Conference on Functional Programming, ICFP 2000, Montreal, Canada, September 18-21, 2000*. SIGPLAN Notices 35(9). ACM, 2000, pp. 233–243. ISBN: 1-58113-202-6.
  - [4] Christian Urban. “Classical Logic and Computation”. Ph.D. Thesis. University of Cambridge, Oct. 2000.

## Socratic Proofs for CPL (2003)

$$\begin{array}{c}
 \frac{?( \Phi ; S' \alpha' T \vdash C ; \Psi )}{?( \Phi ; S' \alpha_1' \alpha_2' T \vdash C ; \Psi )} \mathbf{L}_\alpha \qquad \frac{?( \Phi ; S \vdash \alpha ; \Psi )}{?( \Phi ; S \vdash \alpha_1 ; S \vdash \alpha_2 ; \Psi )} \mathbf{R}_\alpha \\
 \\
 \frac{?( \Phi ; S' \beta' T \vdash C ; \Psi )}{?( \Phi ; S' \beta_1' T \vdash C ; S' \beta_2' T \vdash C ; \Psi )} \mathbf{L}_\beta \\
 \\
 \frac{?( \Phi ; S \vdash \beta ; \Psi )}{?( \Phi ; S' \beta_1^* \vdash \beta_2 ; \Psi )} \mathbf{R}_\beta \qquad \frac{?( \Phi ; S' \neg \neg A' T \vdash C ; \Psi )}{?( \Phi ; S' A' T \vdash C ; \Psi )} \mathbf{L}_{\neg\neg}
 \end{array}$$

Where:

$\alpha$	$\alpha_1$	$\alpha_2$	$\beta$	$\beta_1$	$\beta_2$	$\beta_1^*$
$A \wedge B$	$A$	$B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$A$
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	$A$	$B$	$\neg A$
$\neg(A \rightarrow B)$	$A$	$\neg B$	$A \rightarrow B$	$\neg A$	$B$	$A$

**Clarifications:** The method of Socratic proofs is a method of transforming questions, but these are based on sequences of two-sided, single-conclusion sequents with sequences of formulas in both cedents.  $\Phi, \Psi$  are finite (possibly empty) sequences of sequents.  $S, T$  are finite (possibly empty) sequences of formulas. The semicolon ‘;’ is the concatenation sign for sequences of sequents, whereas ‘’ is the concatenation sign for sequences of formulas. A Socratic proof of sequent ‘ $S \vdash A$ ’ in  $\mathbf{E}^*$  is a finite sequence of questions guided by the rules of  $\mathbf{E}^*$ , starting with ‘ $?(S \vdash A)$ ’ and ending with a question based on a sequence of basic sequents, where a *basic sequent* is a sequent containing the same formula in both of its cedents or containing a formula and its negation in the antecedent.

**History:** The method has been first presented in [2]. Calculus  $\mathbf{E}^*$  is called *erotetic* calculus since it is a calculus of questions (*erotema* means *question* in Greek). Proof-theoretically, it may be viewed as a calculus of hypersequents with ‘;’ understood conjunctively. It is grounded in Inferential Erotetic Logic (cf. [1]).

**Remarks:** A sequent ‘ $S \vdash A$ ’ has a Socratic proof in  $\mathbf{E}^*$  iff  $A$  is CPL-entailed by the set of terms of  $S$ . The rules are invertible.

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- [1] Andrzej Wiśniewski. *Questions, Inferences, and Scenarios*. London: College Publications, 2013.
  - [2] Andrzej Wiśniewski. “Socratic Proofs”. In: *Journal of Philosophical Logic* 33.3 (2004), pp. 299–326.

## Socratic Proofs for Modal Propositional K (2004)

The rules of calculus  $\mathbf{E}^K$ :

$$\frac{?(\Phi; \vdash S' (\alpha)^{\phi(i)} T; \Psi)}{?(\Phi; \vdash S' (\alpha_1)^{\phi(i)} T; \vdash S' (\alpha_2)^{\phi(i)} T; \Psi)} \mathbf{R}_\alpha$$

$$\frac{?(\Phi; \vdash S' (\beta)^{\phi(i)} T; \Psi)}{?(\Phi; \vdash S' (\beta_1)^{\phi(i)} (\beta_2)^{\phi(i)} T; \Psi)} \mathbf{R}_\beta \quad \frac{?(\Phi; \vdash S' (\neg\neg A)^{\phi(i)} T; \Psi)}{?(\Phi; \vdash S' (A)^{\phi(i)} T; \Psi)} \mathbf{R}_{\neg}$$

$$\frac{?(\Phi; \vdash S' (\mu)^{\phi(i)} T; \Psi)}{?(\Phi; \vdash S' (\mu_0)^{\phi(i),j} T; \Psi)} \mathbf{R}_\mu \quad \frac{?(\Phi; \vdash S' (\pi)^{\phi(i)} T; \Psi)}{?(\Phi; \vdash S' (\pi)^{\phi(i)} (\pi_0)^j T; \Psi)} \mathbf{R}_\pi$$

where:

$\alpha$	$\alpha_1$	$\alpha_2$	$\beta$	$\beta_1$	$\beta_2$	$\beta_1^*$	$\mu$	$\mu_0$	$\pi$	$\pi_0$
$A \wedge B$	$A$	$B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$A$	$\Box A$	$A$	$\neg \Box A$	$\neg A$
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	$A$	$B$	$\neg A$	$\neg \Diamond A$	$\neg A$	$\Diamond A$	$A$
$\neg(A \rightarrow B)$	$A$	$\neg B$	$A \rightarrow B$	$\neg A$	$B$	$A$				

$\phi(i)$  is a finite sequence of numerals ending with  $i$  (an index of a formula)

$\phi(i), j$  is a concatenation of  $\phi(i)$  and  $\langle j \rangle$

In  $\mathbf{R}_\mu$ , numeral  $j$  must be new with respect to the sequent distinguished in the premise. In  $\mathbf{R}_\pi$ , the pair  $\langle i, j \rangle$  is present in the premise sequent.

**Clarifications:** The method of Socratic proofs is a method of transforming questions but with a clear proof-theoretic interpretation (see also [22], [??]). The rules act upon right-sided sequents, with sequences of *indexed* formulas in the succedents. The indices store the semantic information. A Socratic proof starts with a question concerning  $?(\vdash (A)^1)$  and ends with a question based on a sequence of basic sequents, where a *basic sequent* is a sequent containing indexed formulas of the forms  $B^{\phi(i)}$ ,  $(\neg B)^{\psi(i)}$ .

**History:** The proof system has been presented in [2], the completeness proof may be found in [1].

**Remarks:** A sequent  $\vdash (A)^1$  has a Socratic proof in  $\mathbf{E}^K$  iff  $A$  is K-valid.

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- [1] Dorota Leszczyńska-Jasion. *The Method of Socratic Proofs for Normal Modal Propositional Logics*. Poznań: Adam Mickiewicz University Press, 2007.
  - [2] Dorota Leszczyńska. "Socratic Proofs for some Normal Modal Propositional Logics". In: *Logique et Analyse* 47.185-188 (2004), pp. 259–285.

## Socratic Proofs for Modal Propositional Logics (2004)

The rules of  $\mathbf{E}^L$  are the rules of  $\mathbf{E}^K$  (see {23}), where the proviso of applicability of  $\mathbf{R}_\mu$  depends on the logic  $L$  and is a combination of some of the following clauses:

1.  $\langle i, j \rangle$  is present in the premise sequent
2.  $i = j$
3.  $\langle j, i \rangle$  is present in the premise sequent
4. there is a sequence  $i_1, \dots, i_n$  such that  $i_1 = i$ ,  $i_n = j$  and each  $\langle i_k, i_{k+1} \rangle$ , where  $1 \leq k \leq n-1$ , is present in the premise sequent
5. there is a sequence  $i_1, \dots, i_n$  such that  $i_1 = i$ ,  $i_n = j$  and for each  $\langle i_k, i_{k+1} \rangle$ , where  $1 \leq k \leq n-1$ ,  $\langle i_k, i_{k+1} \rangle$  or  $\langle i_{k+1}, i_k \rangle$  is present in the premise sequent
6. there are sequences  $i_1, \dots, i_n$  and  $j_1, \dots, j_m$  such that  $i_1 = i$ ,  $i_n = j$ ,  $j_1 = i$ ,  $j_m = j$  and for each  $\langle i_k, i_{k+1} \rangle$ , where  $1 \leq k \leq n-1$ ,  $\langle i_k, i_{k+1} \rangle$  is present in the premise sequent and for each  $\langle j_l, j_{l+1} \rangle$ , where  $1 \leq l \leq m-1$ ,  $\langle j_l, j_{l+1} \rangle$  is present in the premise sequent

L	proviso	L	proviso	L	proviso
K, KD	(1)	K4, KD4	(4)	K5, D5	(6)
KT	(1) or (2)	S4	(2) or (4)	K45, D45	(4) or (6)
KB, KDB	(1) or (3)	KB4	(5)		
KTb	(1) or (2) or (3)	S5	(2) or (5)		

Calculi for logics: KD, KDB, KD4, KD5, KD45 have also the following rule, where  $j$  is new:

$$\frac{?(\Phi ; \vdash S'(\pi)^{\phi(i)} T ; \Psi)}{?(\Phi ; \vdash S'(\pi)^{\phi(i)}(\pi_0)^j T ; \Psi)} \mathbf{R}_{\pi D}$$

**Clarifications:** See {23}, {22}, {??} for more comments.

**History:** The proof system has been presented in [DLJ:2004], the completeness proof may be found in [DLJ:2007], and extensions to some non-basic modal logics in [1].

**Remarks:** A sequent  $\vdash (A)^1$  has a Socratic proof in  $\mathbf{E}^L$  iff  $A$  is  $L$ -valid.

- [1] Dorota Leszczyńska-Jasion. “The Method of Socratic Proofs for Modal Propositional Logics: K5, S4.2, S4.3, S4M, S4F, S4R and G”. In: *Studia Logica* 89.3 (2008), pp. 371–405.

## LK $\mu\tilde{\mu}$ in sequent-free tree form (2005)

STRUCTURAL SUBSYSTEM	
$\frac{\frac{\vdash A}{A \vdash} \text{Focus}_L \quad \frac{A \vdash}{\vdash A} \text{Focus}_R}{\vdash} \text{Cut}$	
INTRODUCTION RULES	
$\frac{A_i \vdash}{A_1 \wedge A_2 \vdash} \wedge_L^i$	$\frac{\vdash A_1 \quad \vdash A_2}{\vdash A_1 \wedge A_2} \wedge_R$
$\frac{A_1 \vdash \quad A_2 \vdash}{A_1 \vee A_2 \vdash} \vee_L$	$\frac{\vdash A_i}{\vdash A_1 \vee A_2} \vee_R^i$
$\frac{\vdash A \quad B \vdash}{A \rightarrow B \vdash} \rightarrow_L$	$\frac{\vdash B}{\vdash A \rightarrow B} \rightarrow_R$
$\frac{A[y] \vdash}{\exists x A[x] \vdash} \exists_L$	$\frac{\vdash A[t]}{\vdash \exists x A[x]} \exists_R$
$\frac{A[t] \vdash}{\forall x A[x] \vdash} \forall_L$	$\frac{\vdash A[y]}{\vdash \forall x A[x]} \forall_R$
$\frac{}{\perp \vdash} \perp_L$	$\frac{}{\vdash \top} \top_R$

**Clarifications:** There are three kinds of nodes,  $\vdash A$  for asserting formulas,  $A \vdash$  for refuting formulas, and  $\vdash$  for expressing a contradiction. Negation  $\neg A$  can be defined as  $A \rightarrow \perp$ . In the rules  $\exists_E$  and  $\forall_R$ ,  $y$  is assumed fresh in all the unbracketed assumption formula upon which that the derivation of  $A(y)$  depends.

**History:** The purpose of this system is to show that the original distinction in Gentzen [Gentzen1935] between natural deduction presented as a tree of formulas and sequent calculus presented as a tree of sequents is no longer relevant. It is known from at least Howard [2] that natural deduction can be presented with sequents. The above formulation shows that systems based on left and right introductions (“sequent-calculus style”) can be presented as a sequent-free tree of formulas [HerbelinHdR].

The terminology “sequent calculus” seems to have become popular from [Prawitz65] followed then e.g. by [3] who were associating the term “sequents” to Gentzen’s LJ and LK systems. The terminology having lost the connection to its etymology, this motivated some authors to use alternative terminologies such as “L” systems [Munch-Maccagnoni09].

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Entry 25 by: Hugo Herbelin

**Remarks:** As pointed out e.g. in [1] in the context of natural deduction, to obtain a computationally non-degenerate proof-as-program correspondence with a presentation of a calculus as a tree of formulas, the bracketed assumptions have to be annotated with the exact occurrence of the rule which bracketed them. Then, annotation by proof-terms can optionally be added as in {21}.

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- [1] Herman Geuvers. “Logics and Type Systems”. Section 3.2. Ph.D. Thesis. Katholieke Universiteit Nijmegen, Sept. 1993.
  - [2] William A. Howard. “The formulae-as-types notion of constructions”. In: *to H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*. Unpublished manuscript of 1969. Academic Press, 1980.
  - [3] Anne S. Troelstra. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*. Vol. 344. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 1973.

## Conditional Labelled Sequent Calculi SeqS (2003-2007)

$\text{(AX)} \Gamma, x : P \vdash \Delta, x : P \quad (P \text{ atomic})$	$\text{(A}\bot\text{)} \Gamma, x : \bot \vdash \Delta$
$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B}{\Gamma \vdash \Delta, x : A \Rightarrow B} \text{(}\Rightarrow \mathbf{R}\text{)} \quad (y \notin \Gamma, \Delta)$	
$\frac{\Gamma, x : A \Rightarrow B \vdash x \xrightarrow{A} y, \Delta \quad \Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B \vdash \Delta} \text{(}\Rightarrow \mathbf{L}\text{)}$	
$\frac{u : A \vdash u : B \quad u : B \vdash u : A}{\Gamma, x \xrightarrow{A} y \vdash x \xrightarrow{B} y, \Delta} \text{(EQ)}$	
$\frac{\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta}{\Gamma, x \xrightarrow{A} y \vdash \Delta} \text{(ID)}$	$\frac{\Gamma \vdash x \xrightarrow{A} x, x : A, \Delta}{\Gamma \vdash x \xrightarrow{A} x, \Delta} \text{(MP)}$
$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A \quad \Gamma[x/u, y/u], u \xrightarrow{A} u \vdash \Delta[x/u, y/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta} \text{(CS)} \quad (x \neq y, u \notin \Gamma, \Delta)$	
$\frac{\Gamma x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z \quad (\Gamma x \xrightarrow{A} y \vdash \Delta)[y/u, z/u]}{\Gamma x \xrightarrow{A} y \vdash \Delta} \text{(CEM)} \quad (y \neq z, u \notin \Gamma, \Delta)$	
<p>Given a sequent <math>\Gamma</math> and labels <math>x</math> and <math>u</math>, <math>\Gamma[x/u]</math> is the sequent obtained by replacing in <math>\Gamma</math> all occurrences of <math>x</math> with <math>u</math>.</p>	

**Clarifications:** Conditional logics extend classical logic with formulas of the form  $A \Rightarrow B$ . SeqS considers the *selection function* semantics:  $A \Rightarrow B$  is true in a world  $w$  if  $B$  is true in the set of worlds selected by the selection function  $f$  for  $A$  and  $w$  (that are most similar to  $w$ ). SeqS manipulate *labelled* formulas, where labels represent worlds, of the form  $x : A$  ( $A$  is true in  $x$ ) and  $x \xrightarrow{A} y$  ( $y$  belongs to  $f(x, A)$ ).

The calculi SeqS consider *normal* conditional logics, such that if  $A$  and  $B$  are true in the same worlds, then  $f(w, A) = f(w, B)$ . The rule **(EQ)** takes care of normality.

Besides the rules shown, SeqS also include standard rules for propositional connectives.

**History:** The calculi SeqS have been introduced in [1]. The theorem prover CondLean, implementing SeqS calculi in Prolog, has been presented in [3, 2].

**Remarks:** Completeness is a consequence of the admissibility of cut. The calculi SeqS can be used to obtain a PSPACE decision procedure for the respective conditional logics and to develop goal-directed proof procedures.

- [1] Nicola Olivetti, Gian Luca Pozzato, and Camilla Schwind. “A sequent calculus and a theorem prover for standard conditional logics”. In: *ACM Transactions on Computational Logic (ToCL)* 8 (4 2007), pp. 1–51. doi: 10.1145/1276920.1276924.
- [2] Nicola Olivetti and Gian Luca Pozzato. “CondLean 3.0: Improving Condlean for Stronger Conditional Logics”. In: *Automated Reasoning with Analytic Tableaux and Related Methods - Proceedings of TABLEAUX 2005 (14th International Conference on Automated Reasoning with Analytic Tableaux and Related Methods)*. Ed. by Bernhard Beckert. Vol. 3702. Lecture Notes in Artificial Intelligence LNAI. Koblenz, Germany: Springer, Sept. 2005, pp. 328–332. doi: 10.1007/11554554\_27.
- [3] Nicola Olivetti and Gian Luca Pozzato. “CondLean: A Theorem Prover for Conditional Logics”. In: *Automated Reasoning with Analytic Tableaux and Related Methods - Proceedings of TABLEAUX 2003 (12th International Conference on Automated Reasoning with Analytic Tableaux and Related Methods)*. Ed. by Marta Cialdea Mayer and Fiora Pirri. Vol. 2796. Lecture Notes in Artificial Intelligence LNAI. Roma, Italy: Springer, Sept. 2003, pp. 264–270. doi: 10.1007/978-3-540-45206-5\_23.



## Preferential Tableau Calculi $\mathcal{TP}^T$ (2005-2009)

$$\begin{array}{c}
 \Gamma, P, \neg P \text{ (AX)} \quad \text{with } P \text{ atomic} \\
 \\
 \frac{\Gamma, \neg(A \vdash B); \Sigma}{A, \Box \neg A, \neg B, \Gamma^{\vdash \pm}; \emptyset} (\vdash^-) \quad \frac{\Gamma, \neg \Box \neg A; \Sigma}{A, \Box \neg A, \Gamma^{\Box}, \Gamma^{\Box \perp}, \Gamma^{\vdash \pm}, \Sigma; \emptyset} (\Box^-) \\
 \\
 \frac{\Gamma, A \vdash B; \Sigma}{\Gamma, \neg A; \Sigma, A \vdash B} (\vdash^+) \quad \frac{\Gamma, A \vdash B; \Sigma}{\Gamma, \neg \Box \neg A; \Sigma, A \vdash B} (\vdash^+) \quad \frac{\Gamma, A \vdash B; \Sigma}{\Gamma, B; \Sigma, A \vdash B} (\vdash^+)
 \end{array}$$

**Clarifications:** According to Kraus, Lehmann and Magidor (KLM) [4], defeasible knowledge is represented by a (finite) set of nonmonotonic conditionals  $A \vdash B$  (normally the  $A$ 's are  $B$ 's). Models are possible-world structures equipped with a preference relation (irreflexive and transitive for **P**) among worlds or states. The meaning of  $A \vdash B$  is that  $B$  holds in the worlds/states where  $A$  holds and that are *minimal* with respect to the preference relation.

The calculus  $\mathcal{TP}^T$  is based on the idea of interpreting the preference relation as an accessibility relation: a conditional  $A \vdash B$  holds in a model if  $B$  is true in all minimal  $A$ -worlds, where a world  $w$  is an  $A$ -world if it satisfies  $A$ , and it is a minimal  $A$ -world if there is no  $A$ -world  $w'$  preferred to  $w$ .

Nodes are pairs  $\Gamma; \Sigma$ , where  $\Gamma$  is a set of formulas and  $\Sigma$  is a set of conditional formulas  $A \vdash B$ .  $\Sigma$  is used to keep track of positive conditionals  $A \vdash B$  to which the rule  $(\vdash^+)$  has already been applied: the idea is that one does not need to apply  $(\vdash^+)$  on the same conditional formula  $A \vdash B$  *more than once in the same world*. When  $(\vdash^+)$  is applied to a formula  $A \vdash B \in \Gamma$ , then  $A \vdash B$  is moved from  $\Gamma$  to  $\Sigma$  in the conclusions of the rule, so that it is no longer available for further applications in the current world. The dynamic rules re-introduce formulas from  $\Sigma$  to  $\Gamma$  in order to allow further applications of  $(\vdash^+)$  in new worlds.

Given  $\Gamma$ , we define:

- $\Gamma^{\Box} = \{\Box \neg A \mid \Box \neg A \in \Gamma\}$
- $\Gamma^{\Box \perp} = \{\neg A \mid \Box \neg A \in \Gamma\}$
- $\Gamma^{\vdash^+} = \{A \vdash B \mid A \vdash B \in \Gamma\}$
- $\Gamma^{\vdash^-} = \{\neg(A \vdash B) \mid \neg(A \vdash B) \in \Gamma\}$
- $\Gamma^{\vdash \pm} = \Gamma^{\vdash^+} \cup \Gamma^{\vdash^-}$

Besides the rules shown above, the calculus  $\mathcal{TP}^T$  also includes standard rules for propositional connectives.

**History:** In [4] Kraus, Lehmann and Magidor proposed a formalization of nonmonotonic reasoning that was early recognized as a landmark. According to their framework, defeasible knowledge is represented by a (finite) set of nonmonotonic conditionals or assertions of the form  $A \vdash B$ , whose reading is *normally (or typically) the  $A$ 's are  $B$ 's*. The operator “ $\vdash$ ” is nonmonotonic, in the sense that  $A \vdash B$  does not imply  $A \wedge C \vdash B$ .

The calculus  $\mathcal{TP}^T$  and extensions for all the logics of the KLM family are proposed in [1]. The theorem provers KLMLearn and FreeP implementing the tableau calculi have been presented at [3, 2].

**Remarks:** The calculus  $\mathcal{TP}^T$  can be used to define a decision procedure and obtain a complexity bound for the preferential logic **P**, namely that it is **coNP**-complete.

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## HO Sequent Calculi $\mathcal{G}_\beta$ and $\mathcal{G}_{\beta\text{fb}}$ (2003-2009)

<b>Basic Rules</b>	$\frac{\Delta, s}{\Delta, \neg\neg s} \mathcal{G}(\neg) \quad \frac{\Delta, \neg s \quad \Delta, \neg t}{\Delta, \neg(s \vee t)} \mathcal{G}(\vee_-) \quad \frac{\Delta, s, t}{\Delta, (s \vee t)} \mathcal{G}(\vee_+)$
	$\frac{\Delta, \neg(s l) \downarrow_\beta \quad l_\alpha \text{ closed term}}{\Delta, \neg \Pi^\alpha s} \mathcal{G}(\Pi_-^l) \quad \frac{\Delta, (sc) \downarrow_\beta \quad c_\delta \text{ new symbol}}{\Delta, \Pi^\alpha s} \mathcal{G}(\Pi_+^c)$
<b>Initialization</b>	$\frac{s \text{ atomic (and } \beta\text{-normal)}}{\Delta, s, \neg s} \mathcal{G}(\text{init}) \quad \frac{\Delta, (s \doteq^o t) \quad s, t \text{ atomic}}{\Delta, \neg s, t} \mathcal{G}(\text{Init}^\pm)$
<b>Extensionality</b>	$\frac{\Delta, (\forall X_\alpha s X \doteq^\beta t X) \downarrow_\beta}{\Delta, (s \doteq^{\alpha \rightarrow \beta} t)} \mathcal{G}(\dagger) \quad \frac{\Delta, \neg s, t \quad \Delta, \neg t, s}{\Delta, (s \doteq^o t)} \mathcal{G}(\text{b})$
<b>Decomposition</b>	$\frac{\Delta, (s^1 \doteq^{\alpha_1} t^1) \dots \Delta, (s^n \doteq^{\alpha_n} t^n) \quad n \geq 1, \beta \in \{o, t\}, h_{\alpha^n \rightarrow \beta} \in \Sigma}{\Delta, (hs^n \doteq^\beta ht^n)} \mathcal{G}(d)$

One-sided sequent calculus  $\mathcal{G}_\beta$  is defined by the rules  $\mathcal{G}(\text{init})$ ,  $\mathcal{G}(\neg)$ ,  $\mathcal{G}(\vee_-)$ ,  $\mathcal{G}(\vee_+)$ ,  $\mathcal{G}(\Pi_-^l)$  and  $\mathcal{G}(\Pi_+^c)$ .  
 Calculus  $\mathcal{G}_{\beta\text{fb}}$  extends  $\mathcal{G}_\beta$  by the additional rules  $\mathcal{G}(\text{b})$ ,  $\mathcal{G}(\dagger)$ ,  $\mathcal{G}(d)$ , and  $\mathcal{G}(\text{Init}^\pm)$ .

**Clarifications:**  $\Delta$  and  $\Delta'$  are finite sets of  $\beta$ -normal closed formulas of classical higher-order logic (HOL; Church's Type Theory) [1].  $\Delta, s$  denotes the set  $\Delta \cup \{s\}$ . Let  $\alpha, \beta, o \in T$ . HOL *terms* are defined by the grammar ( $c_\alpha$  denotes typed constants and  $X_\alpha$  typed variables distinct from  $c_\alpha$ ):  $s, t ::= c_\alpha \mid X_\alpha \mid (\lambda X_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid (\neg_{o \rightarrow o} s_o)_o \mid (s_o \vee_{o \rightarrow o \rightarrow o} t_o)_o \mid (\Pi_{(\alpha \rightarrow o) \rightarrow o} s_{\alpha \rightarrow o})_o$ . *Leibniz equality*  $\doteq^\alpha$  at type  $\alpha$  is defined as  $s_\alpha \doteq^\alpha t_\alpha := \forall P_{\alpha \rightarrow o} (\neg P s \vee P t)$ . For each simply typed  $\lambda$ -term  $s$  there is a unique  $\beta$ -normal form (denoted  $s \downarrow_\beta$ ). HOL formulas are defined as terms of type  $o$ . A *non-atomic formula* is any formula whose  $\beta$ -normal form is of the form  $[cA^n]$  where  $c$  is a logical constant. An *atomic formula* is any other formula.

Theorem proving in these calculi works as follows: In order to prove that a (closed) conjecture formula  $c$  logically follows from a (possibly empty) set of (closed) axioms  $\{a^1, \dots, a^n\}$ , we start from the initial sequent  $\Delta := \{c, \neg a^1, \dots, \neg a^n\}$  and reason backwards by applying the respective calculus rules.

**History:** The calculi have been presented in [2]. Earlier (two-sided) versions and further related sequent calculi for HOL have been presented in [4] and [3].

**Remarks:**  $\mathcal{G}_\beta$  is sound and complete for elementary type theory ( $\mathcal{G}_\beta$  is thus also sound for HOL).  $\mathcal{G}_{\beta\text{fb}}$  is sound and complete for HOL. Moreover, both calculi are cut-free and they do not admit cut-simulation [2].

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## Extensional HO RUE-Resolution (1999-2013)

Normalisation Rules	
$\frac{C \vee [A \vee B]^{\mathfrak{t}}}{C \vee [A]^{\mathfrak{t}} \vee [B]^{\mathfrak{t}}} \vee^{\mathfrak{t}}$	$\frac{C \vee [A \vee B]^{\mathfrak{ff}}}{C \vee [A]^{\mathfrak{ff}} \vee [B]^{\mathfrak{ff}}} \vee^{\mathfrak{ff}}$
$\frac{C \vee [\neg A]^{\mathfrak{t}}}{C \vee [A]^{\mathfrak{ff}}} \neg^{\mathfrak{t}}$	$\frac{C \vee [\neg A]^{\mathfrak{ff}}}{C \vee [A]^{\mathfrak{t}}} \neg^{\mathfrak{ff}}$
$\frac{C \vee [I\tau A]^{\mathfrak{t}} \quad X^{\tau} \text{ fresh variable}}{C \vee [AX]^{\mathfrak{t}}} I\tau^{\mathfrak{t}}$	$\frac{C \vee [I\tau A]^{\mathfrak{ff}} \quad \text{sk}^{\tau} \text{ Skolem term}}{C \vee [A \text{sk}^{\tau}]^{\mathfrak{ff}}} I\tau^{\mathfrak{ff}}$
Resolution, Factorisation and Primitive Substitution	
$\frac{[A]^{p_1} \vee C \quad [B]^{p_2} \vee D \quad p_1 \neq p_2}{C \vee D \vee [A = B]^{\mathfrak{ff}}} \text{res}$	$\frac{C \vee [A]^p \vee [B]^p}{C \vee [A]^p \vee [A = B]^{\mathfrak{ff}}} \text{fac}$
$\frac{[Q_{\tau} \overline{A}^n]^p \vee C \quad P \in \mathcal{AB}_{\tau}^{(k)} \text{ for logic connective } k}{([Q_{\tau} \overline{A}^n]^p \vee C)[P/Q]} \text{prim\_subst}$	
Extensionality and Pre-unification	
$\frac{C \vee [A^{\sigma\tau} = B^{\sigma\tau}]^{\mathfrak{t}} \quad X^{\tau} \text{ fresh variable}}{C \vee [AX = BX]^{\mathfrak{t}}} \text{FUNCPOS}$	$\frac{C \vee [A^o = B^o]^{\mathfrak{t}}}{C \vee [A^o \longleftrightarrow B^o]^{\mathfrak{t}}} \text{BoolPOS}$
$\frac{C \vee [A^{\sigma\tau} = B^{\sigma\tau}]^{\mathfrak{ff}} \quad \text{sk}^{\tau} \text{ Skol. term}}{C \vee [A \text{sk} = B \text{sk}]^{\mathfrak{ff}}} \text{FUNCNEG}$	$\frac{C \vee [A^o = B^o]^{\mathfrak{ff}}}{C \vee [A^o \longleftrightarrow B^o]^{\mathfrak{ff}}} \text{BoolNEG}$
$\frac{C \vee [h^{\sigma\tau} \overline{A}^k = h^{\sigma\tau} \overline{B}^k]^{\mathfrak{ff}}}{C \vee [A_i = B_i]^{\mathfrak{ff}}^{i \leq k}} \text{DEC}$	$\frac{C \vee [X = A]^{\mathfrak{ff}} \quad X \notin \text{FV}(A)}{C[A/X]} \text{SUBST}$
$\frac{C \vee [A = A]^{\mathfrak{ff}}}{C} \text{TRIV}$	$\frac{C \vee [F^{\tau} \overline{A}^n = h \overline{B}^m]^{\mathfrak{ff}} \quad G \in \mathcal{AB}_{\tau}^{(h)}}{C \vee [F = G]^{\mathfrak{ff}} \vee [F \overline{A}^n = h \overline{B}^m]^{\mathfrak{ff}}} \text{FLEXRIGID}$
Choice	
$\frac{C := C' \vee [A[E_{(\alpha \rightarrow o) \rightarrow \alpha} B]]^p \quad \begin{array}{l} \epsilon \in \text{CFs}, E = \epsilon \text{ or } E \in \text{freeVars}(C), \\ \text{freeVars}(\mathbf{B}) \subseteq \text{freeVars}(C), Y \text{ fresh} \end{array}}{[B Y]^{\mathfrak{ff}} \vee [B (\epsilon_{\alpha(o)} \mathbf{B})]^{\mathfrak{t}}} \text{choice}$	
$\frac{[PX]^{\mathfrak{ff}} \vee [P(f_{(\alpha \rightarrow o) \rightarrow \alpha} P)]^{\mathfrak{t}}}{\text{CFs} \leftarrow \text{CFs} \cup \{f_{(\alpha \rightarrow o) \rightarrow \alpha}\}} \text{detectChoiceFn}$	
Optional additional rules include (a) exhaustive universal instantiation rule for (selective) finite domains, (b) detection and removal of Leibniz equations and Andrews equations, and (c) splitting. Like detectChoiceFn these rules are admissible.	

**Clarifications:**  $\mathbf{A}$  and  $\mathbf{B}$  are metavariables ranging over terms of HOL [sep-type-theory-church]; see also [28]). The logical connectives are  $\neg$ ,  $\vee$ ,  $I\tau$  (universal quantification over variables of type  $\tau$ ), and  $=^{\tau}$  (equality on terms of type  $\tau$ ). Types are shown only if unclear in context. For example, in rule choice the variable  $E^{\alpha(o)}$  is of function

type, also written as  $(\alpha \rightarrow o) \rightarrow \alpha$ . Variables like  $F$  are presented as upper case symbols and constant symbols like  $h$  are lower case.  $\alpha$  equality and  $\beta\eta$ -normalisation are treated implicit, meaning that all clauses are implicitly normalised.  $\mathbf{C}$  and  $\mathbf{D}$  are metavariables ranging over clauses, which are disjunctions of literals. These disjunctions are implicitly assumed associative and commutative; the latter also applies to all equations. Literals are formulas shown in square brackets and labelled with a *polarity* (either  $\mathbf{t}$  or  $\mathbf{ff}$ ), e.g.  $[\neg X]^{\mathbf{ff}}$  denotes the negation of  $\neg X$ .  $\text{FV}(\mathbf{A})$  denotes the free variables of term  $\mathbf{A}$ .  $\mathcal{AB}_\tau^{(h)}$  is the set of approximating bindings for head  $h$  and type  $\tau$ .  $\epsilon_{\alpha(\alpha o)}$  is a choice operator and  $\text{CFs}$  is a set of dynamically collected choice functions symbols;  $\text{CFs}$  is initialised with a single choice function.

**History:** The original calculus (without choice) has been presented in [4] and [3]. Recent modifications and extensions (e.g. choice) are discussed in [2] and [1]. The calculus is inspired by and extends Huet’s constrained resolution [6, 7] and the extensional resolution calculus in [5].

**Remarks:** The calculus works for classical higher-order logic with Henkin semantics and choice. Soundness and completeness has been discussed in [4] and [3]. In the prover LEO-II, the factorisation rule is for performance reasons restricted to binary clauses and a (parametrisable) depth limit is employed for pre-unification. Such restrictions are a (deliberate) source for incompleteness.

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  - [5] Christoph Benzmüller and Michael Kohlhase. “Extensional Higher-Order Resolution”. In: *Automated Deduction - CADE-15*. LNAI 1421. Springer, 1998, pp. 56–71.
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## Full Intuitionistic Logic (FIL) (2005)

$\overline{A(n) \Rightarrow A/\{n\}} \text{ } ax$	$\overline{\perp (n) \Rightarrow A_1/\{n\}, \dots, A_k/\{n\}} \perp \Rightarrow$
$\frac{\Gamma_1 \Rightarrow \Delta_1, A/S \quad A(n), \Gamma_1 \Rightarrow \Delta_1}{\Gamma_1, \Gamma_1 \Rightarrow \Delta_1, \Delta_1^*} cut$	$\frac{\Gamma_1, A(m), B(n), \Gamma_1 \Rightarrow \Delta}{\Gamma_1, B(n), A(m), \Gamma_1 \Rightarrow \Delta} perm \Rightarrow$
$\frac{\Gamma \Rightarrow \Delta_1, A/S_1, B/S_1, \Delta_1}{\Gamma \Rightarrow \Delta_1, B/S_1, A/S_1, \Delta_1} \Rightarrow perm$	$\frac{\Gamma \Rightarrow \Delta}{A(n), \Gamma \Rightarrow \Delta^*} weak \Rightarrow$
$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A/\{ \}} \Rightarrow weak$	$\frac{\Gamma, A(n), A(m) \Rightarrow \Delta}{\Gamma, A(k) \Rightarrow \Delta^*} cont \Rightarrow$
$\frac{\Gamma \Rightarrow \Delta, A/S_1, A/S_1}{\Gamma \Rightarrow \Delta, A/S_1 \cup S_1} \Rightarrow cont$	$\frac{\Gamma_1, A(n) \Rightarrow \Delta_1 \quad \Gamma_1, B(m) \Rightarrow \Delta_1}{\Gamma_1, \Gamma_1, (A \vee B)(k) \Rightarrow \Delta_1^*, \Delta_1^*} \vee \Rightarrow$
$\frac{\Gamma \Rightarrow \Delta, A/S_1, B/S_1}{\Gamma \Rightarrow \Delta, (A \vee B)/S_1 \cup S_1} \Rightarrow \vee$	$\frac{\Gamma, A(n), B(m) \Rightarrow \Delta}{\Gamma, (A \wedge B)(k) \Rightarrow \Delta^*} \wedge \Rightarrow$
$\frac{\Gamma \Rightarrow \Delta, A/S_1 \quad \Gamma \Rightarrow \Delta, B/S_1}{\Gamma \Rightarrow \Delta, (A \wedge B)/S_1 \cup S_1} \Rightarrow \wedge$	$\frac{\Gamma_1 \Rightarrow \Delta_1, A/S \quad B(n), \Gamma_1 \Rightarrow \Delta_1}{(A \longrightarrow B)(n), \Gamma_1, \Gamma_1 \Rightarrow \Delta_1, \Delta_1^*} \longrightarrow \Rightarrow$
$\frac{\Gamma, A(n) \Rightarrow \Delta, B/S}{\Gamma \Rightarrow \Delta, (A \longrightarrow B)/S - \{n\}} \Rightarrow \longrightarrow$	

**Clarifications:** TODO

**History:** TODO

## Focused LK (2007)

### ASYNCHRONOUS INTRODUCTION RULES

$$\frac{}{\vdash \Gamma \uparrow t^-, \Theta} \quad \frac{\vdash \Gamma \uparrow B_1, \Theta \quad \vdash \Gamma \uparrow B_2, \Theta}{\vdash \Gamma \uparrow B_1 \wedge^- B_2, \Theta} \quad \frac{\vdash \Gamma \uparrow \Theta}{\vdash \Gamma \uparrow f^-, \Theta} \quad \frac{\vdash \Gamma \uparrow B_1, B_2, \Theta}{\vdash \Gamma \uparrow B_1 \vee^- B_1, \Theta}$$

$$\frac{\vdash \Gamma \uparrow [y/x]B, \Theta}{\vdash \Gamma \uparrow \forall x.B, \Theta}$$

### SYNCHRONOUS INTRODUCTION RULES

$$\frac{}{\vdash \Gamma \Downarrow t^+} \quad \frac{\vdash \Gamma \Downarrow B_1 \quad \vdash \Gamma \Downarrow B_2}{\vdash \Gamma \Downarrow B_1 \wedge^+ B_2} \quad \frac{\vdash \Gamma \Downarrow B_i}{\vdash \Gamma \Downarrow B_1 \vee^+ B_2} \quad i \in \{1, 2\} \quad \frac{\vdash \Gamma \Downarrow [t/x]B}{\vdash \Gamma \Downarrow \exists x.B}$$

### IDENTITY RULES

$$\frac{P \text{ atomic}}{\vdash \neg P, \Gamma \Downarrow P} \text{ init} \quad \frac{\vdash \Gamma \uparrow B \quad \vdash \Gamma \uparrow \neg B}{\vdash \Gamma \uparrow \cdot} \text{ cut}$$

### STRUCTURAL RULES

$$\frac{\vdash \Gamma, C \uparrow \Theta}{\vdash \Gamma \uparrow C, \Theta} \text{ store} \quad \frac{\vdash \Gamma \uparrow N}{\vdash \Gamma \Downarrow N} \text{ release} \quad \frac{\vdash P, \Gamma \Downarrow P}{\vdash P, \Gamma \uparrow \cdot} \text{ decide}$$

Here,  $\Gamma$  ranges over multisets of polarized formulas;  $\Theta$  ranges over lists of polarized formulas;  $P$  denotes a positive formula;  $N$  denotes a negative formula;  $C$  denotes either a negative formula or a positive atom; and  $B$  denotes an unrestricted polarized formula. The negation in  $\neg B$  denotes the negation normal form of the de Morgan dual of  $B$ . The right introduction rule for  $\forall$  has the usual eigenvariable restriction that  $y$  is not free in any formula in the conclusion sequent.

**Clarifications:** This proof system involves *polarized* (negative normal) formulas of first-order classical logic: in order to polarize a formula  $B$ , one must assign the status of “positive” or “negative” bias to all atomic formulas and replace all occurrences of truth with either  $t^+$  or  $t^-$  and replace all occurrences of conjunctions with either  $\wedge^+$  or  $\wedge^-$ ; similarly, all occurrences of false and disjunctions must be polarized into  $f^+$ ,  $f^-$ ,  $\vee^+$ , and  $\vee^-$ . If there are  $n$  occurrences of propositional connectives in  $B$ , there are  $2^n$  ways to polarize  $B$ . The *positive connectives* are  $f^+$ ,  $\vee^+$ ,  $t^+$ ,  $\wedge^+$ , and  $\exists$  while the *negative connectives* are  $t^-$ ,  $\wedge^-$ ,  $f^-$ ,  $\vee^-$ , and  $\forall$ . A formula is *positive* if it is a positive atom or has a top-level positive connective; similarly a formula is *negative* if it is a negative atom or has a top-level negative connective.

There are two kinds of sequents in this proof system, namely,  $\vdash \Gamma \uparrow \Theta$  and  $\vdash \Gamma \Downarrow B$ , where  $\Gamma$  is a multiset of polarized formulas,  $B$  is a polarized formula, and  $\Theta$  is a list of polarized formulas. The list structure of  $\Theta$  can be replaced by a multiset.



**History:** This focused proof system is a slight variation of the proof systems in [1, 2]. A multifocus variant of **LKF** has been described in [chaudhuri14jlc]. The design of **LKF** borrows strongly by Andreoli’s focused proof system for linear logic [andreoli92jlc] and Girard’s LC proof system [girard91mscs]. The first-order versions of the LKT and LKQ proof systems of [danos93wll] can be seen subsystems of **LKF**.

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## Focused LJ (2007)

### ASYNCHRONOUS INTRODUCTION RULES

$$\begin{array}{c}
\frac{\Gamma \uparrow B_1 \vdash B_2 \uparrow}{\Gamma \uparrow \cdot \vdash B_1 \supset B_2 \uparrow} \quad \frac{\Gamma \uparrow \cdot \vdash B_1 \uparrow \quad \Gamma \uparrow \cdot \vdash B_2 \uparrow}{\Gamma \uparrow \cdot \vdash B_1 \wedge^- B_2 \uparrow} \quad \frac{}{\Gamma \uparrow \cdot \vdash t^- \uparrow} \\
\\
\frac{\Gamma \uparrow \cdot \vdash [y/x]B \uparrow}{\Gamma \uparrow \cdot \vdash \forall x.B \uparrow} \quad \frac{\Gamma \uparrow [y/x]B, \Theta \vdash \mathcal{R}}{\Gamma \uparrow \exists x.B, \Theta \vdash \mathcal{R}} \quad \frac{}{\Gamma \uparrow f^+, \Theta \vdash \mathcal{R}} \\
\\
\frac{\Gamma \uparrow B_1, B_2, \Theta \vdash \mathcal{R}}{\Gamma \uparrow B_1 \wedge^+ B_2, \Theta \vdash \mathcal{R}} \quad \frac{\Gamma \uparrow \Theta \vdash \mathcal{R}}{\Gamma \uparrow t^+, \Theta \vdash \mathcal{R}} \quad \frac{\Gamma \uparrow B_1, \Theta \vdash \mathcal{R} \quad \Gamma \uparrow B_2, \Theta \vdash \mathcal{R}}{\Gamma \uparrow B_1 \vee^+ B_2, \Theta \vdash \mathcal{R}}
\end{array}$$

### SYNCHRONOUS INTRODUCTION RULES

$$\begin{array}{c}
\frac{\Gamma \vdash B_1 \Downarrow \quad \Gamma \Downarrow B_2 \vdash E}{\Gamma \Downarrow B_1 \supset B_2 \vdash E} \quad \frac{\Gamma \Downarrow [t/x]B \vdash E}{\Gamma \Downarrow \forall x.B \vdash E} \quad \frac{\Gamma \Downarrow B_i \vdash E}{\Gamma \Downarrow B_1 \wedge^- B_2 \vdash E} \quad i \in \{1, 2\} \\
\\
\frac{\Gamma \vdash B_i \Downarrow}{\Gamma \vdash B_1 \vee^+ B_2 \Downarrow} \quad \frac{}{\Gamma \vdash t^+ \Downarrow} \quad \frac{\Gamma \vdash B_1 \Downarrow \quad \Gamma \vdash B_2 \Downarrow}{\Gamma \vdash B_1 \wedge^+ B_2 \Downarrow} \quad \frac{\Gamma \vdash [t/x]B \Downarrow}{\Gamma \vdash \exists x.B \Downarrow}
\end{array}$$

### IDENTITY RULES

$$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N} I_l \quad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow} I_r \quad \frac{\Gamma \uparrow \cdot \vdash B \uparrow \cdot \quad \Gamma \uparrow B \vdash \cdot \uparrow E}{\Gamma \uparrow \cdot \vdash \cdot \uparrow E} Cut$$

### STRUCTURAL RULES

$$\begin{array}{c}
\frac{\Gamma, N \Downarrow N \vdash E}{\Gamma, N \uparrow \cdot \vdash \cdot \uparrow E} D_l \quad \frac{\Gamma \vdash P \Downarrow}{\Gamma \uparrow \cdot \vdash \cdot \uparrow P} D_r \quad \frac{\Gamma \uparrow P \vdash \cdot \uparrow E}{\Gamma \Downarrow P \vdash E} R_l \quad \frac{\Gamma \uparrow \cdot \vdash N \uparrow \cdot}{\Gamma \vdash N \Downarrow} R_r \\
\\
\frac{C, \Gamma \uparrow \Theta \vdash \mathcal{R}}{\Gamma \uparrow C, \Theta \vdash \mathcal{R}} S_l \quad \frac{\Gamma \uparrow \cdot \vdash \cdot \uparrow E}{\Gamma \uparrow \cdot \vdash E \uparrow \cdot} S_r
\end{array}$$

Here,  $\Theta$  ranges over multisets of polarized formulas;  $\Gamma$  ranges over lists of polarized formulas;  $P$  denotes a positive formula;  $N$  denotes a negative formula;  $C$  denotes either a negative formula or a positive atom; and  $E$  denotes either a positive formula or a negative atom; and  $B$  denotes an unrestricted polarized formula. The introduction rule for  $\forall$  has the usual eigenvariable restriction that  $y$  is not free in any formula in the conclusion sequent.

**Clarifications:** This proof system involves *polarized* formulas of first-order intuitionistic logic: in order to polarize a formula  $B$ , one must assign the status of “positive” or “negative” bias to all atomic formulas and replace all occurrences of truth with either  $t^+$  or  $t^-$  and all occurrences of conjunction with either  $\wedge^+$  or  $\wedge^-$ . If there are  $n$  occurrences of truth and conjunction in  $B$ , there are  $2^n$  ways to do this replacement. Similarly, we replace the false and disjunction with  $f^+$  and  $\vee^+$  since only the

positive polarization for these connectives are available in **LJF**. (Assigning polarization in classical logic is different: see the **LKF** proof system [31].) The *positive connectives* are  $f^+$ ,  $\vee^+$ ,  $t^+$ ,  $\wedge^+$ , and  $\exists$  while the *negative connectives* are  $t^-$ ,  $\wedge^-$ ,  $\supset$ , and  $\forall$ . A formula is *positive* if it is a positive atom or has a top-level positive connective; similarly a formula is *negative* if it is a negative atom or has a top-level negative connective.

There are two kinds of sequents in this proof system. One kind contains a single  $\Downarrow$  on either the right or the left of the turnstile ( $\vdash$ ) and are of the form  $\Gamma \Downarrow B \vdash E$  or  $\Gamma \vdash B \Downarrow$ ; in both of these cases, the formula  $B$  is the *focus* of the sequent. The other kind of sequent has an occurrence of  $\Uparrow$  on each side of the turnstile, eg.,  $\Gamma \Uparrow \Theta \vdash \Delta_1 \Uparrow \Delta_2$ , and is such that the union of the two multisets  $\Delta_1$  and  $\Delta_2$  contains exactly one formula: that is, one of these multisets is empty and the other is a singleton. When writing asynchronous rules that introduce a connective on the left-hand side, we write  $\mathcal{R}$  to denote  $\Delta_1 \Downarrow \Delta_2$ .

Note that in the asynchronous phase, a right introduction rule is applied only when the left asynchronous zone  $\Gamma$  is empty. Similarly, a left-introduction rule in the async phase introduces the connective at the top-level of the first formula in that context. The scheduling of introduction rules during this phase can be assigned arbitrarily and the zone  $\Gamma$  can be interpreted as a multiset instead of a list.

The choice of how to polarize an unpolarized formula does not affect provability in LJF but can make a big impact on the structure of LJF proofs that can be built.

**History:** This focused proof system is a slight variation of the proof system in [liang09tcs, liang07csl]. **LJF** can be seen as a generalization to the MJ sequent system of Howe [3]. Other focused proof systems, such as LJT [4], LJQ/LJQ' [1], and  $\lambda$ RCC [2] can be directly emulated within **LJF** by making the appropriate choice of polarization.

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  - [2] Radha Jagadeesan, Gopalan Nadathur, and Vijay Saraswat. “Testing concurrent systems: An interpretation of intuitionistic logic”. In: *FSTTCS05*. Vol. 3821. LNCS. Hyderabad, India: Springer, 2005, pp. 517–528.
  - [3] J. M. Howe. “Proof Search Issues in Some Non-Classical Logics”. Available as University of St Andrews Research Report CS/99/1. PhD thesis. University of St Andrews, Dec. 1998.
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# Counterfactual Sequent Calculi I

(1983,1992,2012,2013)

$$\begin{array}{c}
 \frac{\{ B_k \vdash A_1, \dots, A_n, D_1, \dots, D_m \mid k \leq n \} \cup \{ C_k \vdash A_1, \dots, A_n, D_1, \dots, D_{k-1} \mid k \leq m \}}{\Gamma, (C_1 \leq D_1), \dots, (C_m \leq D_m) \vdash \Delta, (A_1 \leq B_1), \dots, (A_n \leq B_n)} R_{n,m} \\
 \\
 \frac{\{ C_k \vdash D_1, \dots, D_{k-1} \mid k \leq m \} \quad \Gamma \vdash \Delta, D_1, \dots, D_m}{\Gamma, (C_1 \leq D_1), \dots, (C_m \leq D_m) \vdash \Delta} T_m \\
 \\
 \frac{\{ C_k \vdash A_1, \dots, A_n, D_1, \dots, D_{k-1} \mid k \leq m \} \quad \Gamma \vdash \Delta, A_1, \dots, A_n, D_1, \dots, D_m}{\Gamma, (C_1 \leq D_1), \dots, (C_m \leq D_m) \vdash \Delta, (A_1 \leq B_1), \dots, (A_n \leq B_n)} W_{n,m} \\
 \\
 \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, (A \leq B)} R_{C1} \quad \frac{\Gamma, A \vdash \Delta \quad \Gamma \vdash \Delta, B}{\Gamma, (A \leq B) \vdash \Delta} R_{C2} \\
 \\
 \frac{\{ \Gamma^{\leq}, B_k \vdash \Delta^{\leq}, A_1, \dots, A_n, D_1, \dots, D_m \mid k \leq n \} \cup \{ \Gamma^{\leq}, C_k \vdash \Delta^{\leq}, A_1, \dots, A_n, D_1, \dots, D_{k-1} \mid k \leq m \}}{\Gamma, (C_1 \leq D_1), \dots, (C_m \leq D_m) \vdash \Delta, (A_1 \leq B_1), \dots, (A_n \leq B_n)} A_{n,m} \\
 \\
 \mathcal{R}_{\forall \leq} = \{ R_{n,m} \mid n \geq 1, m \geq 0 \} \\
 \mathcal{R}_{\forall N \leq} = \{ R_{n,m} \mid n+m \geq 1 \} \quad \mathcal{R}_{\forall C \leq} = \mathcal{R}_{\forall} \cup \{ R_{C1}, R_{C2} \} \\
 \mathcal{R}_{\forall T \leq} = \mathcal{R}_{\forall \leq} \cup \{ T_m \mid m \geq 1 \} \quad \mathcal{R}_{\forall A \leq} = \{ A_{n,m} \mid n \geq 1, m \geq 0 \} \\
 \mathcal{R}_{\forall W \leq} = \mathcal{R}_{\forall \leq} \cup \{ W_{n,m} \mid n+m \geq 1 \} \quad \mathcal{R}_{\forall NA \leq} = \{ A_{n,m} \mid n+m \geq 1 \}
 \end{array}$$

**Clarifications:** Sequents are based on multisets. The rules  $\mathcal{R}_{\mathcal{L}^{\leq}}$  form a calculus for a counterfactual logic  $\mathcal{L}^{\leq}$  described in [6], where  $\leq$  is the *comparative plausibility* operator. Besides the rules shown above, these calculi also include the propositional rules of **G3c** [17] and contraction rules. The contexts  $\Gamma^{\leq}$  and  $\Delta^{\leq}$  contain all formulae of resp.  $\Gamma$  and  $\Delta$  of the form  $A \leq B$ .

**History:** The calculus for  $\forall C$  was introduced in the tableaux setting [5, 4]. The remaining calculi were introduced in [3, 2] and corrected in [1].

**Remarks:** Soundness and completeness are shown by proving equivalence to Hilbert-style calculi and (syntactical) cut elimination. These calculi yield PSPACE decision procedures (EXPTIME for  $\forall A_{\leq}$  and  $\forall NA_{\leq}$ ) and, in most cases, enjoy Craig Interpolation. Contraction can be made admissible.

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  - [6] David Lewis. *Counterfactuals*. Blackwell, 1973.

## Counterfactual Sequent Calculi II (2012, 2013)

$$\begin{array}{c}
 \frac{\{ C_k, \mathbf{B}^I \vdash \mathbf{A}^{[n] \setminus I}, \mathbf{C}^J, \mathbf{D}^{[k-1] \setminus J} \mid 1 \leq k \leq m, I \subseteq [n], J \subseteq [k-1] \} \cup \{ A_k, B_k, \mathbf{B}^I \vdash \mathbf{A}^{[n] \setminus I}, \mathbf{C}^J, \mathbf{D}^{[m] \setminus J} \mid k \leq n, I \subseteq [n], J \subseteq [m] \}}{\Gamma, (A_1 \BoxRightarrow B_1), \dots, (A_n \BoxRightarrow B_n) \vdash \Delta, (C_1 \BoxRightarrow D_1), \dots, (C_m \BoxRightarrow D_m)} R_{n,m} \\
 \\
 \frac{\{ \Gamma \vdash \Delta, \mathbf{C}^J, \mathbf{D}^{[m] \setminus J} \mid J \subseteq [m] \} \cup \{ C_k \vdash D_k, \mathbf{C}^J, \mathbf{D}^{[k-1] \setminus J} \mid 1 \leq k \leq m, J \subseteq [k-1] \}}{\Gamma \vdash \Delta, (C_1 \BoxRightarrow D_1), \dots, (C_m \BoxRightarrow D_m)} T_m \\
 \\
 \frac{\{ C_k, \mathbf{B}^I \vdash \mathbf{A}^{[n] \setminus I}, \mathbf{C}^J, \mathbf{D}^{[k-1] \setminus J} \mid 1 \leq k \leq m, I \subseteq [n], J \subseteq [k-1] \} \cup \{ \Gamma, \mathbf{B}^I \vdash \mathbf{A}^{[n] \setminus I}, \mathbf{C}^J, \mathbf{D}^{[m] \setminus J} \mid I \subseteq [n], J \subseteq [m] \}}{\Gamma, (A_1 \BoxRightarrow B_1), \dots, (A_n \BoxRightarrow B_n) \vdash \Delta, (C_1 \BoxRightarrow D_1), \dots, (C_m \BoxRightarrow D_m)} W_{n,m} \\
 \\
 \frac{\Gamma \vdash \Delta, A \quad \Gamma, B \vdash \Delta}{\Gamma, (A \BoxRightarrow B) \vdash \Delta} R_{C1} \quad \frac{\Gamma \vdash \Delta, A \quad \Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, (A \BoxRightarrow B)} R_{C2}
 \end{array}$$

For  $n > 0$  the set  $[n]$  is  $\{1, \dots, n\}$  and  $[0]$  is  $\emptyset$ . For a set  $I$  of indices,  $\mathbf{A}^I$  contains all  $A_i$  with  $i \in I$ .

$$\begin{array}{ll}
 \mathcal{R}_{\mathbf{V}\BoxRightarrow} = \{R_{n,m} \mid n \geq 1, m \geq 0\} & \\
 \mathcal{R}_{\mathbf{V}\mathbf{N}\BoxRightarrow} = \{R_{n,m} \mid n+m \geq 1\} & \mathcal{R}_{\mathbf{V}\mathbf{W}\BoxRightarrow} = \mathcal{R}_{\mathbf{V}\mathbf{T}\BoxRightarrow} \cup \{W_{n,m} \mid n+m \geq 1\} \\
 \mathcal{R}_{\mathbf{V}\mathbf{T}\BoxRightarrow} = \mathcal{R}_{\mathbf{V}\BoxRightarrow} \cup \{T_m \mid m \geq 1\} & \mathcal{R}_{\mathbf{V}\mathbf{C}\BoxRightarrow} = \mathcal{R}_{\mathbf{V}\BoxRightarrow} \cup \{R_{C1}, R_{C2}\}
 \end{array}$$

**Clarifications:** Sequents are based on multisets. The rules  $\mathcal{R}_{\mathcal{L}\BoxRightarrow}$  form a calculus for a counterfactual logic  $\mathcal{L}$  described in [Lewis:1973uq], where  $\BoxRightarrow$  is the *strong counterfactual implication* operator. Besides the rules shown above, these calculi also include the propositional rules of **G3c** {17} and contraction rules.

**History:** These calculi were introduced in [Lellmann:2012fk] and corrected in [Lellmann:2013].

**Remarks:** The calculi are translations of the calculi in {33} to the language with  $\BoxRightarrow$ . They inherit cut elimination and yield PSPACE decision procedures. Contraction can be made admissible.

## Conditional Nested Sequents $\mathcal{NS}$ (2012-2014)

$\frac{\Gamma(P, \neg P)}{P \text{ atomic}} (AX)$	$\Gamma(\top) (AX_{\top})$	$\Gamma(\neg \perp) (AX_{\perp})$
$\frac{\Gamma(A)}{\Gamma(\neg \neg A)} (\neg)$	$\frac{\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B]) \quad A, \neg A' \quad A', \neg A}{\Gamma(\neg(A \Rightarrow B), [A' : \Delta])} (\Rightarrow^-)$	$\frac{\Gamma([A : B])}{\Gamma(A \Rightarrow B)} (\Rightarrow^+)$
$\frac{\Gamma([A : \Delta, \neg A])}{\Gamma([A : \Delta])} (ID)$	$\frac{\Gamma([A : \Delta, \Sigma], [B : \Sigma]) \quad A, \neg B \quad B, \neg A}{\Gamma([A : \Delta], [B : \Sigma])} (CEM)$	
	$\frac{\Gamma(\neg(A \Rightarrow B), A) \quad \Gamma(\neg(A \Rightarrow B), \neg B)}{\Gamma(\neg(A \Rightarrow B))} (MP)$	
	$\frac{\Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg D] \quad \Gamma, \neg(C \Rightarrow D), [A : C] \quad \Gamma, \neg(C \Rightarrow D), [C : A]}{\Gamma, \neg(C \Rightarrow D), [A : \Delta]} (CSO)$	

**Clarifications:** Conditional logics extend classical logic with formulas of the form  $A \Rightarrow B$ : intuitively,  $A \Rightarrow B$  is true in a world  $x$  if  $B$  is true in the set of worlds where  $A$  is true and that are most similar to  $x$ . The calculi  $\mathcal{NS}$  manipulate *nested* sequents, a generalization of ordinary sequent calculi where sequents are allowed to occur within sequents. A nested sequent

$$A_1, \dots, A_m, [B_1 : \Gamma_1], \dots, [B_n : \Gamma_n]$$

is inductively defined by the formula

$$\mathcal{F}(\Gamma) = A_1 \vee \dots \vee A_m \vee (B_1 \Rightarrow \mathcal{F}(\Gamma_1)) \vee \dots \vee (B_n \Rightarrow \mathcal{F}(\Gamma_n)).$$

$\Gamma(\Delta)$  represents a sequent  $\Gamma$  containing a *context* (a unique empty position) filled by the (nested) sequent  $\Delta$ . Besides the rules shown above, the calculi  $\mathcal{NS}$  also include standard rules for propositional connectives.

**History:** The calculi  $\mathcal{NS}$  have been introduced in [3] and extended in [2]. The theorem prover NESCOND, implementing  $\mathcal{NS}$  in Prolog, has been presented in [1].

**Remarks:** Completeness is a consequence of the admissibility of cut. The calculi  $\mathcal{NS}$  can be used to obtain a PSPACE decision procedure for the respective conditional logics (optimal for CK and extensions with ID and MP).

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- [2] Régis Alenda, Nicola Olivetti, and Gian Luca Pozzato. “Nested Sequent Calculi for Normal Conditional Logics”. In: *Journal of Logic and Computation* (2013). doi: doi:10.1093/logcom/ext034.
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## FILL Deep Nested Sequent Calculus [1] (2013)

Propagation rules:

$$\begin{array}{c} \frac{X[S \Rightarrow (A, S' \Rightarrow \mathcal{T}'), \mathcal{T}]}{X[S, A \Rightarrow (S' \Rightarrow \mathcal{T}'), \mathcal{T}]} pl_1 \quad \frac{X[(S \Rightarrow \mathcal{T}, A), S' \Rightarrow \mathcal{T}']}{X[(S \Rightarrow \mathcal{T}), S' \Rightarrow A, \mathcal{T}']} pr_1 \\[10pt] \frac{X[S, A, (S' \Rightarrow \mathcal{T}') \Rightarrow \mathcal{T}]}{X[S, (S', A \Rightarrow \mathcal{T}') \Rightarrow \mathcal{T}]} pl_2 \quad \frac{X[S \Rightarrow \mathcal{T}, A, (S' \Rightarrow \mathcal{T}')] }{X[S \Rightarrow \mathcal{T}, (S' \Rightarrow \mathcal{T}', A)]} pr_2 \end{array}$$

Identity and logical rules: In branching rules,  $X[\ ] \in X_1[\ ] \bullet X_2[\ ]$ ,  $S \in S_1 \bullet S_2$  and  $\mathcal{T} \in \mathcal{T}_1 \bullet \mathcal{T}_2$ .

$$\begin{array}{c} \frac{X[\ ], \mathcal{U} \text{ and } \mathcal{V} \text{ are hollow.}}{X[\mathcal{U}, p \Rightarrow p, \mathcal{V}]} id^d \quad \frac{X[\ ], \mathcal{U} \text{ and } \mathcal{V} \text{ are hollow.}}{X[\perp, \mathcal{U} \Rightarrow \mathcal{V}]} \perp_l^d \quad \frac{X[S \Rightarrow \mathcal{T}]}{X[S \Rightarrow \mathcal{T}, \perp]} \perp_r^d \\[10pt] \frac{X[S \Rightarrow \mathcal{T}]}{X[S, I \Rightarrow \mathcal{T}]} I_l^d \quad \frac{X[\ ], \mathcal{U} \text{ and } \mathcal{V} \text{ are hollow.}}{X[\mathcal{U} \Rightarrow I, \mathcal{V}]} I_r^d \\[10pt] \frac{X[S, A, B \Rightarrow \mathcal{T}]}{X[S, A \otimes B \Rightarrow \mathcal{T}]} \otimes_l^d \quad \frac{X_1[S_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[S_2 \Rightarrow B, \mathcal{T}_2]}{X[S \Rightarrow A \otimes B, \mathcal{T}]} \otimes_r^d \\[10pt] \frac{X_1[S_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[S_2, B \Rightarrow \mathcal{T}_2]}{X[S, A \multimap B \Rightarrow \mathcal{T}]} \multimap_l^d \quad \frac{X[S \Rightarrow \mathcal{T}, (A \Rightarrow B)]}{X[S \Rightarrow \mathcal{T}, A \multimap B]} \multimap_r^d \\[10pt] \frac{X_1[S_1, A \Rightarrow \mathcal{T}_1] \quad X_2[S_2, B \Rightarrow \mathcal{T}_2]}{X[S, A \wp B \Rightarrow \mathcal{T}]} \wp_l^d \quad \frac{X[S \Rightarrow A, B, \mathcal{T}]}{X[S \Rightarrow A \wp B, \mathcal{T}]} \wp_r^d \\[10pt] \frac{X[S, (A \Rightarrow B) \Rightarrow \mathcal{T}]}{X[S, A \prec B \Rightarrow \mathcal{T}]} \prec_l^d \quad \frac{X_1[S_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[S_2, B \Rightarrow \mathcal{T}_2]}{X[S \Rightarrow A \prec B, \mathcal{T}]} \prec_r^d \end{array}$$

**Clarifications:** Following Kashima [3], nested sequents are defined as below where  $A_i$  and  $B_j$  are formulae [2]:

$$S \ T ::= S_1, \dots, S_k, A_1, \dots, A_m \Rightarrow B_1, \dots, B_n, T_1, \dots, T_l$$

$\Gamma$  and  $\Delta$  are multisets of formulae and  $P, Q, S, T, X, Y$ , etc., are nested sequents, and  $S, X$ , etc., are multisets of nested sequents and formulae.

Inference rules in BiLL<sub>dn</sub> are applied in a *context*, i.e., a nested sequent with a hole  $[ \ ]$ . Notice that BiLL<sub>dn</sub> contain no structural rules. The branching rules require operations to merge contexts and nested sequents, which are explained below. The zero-premise rules require that certain sequents or contexts are *hollow*, i.e., containing no occurrences of formulae.

The *merge set*  $X_1 \bullet X_2$  of two sequents  $X_1$  and  $X_2$  is defined as:

$$\begin{aligned} X_1 \bullet X_2 = \{ & (\Gamma_1, \Gamma_2, Y_1, \dots, Y_m \Rightarrow \Delta_1, \Delta_2, Z_1, \dots, Z_n) \mid \\ & X_1 = (\Gamma_1, P_1, \dots, P_m \Rightarrow \Delta_1, Q_1, \dots, Q_n) \text{ and} \\ & X_2 = (\Gamma_2, S_1, \dots, S_m \Rightarrow \Delta_2, T_1, \dots, T_n) \text{ and} \\ & Y_i \in P_i \bullet S_i \text{ for } 1 \leq i \leq m \text{ and } Z_j \in Q_j \bullet T_j \text{ for } 1 \leq j \leq n \} \end{aligned}$$

When  $X \in X_1 \bullet X_2$ , we say that  $X_1$  and  $X_2$  are a *partition* of  $X$ .

The merge set  $X_1[ ] \bullet X_2[ ]$  of two contexts  $X_1[ ]$  and  $X_2[ ]$  is defined in [1]. If  $X[ ] = X_1[ ] \bullet X_2[ ]$  we say  $X_1[ ]$  and  $X_2[ ]$  are a *partition* of  $X[ ]$ . The notion of a merge set between multisets of formulae and sequents is as follows. Given  $\mathcal{X} = \Gamma \cup \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \Delta \cup \{Y_1, \dots, Y_n\}$  their merge set contains all multisets of the form:  $\Gamma \cup \Delta \cup \{Z_1, \dots, Z_n\}$  where  $Z_i \in X_i \bullet Y_i$ .

**History:** The sequent calculus arose from an attempt to give a display calculus for full intuitionistic linear logic (FILL). As usual for display calculi, a detour is necessary through an extension of FILL with an “exclusion” connective  $\multimap$  which forms an adjunction with  $\wp$ . The resulting logic is called Bi-intuitionistic Linear Logic (BiILL). Although sound and complete for BiILL, the resulting display calculus is bad for backward proof search. Following Kashima [3], Alwen Tiu first obtained a shallow nested sequent calculus for BiILL, and then refined that into a deep nested sequent calculus for BiILL. The proof of cut-elimination for the shallow calculus, and the equivalence of the shallow and deep calculi requires over 615 different cases!

**Remarks:** The calculus shown is for Bi-Intuitionistic Linear Logic [2]. It is sound and complete. The soundness w.r.t. the categorical semantics is via the shallow nested sequent calculus for BiILL. A nested sequent is a (nested) *FILL-sequent* if it has no nesting of sequents on the left of  $\Rightarrow$ , and no occurrences of  $\multimap$  at all. Only in the deep nested sequent calculus is it obvious that a derivation of a FILL-sequent encounters FILL-sequents only. The deep sequent calculus enjoys the subformula property and terminating backward-proof search. The validity problem for FILL is co-NP complete and BiILL is conservative over FILL [2]. All of these proofs were eventually formalised in Isabelle [1] by Jeremy Dawson. As far as is known, it is the only sequent calculus for FILL ({12}) that does not require (type) annotations.

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  - [3] Ryo Kashima. “Cut-free sequent calculi for some tense logics”. In: *Studia Logica* 53.1 (1994), pp. 119–136.

## Contextual Natural Deduction (2013)

$$\begin{array}{c}
 \overline{\Gamma, a : A \vdash a : A} \\
 \\
 \frac{\Gamma, a : A \vdash b : C_\pi[B]}{\Gamma \vdash \lambda_\pi a^A. b : C_\pi[A \rightarrow B]} \rightarrow_I(\pi) \\
 \\
 \frac{\Gamma \vdash f : C_{\pi_1}^1[A \rightarrow B] \quad \Gamma \vdash x : C_{\pi_2}^2[A]}{\Gamma \vdash (f\ x)_{(\pi_1; \pi_2)}^{\rightarrow} : C_{\pi_1}^1[C_{\pi_2}^2[B]]} \rightarrow_E^{\rightarrow}(\pi_1; \pi_2) \\
 \\
 \frac{\Gamma \vdash f : C_{\pi_1}^1[A \rightarrow B] \quad \Gamma \vdash x : C_{\pi_2}^2[A]}{\Gamma \vdash (f\ x)_{(\pi_1; \pi_2)}^{\leftarrow} : C_{\pi_1}^2[C_{\pi_2}^1[B]]} \rightarrow_E^{\leftarrow}(\pi_1; \pi_2)
 \end{array}$$

$\pi, \pi_1$  and  $\pi_2$  must be positive positions.  $a$  is allowed to occur in  $b$  only if  $\pi$  is strongly positive.

**Clarifications:**  $C_\pi[F]$  denotes a formula with  $F$  occurring in the hole of a *context*  $C_\pi[]$ .  $\pi$  is the position of the hole. It is: *positive* iff it is in the left side of an even number of implications; *strongly positive* iff this number is zero.

**History:** Contextual Natural Deduction [1] combines the idea of deep inference with Gentzen's natural deduction {1}.

**Remarks:** Soundness and completeness w.r.t. minimal logic are proven [1] by providing translations between  $\mathbf{ND}^c$  and the minimal fragment of  $\mathbf{NJ}$  {1}.  $\mathbf{ND}^c$  proofs can be quadratically shorter than proofs in the minimal fragment of  $\mathbf{NJ}$ .

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- [1] Bruno Woltzenlogel Paleo. "Contextual Natural Deduction". In: *Logical Foundations of Computer Science, International Symposium, LFCS 2013, San Diego, CA, USA, January 6-8, 2013. Proceedings*. Ed. by Sergei N. Artëmov and Anil Nerode. Vol. 7734. Lecture Notes in Computer Science. Springer, 2013, pp. 372–386. ISBN: 978-3-642-35721-3. DOI: 10.1007/978-3-642-35722-0\_27. URL: [http://dx.doi.org/10.1007/978-3-642-35722-0\\_27](http://dx.doi.org/10.1007/978-3-642-35722-0_27).

## IR (2014)

$C$  is a non-tautological clause from the matrix.

$\tau = \{0/u \mid u \text{ is universal in } C\}$ , where the notation  $0/u$  for literals  $u$  is shorthand for  $0/y$  if  $u = y$  and  $1/y$  if  $u = \neg y$ . We define  $\text{restr}(\tau, x)$  as  $\{c/u \mid c/u \in \tau, \text{lv}(u) < \text{lv}(x)\}$ .

$\tau$  is a partial assignment to universal variables with  $\text{rng}(\tau) \subseteq \{0, 1\}$ .  $\xi = \sigma \cup \{c/u \mid c/u \in \text{restr}(\tau, x), u \notin \text{dom}(\sigma)\}$

The rules of IR [2]

**Clarifications:** The calculus aims to refute a quantified Boolean formula (QBF) of the form  $Q_1 x_1 \dots Q_n x_n. \varphi$  where  $Q_i \in \{\forall, \exists\}$  and  $\varphi$  is a Boolean formula in conjunctive normal form (CNF). The formula  $\varphi$  is referred to as the *matrix*. We write  $\text{lv}(x)$  for the *quantification level* of  $x$ , i.e.  $\text{lv}(x_i) = i$ . A variable  $x_i$  is *existential* (resp. *universal*) if  $Q_i = \exists$  (resp.  $Q_i = \forall$ ).

The calculus works by introducing clauses as *annotated clauses*, which are sets of annotated literals. Annotated literals consist of an existential literal and an annotation – a partial assignment to universal variables in  $\{0, 1\}$ . Two literals are identical if and only if both the existential literal and annotation are equal. The calculus enables deriving the empty clause if and only if the given formula is false.

**Remarks:** Soundness was shown by extracting valid Herbrand functions. Completeness is shown by p-simulation of another known QBF system Q-Resolution.

**History:** The name of the calculus comes from the two pivotal operations *instantiation* and *resolution*. The calculus naturally generalizes an older calculus  $\forall\text{Exp}+\text{Res}$  [1], which requires all clauses to be introduced into the proof by using a complete assignment.

- 
- [1] Mikoláš Janota and Joao Marques-Silva. “Expansion-based {QBF} solving versus Q-resolution”. In: *Theoretical Computer Science* 577 (2015), pp. 25–42. doi: <http://dx.doi.org/10.1016/j.tcs.2015.01.048>.
  - [2] Olaf Beyersdorff, Leroy Chew, and Mikoláš Janota. “On Unification of QBF Resolution-Based Calculi”. In: *Mathematical Foundations of Computer Science (MFCS)*. 2014.

## Sequent Calculus SKMlin for Superintuitionistic Modal Logic KMlin (2014)

$\top R \frac{}{\Gamma \vdash \top, \Delta}$	$id \frac{}{\Gamma, \varphi \vdash \varphi, \Delta}$	$\perp L \frac{}{\Gamma, \perp \vdash \Delta}$
$\vee L \frac{\Gamma, \varphi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta}$	$\vee R \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}$	
$\wedge L \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}$	$\wedge R \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}$	
$\rightarrow L \frac{\Gamma, \varphi \rightarrow \psi \vdash \varphi, \Delta \quad \Gamma, \varphi \rightarrow \psi, \psi \vdash \Delta}{\Gamma, \varphi \rightarrow \psi \vdash \Delta}$	$\rightarrow R \frac{\Gamma, \varphi \vdash \psi, \Delta \quad \Gamma \vdash \varphi \rightarrow \psi, \Delta}{\Gamma \vdash \varphi \rightarrow \psi, \Delta}$	
$STEP \frac{Prem_1 \quad \cdots \quad Prem_k \quad Prem_{k+1} \quad \cdots \quad Prem_{k+n}}{\Sigma_l, \Theta^\triangleright, \Gamma^{\rightarrow} \vdash \Delta^{\rightarrow}, \Phi^\triangleright, \Sigma_r} \dagger$		
$Prem_{1 \leq i \leq k} = \Sigma_l, \Theta, \Theta^\triangleright, \Gamma^{\rightarrow}, \varphi_i \rightarrow \psi_i, \varphi_i \vdash \psi_i, \Delta_{-i}^{\rightarrow}, \Phi$		
$Prem_{k+1 \leq i \leq k+n} = \Sigma_l, \Theta, \Theta^\triangleright, \Gamma^{\rightarrow}, \triangleright \varphi_{i-k} \vdash \Delta^{\rightarrow}, \Phi$		
$\Theta^\triangleright = \triangleright \theta_1, \dots, \triangleright \theta_j$ $\Gamma^{\rightarrow} = \{\alpha_1 \rightarrow \beta_1, \dots, \alpha_l \rightarrow \beta_l\}$ $\Delta^{\rightarrow} = \{\varphi_1 \rightarrow \psi_1, \dots, \varphi_k \rightarrow \psi_k\}$ $\Delta_{-i}^{\rightarrow} = \Delta^{\rightarrow} \setminus \{\varphi_i \rightarrow \psi_i\}$ $\Phi^\triangleright = \triangleright \varphi_1, \dots, \triangleright \varphi_n$	$\Theta = \theta_1, \dots, \theta_j$ $\Gamma^{\rightarrow} = \{\alpha_1 \rightarrow \beta_1, \dots, \alpha_l \rightarrow \beta_l\}$ $\Delta^{\rightarrow} = \{\varphi_1 \rightarrow \psi_1, \dots, \varphi_k \rightarrow \psi_k\}$ $\Phi = \varphi_1, \dots, \varphi_n$	
<p>where <math>\dagger</math> means that the conditions C0, C1 and C2 below must hold</p> <p>(C0) <math>\Delta^{\rightarrow} \cup \Phi^\triangleright \neq \emptyset</math></p> <p>(C1) <math>\perp \notin \Sigma_l</math> and <math>\top \notin \Sigma_r</math> and <math>(\Sigma_l \cup \Theta^\triangleright \cup \Gamma^{\rightarrow}) \cap (\Delta^{\rightarrow} \cup \Phi^\triangleright \cup \Sigma_r) = \emptyset</math></p> <p>(C2) <math>\Sigma_l</math> and <math>\Sigma_r</math> each contain atomic formulae only</p> <p>Explanations for the conditions:</p> <p>(C0) there is at least one <math>\triangleright</math>- or <math>\rightarrow</math>-formula in the succedent of the conclusion</p> <p>(C1) none of the rules <math>\perp L, \top R, id</math> are applicable to the conclusion</p> <p>(C2) none of the rules <math>\vee L, \vee R, \wedge L, \wedge R, \rightarrow L, \rightarrow R</math> are applicable to the conclusion</p>		

**Clarifications:** There is a unary modal connective  $\triangleright$  to be read as “later”. Its semantics are box-like in terms of the underlying intuitionistic Kripke relation, which is irreflexive! There is also a new connective  $\rightarrow$  corresponding to an irreflexive version of intuitionistic implication, which can be defined as  $\triangleright(\varphi \rightarrow \psi)$ . The  $\rightarrow L$  rule is **LK**-like {2} in that it is multi-conclusioned and has one branch for each subformula

of  $\varphi \wedge \psi$ , but it also converts  $\varphi \rightarrow \psi$  to  $\varphi \rightarrow \psi$ , read upwards, building in a form of contraction on such formulae. The  $\rightarrow R$  rule is unusual in that it has two premises: the left one is LK-like in that it does not delete  $\Delta$ , read upwards, while the right one converts  $\varphi \rightarrow \psi$  to  $\varphi \rightarrow \psi$  read upwards building in a form of contraction on such formulae. The STEP rule has an indeterminate number of premises, one for each  $\varphi_i \rightarrow \psi_i \in \Delta^{\rightarrow}$ , and one for each  $\varphi_i \in \Phi^{\triangleright}$ . For each such “eventuality”, the rule creates a premise that contains the subformula on an appropriate side, but also creates a copy of the principal formula in the antecedent of that premise, thus building in aspects of the standard sequent calculus for Gödel-Löb logic.

**History:** The superintuitionistic modal logic KMLin is obtained from Kuznetsov-Muravitsky logic KM [1] by demanding that the underlying Kripke frames be linear. The semantics of the unary modality  $\triangleright$  becomes “true in all strict successors”.

Clouston and Goré [2] defined this sequent calculus. Their rules are inspired by those of: Mauro Ferrari, Camillo Fiorentini and Guido Fiorino for a sequent calculus with compartments for intuitionistic logic [3]; Giovanna Corsi for her sequent calculus for (quantified) Gödel-Dummett logic LC [5]; and George Boolos for his sequent calculus for Gödel-Löb logic GL [4].

**Remarks:** Clouston and Goré gave semantic proofs of soundness, cut-free completeness and the finite model property, thus giving decidability. They showed that the validity problem for this logic is coNP-complete. They also showed that all rules are invertible, so the sequent calculus can be used for backtrack-free and terminating backward proof search via the following strategy for rule applications: apply any applicable rule backwards, always preferring zero-premise rules if possible!

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- [1] Tadeusz Litak. “Constructive modalities with provability smack”. Trends in Logic, to appear.
  - [2] Ranald Clouston and Rajeev Goré. “Sequent Calculus in the Topos of Trees”. In: *Proc. Foundations of Software Science and Computation Structures - 18th International Conference, FoSSaCS 2015*. 2015, pp. 133–147.
  - [3] Mauro Ferrari, Camillo Fiorentini, and Guido Fiorino. “Contraction-Free Linear Depth Sequent Calculi for Intuitionistic Propositional Logic with the Subformula Property and Minimal Depth Counter-Models”. In: *J. Autom. Reason.* 51.2 (2013), pp. 129–149.
  - [4] George Boolos. *The logic of provability*. CUP, 1995.
  - [5] Giovanna Corsi. “A cut-free calculus for Dummett’s LC quantified”. In: *Zeitschr. f. math. Logik und Grundlagen d. Math.* 35 (1989), pp. 289–301.

## Erotetic Dual Resolution for Classical Propositional Logic (2014)

$$\begin{array}{c}
 \frac{?( \Phi ; \vdash S' \beta' T ; \Psi )}{?( \Phi ; \vdash S' \beta_1' T ; \vdash S' \beta_2' T ; \Psi )} \mathbf{R}_\beta \\
 \frac{?( \Phi ; \vdash S' \alpha' T ; \Psi )}{?( \Phi ; \vdash S' \alpha_1' \alpha_2' T ; \Psi )} \mathbf{R}_\alpha \quad \frac{?( \Phi ; \vdash S' \neg \neg A' T ; \Psi )}{?( \Phi ; \vdash S' A' T ; \Psi )} \mathbf{R}_{\neg \neg} \\
 \frac{?( \Phi ; \vdash S' A' T ; \Psi ; \vdash U' \bar{A}' V ; \Omega )}{?( \vdash \underline{S}' \underline{T}' \underline{U}' \underline{V}' ; \Phi ; \Psi ; \Omega ; \vdash S' A' T ; \vdash U' \bar{A}' V )} \mathbf{R}_{res}
 \end{array}$$

In  $\mathbf{R}_{res}$ :  $A$  and  $\bar{A}$  must be complementary, that is either  $A = \neg \bar{A}$  or  $\bar{A} = \neg A$ .  $\underline{S}, \underline{T}$  are obtained from  $S, T$  by deleting all occurrences of  $A$  and  $\underline{U}, \underline{V}$  are obtained from  $U, V$  by deleting all occurrences of  $\neg A$ . For the  $\alpha, \beta$  notation see entry {22}.

**Clarifications:** The calculus is built in the framework of Inferential Erotetic Logic ([2]) and is designed to transform questions concerning refutability (falsifiability) of a formula. However, the rules act upon *reversed sequents*. The sequents are right-sided with sequences of formulas in the succedent.  $\Phi, \Psi$  are finite (possibly empty) sequences of sequents.  $S, T$  are finite (possibly empty) sequences of formulas. The semicolon ‘;’ is the concatenation sign for sequences of sequents, whereas ‘,’ is the concatenation sign for sequences of formulas. The resolution rule is non-clausal (all the formulas in the premise,  $A$  included, may be compound) and dual with respect to the standard resolution (instead of CNF of  $\neg A$ , DNF of  $A$  is derived). A Socratic refutation of a sequent of the form ‘ $\vdash A$ ’ is a kind of a resolution refutation of  $\neg A$  and it ends when the empty sequent (a counterpart of the empty clause) is arrived at.

**History:** The calculus  $\mathbf{E}_{res}^{CPL}$  has been presented in [1] together with extensions to some paraconsistent logics (see {41}). Compare also {22}.

**Remarks:** A formula  $A$  is CPL-valid iff  $\vdash A$  has a Socratic refutation in  $\mathbf{E}_{res}^{CPL}$ . Similar results are obtained with respect to CLuN, CLuNs and mbC.

- 
- [1] Szymon Chlebowski and Dorota Leszczyńska-Jasion. “Dual Erotetic Calculi and the Minimal LFI”. In: *accepted to Studia Logica* (2015).
  - [2] Andrzej Wiśniewski. *Questions, Inferences, and Scenarios*. London: College Publications, 2013.

## Erotetic Dual Resolution for mbC (2014)

The rules of  $E_{res}^{CPL}$  (see {40}) and the following rules (‘ $\neg$ ’ is used for the classical negation and ‘ $\sim$ ’ for the paraconsistent one):

$$\frac{?(\Phi; \vdash S' \sim A' T; \Psi)}{?(\Phi; \vdash S' \neg A' T; \vdash S' \chi \sim A' T; \Psi)} \mathbf{R}_{\sim} \quad \frac{?(\Phi; \vdash S' \neg \sim A' T; \Psi)}{?(\Phi; \vdash S' A' \neg \chi \sim A' T; \Psi)} \mathbf{R}_{\neg \sim}$$

$$\frac{?(\Phi; \vdash S' \circ A' T; \Psi)}{?(\Phi; \vdash S' \neg A' \chi \circ A' T; \vdash S' \neg \sim A' \chi \circ A' T; \Psi)} \mathbf{R}_{\circ}$$

$$\frac{?(\Phi; \vdash S' \neg \circ A' T; \Psi)}{?(\Phi; \vdash S' A' \sim A' T; \vdash S' \neg \chi \circ A' T; \Psi)} \mathbf{R}_{\neg \circ}$$

**Clarifications:** See {40} for notational conventions.

The calculus is worded in a language being an extension of the language of mbC, where the role of the additional  $\chi$  operator is to syntactically express the fact that certain formulas (e.g. of the form ‘ $\sim A$ ’, ‘ $\circ A$ ’) may have a logical value independent of the value of  $A$ .

**History:** The calculus  $E_{res}^{mbC}$  has been presented in [1] together with similar calculi for CLuN, CLuNs and for Classical Propositional Logic. The idea to use  $\chi$  operator was taken from [2], where the authors presented erotetic calculi for logics CLuN and CLuNs in a non-resolution account.

**Remarks:** A formula  $A$  is mbC-valid iff  $\vdash A$  has a Socratic refutation in  $E_{res}^{mbC}$ . A formula  $A$  is CLuN-valid iff  $\vdash A$  has a Socratic refutation constructed without the use of rules  $\mathbf{R}_{\circ}$ ,  $\mathbf{R}_{\neg \circ}$ . Similar results are obtained with respect to CLuNs and the classical case.

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- [1] Szymon Chlebowski and Dorota Leszczyńska-Jasion. “Dual Erotetic Calculi and the Minimal LFI”. In: *accepted to Studia Logica* (2015).
  - [2] Andrzej Wiśniewski, Guido Vanackere, and Dorota Leszczyńska. “Socratic Proofs and Paraconsistency: A Case Study”. In: *Studia Logica* 80.2-3 (2004), pp. 433–468.



**Part III**  
***Indexes***



## **List of Contributors**



## **Appendix A**

### **Contributors**

ToDo: Use an index instead.



## **Appendix B**

### **Authors**

ToDo: Use an index instead.





## Appendix C

### Acronyms

Use the template *acronym.tex* together with the Springer document class SVMono (monograph-type books) or SVMult (edited books) to style your list(s) of abbreviations or symbols in the Springer layout.

Lists of abbreviations symbols and the like are easily formatted with the help of the Springer-enhanced `description` environment.

ABC	Spelled-out abbreviation and definition
BABI	Spelled-out abbreviation and definition
CABR	Spelled-out abbreviation and definition

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