Preface

The **Encyclopedia of Proof Systems** aims at providing a reliable, technically informative, historically accurate, concise and convenient central repository of proof systems for various logics. The goal is to facilitate the exchange of information among logicians with mathematical, computational or philosophical backgrounds; in order to foster and accelerate the development of new proof systems and automated deduction tools that rely on them.

Preparatory work for the creation of the Encyclopedia, such as the implementation of the LaTeX template and the setup of the Github repository, started in October 2014, triggered by the call for workshop proposals for the 25th Conference on Automated Deduction (CADE). Christoph Benzmüller, CADE's conference chair, and Jasmin Blanchette, CADE's workshop co-chair, encouraged me to submit a workshop proposal and supported my alternative idea to organize instead a special poster session based on encyclopedia entries. I am thankful for their encouragement and support.

In December 2014, Björn Lellman, Giselle Reis and Martin Riener kindly accepted my request to beta-test the template and the instructions I had created. They submitted the first few example entries to the encyclopedia and provided valuable feedback, for which I am grateful. Their comments were essential for improving the templates and instructions before the public announcement of the encyclopedia.

January 2015 *Bruno Woltzenlogel Paleo*

Contents

Intuitionistic Natural Deduction NJ (1935)

$$
\frac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A}\&\mathfrak{B}}\,UE \qquad \frac{\mathfrak{A}\&\mathfrak{B}}{\mathfrak{A}}\,UB \qquad \frac{\mathfrak{A}\&\mathfrak{B}}{\mathfrak{B}}\,UB
$$

$$
\frac{\mathfrak{A}}{\mathfrak{A}\vee\mathfrak{B}}\,OE \qquad \frac{\mathfrak{B}}{\mathfrak{A}\vee\mathfrak{B}}\,OE \qquad \frac{\mathfrak{A}\vee\mathfrak{B} \quad \begin{array}{c}[\mathfrak{A}]\\ \vdots \\ \mathfrak{C}\end{array} \quad \begin{array}{c}[\mathfrak{B}]\\ \vdots \\ \mathfrak{C}\end{array}}{\mathfrak{C}}\,OB
$$

$$
\frac{\mathfrak{F}\mathfrak{a}}{\forall\mathfrak{x}\mathfrak{F}\mathfrak{x}}\,AE \qquad \frac{\forall\mathfrak{x}\mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}\,AB \qquad \frac{\mathfrak{F}\mathfrak{a}}{\exists\mathfrak{x}\mathfrak{F}\mathfrak{x}}\,EE \qquad \frac{\exists\mathfrak{x}\mathfrak{F}\mathfrak{x} \quad \begin{array}{c}[\mathfrak{F}\mathfrak{a}]\\ \vdots \\ \mathfrak{C}\end{array}}{\mathfrak{C}}\,EB
$$

$$
\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{\mathfrak{A}\supset\mathfrak{B}}\,FE \qquad \frac{\mathfrak{A} \quad \mathfrak{A}\supset\mathfrak{B}}{\mathfrak{B}}\,FB \qquad \frac{\begin{array}{c}[\mathfrak{A}]\\ \vdots \\ \curlywedge\end{array}}{\neg\mathfrak{A}}\,NE \qquad \frac{\mathfrak{A} \quad \neg\mathfrak{A}}{\curlywedge}\,NB \qquad \frac{\curlywedge}{\mathfrak{D}}
$$

The eigenvariable $\mathfrak{a}$ of an AE must not occur in the formula designated in the schema by $\forall\mathfrak{x}\mathfrak{F}\mathfrak{x}$; nor in any assumption formula upon which that formula depends. The eigenvariable $\mathfrak{a}$ of an EB must not occur in the formula designated in the schema by $\exists\mathfrak{x}\mathfrak{F}\mathfrak{x}$; nor in any assumption formula upon which that formula depends, with the exception of the assumption formulae designated by $\mathfrak{F}\mathring{\mathfrak{a}}$.

Clarifications: The names of the rules are those originally given by Gentzen [**Gentzen1935**]:
U = und (and), O = oder (or), A = all, E = es-gibt (exists), F = folgt (follows), N = nicht (not), E = Einführung (introduction), B = Beseitigung (elimination).

History: The main novelty introduced by Gentzen in this proof system is its *assumption* handling mechanism, which allows formal proofs to reflect more naturally the logical reasoning involved in mathematical proofs.

Remarks: In [**Gentzen1935**], completeness of **NJ** is proven by showing how to translate proofs in the Hilbert-style calculus **LHJ** to **NJ**-proofs, and soundness is proven by showing how to translate **NJ**-proofs to **LJ**-proofs {3}.

Entry 1 by: Bruno Woltzenlogel Paleo

Classical Sequent Calculus LK (1935)

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash \Lambda, A \quad A, \Delta \vdash \Theta}{\Gamma, \Delta \vdash \Lambda, \Theta}\ cut$$

$$\frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ w_l \qquad \frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, A}\ w_r$$

$$\frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta}\ e_l \quad \frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ c_l \qquad \frac{\Gamma \vdash \Theta, B, A, \Delta}{\Gamma \vdash \Theta, A, B, \Delta}\ e_r \quad \frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A}\ c_r$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\ \neg_l \qquad \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\ \neg_r$$

$$\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta}\ \wedge_l \qquad \frac{\Gamma \vdash \Theta, A \quad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \wedge B}\ \wedge_r$$

$$\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\ \vee_l \qquad \frac{\Gamma \vdash \Theta, A_i}{\Gamma \vdash \Theta, A_1 \vee A_2}\ \vee_r$$

$$\frac{\Gamma \vdash \Lambda, A \quad B, \Delta \vdash \Theta}{A \to B, \Gamma, \Delta \vdash \Lambda, \Theta}\ \to_l \qquad \frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \to B}\ \to_r$$

$$\frac{A[\alpha], \Gamma \vdash \Theta}{\exists x.A[x], \Gamma \vdash \Theta}\ \exists_l \quad \frac{A[t], \Gamma \vdash \Theta}{\forall x.A[x], \Gamma \vdash \Theta}\ \forall_l \quad \frac{\Gamma \vdash \Theta, A[\alpha]}{\Gamma \vdash \Theta, \forall x.A[x]}\ \forall_r \quad \frac{\Gamma \vdash \Theta, A[t]}{\Gamma \vdash \Theta, \exists x.A[x]}\ \exists_r$$

The eigenvariable α should not occur in Γ, Θ or $A[x]$.
The term t should not contain variables bound in $A[t]$.

History: This is a modern presentation of Gentzen's original **LK** calculus[**lk:Gentzen1935**], using modern notations and rule names.

Remarks: **LK** is complete relative to **NK** (i.e. **NJ** {1} with the axiom of excluded middle) and sound relative to a Hilbert-style calculus **LHK** [**lk:Gentzen1935a**]. Cut is eliminable (*Hauptsatz* [**lk:Gentzen1935**]), and hence classical predicate logic is consistent. Any *prenex* cut-free proof may be further transformed into a shape with only propositional inferences above and only quantifier and structural inferences below a *midsequent* [**lk:Gentzen1935a**].

Entry 2 by: Martin Riener

Intuitionistic Sequent Calculus LJ (1935)

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \quad A,\Delta \vdash \Theta}{\Gamma,\Delta \vdash \Theta}\ cut$$

$$\frac{\Gamma \vdash \Theta}{A,\Gamma \vdash \Theta}\ w_l \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ w_r$$

$$\frac{\Gamma,B,A,\Delta \vdash \Theta}{\Gamma,A,B,\Delta \vdash \Theta}\ e_l \qquad \frac{A,A,\Gamma \vdash \Theta}{A,\Gamma \vdash \Theta}\ c_l$$

$$\frac{\Gamma \vdash A}{\neg A,\Gamma \vdash}\ \neg_l \qquad \frac{A,\Gamma \vdash}{\Gamma \vdash \neg A}\ \neg_r$$

$$\frac{A_i,\Gamma \vdash \Theta}{A_1 \wedge A_2,\Gamma \vdash \Theta}\ \wedge_l \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_r$$

$$\frac{A,\Gamma \vdash \Theta \quad B,\Gamma \vdash \Theta}{A \vee B,\Gamma \vdash \Theta}\ \vee_l \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ \vee_r$$

$$\frac{\Gamma \vdash A \quad B,\Delta \vdash \Theta}{A \rightarrow B,\Gamma,\Delta \vdash \Theta}\ \rightarrow_l \qquad \frac{A,\Gamma \vdash B}{\Gamma \vdash A \rightarrow B}\ \rightarrow_r$$

$$\frac{A[\alpha],\Gamma \vdash \Theta}{\exists x.A[x],\Gamma \vdash \Theta}\ \exists_l \qquad \frac{\Gamma \vdash A[t]}{\Gamma \vdash \exists x.A[x]}\ \exists_r$$

$$\frac{A[t],\Gamma \vdash \Theta}{\forall x.A[x],\Gamma \vdash \Theta}\ \forall_l \qquad \frac{\Gamma \vdash A[\alpha]}{\Gamma \vdash \forall x.A[x]}\ \forall_r$$

The eigenvariable α should not occur in Γ, Θ or $A[x]$.
The term t should not contain variables bound in $A[t]$.

Clarifications: LJ and **LK** {2} have exactly the same rules, but in **LJ** the succedent of every sequent may have at most one formula. This restriction is equivalent to forbidding the axiom of excluded middle in natural deduction.

Remarks: The cut rule is eliminable (*Hauptsatz* [**Gentzen1935**]), and hence intuitionistic predicate logic is consistent and its propositional fragment is decidable [**Gentzen1935a**]. **LJ** is complete relative to **NJ** {1} and sound relative to the Hilbert-style calculus **LHJ** [**Gentzen1935a**].

Entry 3 by: Giselle Reis

Multi-Conclusion Sequent Calculus LJ′ (1954)

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash \Theta, A \quad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Theta, \Lambda}\ cut$$

$$\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta}\ \wedge_l \qquad \frac{\Gamma \vdash \Theta, A \quad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \wedge B}\ \wedge_r$$

$$\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\ \vee_l \qquad \frac{\Gamma \vdash \Theta, A_i}{\Gamma \vdash \Theta, A_1 \vee A_2}\ \vee_r$$

$$\frac{\Gamma \vdash \Theta, A \quad B, \Delta \vdash \Lambda}{A \rightarrow B, \Gamma, \Delta \vdash \Theta, \Lambda}\ \rightarrow_l \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B}\ \rightarrow_r$$

$$\frac{A\alpha, \Gamma \vdash \Theta}{\exists x.Ax, \Gamma \vdash \Theta}\ \exists_l \quad \frac{\Gamma \vdash \Theta, At}{\Gamma \vdash \Theta, \exists x.Ax}\ \exists_r \quad \frac{At, \Gamma \vdash \Theta}{\forall x.Ax, \Gamma \vdash \Theta}\ \forall_l \quad \frac{\Gamma \vdash A\alpha}{\Gamma \vdash \forall x.Ax}\ \forall_r$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\ \neg_l \quad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A}\ \neg_r \quad \frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta}\ e_l \quad \frac{\Gamma\Delta \vdash \Theta, B, A, \Lambda}{\Gamma \vdash \Theta, A, B, \Lambda}\ e_r$$

$$\frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ c_l \quad \frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A}\ c_r \quad \frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ w_l \quad \frac{\Gamma \vdash}{\Gamma \vdash A}\ w_r$$

The eigenvariable α should not occur in Γ, Θ or $A[x]$.
The term t should not contain variables bound in $A[t]$.

Clarifications: While **LJ** {3} is defined by restricting **LK** {2} to a single conclusion (i.e. at most one formula per succedent), in **LJ′** only the rules $\neg_r$, $\rightarrow_r$ and $\forall_r$ have this restriction.

History: **LJ′** was proposed in [**Maehara1954**] and used to prove the completeness of **LJ** {3} in [**Takeuti1987**]. The same calculus (and other multi-conclusion calculi for intuitionistic logic) was rediscovered in [**Nadathur1998**] while analyzing classical and intuitionistic provability.

Remarks: **LJ′** is equivalent to **LJ**, and this is established by translating sequents of the form $\Gamma \vdash A_1, ..., A_n$ into sequents of the form $\Gamma \vdash A_1 \vee ... \vee A_n$. Cut is eliminable and this can be proven by using a combination of the rewriting rules for cut-elimination in **LJ** and **LK**.

Entry 4 by: Giselle Reis

Second Order λ-Calculus (System F) (1971)

$$\frac{(x:T)\in E}{\Gamma;E\vdash x:T}\ \text{assumption}$$

$$\frac{\Gamma;E,(x:T)\vdash e:S}{\Gamma;E\vdash \lambda(x:T.e):(T\to S)}\ \to I \qquad \frac{\Gamma;E\vdash f:(T\to S)\quad \Gamma;E\vdash e:T}{\Gamma;E\vdash (fe):\tau}\ \to E$$

$$\frac{\Gamma X;E\vdash e:T}{\Gamma;E\vdash (\Lambda X:Tp.e):(\forall X:Tp.T)}\ \forall I^{*} \qquad \frac{\Gamma;E\vdash f:(\forall X:Tp.T)\quad \Gamma\vdash S:Tp}{\Gamma;E\vdash fS:[S/X]T}\ \forall E$$

* X must be not free in the type of any free term variable in E.

Clarifications: The presentation from [**AspLongo:91**] with minor corrections is used. Below $X,Y,Z...$ are type-variables and $x,y,...$ term variables.

Type expressions: $T := X|(T\to S)|(\forall X:Tp.T)$.

Term expressions: $e := x|(ee)|(eT)|(\lambda x:T.e)|(\Lambda X:Tp.e)$.

$\forall,\Lambda$ and λ are variable binders. All expressions are considered up to renaming of bound variables (α-conversion). An unbound variable is free. $FV(R)$ is the set of free variables for any (type or term) expression; $[e/x]$, $[S/X]$ mean capture-avoiding substitution in term- and type-expressions respectively (defined by induction). A context is a finite set Γ of type variables; ΓX stands for $\Gamma\cup X$. A type T is legal in Γ iff $FV(T)\subseteq FV(\Gamma)$. A type assignment in Γ is a finite list $E=(x_1:T_1),...,(x_n:T_n)$ where any T_i is legal in Γ. The typing relation $\Gamma;E\vdash e:T$, where E is a type assignmentlegal in Γ, e is a term expression and T is a type expression, is defined by the rules above.

The *conversion relation* between well-typed terms is very important. It is defined by the following axioms: (β) $(\lambda x:T.f)e=[e/x]f$; (β_2) $(\Lambda X:Tp.e)S=[S/X]e$; (η) $\lambda x:T.(ex)=e$ if $x\notin FV(e)$; (η_2) $\Lambda X:Tp.(eX)=e$ if $X\notin FV(e)$, and by usual rules that turn "=" into congruence. The system $\mathbf{F}_c$ is obtained if one more equality axiom is added: (**C**) $\quad eT=eT'$ for $\Gamma;E\vdash e:\forall X.S$ and $X\notin FV(S)$.

History: Introduced by Girard [**Gir:71**] and Reynolds [**Rey:74**]. Inspired works on higher order type systems. Included by Barendregt in his λ-cube [**Bar:91**]. Various extensions were considered, for example, $\mathbf{F}_c$ [**LMS:93**], **F** with subtyping [**CMMS:91**, **LMS:00**]. Important for functional programming languages.

Remarks: A strong normalization theorem for **F** was proved by Girard [**Gir:72**]. It implies a normalization theorem and consistency for second order arithmetic PA_2. For $\mathbf{F}_c$, a *genericity theorem* holds [**LMS:93**].

Entry 5 by: Giuseppe Longo, Kathleen Milsted, Sergei Soloviev

Expansion Proofs (1983)

Expansion trees, *eigenvariables*, and the function $\mathsf{Sh}(-)$ (read *shallow formula of*), that maps an expansion tree to a formula, are defined as follows:

1. If A is $\top$ (true), $\perp$ (false), or a literal, then A is an expansion tree with top node A, and $\mathsf{Sh}(A) = A$.
2. If E is an expansion tree with $\mathsf{Sh}(E) = [y/x]A$ and y is not an eigenvariable of any node in E, then $E' = \forall x.A +^y E$ is an expansion tree with top node $\forall x.A$ and $\mathsf{Sh}(E') = \forall x.A$. The variable y is called an *eigenvariable* of (the top node of) E'. The set of eigenvariables of all nodes in an expansion tree is called the *eigenvariables of* the tree.
3. If $\{t_1, \ldots, t_n\}$ (with $n \geq 0$) is a set of terms and $E_1, \ldots, E_n$ are expansion trees with pairwise disjoint eigenvariable sets and with $\mathsf{Sh}(E_i) = [t_i/x]A$ for $i \in \{1, \ldots, n\}$, then $E' = \exists x.A +^{t_1} E_1 \ldots +^{t_n} E_n$ is an expansion tree with top node $\exists x.A$ and $\mathsf{Sh}(E') = \exists x.A$. The terms $t_1, \ldots, t_n$ are known as the *expansion terms* of (the top node of) E'.
4. If E_1 and E_2 are expansion trees that share no eigenvariables and $\circ \in \{\wedge, \vee\}$, then $E_1 \circ E_2$ is an expansion tree with top node $\circ$ and $\mathsf{Sh}(E_1 \circ E_2) = \mathsf{Sh}(E_1) \circ \mathsf{Sh}(E_2)$.

In the expansion tree $\forall x.A +^x E$ (resp. in $\exists x.A +^{t_1} E_1 \ldots +^{t_n} E_n$), we say that x (resp. t_i) *labels* the top node of E (resp. E_i, for any $i \in \{1, \ldots, n\}$). A term t *dominates* a node in an expansion tree if it labels a parent node of that node in the tree.

For an expansion tree E, the quantifier-free formula $\mathsf{Dp}(E)$, called the *deep formula of E*, is defined as:

- $\mathsf{Dp}(E) = E$ if E is $\top$, $\perp$, or a literal;
- $\mathsf{Dp}(E_1 \circ E_2) = \mathsf{Dp}(E_1) \circ \mathsf{Dp}(E_2)$ for $\circ \in \{\wedge, \vee\}$;
- $\mathsf{Dp}(\forall x.A +^y E) = \mathsf{Dp}(E)$; and
- $\mathsf{Dp}(\exists x.A +^{t_1} E_1 \cdots +^{t_n} E_n) = \mathsf{Dp}(E_1) \vee \ldots \vee \mathsf{Dp}(E_n)$ if $n > 0$, and $\mathsf{Dp}(\exists x.A) = \perp$.

Let $\mathcal{E}$ be an expansion tree and let $<^0_{\mathcal{E}}$ be the binary relation on the occurrences of expansion terms in $\mathcal{E}$ defined by $t <^0_{\mathcal{E}} s$ if there is an x which is free in s and which is the eigenvariable of a node dominated by t. Then $<_{\mathcal{E}}$, the transitive closure of $<^0_{\mathcal{E}}$, is called the *dependency relation* of $\mathcal{E}$.

An expansion tree $\mathcal{E}$ is said to be an *expansion proof* if $<_{\mathcal{E}}$ is acyclic and $\mathsf{Dp}(\mathcal{E})$ is a tautology; in particular, $\mathcal{E}$ is an *expansion proof of* $\mathsf{Sh}(\mathcal{E})$.

Clarifications: The soundness and completeness theorem for expansion trees is the following. A formula B is a theorem of first-order logic if and only if there is an expansion proof Q such that $\mathsf{Sh}(Q) = B$.

History: Expansion trees and expansion proofs [**miller87sl**, **miller83**] provide a simple generalization of both Herbrand's disjunctions and Gentzen's mid-sequent theorem to formulas that are not necessarily in prenex-normal form. These proof

---

Entry 6 by: Dale Miller

structures were originally defined for higher-order classical logic and used to provide a generalization of Herbrand's theorem for higher-order logic as well as a soundness proof for skolemization in the presence of higher-order quantification. Expansion trees are an early example of a matrix-based proof system that emphasizes parallelism within proof structures in a manner similar to that found in linear logic proof nets [**girard87tcs**]. That parallelism is explicitly analyzed in [**chaudhuri14jlc**] using a multi-focused version of LKF {12}.

Natural Knowledge Bases - Muscadet (1984)

Some of the rules given to the system :
To prove $A \wedge B$, prove A and prove B
To prove $\forall X P(X)$, take any X_1 and prove $P(X_1)$
To prove $\exists X P(X)$, search for an X such that $P(X)$
To prove $A \Rightarrow B$, assume A and prove B
To prove C, if $A \vee B$, then prove $A \Rightarrow C \wedge B \Rightarrow C$
Flatten : Replace $P(f(X))$ by $\exists Y(Y : f(X) \wedge P(Y))$ or by $\forall Y(Y : f(X) \Rightarrow P(Y)$
To prove $\neg A$, assume A and search for a contradiction (i.e. prove $false$)
Some of the rules automatically built by metarules from definitions :
If $A \subset B$ and $X \in A$ then $X \in B$
If $C : A \cap B$ and $X \in C$, then $X \in A$
If $C : A \cap B$, $X \in A$ and $X \in B$, then $X \in C$

Clarifications: $C : A \cap B$ expresses that C is $A \cap B$ which has already been introduced. Rules are conditional actions. Actions may be defined by packs of rules. Metarules builds rules from definitions, lemmas and universal hypotheses.

History: Muscadet [**pastre:1989**, **pastre:1993**] is a knowledge-based system. It uses natural deduction (following the terminology of Bledsoe ([**bledsoe:71**, **bledsoe77**]), that is natural humanlike methods). Facts are hypotheses and the conclusion of a theorem or a sub-theorem to be proved, and all sorts of facts which give relevant information during the proof searching process. Universal hypotheses are handled as local definitions (no skolemisation). **Muscadet** worked in set theory, mappings and relations, topology and topological linear spaces, elementary geometry, discrete geometry, cellular automata, and TPTP problems. It attended CASC competitions.

Remarks: The system is sound but not complete (because of the use of many selective rules heuristics). It displays proofs easily readable by a human reader.

Entry 7 by: Dominique Pastre

Z. Luo's LF (1994)

$$\frac{}{<>\vdash \textbf{valid}} \quad \frac{\Gamma\vdash K\textbf{kind} \quad x\notin FV(\Gamma)}{\Gamma, x:K\vdash \textbf{valid}} \quad \frac{\Gamma, x:K, \Gamma'\vdash \textbf{valid}}{\Gamma, x:K, \Gamma'\vdash x:K} \quad (1)$$

$$\frac{\Gamma\vdash k:K \quad \Gamma\vdash K=K'}{\Gamma\vdash k:K'} \quad \frac{\Gamma\vdash k=k':K \quad \Gamma\vdash K=K'}{\Gamma\vdash k=k':K'} \quad (2)^{*}$$

$$\frac{\Gamma, x:K, \Gamma'\vdash J \quad \Gamma\vdash k:K}{\Gamma, [k/x]\Gamma'\vdash [k/x]J} \quad (3)^{**}$$

$$\frac{\Gamma\vdash K\textbf{kind} \quad \Gamma, x:K\vdash K'\textbf{kind}}{\Gamma\vdash (x:K)K'\textbf{kind}} \quad \frac{\Gamma\vdash K_1=K_2 \quad \Gamma, x:K_1\vdash K_1'=K_2'}{\Gamma\vdash (x:K_1)K_1'=(x:K_2)K_2'}$$

$$\frac{\Gamma\vdash K\textbf{kind} \quad \Gamma, x:K\vdash K'\textbf{kind}}{\Gamma\vdash (x:K)K'\textbf{kind}} \quad \frac{\Gamma\vdash K_1=K_2 \quad \Gamma, x:K_1\vdash K_1'=K_2'}{\Gamma\vdash (x:K_1)K_1'=(x:K_2)K_2'}$$

$$\frac{\Gamma\vdash K\textbf{kind} \quad \Gamma, x:K\vdash K'\textbf{kind}}{\Gamma\vdash (x:K)K'\textbf{kind}} \quad \frac{\Gamma\vdash K_1=K_2 \quad \Gamma, x:K_1\vdash K_1'=K_2'}{\Gamma\vdash (x:K_1)K_1'=(x:K_2)K_2'}$$

$$\frac{\Gamma\vdash K\textbf{kind} \quad \Gamma, x:K\vdash K'\textbf{kind}}{\Gamma\vdash (x:K)K'\textbf{kind}} \quad \frac{\Gamma\vdash K_1=K_2 \quad \Gamma, x:K_1\vdash K_1'=K_2'}{\Gamma\vdash (x:K_1)K_1'=(x:K_2)K_2'} \quad (4)$$

$$\frac{\Gamma\vdash \textbf{valid}}{\Gamma\vdash \textbf{Type}\,\textbf{kind}} \quad \frac{\Gamma\vdash A:\textbf{Type}}{\Gamma\vdash El(A)\textbf{kind}} \quad (5)$$

Clarifications: We follow [**Luo:94**]. Terms of **LF** are of the forms **Type**, $El(A)$, $(x:K)K'$ (dependent product), $[x:K]K'$ (abstraction), $f(k)$, and judgements of the forms $\Gamma\vdash$ **valid** (validity of context), $\Gamma\vdash K$**kind**, $\Gamma\vdash k:K$, $\Gamma\vdash k=k':K$, $\Gamma\vdash K=K'$. Rule groups: (1) rules for contexts and assumptions; (2)* equality rules (reflexivity, symmetry and transitivity rules are ommitted); (3)** substitution rules (J denotes the right side of any of the five forms of judgement); (4) rules for dependent product kinds; (5) and the kind **Type**.

History: The calculus was defined in [**Luo:94**], ch. 9. LF is a typed version of Martin-Löf's logical framework [**NPS:90**]. Type theories specified in **LF** were used as basis of proof-assistants Lego and Plastic. Later the system was extended to include coercive subtyping [**SolLuo:02**, **LuoSolXue:14**].

Remarks: The proof-theoretical analysis of LF above was used in meta-theoretical studies of larger theories defined on its basis, *e.g.*, UTT (Universal Type Theory) that includes inductive schemata, second order logic SOL with impredicative type *Prop* and a hierarchy of predicative universes [**Luo:94**]. H. Goguen defined a typed operational semantics for UTT and proved strong normalization theorem [**HG:94**]. For **LF** with coercive subtyping conservativity results were obtained [**SolLuo:02**, **LuoSolXue:14**].

Entry 8 by: Zhaohui Luo, Sergei Soloviev

Sequent Calculus G3c (1996)

$$\frac{}{P, \Gamma \vdash \Delta, P}\ \mathrm{Ax} \qquad \frac{}{\bot, \Gamma \vdash \Delta}\ \mathrm{L}\bot$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \mathrm{L}\wedge \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ \mathrm{R}\wedge$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ \mathrm{L}\vee \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\ \mathrm{R}\vee$$

$$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta}\ \mathrm{L}\to \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ \mathrm{R}\to$$

$$\frac{\forall x A, A[x/t], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta}\ \mathrm{L}\forall \qquad \frac{\Gamma \vdash \Delta, A[x/y]}{\Gamma \vdash \Delta, \forall x A}\ \mathrm{R}\forall$$

$$\frac{A[x/y], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta}\ \mathrm{L}\exists \qquad \frac{\Gamma \vdash \Delta, A[x/t], \exists x A}{\Gamma \vdash \Delta, \exists x A}\ \mathrm{R}\exists$$

P should be atomic in Ax and y should not be free in the conclusion of R∀ and L∃

Clarifications: Sequents are based on multisets. A formula $A[x/t]$ is the result of uniformly substituting the term t for the variable x in A, renaming bound variables to prevent clashes with the variables in t.

Remarks: G3c is sound and complete w.r.t. classical first-order logic. Weakening and contraction are depth-preserving admissible and all rules are depth-preserving invertible.

Entry 9 by: Björn Lellmann

One-Sided Higher-Order Sequent Calculi $\mathcal{G}_\beta$ and $\mathcal{G}_{\beta\mathfrak{f}\mathfrak{b}}$ (2004-2009)

Basic Rules

$$\frac{\Delta, s}{\Delta, \neg\neg s}\,\mathcal{G}(\neg) \qquad \frac{\Delta, \neg s \quad \Delta, \neg t}{\Delta, \neg(s \vee t)}\,\mathcal{G}(\vee_-) \qquad \frac{\Delta, s, t}{\Delta, (s \vee t)}\,\mathcal{G}(\vee_+)$$

$$\frac{\Delta, \neg(sl){\downarrow_\beta} \quad l_\alpha \text{ closed term}}{\Delta, \neg\Pi^\alpha s}\,\mathcal{G}(\Pi^l_-) \qquad \frac{\Delta, (sc){\downarrow_\beta} \quad c_\delta \text{ new symbol}}{\Delta, \Pi^\alpha s}\,\mathcal{G}(\Pi^c_+)$$

Initialization

$$\frac{s \text{ atomic (and } \beta\text{-normal)}}{\Delta, s, \neg s}\,\mathcal{G}(init)$$

$$\frac{\Delta, (s \doteq^o t) \quad s,t \text{ atomic}}{\Delta, \neg s, t}\,\mathcal{G}(Init^{\doteq})$$

Extensionality

$$\frac{\Delta, (\forall X_\alpha sX \doteq^\beta tX){\downarrow_\beta}}{\Delta, (s \doteq^{\alpha\to\beta} t)}\,\mathcal{G}(\mathfrak{f}) \qquad \frac{\Delta, \neg s, t \quad \Delta, \neg t, s}{\Delta, (s \doteq^o t)}\,\mathcal{G}(\mathfrak{b})$$

Decomposition

$$\frac{\Delta, (s^1 \doteq^{\alpha_1} t^1) \cdots \Delta, (s^n \doteq^{\alpha_n} t^n) \quad n \geq 1, \beta \in \{o, \iota\},\ h_{\overline{\alpha^n}\to\beta} \in \Sigma}{\Delta, (h\overline{s^n} \doteq^\beta h\overline{t^n})}\,\mathcal{G}(d)$$

Calculus $\mathcal{G}_\beta$ is defined by $\mathcal{G}(init)$, $\mathcal{G}(\neg)$, $\mathcal{G}(\vee_-)$, $\mathcal{G}(\vee_+)$, $\mathcal{G}(\Pi^l_-)$ and $\mathcal{G}(\Pi^c_+)$.
Calculus $\mathcal{G}_{\beta\mathfrak{f}\mathfrak{b}}$ extends $\mathcal{G}_\beta$ by the additional rules $\mathcal{G}(\mathfrak{b})$, $\mathcal{G}(\mathfrak{f})$, $\mathcal{G}(d)$, and $\mathcal{G}(Init^{\doteq})$.

Clarifications: Δ and Δ' are finite sets of β-normal closed formulas of classical higher-order logic (HOL) [**sep-type-theory-church**] and let Δ, s denote the set $\Delta \cup \{s\}$. HOL terms and formulas: Let $\alpha, \beta, o \in T$. The *terms* of HOL are defined by the grammar (c_α denotes typed constants and X_α typed variables distinct from c_α): $s,t ::= c_\alpha \mid X_\alpha \mid (\lambda X_\alpha s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta} t_\alpha)_\beta \mid (\neg_{o\to o}\ s_o)_o \mid (s_o \vee_{o\to o\to o} t_o)_o \mid (\Pi_{(\alpha\to o)\to o}\ s_{\alpha\to o})_o$. *Leibniz equality* $\doteq^\alpha$ at type α is defined as $s_\alpha \doteq^\alpha t_\alpha := \forall P_{\alpha\to o}(\neg Ps \vee Pt)$. For each simply typed λ-term s there is a unique *β-normal form* (denoted $s{\downarrow_\beta}$) and a unique $\beta\eta$-normal form (denoted $s{\downarrow_{\beta\eta}}$). HOL formulas are defined as terms of type o. A *non-atomic formula* is any formula whose β-normal form is of the form $[c\overline{\mathbf{A}^n}]$ where c is a logical constant. An *atomic formula* is any other formula.

History: The calculi have been presented in [**J18**]. Earlier versions and further related sequent calculi for HOL are presented in [**R19**].

Entry 10 by: Christoph Benzmüller

Remarks: $\mathcal{G}_\beta$ is sound and complete for elementary type theory (ETT): $\models^{\mathrm{ETT}} c$ if and only if $\vdash^{\mathcal{G}_\beta} \{c\}$. $\mathcal{G}_\beta$ is thus also sound for HOL. $\mathcal{G}_{\beta\mathfrak{f}\mathfrak{b}}$ is sound and complete for HOL: $\models^{\mathrm{HOL}} c$ if and only if $\vdash^{\mathcal{G}_{\beta\mathfrak{f}\mathfrak{b}}} \{c\}$

Extensional HO RUE-Resolution (1999-2013)

Normalisation Rules

$$\frac{\mathbf{C} \vee [\mathbf{A} \vee \mathbf{B}]^{\mathtt{tt}}}{\mathbf{C} \vee [\mathbf{A}]^{\mathtt{tt}} \vee [\mathbf{B}]^{\mathtt{tt}}}\ \vee^{\mathtt{tt}} \qquad \frac{\mathbf{C} \vee [\mathbf{A} \vee \mathbf{B}]^{\mathtt{ff}}}{\begin{array}{c}\mathbf{C} \vee [\mathbf{A}]^{\mathtt{ff}} \\ \mathbf{C} \vee [\mathbf{B}]^{\mathtt{ff}}\end{array}}\ \vee^{\mathtt{ff}} \qquad \frac{\mathbf{C} \vee [\neg \mathbf{A}]^{\mathtt{tt}}}{\mathbf{C} \vee [\mathbf{A}]^{\mathtt{ff}}}\ \neg^{\mathtt{tt}} \qquad \frac{\mathbf{C} \vee [\neg \mathbf{A}]^{\mathtt{ff}}}{\mathbf{C} \vee [\mathbf{A}]^{\mathtt{tt}}}\ \neg^{\mathtt{ff}}$$

$$\frac{\mathbf{C} \vee [\Pi^{\tau}\mathbf{A}]^{\mathtt{tt}} \quad X^{\tau} \text{ fresh variable}}{\mathbf{C} \vee [\mathbf{A}\,X]^{\mathtt{tt}}}\ \Pi^{\mathtt{tt}} \qquad \frac{\mathbf{C} \vee [\Pi^{\tau}\mathbf{A}]^{\mathtt{ff}} \quad \mathsf{sk}^{\tau} \text{ Skolem term}}{\mathbf{C} \vee [\mathbf{A}\,\mathsf{sk}^{\tau}]^{\mathtt{ff}}}\ \Pi^{\mathtt{ff}}$$

Resolution, Factorisation and Primitive Substitution

$$\frac{[\mathbf{A}]^{p_1} \vee \mathbf{C} \quad [\mathbf{B}]^{p_2} \vee \mathbf{D} \quad p_1 \neq p_2}{\mathbf{C} \vee \mathbf{D} \vee [\mathbf{A} = \mathbf{B}]^{\mathtt{ff}}}\ \mathsf{res} \qquad \frac{\mathbf{C} \vee [\mathbf{A}]^{p} \vee [\mathbf{B}]^{p}}{\mathbf{C} \vee [\mathbf{A}]^{p} \vee [\mathbf{A} = \mathbf{B}]^{\mathtt{ff}}}\ \mathsf{fac}$$

$$\frac{[Q_{\tau}\overline{\mathbf{A}}^{n}]^{p} \vee \mathbf{C} \quad \mathbf{P} \in \mathcal{AB}_{\tau}^{(k)} \text{ for logic connective } k}{([Q_{\tau}\overline{\mathbf{A}}^{n}]^{p} \vee \mathbf{C})[\mathbf{P}/Q]}\ \mathsf{prim\_subst}$$

Extensionality and Pre-unification

$$\frac{\mathbf{C} \vee [\mathbf{A}^{\sigma\tau} = \mathbf{B}^{\sigma\tau}]^{\mathtt{tt}} \quad X^{\tau} \text{ fresh variable}}{\mathbf{C} \vee [\mathbf{A}\,X = \mathbf{B}\,X]^{\mathtt{tt}}}\ \textsc{FuncPos} \qquad \frac{\mathbf{C} \vee [\mathbf{A}^{o} = \mathbf{B}^{o}]^{\mathtt{tt}}}{\mathbf{C} \vee [\mathbf{A}^{o} \longleftrightarrow \mathbf{B}^{o}]^{\mathtt{tt}}}\ \textsc{BoolPos}$$

$$\frac{\mathbf{C} \vee [\mathbf{A}^{\sigma\tau} = \mathbf{B}^{\sigma\tau}]^{\mathtt{ff}} \quad \mathsf{sk}^{\tau} \text{ Skol. term}}{\mathbf{C} \vee [\mathbf{A}\,\mathsf{sk} = \mathbf{B}\,\mathsf{sk}]^{\mathtt{ff}}}\ \textsc{FuncNeg} \qquad \frac{\mathbf{C} \vee [\mathbf{A}^{o} = \mathbf{B}^{o}]^{\mathtt{ff}}}{\mathbf{C} \vee [\mathbf{A}^{o} \longleftrightarrow \mathbf{B}^{o}]^{\mathtt{ff}}}\ \textsc{BoolNeg}$$

$$\frac{\mathbf{C} \vee [h^{\sigma\tau}\overline{\mathbf{A}^{\alpha}}^{k} \ = \ h^{\sigma\tau}\overline{\mathbf{B}^{\alpha}}^{k}]^{\mathtt{ff}}}{\mathbf{C} \vee \overline{[\mathbf{A}_i \ = \ \mathbf{B}_i]^{\mathtt{ff}}}^{i \leq k}}\ \textsc{Dec} \qquad \frac{\mathbf{C} \vee [X = \mathbf{A}]^{\mathtt{ff}} \quad X \notin \mathrm{FV}(\mathbf{A})}{\mathbf{C}[\mathbf{A}/X]}\ \textsc{Subst}$$

$$\frac{\mathbf{C} \vee [\mathbf{A} = \mathbf{A}]^{\mathtt{ff}}}{\mathbf{C}}\ \textsc{Triv} \qquad \frac{\mathbf{C} \vee [F^{\tau}\overline{\mathbf{A}}^{n} \ = \ h\overline{\mathbf{B}}^{m}]^{\mathtt{ff}} \quad \mathbf{G} \in \mathcal{AB}_{\tau}^{(h)}}{\mathbf{C} \vee [F = \mathbf{G}]^{\mathtt{ff}} \ \vee \ [F\overline{\mathbf{A}}^{n} = h\overline{\mathbf{B}}^{m}]^{\mathtt{ff}}}\ \textsc{FlexRigid}$$

Choice

$$\frac{C := \mathbf{C}' \vee [\mathbf{A}[E_{(\alpha\to o)\to\alpha}\mathbf{B}]]^{p} \quad \begin{array}{c}\epsilon \in \mathsf{CFs},\ E = \epsilon \text{ or } E \in \mathit{freeVars}(C), \\ \mathit{freeVars}(\mathbf{B}) \subseteq \mathit{freeVars}(C), Y \mathit{fresh}\end{array}}{[\mathbf{B}\ Y]^{\mathtt{ff}} \vee [\mathbf{B}\ (\epsilon_{\alpha(\alpha o)}\mathbf{B})]^{\mathtt{tt}}}\ \mathsf{choice}$$

$$\frac{[PX]^{\mathtt{ff}} \vee [P(f_{(\alpha\to o)\to\alpha}P)]^{\mathtt{tt}}}{\mathsf{CFs} \longleftarrow \mathsf{CFs} \cup \{f_{(\alpha\to o)\to\alpha}\}}\ \mathsf{detectChoiceFn}$$

Optional additional rules include (a) exhaustive universal instantion rule for (selective) finite domains, (b) detection and removal of Leibniz equations and Andrews equations, and (c) splitting. Like detectChoiceFn these rules are admissible.

Clarifications: **A** and **B** are metavariables ranging over terms of classical higher-order logic (resp. Church's type theory). The logical connectives are $\neg$, $\vee$, Π^{τ} (universal quantification over variables of type τ), and $=^{\tau}$ (equality on terms of type τ). Types are shown only if unclear in context. For example, in rule choice the variable

Entry 11 by: Christoph Benzmüller

$E^{\alpha(\alpha o)}$ is of function type, also written as $(\alpha \rightarrow o) \rightarrow \alpha$. Variables like F are presented as upper case symbols and constant symbols like h are lower case. α equality and $\beta\eta$-normalisation are treated implicit, meaning that all clauses are implicitly normalised. **C** and **D** are metavariables ranging over clauses, which are disjunctions of literals. These disjunctions are implicitly assumed associative and commutative; the latter also applies to all equations. Literals are formulas shown in square brackets and labelled with a *polarity* (either $\mathtt{tt}$ or $\mathtt{ff}$), e.g. $[\neg X]^{\mathtt{ff}}$ denotes the negation of $\neg X$. FV(**A**) denotes the free variables of term **A**. $\mathcal{AB}_{\tau}^{(h)}$ is the set of approximating bindings for head h and type τ. $\epsilon_{\alpha(\alpha o)}$ is a choice operator and CFs is a set of dynamically collected choice functions symbols; CFs is initialised with a single choice function.

History: The original calculus (without choice) has been presented in [**C5**] and [**J5**]. Recent modifications and extensions (e.g. choice) are discussed in [**W47**] and [**EasyChair:215**]. The calculus is inspired by and extends Huet's constrained resolution [**Huet:amott73**, **Huet:cracmfhol72**] and the extensional resolution calculus in [**C2**].

Remarks: The calculus works for classical higher-order logic with Henkin semantics and choice. Soundness and completeness has been discussed in [**C5**] and [**J5**]. In the prover LEO-II, the factorisation rule is for performance reasons restricted to binary clauses and a (parametrisable) depth limit is employed for pre-unification. Such restrictions are a (deliberate) source for incompleteness.

Focused LK (2007)

Asynchronous introduction rules

$$\frac{}{\vdash \Gamma \Uparrow t^-, \Theta} \qquad \frac{\vdash \Gamma \Uparrow B_1, \Theta \quad \vdash \Gamma \Uparrow B_2, \Theta}{\vdash \Gamma \Uparrow B_1 \wedge^- B_2, \Theta} \qquad \frac{\vdash \Gamma \Uparrow \Theta}{\vdash \Gamma \Uparrow f^-, \Theta} \qquad \frac{\vdash \Gamma \Uparrow B_1, B_2, \Theta}{\vdash \Gamma \Uparrow B_1 \vee^- B_1, \Theta}$$

$$\frac{\vdash \Gamma \Uparrow [y/x]B, \Theta}{\vdash \Gamma \Uparrow \forall x.B, \Theta}$$

Synchronous introduction rules

$$\frac{}{\vdash \Gamma \Downarrow t^+} \qquad \frac{\vdash \Gamma \Downarrow B_1 \quad \vdash \Gamma \Downarrow B_2}{\vdash \Gamma \Downarrow B_1 \wedge^+ B_2} \qquad \frac{\vdash \Gamma \Downarrow B_i}{\vdash \Gamma \Downarrow B_1 \vee^+ B_2}\ i \in \{1,2\} \qquad \frac{\vdash \Gamma \Downarrow [t/x]B}{\vdash \Gamma \Downarrow \exists x.B}$$

Identity rules

$$\frac{P \text{ atomic}}{\vdash \neg P, \Gamma \Downarrow P}\ \mathit{init} \qquad \frac{\vdash \Gamma \Uparrow B \quad \vdash \Gamma \Uparrow \neg B}{\vdash \Gamma \Uparrow \cdot}\ \mathit{cut}$$

Structural rules

$$\frac{\vdash \Gamma, C \Uparrow \Theta}{\vdash \Gamma \Uparrow C, \Theta}\ \mathit{store} \qquad \frac{\vdash \Gamma \Uparrow N}{\vdash \Gamma \Downarrow N}\ \mathit{release} \qquad \frac{\vdash P, \Gamma \Downarrow P}{\vdash P, \Gamma \Uparrow \cdot}\ \mathit{decide}$$

Here, Γ ranges over multisets of polarized formulas; Θ ranges over lists of polarized formulas; P denotes a positive formula; N denotes a negative formula; C denotes either a negative formula or a positive atom; and B denotes an unrestricted polarized formula. The negation in $\neg B$ denotes the negation normal form of the de Morgan dual of B. The right introduction rule for $\forall$ has the the usual eigenvariable restriction that y is not free in any formula in the conclusion sequent.

Clarifications: This proof system involves *polarized* (negative normal) formulas of first-order classical logic: in order to polarize a formula B, one must assign the status of "positive" or "negative" bias to all atomic formulas and replace all occurrences of truth with either t^+ or t^+ and replace all occurrences of conjunctions with either $\wedge^+$ or $\wedge^-$; similarly, all occurrences of false and disjunctions must be polarized into f^+, f^-, $\vee^+$, and $\vee^-$. If there are n occurrences of propositional connectives in B, there are 2^n ways to polarize B. The *positive connectives* are f^+, $\vee^+$, t^+, $\wedge^+$, and $\exists$ while the *negative connectives* are t^-, $\wedge^-$, f^-, $\vee^-$, and $\forall$. A formula is *positive* it is a positive atom or has a top-level positive connective; similarly a formula is *negative* it is a negative atom or has a top-level negative connective.

There are two kinds of sequents in this proof system, namely, $\vdash \Gamma \Uparrow \Theta$ and $\vdash \Gamma \Downarrow B$, where Γ is a multiset of polarized formulas, B is a polarized formula, and Θ is a list of polarized formulas. The list structure of Θ can be replaced by a multiset.

Entry 12 by: Dale Miller

History: This focused proof system is a slight variation of the proof systems in [**liang09tcs**, **liang07csl**]. A multifocus variant of **LKF** has been described in [**chaudhuri14jlc**]. The design of **LKF** borrows strongly by Andreoli's focused proof system for linear logic [**andreoli92jlc**] and Girard's LC proof system [**girard91mscs**]. The first-order versions of the LKT and LKQ proof systems of [**danos93wll**] can be seen subsystems of **LKF**.

Focused LJ (2007)

ASYNCHRONOUS INTRODUCTION RULES

$$\frac{\Gamma \Uparrow B_1 \vdash B_2 \Uparrow}{\Gamma \Uparrow \cdot \vdash B_1 \supset B_2 \Uparrow} \qquad \frac{\Gamma \Uparrow \cdot \vdash B_1 \Uparrow \quad \Gamma \Uparrow \cdot \vdash B_2 \Uparrow}{\Gamma \Uparrow \cdot \vdash B_1 \wedge^- B_2 \Uparrow} \qquad \frac{}{\Gamma \Uparrow \cdot \vdash t^- \Uparrow}$$

$$\frac{\Gamma \Uparrow \cdot \vdash [y/x]B \Uparrow}{\Gamma \Uparrow \cdot \vdash \forall x.B \Uparrow} \qquad \frac{\Gamma \Uparrow [y/x]B, \Theta \vdash \mathcal{R}}{\Gamma \Uparrow \exists x.B, \Theta \vdash \mathcal{R}} \qquad \frac{}{\Gamma \Uparrow f^+, \Theta \vdash \mathcal{R}}$$

$$\frac{\Gamma \Uparrow B_1, B_2, \Theta \vdash \mathcal{R}}{\Gamma \Uparrow B_1 \wedge^+ B_2, \Theta \vdash \mathcal{R}} \qquad \frac{\Gamma \Uparrow \Theta \vdash \mathcal{R}}{\Gamma \Uparrow t^+, \Theta \vdash \mathcal{R}} \qquad \frac{\Gamma \Uparrow B_1, \Theta \vdash \mathcal{R} \quad \Gamma \Uparrow B_2, \Theta \vdash \mathcal{R}}{\Gamma \Uparrow B_1 \vee^+ B_2, \Theta \vdash \mathcal{R}}$$

SYNCHRONOUS INTRODUCTION RULES

$$\frac{\Gamma \vdash B_1 \Downarrow \quad \Gamma \Downarrow B_2 \vdash E}{\Gamma \Downarrow B_1 \supset B_2 \vdash E} \qquad \frac{\Gamma \Downarrow [t/x]B \vdash E}{\Gamma \Downarrow \forall x.B \vdash E} \qquad \frac{\Gamma \Downarrow B_i \vdash E}{\Gamma \Downarrow B_1 \wedge^- B_2 \vdash E}\ i \in \{1,2\}$$

$$\frac{\Gamma \vdash B_i \Downarrow}{\Gamma \vdash B_1 \vee^+ B_2 \Downarrow} \qquad \frac{}{\Gamma \vdash t^+ \Downarrow} \qquad \frac{\Gamma \vdash B_1 \Downarrow \quad \Gamma \vdash B_2 \Downarrow}{\Gamma \vdash B_1 \wedge^+ B_2 \Downarrow} \qquad \frac{\Gamma \vdash [t/x]B \Downarrow}{\Gamma \vdash \exists x.B \Downarrow}$$

IDENTITY RULES

$$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N}\ I_l \qquad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow}\ I_r \qquad \frac{\Gamma \Uparrow \cdot \vdash B \Uparrow \cdot \qquad \Gamma \Uparrow B \vdash \cdot \Uparrow E}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow E}\ Cut$$

STRUCTURAL RULES

$$\frac{\Gamma, N \Downarrow N \vdash E}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow E}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P}\ D_r \qquad \frac{\Gamma \Uparrow P \vdash \cdot \Uparrow E}{\Gamma \Downarrow P \vdash E}\ R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \cdot}{\Gamma \vdash N \Downarrow}\ R_r$$

$$\frac{C, \Gamma \Uparrow \Theta \vdash \mathcal{R}}{\Gamma \Uparrow C, \Theta \vdash \mathcal{R}}\ S_l \qquad \frac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow E}{\Gamma \Uparrow \cdot \vdash E \Uparrow \cdot}\ S_r$$

Here, Θ ranges over multisets of polarized formulas; Γ ranges over lists of polarized formulas; P denotes a positive formula; N denotes a negative formula; C denotes either a negative formula or a positive atom; and E denotes either a positive formula or a negative atom; and B denotes an unrestricted polarized formula. The introduction rule for $\forall$ has the usual eigenvariable restriction that y is not free in any formula in the conclusion sequent.

Clarifications: This proof system involves *polarized* formulas of first-order intuitionistic logic: in order to polarize a formula B, one must assign the status of "positive" or "negative" bias to all atomic formulas and replace all occurrences of truth with either t^+ or t^- and all occurrences of conjunction with either $\wedge^+$ or $\wedge^-$. If there are n occurrences of truth and conjunction in B, there are 2^n ways to do this replacement. Similarly, we replace the false and disjunction with f^+ and $\vee^+$ since only the

Entry 13 by: Dale Miller

positive polarization for these connectives are available in **LJF**. (Assigning polarization in classical logic is different: see the **LKF** proof system {12}.) The *positive connectives* are f^+, $\vee^+$, t^+, $\wedge^+$, and $\exists$ while the *negative connectives* are t^-, $\wedge^-$, $\supset$, and $\forall$. A formula is *positive* if it is a positive atom or has a top-level positive connective; similarly a formula is *negative* if it is a negative atom or has a top-level negative connective.

There are two kinds of sequents in this proof system. One kind contains a single $\Downarrow$ on either the right or the left of the turnstyle ($\vdash$) and are of the form $\Gamma \Downarrow B \vdash E$ or $\Gamma \vdash B \Downarrow$: in both of these cases, the formula B is the *focus* of the sequent. The other kind of sequent has an occurrence of $\Uparrow$ on each side of the turnstyle, eg., $\Gamma \Uparrow \Theta \vdash \Delta_1 \Uparrow \Delta_2$, and is such that the union of the two multisets Δ_1 and Δ_2 contains exactly one formula: that is, one of these multisets is empty and the other is a singleton. When writing asynchronous rules that introduce a connective on the left-hand side, we write $\mathcal{R}$ to denote $\Delta_1 \Downarrow \Delta_2$.

Note that in the asynchronous phase, a right introduction rule is applied only when the left asynchronous zone Γ is empty. Similarly, a left-introduction rule in the async phase introduces the connective at the top-level of the first formula in that context. The scheduling of introduction rules during this phase can be assigned arbitrarily and the zone Γ can be interpreted as a multiset instead of a list.

History: This focused proof system is a slight variation of the proof system in [**liang09tcs**, **liang07csl**]. The choice of how to polarize an unpolarized formula does not affect provability in LJF but can make a big impact on the structure of LJF proofs that can be built. **LJF** can be seen as a generalization to the MJ sequent system of Howe [**howe98phd**]. Other focused proof systems, such as LJT [**herbelin95phd**], LJQ/LJQ' [**dyckhoff06cie**], and λRCC [**jagadeesan05fsttcs**] can be directly emulated within **LJF** by making the appropriate choice of polarization.

Counterfactual Sequent Calculi I (1983,1992,2012,2013)

$$\frac{\begin{array}{c}\{\, B_k \vdash A_1,\dots,A_n,D_1,\dots,D_m \mid k \le n \,\}\\ \cup\, \{\, C_k \vdash A_1,\dots,A_n,D_1,\dots,D_{k-1} \mid k \le m \,\}\end{array}}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1 \preccurlyeq B_1),\dots,(A_n \preccurlyeq B_n)}\ R_{n,m}$$

$$\frac{\{\, C_k \vdash D_1,\dots,D_{k-1} \mid k \le m \,\} \qquad \Gamma \vdash \Delta, D_1,\dots,D_m}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta}\ T_m$$

$$\frac{\{\, C_k \vdash A_1,\dots,A_n,D_1,\dots,D_{k-1} \mid k \le m \,\} \qquad \Gamma \vdash \Delta, A_1,\dots,A_n,D_1,\dots,D_m}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1 \preccurlyeq B_1),\dots,(A_n \preccurlyeq B_n)}\ W_{n,m}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta,(A \preccurlyeq B)}\ R_{C1} \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash \Delta, B}{\Gamma,(A \preccurlyeq B) \vdash \Delta}\ R_{C2}$$

$$\frac{\begin{array}{c}\{\, \Gamma^{\preccurlyeq}, B_k \vdash \Delta^{\preccurlyeq}, A_1,\dots,A_n,D_1,\dots,D_m \mid k \le n \,\}\\ \cup\, \{\, \Gamma^{\preccurlyeq}, C_k \vdash \Delta^{\preccurlyeq}, A_1,\dots,A_n,D_1,\dots,D_{k-1} \mid k \le m \,\}\end{array}}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1, \preccurlyeq B_1),\dots,(A_n \preccurlyeq B_n)}\ A_{n,m}$$

$$\mathcal{R}_{\mathbb{V}_{\preccurlyeq}} = \{R_{n,m} \mid n \ge 1, m \ge 0\}$$

$$\begin{array}{llll}
\mathcal{R}_{\mathbb{VN}_{\preccurlyeq}} & = \{R_{n,m} \mid n+m \ge 1\} & \mathcal{R}_{\mathbb{VC}_{\preccurlyeq}} & = \mathcal{R}_{\mathbb{V}} \cup \{R_{C1}, R_{C2}\}\\
\mathcal{R}_{\mathbb{VT}_{\preccurlyeq}} & = \mathcal{R}_{\mathbb{V}_{\preccurlyeq}} \cup \{T_m \mid m \ge 1\} & \mathcal{R}_{\mathbb{VA}_{\preccurlyeq}} & = \{A_{n,m} \mid n \ge 1, m \ge 0\}\\
\mathcal{R}_{\mathbb{VW}_{\preccurlyeq}} & = \mathcal{R}_{\mathbb{V}_{\preccurlyeq}} \cup \{W_{n,m} \mid n+m \ge 1\} & \mathcal{R}_{\mathbb{VNA}_{\preccurlyeq}} & = \{A_{n,m} \mid n+m \ge 1\}
\end{array}$$

Clarifications: Sequents are based on multisets. The rules $\mathcal{R}_{\mathcal{L}_{\preccurlyeq}}$ form a calculus for a counterfactual logic $\mathcal{L}$ described in [**Lewis:1973uq**], where $\preccurlyeq$ is the *comparative plausibility* operator. Besides the rules shown above, these calculi also include the propositional rules of **G3c** {9} and contraction rules. The contexts $\Gamma^{\preccurlyeq}$ and $\Delta^{\preccurlyeq}$ contain all formulae of resp. Γ and Δ of the form $A \preccurlyeq B$.

History: The calculus for $\mathbb{VC}$ was introduced in the tableaux setting [**Swart:1983uq**, **Gent:1992p3090**]. The remaining calculi were introduced in [**Lellmann:2012fk**, **Lellmann:2013fk**] and corrected in [**Lellmann:2013**].

Remarks: Soundness and completeness are shown by proving equivalence to Hilbert-style calculi and (syntactical) cut elimination. These calculi yield PSPACE decision procedures (EXPTIME for $\mathbb{VA}_{\preccurlyeq}$ and $\mathbb{VNA}_{\preccurlyeq}$) and, in most cases, enjoy Craig Interpolation. Contraction can be made admissible.

Entry 14 by: Björn Lellmann

Counterfactual Sequent Calculi II (2012, 2013)

$$\frac{\begin{array}{c}\{\, C_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m, I \subseteq [n], J \subseteq [k-1] \,\} \\ \cup \{\, A_k, B_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid k \le n, I \subseteq [n], J \subseteq [m] \,\}\end{array}}{\Gamma, (A_1 \mathrel{\Box\!\!\Rightarrow} B_1), \dots, (A_n \mathrel{\Box\!\!\Rightarrow} B_n) \vdash \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \dots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)}\ R_{n,m}$$

$$\frac{\{\, \Gamma \vdash \Delta, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid J \subseteq [m] \,\} \cup \{\, C_k \vdash D_k, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m, J \subseteq [k-1] \,\}}{\Gamma \vdash \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \dots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)}\ T_m$$

$$\frac{\begin{array}{c}\{\, C_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m, I \subseteq [n], J \subseteq [k-1] \,\} \\ \cup \{\, \Gamma, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid I \subseteq [n], J \subseteq [m] \,\}\end{array}}{\Gamma, (A_1 \mathrel{\Box\!\!\Rightarrow} B_1), \dots, (A_n \mathrel{\Box\!\!\Rightarrow} B_n) \vdash \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \dots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)}\ W_{n,m}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, (A \mathrel{\Box\!\!\Rightarrow} B) \vdash \Delta}\ R_{C1} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, (A \mathrel{\Box\!\!\Rightarrow} B)}\ R_{C2}$$

For $n > 0$ the set $[n]$ is $\{1, \dots, n\}$ and $[0]$ is $\emptyset$. For a set I of indices, $\mathbf{A}^I$ contains all A_i with $i \in I$.

$$\mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} = \{R_{n,m} \mid n \ge 1, m \ge 0\}$$

$$\mathcal{R}_{\mathbb{VN}_{\Box\Rightarrow}} = \{R_{n,m} \mid n+m \ge 1\} \qquad \mathcal{R}_{\mathbb{VW}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{VT}_{\Box\Rightarrow}} \cup \{W_{n,m} \mid n+m \ge 1\}$$

$$\mathcal{R}_{\mathbb{VT}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{T_m \mid m \ge 1\} \qquad \mathcal{R}_{\mathbb{VC}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{R_{C1}, R_{C2}\}$$

Clarifications: Sequents are based on multisets. The rules $\mathcal{R}_{\mathcal{L}_{\Box\Rightarrow}}$ form a calculus for a counterfactual logic $\mathcal{L}$ described in [**Lewis:1973uq**], where $\mathrel{\Box\!\!\Rightarrow}$ is the *strong counterfactual implication* operator. Besides the rules shown above, these calculi also include the propositional rules of **G3c** {9} and contraction rules.

History: These calculi were introduced in [**Lellmann:2012fk**] and corrected in [**Lellmann:2013**].

Remarks: The calculi are translations of the calculi in {14} to the language with $\mathrel{\Box\!\!\Rightarrow}$. They inherit cut elimination and yield PSPACE decision procedures. Contraction can be made admissible.

Entry 15 by: Björn Lellmann

Contextual Natural Deduction (2013)

$$\overline{\Gamma, a : A \vdash a : A}$$

$$\frac{\Gamma, a : A \vdash b : C_\pi[B]}{\Gamma \vdash \lambda_\pi a^A.b : C_\pi[A \to B]} \to_I (\pi)$$

$$\frac{\Gamma \vdash f : C^1_{\pi_1}[A \to B] \quad \Gamma \vdash x : C^2_{\pi_2}[A]}{\Gamma \vdash (f\ x)^{\rightharpoonup}_{(\pi_1;\pi_2)} : C^1_{\pi_1}[C^2_{\pi_2}[B]]} \to^{\rightharpoonup}_E (\pi_1;\pi_2)$$

$$\frac{\Gamma \vdash f : C^1_{\pi_1}[A \to B] \quad \Gamma \vdash x : C^2_{\pi_2}[A]}{\Gamma \vdash (f\ x)^{\leftharpoonup}_{(\pi_1;\pi_2)} : C^2_{\pi_1}[C^1_{\pi_2}[B]]} \to^{\leftharpoonup}_E (\pi_1;\pi_2)$$

π, π_1 and π_2 must be positive positions. a is allowed to occur in b only if π is strongly positive.

Clarifications: $C_\pi[F]$ denotes a formula with F occurring in the hole of a *context* $C_\pi[]$. π is the position of the hole. It is: *positive* iff it is in the left side of an even number of implications; *strongly positive* iff this number is zero.

History: Contextual Natural Deduction [**ContextualND**] combines the idea of deep inference with Gentzen's natural deduction {1}.

Remarks: Soundness and completeness w.r.t. minimal logic are proven [**ContextualND**] by providing translations between **ND^c** and the minimal fragment of **NJ** {1}. **ND^c** proofs can be quadratically shorter than proofs in the minimal fragment of **NJ**.

Entry 16 by: Bruno Woltzenlogel Paleo

IR (2014)

C is a non-tautological clause from the matrix. $\tau = \{0/u \mid u \text{ is universal in } C\}$, where the notation $0/u$ for literals u is shorthand for $0/y$ if $u = y$ and $1/y$ if $u = \neg y$. We define $\mathsf{restr}(\tau, x)$ as $\{c/u \mid c/u \in \tau, \mathsf{lv}(u) < \mathsf{lv}(x)\}$. τ is a partial assignment to universal variables with $\mathsf{rng}(\tau) \subseteq \{0,1\}$. $\xi = \sigma \cup \{c/u \mid c/u \in \mathsf{restr}(\tau, x), u \notin \mathsf{dom}(\sigma)\}$ The rules of IR [**MFCS14**]

Clarifications: The calculus aims to refute a quantified Boolean formula (QBF) of the form $Q_1 x_1 \dots Q_n x_n . \varphi$ where $Q_i \in \{\forall, \exists\}$ and φ is a Boolean formula in conjunctive normal form (CNF). The formula φ is referred to as the *matrix*. We write $\mathsf{lv}(x)$ for the *quantification level* of x, i.e. $\mathsf{lv}(x_i) = i$. A variable x_i is *existential* (resp. *universal*) if $Q_i = \exists$ (resp. $Q_i = \forall$).

The calculus works by introducing clauses as *annotated clauses*, which are sets of annotated literals. Annotated literals consist of an existential literal and an annotation – a partial assignment to universal variables in $\{0,1\}$. Two literals are identical if and only if both the existential literal and annotation are equal. The calculus enables deriving the empty clause if and only if the given formula is false.

Remarks: Soundness was shown by extracting valid Herbrand functions. Completeness is shown by p-simulation of another known QBF system Q-Resolution .

History: The name of the calculus comes from the two pivotal operations *instantiation* and *resolution*. The calculus naturally generalizes an older calculus $\forall$Exp+Res [**JanotaTCS15**], which requires all clauses to be introduced into the proof by using a complete assignment.

Entry 17 by: Leroy Chew, Mikoláš Janota

Logics

Proof Systems Grouped by Type