Contents

G3c – G3c (1996)

$$\frac{}{P,\Gamma \vdash \Delta, P}\ \text{Ax} \qquad \frac{}{\bot,\Gamma \vdash \Delta}\ \text{L}\bot$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \text{L}\wedge \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ \text{R}\wedge$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ \text{L}\vee \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\ \text{R}\vee$$

$$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta}\ \text{L}\to \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ \text{R}\to$$

$$\frac{\forall x A, A[x/t], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta}\ \text{L}\forall \qquad \frac{\Gamma \vdash \Delta, A[x/y]}{\Gamma \vdash \Delta, \forall x A}\ \text{R}\forall$$

$$\frac{A[x/y], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta}\ \text{L}\exists \qquad \frac{\Gamma \vdash \Delta, A[x/t], \exists x A}{\Gamma \vdash \Delta, \exists x A}\ \text{R}\exists$$

with P in Ax atomic and y not free in the conclusion in R∀, L∃

Clarifications: Sequents are based on multisets. A formula $A[x/t]$ is the result of uniformly substituting the term t for the variable x in A.

Technicalities: Sound and complete wrt. classical first-order logic. Weakening and contraction are depth-preserving admissible and all the rules are depth-preserving invertible.

References

[1] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. 2nd ed. Vol. 43. Cambridge Tracts In Theoretical Computer Science. Cambridge University Press, 2000.

Entry by:

Calculi for Lewis' Counterfactual Logics II – (2012, 2013)

$$\frac{\begin{array}{c}\{\, C_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m,\, I \subseteq [n],\, J \subseteq [k-1] \,\}\\ \cup \{\, A_k, B_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid k \le n, I \subseteq [n], J \subseteq [m] \,\}\end{array}}{\Gamma, (A_1 \,\Box\!\!\Rightarrow B_1), \dots, (A_n \,\Box\!\!\Rightarrow B_n) \vdash \Delta, (C_1 \,\Box\!\!\Rightarrow D_1), \dots, (C_m \,\Box\!\!\Rightarrow D_m)}\ R_{n,m}$$

$$\frac{\{\, \Gamma \vdash \Delta, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid J \subseteq [m] \,\} \cup \{\, C_k \vdash D_k, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m, J \subseteq [k-1] \,\}}{\Gamma \vdash \Delta, (C_1 \,\Box\!\!\Rightarrow D_1), \dots, (C_m \,\Box\!\!\Rightarrow D_m)}\ T_m$$

$$\frac{\begin{array}{c}\{\, C_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m,\, I \subseteq [n],\, J \subseteq [k-1] \,\}\\ \cup \{\, \Gamma, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid I \subseteq [n], J \subseteq [m] \,\}\end{array}}{\Gamma, (A_1 \,\Box\!\!\Rightarrow B_1), \dots, (A_n \,\Box\!\!\Rightarrow B_n) \vdash \Delta, (C_1 \,\Box\!\!\Rightarrow D_1), \dots, (C_m \,\Box\!\!\Rightarrow D_m)}\ W_{n,m}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, (A \,\Box\!\!\Rightarrow B) \vdash \Delta}\ R_{C1} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, (A \,\Box\!\!\Rightarrow B)}\ R_{C2}$$

For $n > 0$ the set $[n]$ is $\{1, \dots, n\}$ and $[0]$ is $\emptyset$. For a set I of indices, $\mathbf{A}^I$ contains all A_i with $i \in I$.

$$\mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} = \{R_{n,m} \mid n \ge 1, m \ge 0\}$$

$$\begin{array}{ll}\mathcal{R}_{\mathbb{VN}_{\Box\Rightarrow}} = \{R_{n,m} \mid n+m \ge 1\} & \mathcal{R}_{\mathbb{VW}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{VT}_{\Box\Rightarrow}} \cup \{W_{n,m} \mid n+m \ge 1\}\\ \mathcal{R}_{\mathbb{VT}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{T_m \mid m \ge 1\} & \mathcal{R}_{\mathbb{VC}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{R_{C1}, R_{C2}\}\end{array}$$

Clarifications: Sequents are based on multisets. The propositional part is that of **G3c** {1}. Also includes the contraction rules. Rules $\mathcal{R}_{\mathcal{L}_{\Box\Rightarrow}}$ are for the logic $\mathcal{L}$ in terms of the *strong counterfactual implication* $\Box\!\!\Rightarrow$ from [3].

History: Introduced in [2], corrected in [1].

Technicalities: Translations of the calculi {**??**} to the language with $\Box\!\!\Rightarrow$. Inherit cut elimination and yield PSPACE decision procedures. Contraction can be made admissible.

References

[1] Björn Lellmann. "Sequent Calculi with Context Restrictions and Applications to Conditional Logic". PhD thesis. Imperial College London, 2013. URL: `http://hdl.handle.net/10044/1/18059`.

[2] Björn Lellmann and Dirk Pattinson. "Sequent Systems for Lewis' Conditional Logics". In: *JELIA 2012*. Ed. by Luis Fariñas del Cerro, Andreas Herzig, and Jerome Mengin. Vol. 7519. LNCS. Springer-Verlag Berlin Heidelberg, 2012, pp. 320–332.

[3] David Lewis. *Counterfactuals*. Blackwell, 1973.

Entry by: Björn Lellmann

Calculi for Lewis' Counterfactual Logics I – (1983,1992,2012,2013)

$$\frac{\{\, B_k \vdash A_1,\dots,A_n,D_1,\dots,D_m \mid k \le n \,\} \cup \{\, C_k \vdash A_1,\dots,A_n,D_1,\dots,D_{k-1} \mid k \le m \,\}}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1 \preccurlyeq B_1),\dots,(A_n \preccurlyeq B_n)}\ R_{n,m}$$

$$\frac{\{\, C_k \vdash D_1,\dots,D_{k-1} \mid k \le m \,\} \qquad \Gamma \vdash \Delta, D_1,\dots,D_m}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta}\ T_m$$

$$\frac{\{\, C_k \vdash A_1,\dots,A_n,D_1,\dots,D_{k-1} \mid k \le m \,\} \qquad \Gamma \vdash \Delta, A_1,\dots,A_n,D_1,\dots,D_m}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1 \preccurlyeq B_1),\dots,(A_n \preccurlyeq B_n)}\ W_{n,m}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta,(A \preccurlyeq B)}\ R_{C1} \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash \Delta, B}{\Gamma,(A \preccurlyeq B) \vdash \Delta}\ R_{C2}$$

$$\frac{\begin{array}{c}\{\, \Gamma^{\preccurlyeq}, B_k \vdash \Delta^{\preccurlyeq}, A_1,\dots,A_n,D_1,\dots,D_m \mid k \le n \,\} \\ \cup \{\, \Gamma^{\preccurlyeq}, C_k \vdash \Delta^{\preccurlyeq}, A_1,\dots,A_n,D_1,\dots,D_{k-1} \mid k \le m \,\}\end{array}}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1, \preccurlyeq B_1),\dots,(A_n \preccurlyeq B_n)}\ A_{n,m}$$

$$\mathcal{R}_{\mathbb{V}_\preccurlyeq} = \{R_{n,m} \mid n \ge 1, m \ge 0\}$$

$$\begin{array}{llll}
\mathcal{R}_{\mathbb{VN}_\preccurlyeq} & = \{R_{n,m} \mid n+m \ge 1\} & \mathcal{R}_{\mathbb{VC}_\preccurlyeq} & = \mathcal{R}_{\mathbb{V}} \cup \{R_{C1}, R_{C2}\} \\
\mathcal{R}_{\mathbb{VT}_\preccurlyeq} & = \mathcal{R}_{\mathbb{V}_\preccurlyeq} \cup \{T_m \mid m \ge 1\} & \mathcal{R}_{\mathbb{VA}_\preccurlyeq} & = \{A_{n,m} \mid n \ge 1, m \ge 0\} \\
\mathcal{R}_{\mathbb{VW}_\preccurlyeq} & = \mathcal{R}_{\mathbb{V}_\preccurlyeq} \cup \{W_{n,m} \mid n+m \ge 1\} & \mathcal{R}_{\mathbb{VNA}_\preccurlyeq} & = \{A_{n,m} \mid n+m \ge 1\}
\end{array}$$

Clarifications: Sequents are based on multisets. The propositional part is that of **G3c** {1}. Also include the contraction rules. Rules $\mathcal{R}_{\mathcal{L}_\preccurlyeq}$ are for the logic $\mathcal{L}$ in terms of the *comparative plausibility* operator $\preccurlyeq$ from [6]. The contexts $\Gamma^{\preccurlyeq}$ resp. $\Delta^{\preccurlyeq}$ contain all formulae of Γ resp. Δ of the form $A \preccurlyeq B$.

History: The calculus for $\mathbb{VC}$ was introduced in the tableaux setting [5, 4]. The remaining calculi were introduced in [3, 2] and corrected in [1].

Technicalities: Soundness and completeness via equivalence to Hilbert-style calculi and (syntactical) cut elimination. Yield PSPACE decision procedures (resp. EXPTIME for $\mathbb{VA}_\preccurlyeq$ and $\mathbb{VNA}_\preccurlyeq$) and in most cases Craig Interpolation. Contraction can be made admissible. [3, 1]

References

[1] Björn Lellmann. "Sequent Calculi with Context Restrictions and Applications to Conditional Logic". PhD thesis. Imperial College London, 2013. URL: http://hdl.handle.net/10044/1/18059.

Entry by: Björn Lellmann

[2] Björn Lellmann and Dirk Pattinson. "Constructing Cut Free Sequent Systems With Context Restrictions Based on Classical or Intuitionistic Logic". In: *ICLA 2013*. Ed. by Kamal Lodaya. Vol. 7750. LNAI. Springer-Verlag Berlin Heidelberg, 2013, pp. 148–160.

[3] Björn Lellmann and Dirk Pattinson. "Sequent Systems for Lewis' Conditional Logics". In: *JELIA 2012*. Ed. by Luis Fariñas del Cerro, Andreas Herzig, and Jerome Mengin. Vol. 7519. LNCS. Springer-Verlag Berlin Heidelberg, 2012, pp. 320–332.

[4] Ian P. Gent. "A Sequent- or Tableau-style System for Lewis's Counterfactual Logic VC". In: *Notre Dame J. Form. Log.* 33.3 (1992), pp. 369–382.

[5] Harrie C.M. de Swart. "A Gentzen- or Beth-Type System, a Practical Decision Procedure and a Constructive Completeness Proof for the Counterfactual Logics VC and VCS". In: *J. Symb. Log.* 48.1 (1983), pp. 1–20.

[6] David Lewis. *Counterfactuals*. Blackwell, 1973.

Contextual Natural Deduction – $\mathbf{ND^c}$ (2013)

$$\frac{}{\Gamma, a : A \vdash a : A}$$

$$\frac{\Gamma, a : A \vdash b : C_\pi[B]}{\Gamma \vdash \lambda_\pi a^A.b : C_\pi[A \to B]} \to_I (\pi)$$

$$\frac{\Gamma \vdash f : C^1_{\pi_1}[A \to B] \quad \Gamma \vdash x : C^2_{\pi_2}[A]}{\Gamma \vdash (f\ x)^{\rightharpoonup}_{(\pi_1;\pi_2)} : C^1_{\pi_1}[C^2_{\pi_2}[B]]} \to^{\rightharpoonup}_E (\pi_1;\pi_2)$$

$$\frac{\Gamma \vdash f : C^1_{\pi_1}[A \to B] \quad \Gamma \vdash x : C^2_{\pi_2}[A]}{\Gamma \vdash (f\ x)^{\leftharpoonup}_{(\pi_1;\pi_2)} : C^2_{\pi_1}[C^1_{\pi_2}[B]]} \to^{\leftharpoonup}_E (\pi_1;\pi_2)$$

π, π_1 and π_2 must be positive positions. a is allowed to occur in b only if π is strongly positive.

Clarifications: $C_\pi[F]$ denotes a formula with F occurring in the hole of a *context* $C_\pi[]$. π is the position of the hole. It is: *positive* iff it is in the left side of an even number of implications; *strongly positive* iff this number is zero.

History: Contextual Natural Deduction [1] combines the idea of deep inference {5} with Gentzen's natural deduction {6}.

Technicalities: Soundness and completeness w.r.t. minimal logic are proven [1] by providing translations between $\mathbf{ND^c}$ and Gentzen's natural deduction. Proofs in $\mathbf{ND^c}$ can be quadratically shorter than proofs in Gentzen's natural deduction.

References

[1] Bruno Woltzenlogel Paleo. "Contextual Natural Deduction". In: *Logical Foundations of Computer Science, International Symposium, LFCS 2013, San Diego, CA, USA, January 6-8, 2013. Proceedings*. Ed. by Sergei N. Artëmov and Anil Nerode. Vol. 7734. Lecture Notes in Computer Science. Springer, 2013, pp. 372–386. ISBN: 978-3-642-35721-3. DOI: 10.1007/978-3-642-35722-0_27. URL: http://dx.doi.org/10.1007/978-3-642-35722-0_27.

Entry by: Bruno Woltzenlogel Paleo

ToDo – (ToDo)

ToDo

Entry by: ToDo

Natural Deduction – ToDo(1934)

ToDo

Clarifications:

History:

Technicalities:

Entry by: Bruno Woltzenlogel Paleo

Sequent Calculus for Classical Logic – LK (1935)

$$\frac{}{D \vdash D}\ \mathit{Axiom} \qquad\qquad \frac{\Gamma \vdash \Theta, D \qquad \Gamma, D \vdash \Theta}{\Gamma \vdash \Theta}\ \mathit{cut}$$

$$\frac{\Gamma \vdash \Theta}{D, \Gamma \vdash \Theta}\ w:l \qquad \frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, D}\ w:r \qquad \frac{D, D, \Gamma \vdash \Theta}{D, \Gamma \vdash \Theta}\ c:l$$

$$\frac{\Gamma, D, E, \Delta \vdash \Theta}{\Gamma, E, D, \Delta \vdash \Theta}\ e:l \qquad \frac{\Gamma \vdash \Theta, D, E, \Lambda}{\Gamma \vdash \Theta, E, D, \Lambda}\ e:r \qquad \frac{\Gamma \vdash \Theta, D, D}{\Gamma \vdash \Theta, D}\ c:r$$

$$\frac{A, \Gamma \vdash \Theta}{A \wedge B, \Gamma \vdash \Theta}\ \wedge:l \qquad \frac{B, \Gamma \vdash \Theta}{A \wedge B, \Gamma \vdash \Theta}\ \wedge:l \qquad \frac{\Gamma \vdash \Theta, A}{\Gamma \vdash \Theta, A \vee B}\ \vee:r$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\ \neg:l \qquad \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\ \neg:r \qquad \frac{\Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \vee B}\ \vee:r$$

$$\frac{A, \Gamma \vdash \Theta \qquad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\ \vee:l \qquad\qquad \frac{\Gamma \vdash \Theta, A \qquad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \wedge B}\ \wedge:r$$

$$\frac{\Gamma \vdash \Theta, A \qquad B, \Delta \vdash \Lambda}{A \to B, \Gamma \vdash \Theta}\ \to:l \qquad\qquad \frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \to B}\ \to:r$$

$$\frac{Fa, \Gamma \vdash \Theta}{\forall x Fx, \Gamma \vdash \Theta}\ \forall:l \qquad\qquad \frac{\Gamma \vdash \Theta, Fa}{\Gamma \vdash \Theta, \exists x Fx}\ \exists:r$$

$$\frac{Fa, \Gamma \vdash \Theta}{\exists x Fx, \Gamma \vdash \Theta}\ \exists:l(*) \qquad\qquad \frac{\Gamma \vdash \Theta, Fa}{\Gamma \vdash \Theta, \forall x Fx}\ \forall:r(*)$$

$(*)$: Eigenvariable condition: x does not occur in Γ, Δ and $\forall x Fx/\exists x Fx$.

Clarifications: In all rules, A, B, D, E, F are formulas, $\Gamma, \Theta, \Delta, \Lambda$ are lists of formulas, a is a free and x a bound variable. Within the quantifier rules, Fa is obtained from Fx by applying the substitution $\{a/x\}$.

History: This is Gentzen's original formulation[2] where the term language has only free and bound variables. Takeuti's version[1] adds individual and function constants. An invertible variant of **LK** is **LK′**, an intuitionistic version is **LJ** {8}.

Technicalities: Soundness is obtained by a translation to **ND**, completeness by cut-elimination i.e. Gentzen's Hauptsatz[2].

References

[1] Gaisi Takeuti. *Proof Theory*. English. North-Holland Pub. Co. ; American Elsevier Pub. Co Amsterdam : New York, 1975, 372 p. ISBN: 0444104925.

[2] Gerhard Gentzen. "Untersuchungen über das logische Schließen I". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.

Entry by: Martin Riener

Sequent Calculus LJ – (1935)

$$\frac{}{A \vdash A}\ init \qquad \frac{\Gamma_1 \vdash P \quad \Gamma_2, P \vdash C}{\Gamma_1, \Gamma_2 \vdash C}\ cut$$

$$\frac{\Gamma \vdash P}{\Gamma, \neg P \vdash}\ \neg_l \qquad \frac{\Gamma, P \vdash}{\Gamma \vdash \neg P}\ \neg_r$$

$$\frac{P_i, \Gamma \vdash C}{P_1 \wedge P_2, \Gamma \vdash C}\ \wedge_{li} \qquad \frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\ \wedge_r$$

$$\frac{P, \Gamma \vdash C \quad Q, \Gamma \vdash C}{P \vee Q, \Gamma \vdash C}\ \vee_l \qquad \frac{\Gamma \vdash P_i}{\Gamma \vdash P_1 \vee P_2}\ \vee_{ri}$$

$$\frac{\Gamma_1 \vdash P \quad Q, \Gamma_2 \vdash C}{P \to Q, \Gamma_1, \Gamma_2 \vdash C}\ \to_l \qquad \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}\ \to_r$$

$$\frac{P\{x \leftarrow \alpha\}, \Gamma \vdash C}{\exists x.P, \Gamma \vdash C}\ \exists_l \qquad \frac{\Gamma \vdash P\{x \leftarrow t\}}{\Gamma \vdash \exists x.P}\ \exists_r$$

$$\frac{P\{x \leftarrow t\}, \Gamma \vdash C}{\forall x.P, \Gamma \vdash C}\ \forall_l \qquad \frac{\Gamma \vdash P\{x \leftarrow \alpha\}}{\Gamma \vdash \forall x.P}\ \forall_r$$

$$\frac{P, P, \Gamma \vdash C}{P, \Gamma \vdash C}\ c_l \qquad \frac{\Gamma \vdash C}{P, \Gamma \vdash C}\ w_l \qquad \frac{\Gamma \vdash}{\Gamma \vdash P}\ w_r$$

Clarifications: Assuming that α is a variable not contained in P, Γ or C, t does not contain variables bound in P and C stands for one formula or the empty set.

History: Proposed by Gentzen in [1] by restricting the succedent of sequents in to have at most one formula. In the original paper, he notes that this restriction is equivalent to removing the principle of excluded middle from the natural deduction system {6} in order to obtain . The cut is admissible in **LJ** and this result is known as *Hauptsatz*.

Technicalities: Soundness and completeness of **LJ** can be proved using a translation of **LJ** derivations into **NJ**. Decidability of the propositional fragment and consistency of intuitionistic logic follows from the cut admissibility in this calculus.

References

[1] Gerhard Gentzen. "Untersuchungen über das logische Schließen I". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.

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