Preface

The **Encyclopedia of Proof Systems** aims at providing a reliable, technically informative, historically accurate, concise and convenient central repository of proof systems for various logics. The goal is to facilitate the exchange of information among logicians with mathematical, computational or philosophical backgrounds; in order to foster and accelerate the development of new proof systems and automated deduction tools that rely on them.

Preparatory work for the creation of the Encyclopedia, such as the implementation of the LaTeX template and the setup of the Github repository, started in October 2014, triggered by the call for workshop proposals for the 25th Conference on Automated Deduction (CADE). Christoph Benzmüller, CADE's conference chair, and Jasmin Blanchette, CADE's workshop co-chair, encouraged me to submit a workshop proposal and supported my alternative idea to organize instead a special poster session based on encyclopedia entries. I am thankful for their encouragement and support.

In December 2014, Björn Lellman, Giselle Reis and Martin Riener kindly accepted my request to beta-test the template and the instructions I had created. They submitted the first few example entries to the encyclopedia and provided valuable feedback, for which I am grateful. Their comments were essential for improving the templates and instructions before the public announcement of the encyclopedia.

January 2015 *Bruno Woltzenlogel Paleo*

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$$\frac{\mathfrak{A}}{\mathfrak{A} \vee \mathfrak{B}}\; OE \qquad \frac{\mathfrak{B}}{\mathfrak{A} \vee \mathfrak{B}}\; OE \qquad \frac{\mathfrak{A} \vee \mathfrak{B} \quad \begin{array}{c}[\mathfrak{A}]\\ \vdots \\ \mathfrak{C}\end{array} \quad \begin{array}{c}[\mathfrak{B}]\\ \vdots \\ \mathfrak{C}\end{array}}{\mathfrak{C}}\; OB$$

$$\frac{\mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x} \mathfrak{F}\mathfrak{x}}\; AE \qquad \frac{\forall \mathfrak{x} \mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}\; AB \qquad \frac{\mathfrak{F}\mathfrak{a}}{\exists \mathfrak{x} \mathfrak{F}\mathfrak{x}}\; EE \qquad \frac{\exists \mathfrak{x} \mathfrak{F}\mathfrak{x} \quad \begin{array}{c}[\mathfrak{F}\mathfrak{a}]\\ \vdots \\ \mathfrak{C}\end{array}}{\mathfrak{C}}\; EB$$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{\mathfrak{A} \supset \mathfrak{B}}\; FE \qquad \frac{\mathfrak{A} \quad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}}\; FB \qquad \frac{\begin{array}{c}[\mathfrak{A}]\\ \vdots \\ \curlywedge\end{array}}{\neg \mathfrak{A}}\; NE \qquad \frac{\mathfrak{A} \quad \neg \mathfrak{A}}{\curlywedge}\; NB \qquad \frac{\curlywedge}{\mathfrak{D}}$$

The eigenvariable $\mathfrak{a}$ of an AE must not occur in the formula designated in the schema by $\forall \mathfrak{x} \mathfrak{F}\mathfrak{x}$; nor in any assumption formula upon which that formula depends. The eigenvariable $\mathfrak{a}$ of an EB must not occur in the formula designated in the schema by $\exists \mathfrak{x} \mathfrak{F}\mathfrak{x}$; nor in any assumption formula upon which that formula depends, with the exception of the assumption formulae designated by $\mathfrak{F}\mathring{\mathfrak{a}}$.

Clarifications: The names of the rules are those originally given by Gentzen [1]: U = und (and), O = oder (or), A = all, E = es-gibt (exists), F = folgt (follows), N = nicht (not), E = Einführung (introduction), B = Beseitigung (elimination).

History: The main novelty introduced by Gentzen in this proof system is its *assumption* handling mechanism, which allows formal proofs to reflect more naturally the logical reasoning involved in mathematical proofs.

Remarks: In [1], completeness of **NJ** is proven by showing how to translate proofs in the Hilbert-style calculus **LHJ** to **NJ**-proofs, and soundness is proven by showing how to translate **NJ**-proofs to **LJ**-proofs {3}.

[1] Gerhard Gentzen. “Untersuchungen über das logische Schließen I”. In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.

Entry 1 by: Bruno Woltzenlogel Paleo

Classical Sequent Calculus LK (1935)

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash \Lambda, A \quad A, \Delta \vdash \Theta}{\Gamma, \Delta \vdash \Lambda, \Theta}\ cut$$

$$\frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ w_l \qquad \frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, A}\ w_r$$

$$\frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta}\ e_l \quad \frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ c_l \qquad \frac{\Gamma \vdash \Theta, B, A, \Delta}{\Gamma \vdash \Theta, A, B, \Delta}\ e_r \quad \frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A}\ c_r$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\ \neg_l \qquad \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\ \neg_r$$

$$\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta}\ \wedge_l \qquad \frac{\Gamma \vdash \Theta, A \quad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \wedge B}\ \wedge_r$$

$$\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\ \vee_l \qquad \frac{\Gamma \vdash \Theta, A_i}{\Gamma \vdash \Theta, A_1 \vee A_2}\ \vee_r$$

$$\frac{\Gamma \vdash \Lambda, A \quad B, \Delta \vdash \Theta}{A \rightarrow B, \Gamma, \Delta \vdash \Lambda, \Theta}\ \rightarrow_l \qquad \frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \rightarrow B}\ \rightarrow_r$$

$$\frac{A[\alpha], \Gamma \vdash \Theta}{\exists x.A[x], \Gamma \vdash \Theta}\ \exists_l \quad \frac{A[t], \Gamma \vdash \Theta}{\forall x.A[x], \Gamma \vdash \Theta}\ \forall_l \quad \frac{\Gamma \vdash \Theta, A[\alpha]}{\Gamma \vdash \Theta, \forall x.A[x]}\ \forall_r \quad \frac{\Gamma \vdash \Theta, A[t]}{\Gamma \vdash \Theta, \exists x.A[x]}\ \exists_r$$

The eigenvariable α should not occur in Γ, Θ or $A[x]$.
The term t should not contain variables bound in $A[t]$.

History: This is a modern presentation of Gentzen's original **LK** calculus[1], using modern notations and rule names.

Remarks: **LK** is complete relative to **NK** (i.e. **NJ** {1} with the axiom of excluded middle) and sound relative to a Hilbert-style calculus **LHK** [2]. Cut is eliminable (*Hauptsatz* [1]), and hence classical predicate logic is consistent. Any *prenex* cut-free proof may be further transformed into a shape with only propositional inferences above and only quantifier and structural inferences below a *midsequent* [2].

[1] Gerhard Gentzen. "Untersuchungen über das logische Schließen I". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.

[2] Gerhard Gentzen. "Untersuchungen über das logische Schließen II". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 405–431.

Entry 2 by: Martin Riener

Intuitionistic Sequent Calculus LJ (1935)

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \quad A, \Delta \vdash \Theta}{\Gamma, \Delta \vdash \Theta}\ cut$$

$$\frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ w_l \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ w_r$$

$$\frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta}\ e_l \qquad \frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ c_l$$

$$\frac{\Gamma \vdash A}{\neg A, \Gamma \vdash}\ \neg_l \qquad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A}\ \neg_r$$

$$\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta}\ \wedge_l \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_r$$

$$\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\ \vee_l \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ \vee_r$$

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash \Theta}{A \to B, \Gamma, \Delta \vdash \Theta}\ \to_l \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}\ \to_r$$

$$\frac{A[\alpha], \Gamma \vdash \Theta}{\exists x.A[x], \Gamma \vdash \Theta}\ \exists_l \qquad \frac{\Gamma \vdash A[t]}{\Gamma \vdash \exists x.A[x]}\ \exists_r$$

$$\frac{A[t], \Gamma \vdash \Theta}{\forall x.A[x], \Gamma \vdash \Theta}\ \forall_l \qquad \frac{\Gamma \vdash A[\alpha]}{\Gamma \vdash \forall x.A[x]}\ \forall_r$$

The eigenvariable α should not occur in Γ, Θ or $A[x]$.
The term t should not contain variables bound in $A[t]$.

Clarifications: **LJ** and **LK** {2} have exactly the same rules, but in **LJ** the succedent of every sequent may have at most one formula. This restriction is equivalent to forbidding the axiom of excluded middle in natural deduction.

Remarks: The cut rule is eliminable (*Hauptsatz* [1]), and hence intuitionistic predicate logic is consistent and its propositional fragment is decidable [2]. **LJ** is complete relative to **NJ** {1} and sound relative to the Hilbert-style calculus **LHJ** [2].

[1] Gerhard Gentzen. "Untersuchungen über das logische Schließen I". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.

[2] Gerhard Gentzen. "Untersuchungen über das logische Schließen II". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 405–431.

Entry 3 by: Giselle Reis

Multi-Conclusion Sequent Calculus LJ′ (1954)

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash \Theta, A \quad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Theta, \Lambda}\ cut$$

$$\frac{A_i, \Gamma \vdash \Theta}{A_1 \wedge A_2, \Gamma \vdash \Theta}\ \wedge_l \qquad \frac{\Gamma \vdash \Theta, A \quad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \wedge B}\ \wedge_r$$

$$\frac{A, \Gamma \vdash \Theta \quad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\ \vee_l \qquad \frac{\Gamma \vdash \Theta, A_i}{\Gamma \vdash \Theta, A_1 \vee A_2}\ \vee_r$$

$$\frac{\Gamma \vdash \Theta, A \quad B, \Delta \vdash \Lambda}{A \to B, \Gamma, \Delta \vdash \Theta, \Lambda}\ \to_l \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}\ \to_r$$

$$\frac{A\alpha, \Gamma \vdash \Theta}{\exists x.Ax, \Gamma \vdash \Theta}\ \exists_l \quad \frac{\Gamma \vdash \Theta, At}{\Gamma \vdash \Theta, \exists x.Ax}\ \exists_r \quad \frac{At, \Gamma \vdash \Theta}{\forall x.Ax, \Gamma \vdash \Theta}\ \forall_l \quad \frac{\Gamma \vdash A\alpha}{\Gamma \vdash \forall x.Ax}\ \forall_r$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\ \neg_l \quad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A}\ \neg_r \quad \frac{\Gamma, B, A, \Delta \vdash \Theta}{\Gamma, A, B, \Delta \vdash \Theta}\ e_l \quad \frac{\Gamma\Delta \vdash \Theta, B, A, \Lambda}{\Gamma \vdash \Theta, A, B, \Lambda}\ e_r$$

$$\frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ c_l \quad \frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A}\ c_r \quad \frac{\Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ w_l \quad \frac{\Gamma \vdash}{\Gamma \vdash A}\ w_r$$

The eigenvariable α should not occur in Γ, Θ or $A[x]$.
The term t should not contain variables bound in $A[t]$.

Clarifications: While **LJ** {3} is defined by restricting **LK** {2} to a single conclusion (i.e. at most one formula per succedent), in **LJ′** only the rules $\neg_r$, $\to_r$ and $\forall_r$ have this restriction.

History: **LJ′** was proposed in [3] and used to prove the completeness of **LJ** {3} in [2]. The same calculus (and other multi-conclusion calculi for intuitionistic logic) was rediscovered in [1] while analyzing classical and intuitionistic provability.

Remarks: **LJ′** is equivalent to **LJ**, and this is established by translating sequents of the form $\Gamma \vdash A_1, ..., A_n$ into sequents of the form $\Gamma \vdash A_1 \vee ... \vee A_n$. Cut is eliminable and this can be proven by using a combination of the rewriting rules for cut-elimination in **LJ** and **LK**.

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[1] Gopalan Nadathur. "Correspondences between Classical, Intuitionistic and Uniform Provability". In: *CoRR* cs.LO/9809015 (1998).

[2] Gaisi Takeuti. *Proof Theory*. Studies in logic and the foundations of mathematics. North-Holland, 1987. ISBN: 9780444879431.

[3] Shôji Maehara. "Eine Darstellung der intuitionistischen Logik in der klassischen". In: *Nagoya Math. J.* 7 (1954), pp. 45–64.

Entry 4 by: Giselle Reis

Sequent Calculus G3c (1996)

$$\frac{}{P,\Gamma \vdash \Delta, P}\ \mathrm{Ax} \qquad \frac{}{\bot,\Gamma \vdash \Delta}\ \mathrm{L}\bot$$

$$\frac{A,B,\Gamma \vdash \Delta}{A \wedge B,\Gamma \vdash \Delta}\ \mathrm{L}\wedge \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ \mathrm{R}\wedge$$

$$\frac{A,\Gamma \vdash \Delta \quad B,\Gamma \vdash \Delta}{A \vee B,\Gamma \vdash \Delta}\ \mathrm{L}\vee \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\ \mathrm{R}\vee$$

$$\frac{\Gamma \vdash \Delta, A \quad B,\Gamma \vdash \Delta}{A \to B,\Gamma \vdash \Delta}\ \mathrm{L}\to \qquad \frac{A,\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ \mathrm{R}\to$$

$$\frac{\forall x A, A[x/t],\Gamma \vdash \Delta}{\forall x A,\Gamma \vdash \Delta}\ \mathrm{L}\forall \qquad \frac{\Gamma \vdash \Delta, A[x/y]}{\Gamma \vdash \Delta, \forall x A}\ \mathrm{R}\forall$$

$$\frac{A[x/y],\Gamma \vdash \Delta}{\exists x A,\Gamma \vdash \Delta}\ \mathrm{L}\exists \qquad \frac{\Gamma \vdash \Delta, A[x/t], \exists x A}{\Gamma \vdash \Delta, \exists x A}\ \mathrm{R}\exists$$

P should be atomic in Ax and y should not be free in the conclusion of R$\forall$ and L$\exists$

Clarifications: Sequents are based on multisets. A formula $A[x/t]$ is the result of uniformly substituting the term t for the variable x in A, renaming bound variables to prevent clashes with the variables in t.

Remarks: **G3c** is sound and complete w.r.t. classical first-order logic. Weakening and contraction are depth-preserving admissible and all rules are depth-preserving invertible.

[1] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. 2nd ed. Vol. 43. Cambridge Tracts In Theoretical Computer Science. Cambridge University Press, 2000.

Entry 5 by: Björn Lellmann

Counterfactual Sequent Calculi I (1983,1992,2012,2013)

$$\frac{\begin{array}{c}\{\, B_k \vdash A_1,\ldots,A_n,D_1,\ldots,D_m \mid k \leq n \,\}\\ \cup \{\, C_k \vdash A_1,\ldots,A_n,D_1,\ldots,D_{k-1} \mid k \leq m \,\}\end{array}}{\Gamma,(C_1 \preccurlyeq D_1),\ldots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1 \preccurlyeq B_1),\ldots,(A_n \preccurlyeq B_n)}\; R_{n,m}$$

$$\frac{\{\, C_k \vdash D_1,\ldots,D_{k-1} \mid k \leq m \,\} \qquad \Gamma \vdash \Delta, D_1,\ldots,D_m}{\Gamma,(C_1 \preccurlyeq D_1),\ldots,(C_m \preccurlyeq D_m) \vdash \Delta}\; T_m$$

$$\frac{\{\, C_k \vdash A_1,\ldots,A_n,D_1,\ldots,D_{k-1} \mid k \leq m \,\} \qquad \Gamma \vdash \Delta,A_1,\ldots,A_n,D_1,\ldots,D_m}{\Gamma,(C_1 \preccurlyeq D_1),\ldots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1 \preccurlyeq B_1),\ldots,(A_n \preccurlyeq B_n)}\; W_{n,m}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta,(A \preccurlyeq B)}\; R_{C1} \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash \Delta, B}{\Gamma,(A \preccurlyeq B) \vdash \Delta}\; R_{C2}$$

$$\frac{\begin{array}{c}\{\, \Gamma^{\preccurlyeq}, B_k \vdash \Delta^{\preccurlyeq},A_1,\ldots,A_n,D_1,\ldots,D_m \mid k \leq n \,\}\\ \cup \{\, \Gamma^{\preccurlyeq}, C_k \vdash \Delta^{\preccurlyeq},A_1,\ldots,A_n,D_1,\ldots,D_{k-1} \mid k \leq m \,\}\end{array}}{\Gamma,(C_1 \preccurlyeq D_1),\ldots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1, \preccurlyeq B_1),\ldots,(A_n \preccurlyeq B_n)}\; A_{n,m}$$

$$\mathcal{R}_{\mathbb{V}_{\preccurlyeq}} = \{R_{n,m} \mid n \geq 1, m \geq 0\}$$

$$\begin{array}{llll}
\mathcal{R}_{\mathbb{VN}_{\preccurlyeq}} & = \{R_{n,m} \mid n+m \geq 1\} & \mathcal{R}_{\mathbb{VC}_{\preccurlyeq}} & = \mathcal{R}_{\mathbb{V}} \cup \{R_{C1}, R_{C2}\}\\
\mathcal{R}_{\mathbb{VT}_{\preccurlyeq}} & = \mathcal{R}_{\mathbb{V}_{\preccurlyeq}} \cup \{T_m \mid m \geq 1\} & \mathcal{R}_{\mathbb{VA}_{\preccurlyeq}} & = \{A_{n,m} \mid n \geq 1, m \geq 0\}\\
\mathcal{R}_{\mathbb{VW}_{\preccurlyeq}} & = \mathcal{R}_{\mathbb{V}_{\preccurlyeq}} \cup \{W_{n,m} \mid n+m \geq 1\} & \mathcal{R}_{\mathbb{VNA}_{\preccurlyeq}} & = \{A_{n,m} \mid n+m \geq 1\}
\end{array}$$

Clarifications: Sequents are based on multisets. The rules $\mathcal{R}_{\mathcal{L}_{\preccurlyeq}}$ form a calculus for a counterfactual logic $\mathcal{L}$ described in [6], where $\preccurlyeq$ is the *comparative plausibility* operator. Besides the rules shown above, these calculi also include the propositional rules of **G3c** {5} and contraction rules. The contexts $\Gamma^{\preccurlyeq}$ and $\Delta^{\preccurlyeq}$ contain all formulae of resp. Γ and Δ of the form $A \preccurlyeq B$.

History: The calculus for $\mathbb{VC}$ was introduced in the tableaux setting [5, 4]. The remaining calculi were introduced in [3, 2] and corrected in [1].

Remarks: Soundness and completeness are shown by proving equivalence to Hilbert-style calculi and (syntactical) cut elimination. These calculi yield PSPACE decision procedures (EXPTIME for $\mathbb{VA}_{\preccurlyeq}$ and $\mathbb{VNA}_{\preccurlyeq}$) and, in most cases, enjoy Craig Interpolation. Contraction can be made admissible.

Entry 6 by: Björn Lellmann

[1] Björn Lellmann. "Sequent Calculi with Context Restrictions and Applications to Conditional Logic". PhD thesis. Imperial College London, 2013. URL: `http://hdl.handle.net/10044/1/18059`.

[2] Björn Lellmann and Dirk Pattinson. "Constructing Cut Free Sequent Systems With Context Restrictions Based on Classical or Intuitionistic Logic". In: *ICLA 2013*. Ed. by Kamal Lodaya. Vol. 7750. LNAI. Springer-Verlag Berlin Heidelberg, 2013, pp. 148–160.

[3] Björn Lellmann and Dirk Pattinson. "Sequent Systems for Lewis' Conditional Logics". In: *JELIA 2012*. Ed. by Luis Fariñas del Cerro, Andreas Herzig, and Jerome Mengin. Vol. 7519. LNCS. Springer-Verlag Berlin Heidelberg, 2012, pp. 320–332.

[4] Ian P. Gent. "A Sequent- or Tableau-style System for Lewis's Counterfactual Logic VC". In: *Notre Dame J. Form. Log.* 33.3 (1992), pp. 369–382.

[5] Harrie C.M. de Swart. "A Gentzen- or Beth-Type System, a Practical Decision Procedure and a Constructive Completeness Proof for the Counterfactual Logics VC and VCS". In: *J. Symb. Log.* 48.1 (1983), pp. 1–20.

[6] David Lewis. *Counterfactuals*. Blackwell, 1973.

Counterfactual Sequent Calculi II (2012, 2013)

$$\frac{\begin{array}{c}\{\, C_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m, I \subseteq [n], J \subseteq [k-1] \,\} \\ \cup \{\, A_k, B_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid k \le n, I \subseteq [n], J \subseteq [m] \,\}\end{array}}{\Gamma, (A_1 \mathrel{\Box\!\!\Rightarrow} B_1), \ldots, (A_n \mathrel{\Box\!\!\Rightarrow} B_n) \vdash \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \ldots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)} \; R_{n,m}$$

$$\frac{\{\, \Gamma \vdash \Delta, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid J \subseteq [m] \,\} \cup \{\, C_k \vdash D_k, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m, J \subseteq [k-1] \,\}}{\Gamma \vdash \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \ldots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)} \; T_m$$

$$\frac{\begin{array}{c}\{\, C_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m, I \subseteq [n], J \subseteq [k-1] \,\} \\ \cup \{\, \Gamma, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid I \subseteq [n], J \subseteq [m] \,\}\end{array}}{\Gamma, (A_1 \mathrel{\Box\!\!\Rightarrow} B_1), \ldots, (A_n \mathrel{\Box\!\!\Rightarrow} B_n) \vdash \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \ldots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)} \; W_{n,m}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, (A \mathrel{\Box\!\!\Rightarrow} B) \vdash \Delta} \; R_{C1} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, (A \mathrel{\Box\!\!\Rightarrow} B)} \; R_{C2}$$

For $n > 0$ the set $[n]$ is $\{1, \ldots, n\}$ and $[0]$ is $\emptyset$. For a set I of indices, $\mathbf{A}^I$ contains all A_i with $i \in I$.

$$\mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} = \{R_{n,m} \mid n \ge 1, m \ge 0\}$$

$$\mathcal{R}_{\mathbb{VN}_{\Box\Rightarrow}} = \{R_{n,m} \mid n + m \ge 1\} \qquad \mathcal{R}_{\mathbb{VW}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{VT}_{\Box\Rightarrow}} \cup \{W_{n,m} \mid n + m \ge 1\}$$

$$\mathcal{R}_{\mathbb{VT}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{T_m \mid m \ge 1\} \qquad \mathcal{R}_{\mathbb{VC}_{\Box\Rightarrow}} = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{R_{C1}, R_{C2}\}$$

Clarifications: Sequents are based on multisets. The rules $\mathcal{R}_{\mathcal{L}_{\Box\Rightarrow}}$ form a calculus for a counterfactual logic $\mathcal{L}$ described in [3], where $\Box\!\!\Rightarrow$ is the *strong counterfactual implication* operator. Besides the rules shown above, these calculi also include the propositional rules of **G3c** {5} and contraction rules.

History: These calculi were introduced in [2] and corrected in [1].

Remarks: The calculi are translations of the calculi in {6} to the language with $\Box\!\!\Rightarrow$. They inherit cut elimination and yield PSPACE decision procedures. Contraction can be made admissible.

[1] Björn Lellmann. "Sequent Calculi with Context Restrictions and Applications to Conditional Logic". PhD thesis. Imperial College London, 2013. URL: `http://hdl.handle.net/10044/1/18059`.

[2] Björn Lellmann and Dirk Pattinson. "Sequent Systems for Lewis' Conditional Logics". In: *JELIA 2012*. Ed. by Luis Fariñas del Cerro, Andreas Herzig, and Jerome Mengin. Vol. 7519. LNCS. Springer-Verlag Berlin Heidelberg, 2012, pp. 320–332.

[3] David Lewis. *Counterfactuals*. Blackwell, 1973.

Entry 7 by: Björn Lellmann

Contextual Natural Deduction (2013)

$$\overline{\Gamma, a : A \vdash a : A}$$

$$\frac{\Gamma, a : A \vdash b : C_\pi[B]}{\Gamma \vdash \lambda_\pi a^A . b : C_\pi[A \to B]} \to_I (\pi)$$

$$\frac{\Gamma \vdash f : C^1_{\pi_1}[A \to B] \quad \Gamma \vdash x : C^2_{\pi_2}[A]}{\Gamma \vdash (f\ x)^{\rightharpoonup}_{(\pi_1;\pi_2)} : C^1_{\pi_1}[C^2_{\pi_2}[B]]} \to_{\overrightarrow{E}} (\pi_1;\pi_2)$$

$$\frac{\Gamma \vdash f : C^1_{\pi_1}[A \to B] \quad \Gamma \vdash x : C^2_{\pi_2}[A]}{\Gamma \vdash (f\ x)^{\leftharpoonup}_{(\pi_1;\pi_2)} : C^2_{\pi_1}[C^1_{\pi_2}[B]]} \to_{\overleftarrow{E}} (\pi_1;\pi_2)$$

π, π_1 and π_2 must be positive positions. a is allowed to occur in b only if π is strongly positive.

Clarifications: $C_\pi[F]$ denotes a formula with F occurring in the hole of a *context* $C_\pi[]$. π is the position of the hole. It is: *positive* iff it is in the left side of an even number of implications; *strongly positive* iff this number is zero.

History: Contextual Natural Deduction [1] combines the idea of deep inference with Gentzen's natural deduction {1}.

Remarks: Soundness and completeness w.r.t. minimal logic are proven [1] by providing translations between $\mathbf{ND^c}$ and the minimal fragment of **NJ** {1}. $\mathbf{ND^c}$ proofs can be quadratically shorter than proofs in the minimal fragment of **NJ**.

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[1] Bruno Woltzenlogel Paleo. "Contextual Natural Deduction". In: *Logical Foundations of Computer Science, International Symposium, LFCS 2013, San Diego, CA, USA, January 6-8, 2013. Proceedings*. Ed. by Sergei N. Artëmov and Anil Nerode. Vol. 7734. Lecture Notes in Computer Science. Springer, 2013, pp. 372–386. ISBN: 978-3-642-35721-3. DOI: 10.1007/978-3-642-35722-0_27. URL: http://dx.doi.org/10.1007/978-3-642-35722-0_27.

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Logics

Proof Systems Grouped by Type