Contents

Contextual Natural Deduction – $\mathbf{ND^c}$ (2013)

$$\overline{\Gamma, a : A \vdash a : A}$$

$$\frac{\Gamma, a : A \vdash b : \mathscr{C}_\pi[B]}{\Gamma \vdash \lambda_\pi a^A.b : \mathscr{C}_\pi[A \to B]} \to_I (\pi)$$

$$\frac{\Gamma \vdash f : \mathscr{C}^1_{\pi_1}[A \to B] \quad \Gamma \vdash x : \mathscr{C}^2_{\pi_2}[A]}{\Gamma \vdash (f\ x)^{\rightharpoonup}_{(\pi_1;\pi_2)} : \mathscr{C}^1_{\pi_1}[\mathscr{C}^2_{\pi_2}[B]]} \to^{\rightharpoonup}_E (\pi_1;\pi_2)$$

$$\frac{\Gamma \vdash f : \mathscr{C}^1_{\pi_1}[A \to B] \quad \Gamma \vdash x : \mathscr{C}^2_{\pi_2}[A]}{\Gamma \vdash (f\ x)^{\leftharpoonup}_{(\pi_1;\pi_2)} : \mathscr{C}^2_{\pi_1}[\mathscr{C}^1_{\pi_2}[B]]} \to^{\leftharpoonup}_E (\pi_1;\pi_2)$$

π, π_1 and π_2 must be positive positions. a is allowed to occur in b only if π is strongly positive.

Clarifications: $\mathscr{C}_\pi[F]$ denotes a formula with F occurring in the hole of a *context* $\mathscr{C}_\pi[]$. π is the position of the hole. It is: *positive* iff it is in the left side of an even number of implications; *strongly positive* iff this number is zero.

History: Contextual Natural Deduction [1] combines the idea of deep inference $\{2\}$ with Gentzen's natural deduction $\{3\}$.

Technicalities: Soundness and completeness w.r.t. minimal logic are proven [1] by providing translations between $\mathbf{ND^c}$ and Gentzen's natural deduction. Proofs in $\mathbf{ND^c}$ can be quadratically shorter than proofs in Gentzen's natural deduction.

References

[1] Bruno Woltzenlogel Paleo. "Contextual Natural Deduction". In: *Logical Foundations of Computer Science, International Symposium, LFCS 2013, San Diego, CA, USA, January 6-8, 2013. Proceedings*. Ed. by Sergei N. Artëmov and Anil Nerode. Vol. 7734. Lecture Notes in Computer Science. Springer, 2013, pp. 372–386. ISBN: 978-3-642-35721-3. DOI: 10.1007/978-3-642-35722-0_27. URL: http://dx.doi.org/10.1007/978-3-642-35722-0_27.

Entry by: Bruno Woltzenlogel Paleo

ToDo – (ToDo)

ToDo

Entry by: ToDo

Natural Deduction – ToDo(1934)

ToDo

Clarifications:

History:

Technicalities:

Entry by: Bruno Woltzenlogel Paleo

Sequent Calculus for Classical Logic – LK (1935)

$$\frac{}{D \vdash D}\;\mathit{Axiom} \qquad \frac{\Gamma \vdash \Theta, D \qquad \Gamma, D \vdash \Theta}{\Gamma \vdash \Theta}\;\mathit{cut}$$

$$\frac{\Gamma \vdash \Theta}{D, \Gamma \vdash \Theta}\;w:l \qquad \frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, D}\;w:r \qquad \frac{D, D, \Gamma \vdash \Theta}{D, \Gamma \vdash \Theta}\;c:l$$

$$\frac{\Gamma, D, E, \Delta \vdash \Theta}{\Gamma, E, D, \Delta \vdash \Theta}\;e:l \qquad \frac{\Gamma \vdash \Theta, D, E, \Lambda}{\Gamma \vdash \Theta, E, D, \Lambda}\;e:r \qquad \frac{\Gamma \vdash \Theta, D, D}{\Gamma \vdash \Theta, D}\;c:r$$

$$\frac{A, \Gamma \vdash \Theta}{A \wedge B, \Gamma \vdash \Theta}\;\wedge:l \qquad \frac{B, \Gamma \vdash \Theta}{A \wedge B, \Gamma \vdash \Theta}\;\wedge:l \qquad \frac{\Gamma \vdash \Theta, A}{\Gamma \vdash \Theta, A \vee B}\;\vee:r$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\;\neg:l \qquad \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\;\neg:r \qquad \frac{\Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \vee B}\;\vee:r$$

$$\frac{A, \Gamma \vdash \Theta \qquad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\;\vee:l \qquad \frac{\Gamma \vdash \Theta, A \qquad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \wedge B}\;\wedge:r$$

$$\frac{\Gamma \vdash \Theta, A \qquad B, \Delta \vdash \Lambda}{A \rightarrow B, \Gamma \vdash \Theta}\;\rightarrow:l \qquad \frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \rightarrow B}\;\rightarrow:r$$

$$\frac{F\,a, \Gamma \vdash \Theta}{\forall x F x, \Gamma \vdash \Theta}\;\forall:l \qquad \frac{\Gamma \vdash \Theta, F\,a}{\Gamma \vdash \Theta, \exists x F x}\;\exists:r$$

$$\frac{F\,a, \Gamma \vdash \Theta}{\exists x F x, \Gamma \vdash \Theta}\;\exists:l\,(*) \qquad \frac{\Gamma \vdash \Theta, F\,a}{\Gamma \vdash \Theta, \forall x F x}\;\forall:r\,(*)$$

$(*)$: Eigenvariable condition: x does not occur in Γ, Δ and $\forall x F x / \exists x F x$.

Clarifications: In all rules, A, B, D, E, F are formulas, $\Gamma, \Theta, \Delta, \Lambda$ are lists of formulas, a is a free and x a bound variable. Within the quantifier rules, Fa is obtained from Fx by applying the substitution $\{a/x\}$.

History: This is Gentzen's original formulation[2] where the term language has only free and bound variables. Takeuti's version[1] adds individual and function constants. An invertible variant of **LK** is **LK**$'$, an intuitionistic version is **LJ** $\{5\}$.

Technicalities: Soundness is obtained by a translation to **ND**, completeness by cut-elimination i.e. Gentzen's Hauptsatz[2].

References

[1] Gaisi Takeuti. *Proof Theory*. English. North-Holland Pub. Co. ; American Elsevier Pub. Co Amsterdam : New York, 1975, 372 p. ISBN: 0444104925.

[2] Gerhard Gentzen. "Untersuchungen über das logische Schließen I". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.

Entry by: Martin Riener

Sequent Calculus LJ – (1935)

$$
\frac{}{A \vdash A}\ init \qquad \frac{\Gamma_1 \vdash P \quad \Gamma_2, P \vdash C}{\Gamma_1, \Gamma_2 \vdash C}\ cut
$$

$$
\frac{\Gamma \vdash P}{\Gamma, \neg P \vdash}\ \neg_l \qquad \frac{\Gamma, P \vdash}{\Gamma \vdash \neg P}\ \neg_r
$$

$$
\frac{P_i, \Gamma \vdash C}{P_1 \wedge P_2, \Gamma \vdash C}\ \wedge_{li} \qquad \frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\ \wedge_r
$$

$$
\frac{P, \Gamma \vdash C \quad Q, \Gamma \vdash C}{P \vee Q, \Gamma \vdash C}\ \vee_l \qquad \frac{\Gamma \vdash P_i}{\Gamma \vdash P_1 \vee P_2}\ \vee_{ri}
$$

$$
\frac{\Gamma_1 \vdash P \quad Q, \Gamma_2 \vdash C}{P \rightarrow Q, \Gamma_1, \Gamma_2 \vdash C}\ \rightarrow_l \qquad \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \rightarrow Q}\ \rightarrow_r
$$

$$
\frac{P\{x \leftarrow \alpha\}, \Gamma \vdash C}{\exists x.P, \Gamma \vdash C}\ \exists_l \qquad \frac{\Gamma \vdash P\{x \leftarrow t\}}{\Gamma \vdash \exists x.P}\ \exists_r
$$

$$
\frac{P\{x \leftarrow t\}, \Gamma \vdash C}{\forall x.P, \Gamma \vdash C}\ \forall_l \qquad \frac{\Gamma \vdash P\{x \leftarrow \alpha\}}{\Gamma \vdash \forall x.P}\ \forall_r
$$

$$
\frac{P, P, \Gamma \vdash C}{P, \Gamma \vdash C}\ c_l \qquad \frac{\Gamma \vdash C}{P, \Gamma \vdash C}\ w_l \qquad \frac{\Gamma \vdash}{\Gamma \vdash P}\ w_r
$$

Clarifications: Assuming that α is a variable not contained in P, Γ or C, t does not contain variables bound in P and C stands for one formula or the empty set.

History: Proposed by Gentzen in [1] by restricting the succedent of sequents in to have at most one formula. In the original paper, he notes that this restriction is equivalent to removing the principle of excluded middle from the natural deduction system $\{3\}$ in order to obtain . The cut is admissible in **LJ** and this result is known as *Hauptsatz*.

Technicalities: Soundness and completeness of **LJ** can be proved using a translation of **LJ** derivations into **NJ**. Decidability of the propositional fragment and consistency of intuitionistic logic follows from the cut admissibility in this calculus.

Entry by: Giselle Reis

References

[1] Gerhard Gentzen. “Untersuchungen über das logische Schließen I”. In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.