Preface

The **Encyclopedia of Proof Systems** aims at providing a reliable, technically informative, historically accurate, concise and convenient central repository of proof systems for various logics. The goal is to facilitate the exchange of information among logicians with mathematical, computational or philosophical backgrounds; in order to foster and accelerate the development of new proof systems and automated deduction tools that rely on them.

Preparatory work for the creation of the Encyclopedia, such as the implementation of the LaTeX template and the setup of the Github repository, started in October 2014, triggered by the call for workshop proposals for the 25th Conference on Automated Deduction (CADE). Christoph Benzmüller, CADE's general chair, and Jasmin Blanchette, CADE's workshop co-chair, encouraged me to submit a workshop proposal and supported my alternative idea to organize instead a special poster session based on encyclopedia entries. I am thankful for their encouragement and support.

In December 2014, Björn Lellman, Giselle Reis and Martin Riener kindly accepted my request to beta-test the template and the instructions I had created. They submitted the first few example entries to the encyclopedia and provided valuable feedback, for which I am grateful. Their comments were essential for improving the templates and instructions before the public announcement of the encyclopedia.

January 2015 *Bruno Woltzenlogel Paleo*

Contents

Intuitionistic Natural Deduction NJ (1935)

$$\frac{A \quad B}{A\&B}\,UE \qquad \frac{A\&B}{A}\,UB \qquad \frac{A\&B}{B}\,UB$$

$$\frac{A}{A \vee B}\,OE \qquad \frac{B}{A \vee B}\,OE \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\,OB$$

$$\frac{Fa}{\forall x F x}\,AE \qquad \frac{\forall x F x}{Fa}\,AB \qquad \frac{Fa}{\exists x F x}\,EE \qquad \frac{\exists x F x \quad \begin{array}{c}[Fa]\\ \vdots \\ C\end{array}}{C}\,EB$$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \supset B}\,FE \qquad \frac{A \quad A \supset B}{B}\,FB \qquad \frac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A}\,NE \qquad \frac{A \quad \neg A}{\bot}\,NB \qquad \frac{\bot}{D}$$

The eigenvariable a of an AE must not occur in the formula designated in the schema by $\forall x F x$; nor in any assumption formula upon which that formula depends. The eigenvariable a of an EB must not occur in the formula designated in the schema by $\exists x F x$; nor in any assumption formula upon which that formula depends, with the exception of the assumption formulae designated by Fa.

Clarifications: The names of the rules are those originally given by Gentzen [1]: U = und (and), O = oder (or), A = all, E = es-gibt (exists), F = folgt (follows), N = nicht (not), E = Einführung (introduction), B = Beseitigung (elimination).

History: The main novelty introduced by Gentzen in this proof system is its *assumption* handling mechanism, which allows formal proofs to reflect more naturally the logical reasoning involved in mathematical proofs.

Remarks: In [1], completeness of **NJ** is proven by showing how to translate proofs in the Hilbert-style calculus **LHJ** to **NJ**-proofs, and soundness is proven by showing how to translate **NJ**-proofs to **LJ**-proofs {3}.

[1] Gerhard Gentzen. "Untersuchungen über das logische Schließen I". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.

Entry 1 by: Bruno Woltzenlogel Paleo

Classical Sequent Calculus LK (1935)

$$\frac{}{D \vdash D}\,Axiom \qquad\qquad \frac{\Gamma \vdash \Theta, D \qquad \Gamma, D \vdash \Theta}{\Gamma \vdash \Theta}\,cut$$

$$\frac{\Gamma \vdash \Theta}{D, \Gamma \vdash \Theta}\,w_l \qquad \frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, D}\,w_r \qquad \frac{D, D, \Gamma \vdash \Theta}{D, \Gamma \vdash \Theta}\,c_l$$

$$\frac{\Gamma, D, E, \Delta \vdash \Theta}{\Gamma, E, D, \Delta \vdash \Theta}\,e_l \qquad \frac{\Gamma \vdash \Theta, D, E, \Lambda}{\Gamma \vdash \Theta, E, D, \Lambda}\,e_r \qquad \frac{\Gamma \vdash \Theta, D, D}{\Gamma \vdash \Theta, D}\,c_r$$

$$\frac{A, \Gamma \vdash \Theta}{A \land B, \Gamma \vdash \Theta}\,\land_l \qquad \frac{B, \Gamma \vdash \Theta}{A \land B, \Gamma \vdash \Theta}\,\land_l \qquad \frac{\Gamma \vdash \Theta, A}{\Gamma \vdash \Theta, A \lor B}\,\lor_r$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\,\neg_l \qquad \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\,\neg_r \qquad \frac{\Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \lor B}\,\lor_r$$

$$\frac{A, \Gamma \vdash \Theta \qquad B, \Gamma \vdash \Theta}{A \lor B, \Gamma \vdash \Theta}\,\lor_l \qquad\qquad \frac{\Gamma \vdash \Theta, A \qquad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \land B}\,\land_r$$

$$\frac{\Gamma \vdash \Theta, A \qquad B, \Delta \vdash \Lambda}{A \to B, \Gamma \vdash \Theta}\,\to_l \qquad\qquad \frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \to B}\,\to_r$$

$$\frac{Fa, \Gamma \vdash \Theta}{\forall x Fx, \Gamma \vdash \Theta}\,\forall_l \qquad\qquad \frac{\Gamma \vdash \Theta, Fa}{\Gamma \vdash \Theta, \exists x Fx}\,\exists_r$$

$$\frac{Fa, \Gamma \vdash \Theta}{\exists x Fx, \Gamma \vdash \Theta}\,\exists_l (*) \qquad\qquad \frac{\Gamma \vdash \Theta, Fa}{\Gamma \vdash \Theta, \forall x Fx}\,\forall_r (*)$$

$(*)$: Eigenvariable condition: x does not occur in Γ, Δ and $\forall x Fx / \exists x Fx$.

Clarifications: In all rules, A, B, D, E, F are formulas, $\Gamma, \Theta, \Delta, \Lambda$ are lists of formulas, a is a free and x a bound variable. Within the quantifier rules, Fa is obtained from Fx by applying the substitution $\{a/x\}$. Inferences over a term t are obtained by substituting a free variable for t.

History: This is Gentzen's original formulation[2], Takeuti[1] has a bit more modern presentation. An invertible variant of **LK** is **LK′**, an intuitionistic version is **LJ** {**??**}.

Remarks: Soundness is obtained by a translation to **ND**, completeness by cut-elimination i.e. Gentzen's Hauptsatz[2].

[1] Gaisi Takeuti. *Proof Theory*. English. North-Holland Pub. Co. ; American Elsevier Pub. Co Amsterdam : New York, 1975, 372 p. ISBN: 0444104925.

[2] Gerhard Gentzen. "Untersuchungen über das logische Schließen I". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.

Entry 2 by: Martin Riener

Intuitionistic Sequent Calculus LJ (1935)

$$\frac{}{A \to A}\ \mathit{init} \qquad \frac{\Gamma \to A \quad A,\Delta \to \Theta}{\Gamma,\Delta \to \Theta}\ \mathit{cut}$$

$$\frac{\Gamma \to A}{\neg A,\Gamma \to}\ \neg_l \qquad \frac{A,\Gamma \to}{\Gamma \to \neg A}\ \neg_r$$

$$\frac{A_i,\Gamma \to \Theta}{A_1 \& A_2,\Gamma \to \Theta}\ \&_{li} \qquad \frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \& B}\ \&_r$$

$$\frac{A,\Gamma \to \Theta \quad B,\Gamma \to \Theta}{A \vee B,\Gamma \to \Theta}\ \vee_l \qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2}\ \vee_{ri}$$

$$\frac{\Gamma \to A \quad B,\Delta \to \Theta}{A \supset B,\Gamma,\Delta \to \Theta}\ \supset_l \qquad \frac{A,\Gamma \to B}{\Gamma \to A \supset B}\ \supset_r$$

$$\frac{A\alpha,\Gamma \to \Theta}{\exists x.Ax,\Gamma \to \Theta}\ \exists_l \qquad \frac{\Gamma \to At}{\Gamma \to \exists x.Ax}\ \exists_r$$

$$\frac{At,\Gamma \to \Theta}{\forall x.Ax,\Gamma \to \Theta}\ \forall_l \qquad \frac{\Gamma \to A\alpha}{\Gamma \to \forall x.Ax}\ \forall_r$$

$$\frac{\Gamma,B,A,\Delta \to \Theta}{\Gamma,A,B,\Delta \to \Theta}\ e_l \qquad \frac{A,A,\Gamma \to \Theta}{A,\Gamma \to \Theta}\ c_l$$

$$\frac{\Gamma \to \Theta}{A,\Gamma \to \Theta}\ w_l \qquad \frac{\Gamma \to}{\Gamma \to A}\ w_r$$

Clarifications: In all rules, A, A_i and B are arbitrary formulas and Θ is a set with at most one formula. In rules $\exists_l$ and $\forall_r$, α is a variable not contained in A, Γ or Θ. In rules $\exists_r$ and $\forall_l$, t does not contain variables bound in A. It is common to consider **LJ** without the exchange rule e_l just by interpreting Γ and Θ as multi-sets of formulas instead of lists. Also, the conjunction & is usually denoted by $\wedge$.

History: Proposed by Gentzen in [1] by restricting the succedent of sequents in to have at most one formula. In the original paper, he notes that this restriction is equivalent to removing the principle of excluded middle from the natural deduction system {1} in order to obtain .

Remarks: Soundness and completeness of **LJ** can be proved using a translation of **LJ** derivations into **NJ**. Decidability of the propositional fragment and consistency of intuitionistic logic follows from cut admissibility in this calculus (*Hauptsatz*).

[1] Gerhard Gentzen. "Untersuchungen über das logische Schließen I". In: *Mathematische Zeitschrift* 39.1 (Dec. 1935), pp. 176–210.

Entry 3 by: Giselle Reis

Multi-conclusion Intuitionistic Sequent Calculus LJ' (1954)

$$\frac{}{A \to A}\ init \qquad \frac{\Gamma \to \Theta, A \quad A, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}\ cut$$

$$\frac{A_i, \Gamma \to \Theta}{A_1 \wedge A_2, \Gamma \to \Theta}\ \wedge_{li} \qquad \frac{\Gamma \to \Theta, A \quad \Gamma \to \Theta, B}{\Gamma \to \Theta, A \wedge B}\ \wedge_r$$

$$\frac{A, \Gamma \to \Theta \quad B, \Gamma \to \Theta}{A \vee B, \Gamma \to \Theta}\ \vee_l \qquad \frac{\Gamma \to \Theta, A_i}{\Gamma \to \Theta, A_1 \vee A_2}\ \vee_{ri}$$

$$\frac{\Gamma \to \Theta, A \quad B, \Delta \to \Lambda}{A \supset B, \Gamma, \Delta \to \Theta, \Lambda}\ \supset_l \qquad \frac{A, \Gamma \to B}{\Gamma \to A \supset B}\ \supset_r$$

$$\frac{A\alpha, \Gamma \to \Theta}{\exists x.Ax, \Gamma \to \Theta}\ \exists_l \quad \frac{\Gamma \to \Theta, At}{\Gamma \to \Theta, \exists x.Ax}\ \exists_r \quad \frac{At, \Gamma \to \Theta}{\forall x.Ax, \Gamma \to \Theta}\ \forall_l \quad \frac{\Gamma \to A\alpha}{\Gamma \to \forall x.Ax}\ \forall_r$$

$$\frac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta}\ \neg_l \quad \frac{A, \Gamma \to}{\Gamma \to \neg A}\ \neg_r \quad \frac{\Gamma, B, A, \Delta \to \Theta}{\Gamma, A, B, \Delta \to \Theta}\ e_l \quad \frac{\Gamma\Delta \to \Theta, B, A, \Lambda}{\Gamma \to \Theta, A, B, \Lambda}\ e_r$$

$$\frac{A, A, \Gamma \to \Theta}{A, \Gamma \to \Theta}\ c_l \quad \frac{\Gamma \to \Theta, A, A}{\Gamma \to \Theta, A}\ c_r \quad \frac{\Gamma \to \Theta}{A, \Gamma \to \Theta}\ w_l \quad \frac{\Gamma \to}{\Gamma \to A}\ w_r$$

Clarifications: In all rules, A, A_i and B are arbitrary formulas. In rules $\exists_l$ and $\forall_r$, α is a variable not contained in A, Γ or Θ. In rules $\exists_r$ and $\forall_l$, t does not contain variables bound in A. It is common to consider **LJ'** without the exchange rules e_l and e_r just by interpreting Γ and Θ as multi-sets of formulas instead of lists. While **LJ**{3} is defined by restricting all rules of **LK**{2} to single conclusion, in **LJ'** this restriction is only for the rules $\neg_r$, $\supset_r$ and $\forall_r$.

History: Proposed by Maehara in [3] and used to prove the completeness of **LJ**{3} in [2]. The same calculus (and other multi-conclusion calculi for intuitionistic logic) is rediscovered in [1] while analysing classical and intuitionistic provability.

Remarks: The equivalence of **LJ'** to **LJ** is established by translating sequents $\Gamma \vdash A_1, ..., A_n$ into $\Gamma \vdash A_1 \vee ... \vee A_n$. Cut is admissible in this calculus via a combination of the rewriting rules for cut-elimination in **LJ** and **LK**.

[1] Gopalan Nadathur. "Correspondences between Classical, Intuitionistic and Uniform Provability". In: *CoRR* cs.LO/9809015 (1998).

[2] Gaisi Takeuti. *Proof Theory*. Studies in logic and the foundations of mathematics. North-Holland, 1987. ISBN: 9780444879431.

[3] Shôji Maehara. "Eine Darstellung der intuitionistischen Logik in der klassischen". In: *Nagoya Math. J.* 7 (1954), pp. 45–64.

Entry 4 by: Giselle Reis

Sequent Calculus G3c (1996)

$$\frac{}{P,\Gamma \vdash \Delta, P}\ \mathrm{Ax} \qquad \frac{}{\bot,\Gamma \vdash \Delta}\ \mathrm{L}\bot$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \mathrm{L}\wedge \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ \mathrm{R}\wedge$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ \mathrm{L}\vee \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\ \mathrm{R}\vee$$

$$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta}\ \mathrm{L}\to \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ \mathrm{R}\to$$

$$\frac{\forall x A, A[x/t], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta}\ \mathrm{L}\forall \qquad \frac{\Gamma \vdash \Delta, A[x/y]}{\Gamma \vdash \Delta, \forall x A}\ \mathrm{R}\forall$$

$$\frac{A[x/y], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta}\ \mathrm{L}\exists \qquad \frac{\Gamma \vdash \Delta, A[x/t], \exists x A}{\Gamma \vdash \Delta, \exists x A}\ \mathrm{R}\exists$$

with P in Ax atomic and y not free in the conclusion in R$\forall$, L$\exists$

Clarifications: Sequents are based on multisets. A formula $A[x/t]$ is the result of uniformly substituting the term t for the variable x in A.

Remarks: Sound and complete wrt. classical first-order logic. Weakening and contraction are depth-preserving admissible and all the rules are depth-preserving invertible.

[1] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. 2nd ed. Vol. 43. Cambridge Tracts In Theoretical Computer Science. Cambridge University Press, 2000.

Entry 5 by: Björn Lellmann

Counterfactual Sequent Calculi I (1983,1992,2012,2013)

$$\frac{\begin{array}{c}\{\, B_k \vdash A_1,\dots,A_n,D_1,\dots,D_m \mid k \le n \,\}\\ \cup \{\, C_k \vdash A_1,\dots,A_n,D_1,\dots,D_{k-1} \mid k \le m \,\}\end{array}}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1 \preccurlyeq B_1),\dots,(A_n \preccurlyeq B_n)}\ R_{n,m}$$

$$\frac{\{\, C_k \vdash D_1,\dots,D_{k-1} \mid k \le m \,\} \qquad \Gamma \vdash \Delta,D_1,\dots,D_m}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta}\ T_m$$

$$\frac{\{\, C_k \vdash A_1,\dots,A_n,D_1,\dots,D_{k-1} \mid k \le m \,\} \qquad \Gamma \vdash \Delta,A_1,\dots,A_n,D_1,\dots,D_m}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1 \preccurlyeq B_1),\dots,(A_n \preccurlyeq B_n)}\ W_{n,m}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta,(A \preccurlyeq B)}\ R_{C1} \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash \Delta, B}{\Gamma,(A \preccurlyeq B) \vdash \Delta}\ R_{C2}$$

$$\frac{\begin{array}{c}\{\, \Gamma^{\preccurlyeq}, B_k \vdash \Delta^{\preccurlyeq},A_1,\dots,A_n,D_1,\dots,D_m \mid k \le n \,\}\\ \cup \{\, \Gamma^{\preccurlyeq}, C_k \vdash \Delta^{\preccurlyeq},A_1,\dots,A_n,D_1,\dots,D_{k-1} \mid k \le m \,\}\end{array}}{\Gamma,(C_1 \preccurlyeq D_1),\dots,(C_m \preccurlyeq D_m) \vdash \Delta,(A_1, \preccurlyeq B_1),\dots,(A_n \preccurlyeq B_n)}\ A_{n,m}$$

$$\mathcal{R}_{\mathbb{V}_{\preccurlyeq}} = \{R_{n,m} \mid n \ge 1, m \ge 0\}$$

$$\begin{array}{llll}\mathcal{R}_{\mathbb{VN}_{\preccurlyeq}} & = \{R_{n,m} \mid n+m \ge 1\} & \mathcal{R}_{\mathbb{VC}_{\preccurlyeq}} & = \mathcal{R}_{\mathbb{V}} \cup \{R_{C1}, R_{C2}\}\\ \mathcal{R}_{\mathbb{VT}_{\preccurlyeq}} & = \mathcal{R}_{\mathbb{V}_{\preccurlyeq}} \cup \{T_m \mid m \ge 1\} & \mathcal{R}_{\mathbb{VA}_{\preccurlyeq}} & = \{A_{n,m} \mid n \ge 1, m \ge 0\}\\ \mathcal{R}_{\mathbb{VW}_{\preccurlyeq}} & = \mathcal{R}_{\mathbb{V}_{\preccurlyeq}} \cup \{W_{n,m} \mid n+m \ge 1\} & \mathcal{R}_{\mathbb{VNA}_{\preccurlyeq}} & = \{A_{n,m} \mid n+m \ge 1\}\end{array}$$

Clarifications: Sequents are based on multisets. The propositional part is that of **G3c** {5}. Also include the contraction rules. Rules $\mathcal{R}_{\mathcal{L}_{\preccurlyeq}}$ are for the logic $\mathcal{L}$ in terms of the *comparative plausibility* operator $\preccurlyeq$ from [6]. The contexts $\Gamma^{\preccurlyeq}$ resp. $\Delta^{\preccurlyeq}$ contain all formulae of Γ resp. Δ of the form $A \preccurlyeq B$.

History: The calculus for $\mathbb{VC}$ was introduced in the tableaux setting [5, 4]. The remaining calculi were introduced in [3, 2] and corrected in [1].

Remarks: Soundness and completeness via equivalence to Hilbert-style calculi and (syntactical) cut elimination. Yield PSPACE decision procedures (resp. EXPTIME for $\mathbb{VA}_{\preccurlyeq}$ and $\mathbb{VNA}_{\preccurlyeq}$) and in most cases Craig Interpolation. Contraction can be made admissible. [3, 1]

Entry 6 by: Björn Lellmann

[1] Björn Lellmann. “Sequent Calculi with Context Restrictions and Applications to Conditional Logic”. PhD thesis. Imperial College London, 2013. URL: `http://hdl.handle.net/10044/1/18059`.

[2] Björn Lellmann and Dirk Pattinson. “Constructing Cut Free Sequent Systems With Context Restrictions Based on Classical or Intuitionistic Logic”. In: *ICLA 2013*. Ed. by Kamal Lodaya. Vol. 7750. LNAI. Springer-Verlag Berlin Heidelberg, 2013, pp. 148–160.

[3] Björn Lellmann and Dirk Pattinson. “Sequent Systems for Lewis’ Conditional Logics”. In: *JELIA 2012*. Ed. by Luis Fariñas del Cerro, Andreas Herzig, and Jerome Mengin. Vol. 7519. LNCS. Springer-Verlag Berlin Heidelberg, 2012, pp. 320–332.

[4] Ian P. Gent. “A Sequent- or Tableau-style System for Lewis’s Counterfactual Logic VC”. In: *Notre Dame J. Form. Log.* 33.3 (1992), pp. 369–382.

[5] Harrie C.M. de Swart. “A Gentzen- or Beth-Type System, a Practical Decision Procedure and a Constructive Completeness Proof for the Counterfactual Logics VC and VCS”. In: *J. Symb. Log.* 48.1 (1983), pp. 1–20.

[6] David Lewis. *Counterfactuals*. Blackwell, 1973.

Counterfactual Sequent Calculi II (2012, 2013)

$$\dfrac{\begin{array}{c}\{\, C_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m, I \subseteq [n], J \subseteq [k-1] \,\} \\ \cup \{\, A_k, B_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid k \le n, I \subseteq [n], J \subseteq [m] \,\}\end{array}}{\Gamma, (A_1 \mathrel{\Box\!\!\Rightarrow} B_1), \ldots, (A_n \mathrel{\Box\!\!\Rightarrow} B_n) \vdash \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \ldots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)}\ R_{n,m}$$

$$\dfrac{\{\, \Gamma \vdash \Delta, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid J \subseteq [m] \,\} \cup \{\, C_k \vdash D_k, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m, J \subseteq [k-1] \,\}}{\Gamma \vdash \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \ldots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)}\ T_m$$

$$\dfrac{\begin{array}{c}\{\, C_k, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[k-1]\setminus J} \mid 1 \le k \le m, I \subseteq [n], J \subseteq [k-1] \,\} \\ \cup \{\, \Gamma, \mathbf{B}^I \vdash \mathbf{A}^{[n]\setminus I}, \mathbf{C}^J, \mathbf{D}^{[m]\setminus J} \mid I \subseteq [n], J \subseteq [m] \,\}\end{array}}{\Gamma, (A_1 \mathrel{\Box\!\!\Rightarrow} B_1), \ldots, (A_n \mathrel{\Box\!\!\Rightarrow} B_n) \vdash \Delta, (C_1 \mathrel{\Box\!\!\Rightarrow} D_1), \ldots, (C_m \mathrel{\Box\!\!\Rightarrow} D_m)}\ W_{n,m}$$

$$\dfrac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, (A \mathrel{\Box\!\!\Rightarrow} B) \vdash \Delta}\ R_{C1} \qquad \dfrac{\Gamma \vdash \Delta, A \qquad \Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, (A \mathrel{\Box\!\!\Rightarrow} B)}\ R_{C2}$$

For $n > 0$ the set $[n]$ is $\{1, \ldots, n\}$ and $[0]$ is $\emptyset$. For a set I of indices, $\mathbf{A}^I$ contains all A_i with $i \in I$.

$$\mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} = \{R_{n,m} \mid n \ge 1, m \ge 0\}$$

$$\begin{array}{llll} \mathcal{R}_{\mathbb{VN}_{\Box\Rightarrow}} & = \{R_{n,m} \mid n + m \ge 1\} & \mathcal{R}_{\mathbb{VW}_{\Box\Rightarrow}} & = \mathcal{R}_{\mathbb{VT}_{\Box\Rightarrow}} \cup \{W_{n,m} \mid n + m \ge 1\} \\ \mathcal{R}_{\mathbb{VT}_{\Box\Rightarrow}} & = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{T_m \mid m \ge 1\} & \mathcal{R}_{\mathbb{VC}_{\Box\Rightarrow}} & = \mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}} \cup \{R_{C1}, R_{C2}\} \end{array}$$

Clarifications: Sequents are based on multisets. The propositional part is that of **G3c** {5}. Also includes the contraction rules. Rules $\mathcal{R}_{\mathcal{L}_{\Box\Rightarrow}}$ are for the logic $\mathcal{L}$ in terms of the *strong counterfactual implication* $\Box\!\!\Rightarrow$ from [3].

History: Introduced in [2], corrected in [1].

Remarks: Translations of the calculi {**??**} to the language with $\Box\!\!\Rightarrow$. Inherit cut elimination and yield PSPACE decision procedures. Contraction can be made admissible.

[1] Björn Lellmann. "Sequent Calculi with Context Restrictions and Applications to Conditional Logic". PhD thesis. Imperial College London, 2013. URL: `http://hdl.handle.net/10044/1/18059`.

[2] Björn Lellmann and Dirk Pattinson. "Sequent Systems for Lewis' Conditional Logics". In: *JELIA 2012*. Ed. by Luis Fariñas del Cerro, Andreas Herzig, and Jerome Mengin. Vol. 7519. LNCS. Springer-Verlag Berlin Heidelberg, 2012, pp. 320–332.

[3] David Lewis. *Counterfactuals*. Blackwell, 1973.

Entry 7 by: Björn Lellmann

Contextual Natural Deduction (2013)

$$\overline{\Gamma, a : A \vdash a : A}$$

$$\frac{\Gamma, a : A \vdash b : C_\pi[B]}{\Gamma \vdash \lambda_\pi a^A . b : C_\pi[A \to B]} \to_I (\pi)$$

$$\frac{\Gamma \vdash f : C^1_{\pi_1}[A \to B] \quad \Gamma \vdash x : C^2_{\pi_2}[A]}{\Gamma \vdash (f\ x)^{\rightharpoonup}_{(\pi_1;\pi_2)} : C^1_{\pi_1}[C^2_{\pi_2}[B]]} \to^{\rightharpoonup}_E (\pi_1;\pi_2)$$

$$\frac{\Gamma \vdash f : C^1_{\pi_1}[A \to B] \quad \Gamma \vdash x : C^2_{\pi_2}[A]}{\Gamma \vdash (f\ x)^{\leftharpoonup}_{(\pi_1;\pi_2)} : C^2_{\pi_1}[C^1_{\pi_2}[B]]} \to^{\leftharpoonup}_E (\pi_1;\pi_2)$$

π, π_1 and π_2 must be positive positions. a is allowed to occur in b only if π is strongly positive.

Clarifications: $C_\pi[F]$ denotes a formula with F occurring in the hole of a *context* $C_\pi[]$. π is the position of the hole. It is: *positive* iff it is in the left side of an even number of implications; *strongly positive* iff this number is zero.

History: Contextual Natural Deduction [1] combines the idea of deep inference with Gentzen's natural deduction {1}.

Remarks: Soundness and completeness w.r.t. minimal logic are proven [1] by providing translations between $\mathbf{ND^c}$ and Gentzen's natural deduction. Proofs in $\mathbf{ND^c}$ can be quadratically shorter than proofs in Gentzen's natural deduction.

[1] Bruno Woltzenlogel Paleo. "Contextual Natural Deduction". In: *Logical Foundations of Computer Science, International Symposium, LFCS 2013, San Diego, CA, USA, January 6-8, 2013. Proceedings*. Ed. by Sergei N. Artëmov and Anil Nerode. Vol. 7734. Lecture Notes in Computer Science. Springer, 2013, pp. 372–386. ISBN: 978-3-642-35721-3. DOI: 10.1007/978-3-642-35722-0_27. URL: http://dx.doi.org/10.1007/978-3-642-35722-0_27.

Entry 8 by: Bruno Woltzenlogel Paleo

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