MATH 4311 Spring 2025

Program 4 Due: May 8, 2025

Question 1 Consider the heat equation, for $0 \le x \le 1$, $0 \le t \le 1$,

$$\begin{cases} u_t = 2u_{xx}, \\ u(x,0) = 2\cosh(x), \\ u(0,t) = 2e^{2t}, \\ u(1,t) = (e^2+1)e^{2t-1}. \end{cases}$$

In homework 8, we have verified $u(x,t) = e^{2t+x} + e^{2t-x}$ is the exact solution.

- Solve by Forward difference method. The numerical solution at (x_i, t_j) is $w_{i,j}$. Verify the error at $t = 1$ is $O(h + k^2)$ by choosing different values of h and k. You might choose the L^2 or L^∞ norm of the error vector.
- Solve by Backward difference method. Verify the error at $t = 1$ is $O(h + k^2)$
- Solve by Crank–Nicolson method. Verify the error at $t = 1$ is $O(h^2 + k^2)$

Question 2 Use finite difference method to solve the Wave equation,

$$\begin{cases} u_{tt} = 16u_{xx}, \\ u(0,t) = 0; \\ u(1,t) = 0; \\ u(x,0) = \sin(\pi x) \\ u_t(x,0) = 0. \end{cases}$$

- Verify $u(x,t) = \sin(\pi x)\cos(4\pi t)$ is the exact solution.
- Make a table of the numerical solution and error at $(x,t) = (1/4, 3/4)$ as a function of step sizes $h = ck = 2^{-p}$ for $p = 4, \cdots, 10$.

Question 3 Consider solving the following equation,

$$\begin{cases} \Delta u + \dfrac{u}{x^2+y^2} = 5, \\ u(x,1) = x^2 + 1, \text{ for } 3 \le x \le 5, \\ u(x,2) = x^2 + 4, \text{ for } 3 \le x \le 5, \\ u(3,y) = y^2 + 9, \text{ for } 1 \le y \le 2, \\ u(5,y) = y^2 + 25, \text{ for } 1 \le y \le 2. \end{cases}$$

(a) Verify $u(x,y) = x^2 + y^2$ is the exact solution.

(b) Solve the system using finite difference method with $h_x = h_y = 2^{-p}$, where $p = 2, 3, \cdots, 7$. The error is defined as $e_{i,j} = u_{i,j} - u(x_i, t_j)$. Plot the error distribution on the xy plane. How does the error decay as h and k gets smaller? How does the time cost increase as h and k gets smaller?

(c) Repeat part (b) using finite element method.

Hint: You could use the matlab code on Canvas. You might need chage the full matrix A to sparse matrix in the program, in case the program is too slow, or run out of memory.