MATH 4311 Spring 2025

Program 1 Due: Feb 13, 2025

This project considers the Global Positioning System (GPS). To make things simple, we start with 4 satellites. At an instant time $t = d$, the receiver collects the synchronized signal from the satellites, which was sent at time t_i from the coordinate (A_i, B_i, C_i). To find out the receiver's position (x, y, z), we have the following system of equations,

$$\begin{cases} r_1 = \sqrt{(x - A_1)^2 + (y - B_1)^2 + (z - C_1)^2} = c(t_1 - d), \\ r_2 = \sqrt{(x - A_2)^2 + (y - B_2)^2 + (z - C_2)^2} = c(t_2 - d), \\ r_3 = \sqrt{(x - A_3)^2 + (y - B_3)^2 + (z - C_3)^2} = c(t_3 - d), \\ r_4 = \sqrt{(x - A_4)^2 + (y - B_4)^2 + (z - C_4)^2} = c(t_4 - d), \end{cases}$$

Here $c \approx 299792.458$ km/sec is the speed of light. Notice d should also be considered to be unknown because the receiver's time is inaccurate.

(I) Write a program to solve (x, y, z, d) with given (A_i, B_i, C_i, t_i), using Multivariable Newton's method. Notice, the system might have 2 solutions, one is near earth, one is far from earth. We only want the near earth solution. The earth center is fixed at $(0, 0, 0)$.

(II) Given $(15600, 7540, 20140, 0.07074)$, $(18760, 2750, 18610, 0.07220)$, $(17610, 14630, 13480, 0.07690)$, $(19170, 610, 18390, 0.07242)$, as the position (in km) and time (in second) of the four satellites. Use the program in part (I) to find the receiver's position and time. Set the initial guess $(0, 0, 6670, 0)$.

(III) The clocks aboard the satellites are correct up to about 10^{-8} second. Change each t_i in Part (II) with $\pm 10^{-8}$, then the position as an output will be changed by $(\Delta x, \Delta y, \Delta z)$. The distance $||(\Delta x, \Delta y, \Delta z)||_2$ tells the error estimate of the position found in Part (II). Notice, some t_i could be changed by 10^{-8}, some could be changed by -10^{-8}. Please try all possible combinations, and find the largest distance (error).

(IV) In order to increase accuracy, we add another 4 satellites, so that we have 8 satellites in total. Design a program to solve the least square problem using Gauss–Newton's method.

(V) Arbitrarily (randomly) set the location (A_i, B_i, C_i) of the eight satellites on a sphere of radius 26570 km. In order for the receiver at $(0, 0, 6670)$ to receive the signal, we require $C_i > 6670$. In order to have good accuracy, do not set the satellites to overlap with each other. The time $t_i = \sqrt{A_i^2 + B_i^2 + (6670 - C_i)^2}/c$. Now the system of eight equations and four unknowns have an exact solution $(0, 0, 6670, 0)$. Estimate the error in position by changing each t_i by $\pm 10^{-8}$ second.