Constraint Networks

Chapter 2 @ Constraint Processing by Rina Dechter

Constraint programming

- Model problem by specifying constraints on acceptable solutions
  - define variables and domains
  - post constraints on these variables
- Solve model
  - choose algorithm
    - incremental assignment / backtracking search
    - complete assignments / stochastic search
  - design heuristics

Mathematical Background: sets

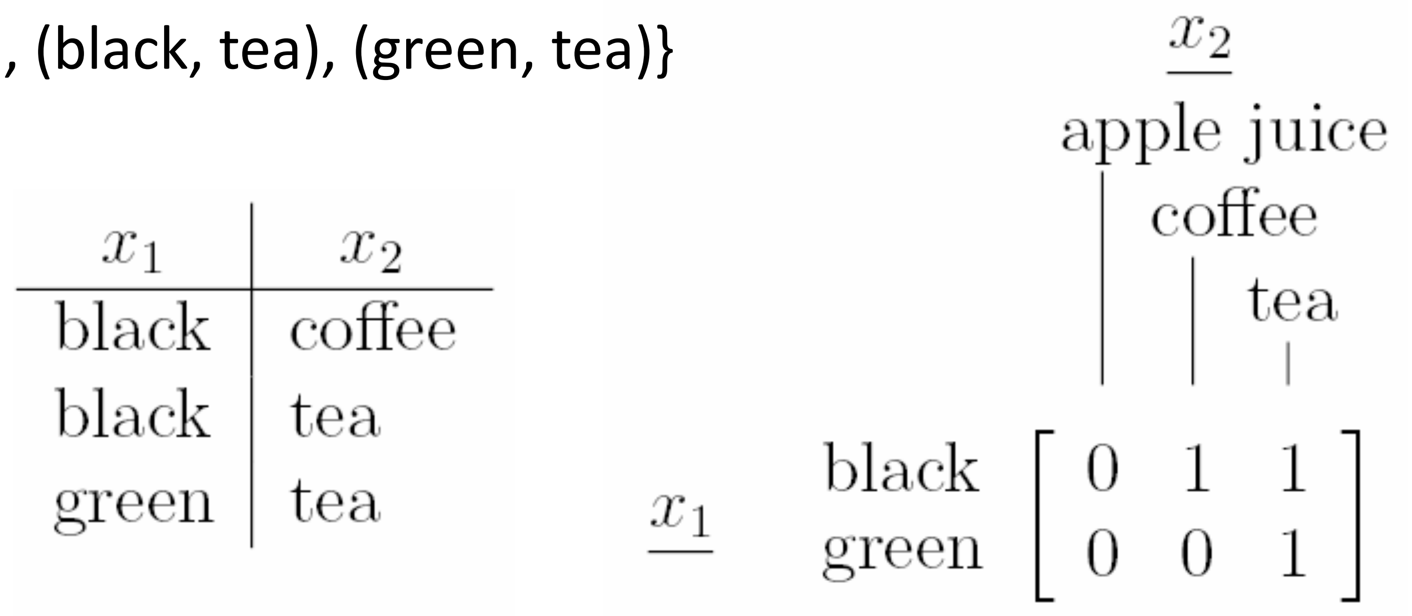
- Sets, Domains and Tuples
  - A set is a collection of objects; objects are not ordered; objects cannot be repeated
  - Set operations: member, subset, intersection, union, difference, etc.
  - The domain of a variable is a set of values
  - Cartesian product of a list of domains – example:
    - D_1 = {black, green}, D_2 = {apple juice, coffee, tea}, D_1 x D_2 = ???

Mathematical Background: relations (I)

- Relations
  - A relation on a set of variables is any subset of the Cartesian product of their domains
  - The scope of a relation is the set of variables on which a relation is defined
    - D_1 (D_2) is the scope of variable x_1 (x_2)
    - The set of tuples {(black, coffee), (black, tea), (green, tea)} is a relation on $\{x_1, x_2\}$, i.e. the scope of the relation is $\{x_1, x_2\}$

Mathematical Background: relations (II)

- Graphical representation
  - Scope {x_1, x_2}
  - D_1 = {black, green}, D_2 = {apple juice, coffee, tea}
  - R_{12} = {(black, coffee), (black, tea), (green, tea)}



x_1	x_2
black	coffee
black	tea
green	tea

(a) table

$$x_1 \quad \begin{array}{c|ccc} & \text{apple juice} & \text{coffee} & \text{tea} \\ \hline \text{black} & 0 & 1 & 1 \\ \text{green} & 0 & 0 & 1 \end{array}$$

x_2 labels the columns.

(b) (0,1)-matrix

Mathematical Background: relations (III)

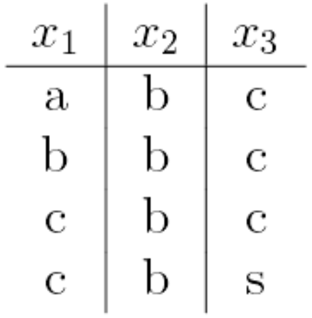
- Relations may be explicitly or implicitly defined
  - Variables x_1 and x_2
  - Domains $D_1 = \{2,4\}$ and $D_2 = \{1,3,5\}$

  - Explicit relation $R_{12} = \{(2,3),(2,5),(4,5)\}$
  - Implicit relation $R'_{12} = (x_1 < x_2)$

  - $R_{12} = R'_{12}$

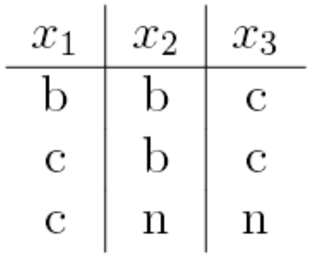
Mathematical Background: relations (IV)

- Intersection, union and difference (for relations on the same scope)



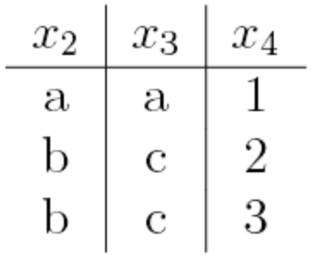
x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s

(a) Relation R



x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

(b) Relation R'



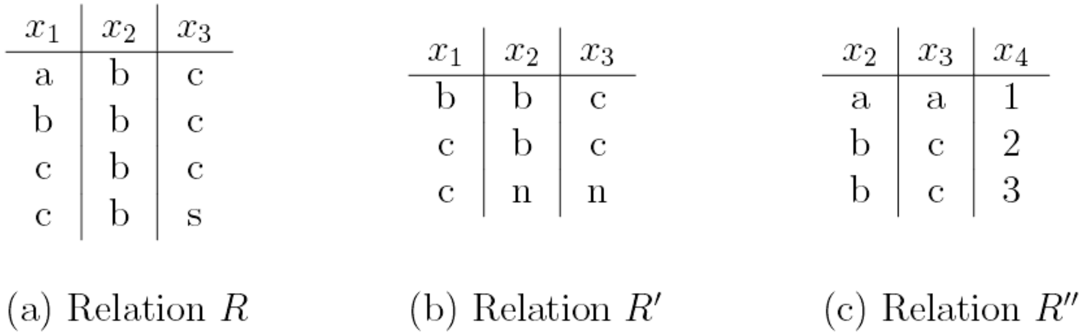
x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

(c) Relation R''

- Intersection(R, R’)? Union(R,R’)? Difference(R,R’)?

You have
2 minutes!

Mathematical Background: relations (IV)



x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s

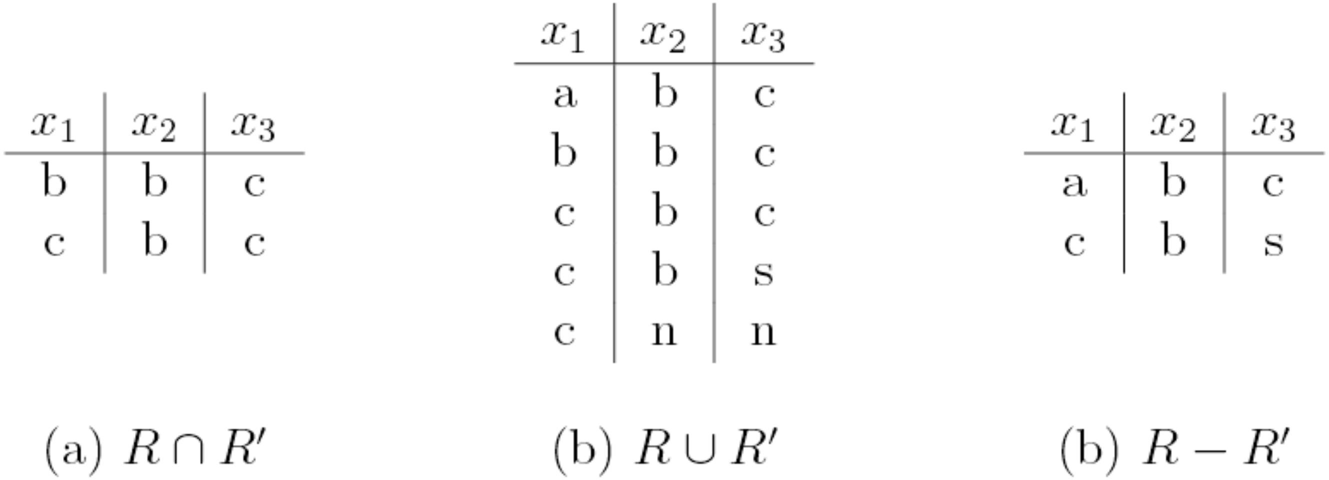
(a) Relation R

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

(b) Relation R'

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

(c) Relation R''



x_1	x_2	x_3
b	b	c
c	b	c

(a) $R \cap R'$

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s
c	n	n

(b) $R \cup R'$

x_1	x_2	x_3
a	b	c
c	b	s

(b) $R - R'$

Mathematical Background: relations (V)

- Selection, projection, and join operations on relations

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s

(a) Relation R

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

(b) Relation R'

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

(c) Relation R''

x_1	x_2	x_3
b	b	c
c	b	c

(a) $\sigma_{x_3=c}(R')$

x_2	x_3
b	c
n	n

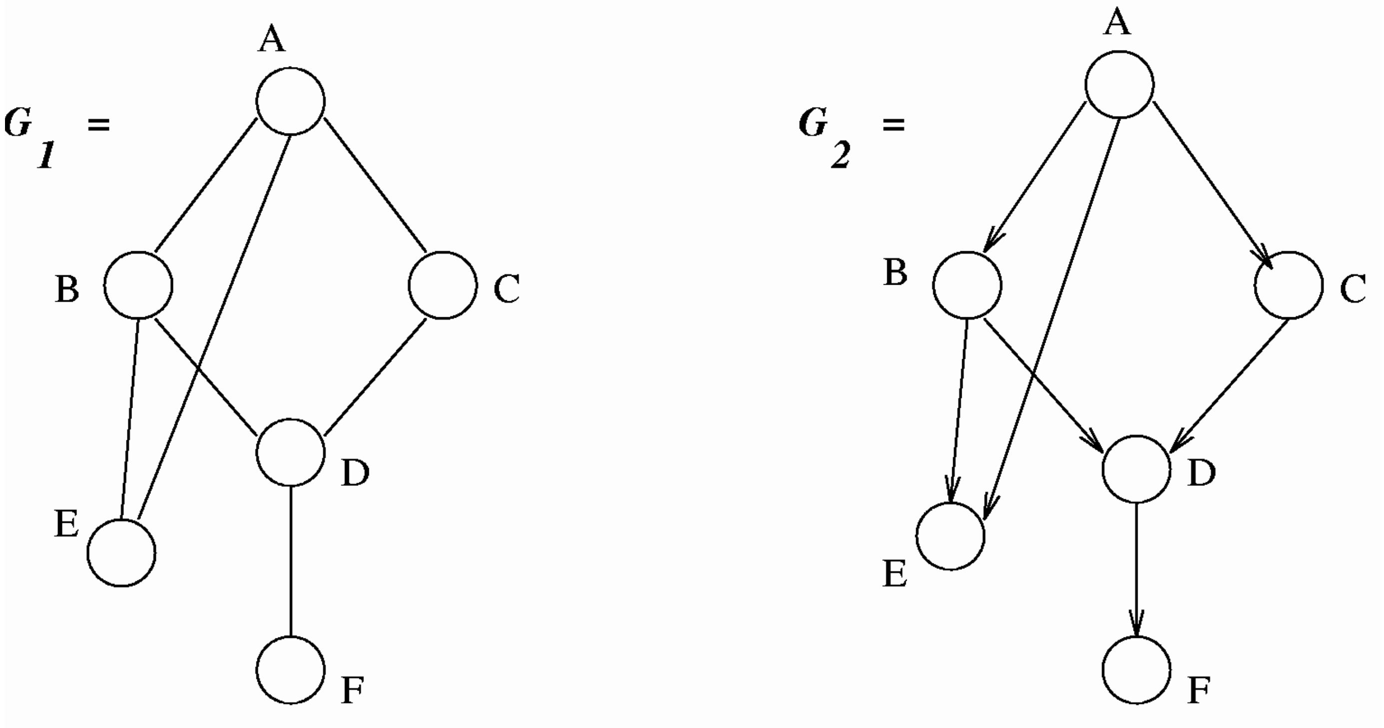
(b) $\pi_{\{x_2,x_3\}}(R')$

x_1	x_2	x_3	x_4
b	b	c	2
b	b	c	3
c	b	c	2
c	b	c	3

(c) $R' \bowtie R''$

Mathematical Background: graphs

- Graph (V,E) with V set of vertices and E set of edges
- Path is a sequence of edges
- Undirected (G_1) and directed (G_2) graphs



Constraint Network / Constraint Problem

- Triple (X,D,C)
- Finite set of variables X and respective set of domains D
  - A domain lists the possible values for each variable
- Set of constraints C
  - A constraint is a relation defined on a subset of variables

N-Queens problem (N=4)

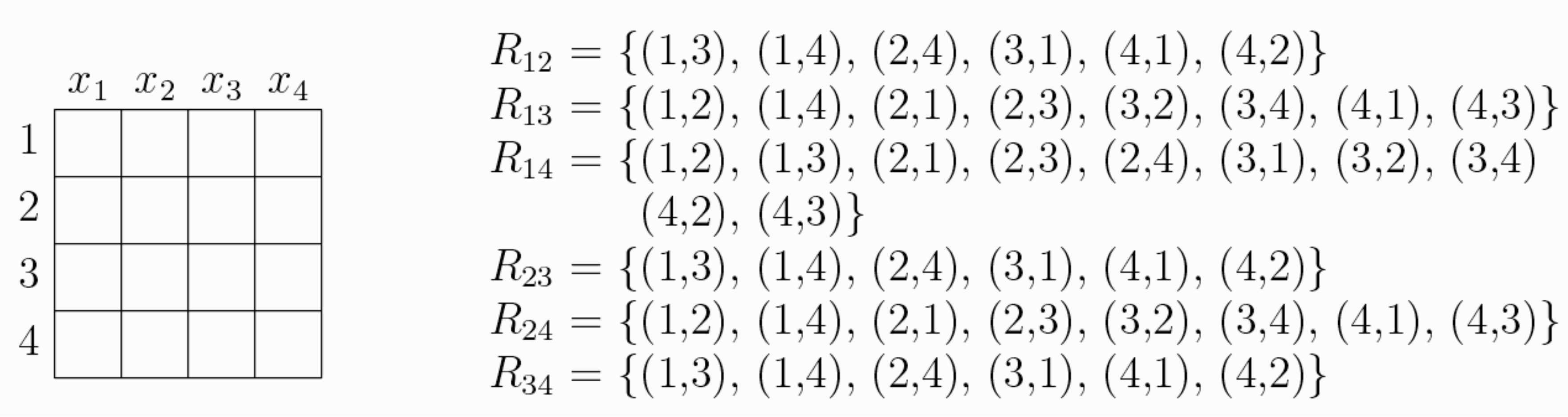
You have
5 minutes!

- Variables?
- Domains?
- Constraints?

		Q	
Q			
			Q
	Q		

N-Queens problem (N=4)

- $X = \{x_1, x_2, x_3, x_4\}$, $D_i = \{1,2,3,4\}$
- Six constraints given by relations R_{ij}

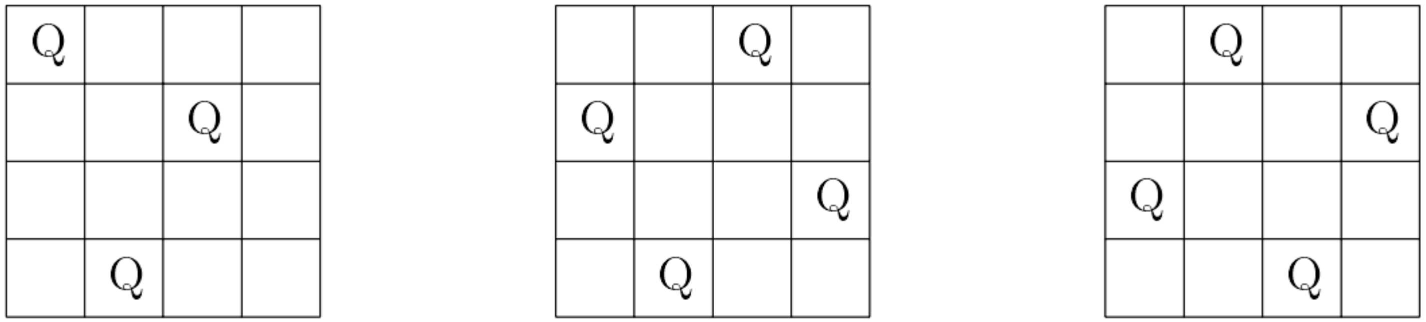


$$
\begin{aligned}
R_{12} &= \{(1{,}3),\ (1{,}4),\ (2{,}4),\ (3{,}1),\ (4{,}1),\ (4{,}2)\}\\
R_{13} &= \{(1{,}2),\ (1{,}4),\ (2{,}1),\ (2{,}3),\ (3{,}2),\ (3{,}4),\ (4{,}1),\ (4{,}3)\}\\
R_{14} &= \{(1{,}2),\ (1{,}3),\ (2{,}1),\ (2{,}3),\ (2{,}4),\ (3{,}1),\ (3{,}2),\ (3{,}4)\\
&\qquad (4{,}2),\ (4{,}3)\}\\
R_{23} &= \{(1{,}3),\ (1{,}4),\ (2{,}4),\ (3{,}1),\ (4{,}1),\ (4{,}2)\}\\
R_{24} &= \{(1{,}2),\ (1{,}4),\ (2{,}1),\ (2{,}3),\ (3{,}2),\ (3{,}4),\ (4{,}1),\ (4{,}3)\}\\
R_{34} &= \{(1{,}3),\ (1{,}4),\ (2{,}4),\ (3{,}1),\ (4{,}1),\ (4{,}2)\}
\end{aligned}
$$

Solutions of a Constraint Network

- An instantiation of a subset of variables is an assignment of each variable to a value in the domain
  - An assignment is represented by a tuple <x,a> or x=a or $<a_1, a_2, \ldots>$
- An instantiation satisfies a constraint iff
  - It is defined over all the variables in the constraints
  - The components of the instantiation are present in the relation
- A partial instantiation is consistent if it satisfies all of the constraints with no uninstantiated variables
- A solution is an instantiation to all variables that satisfies all constraints

4Q: consistent instantiation & 2 solutions



Q			
		Q	
	Q		

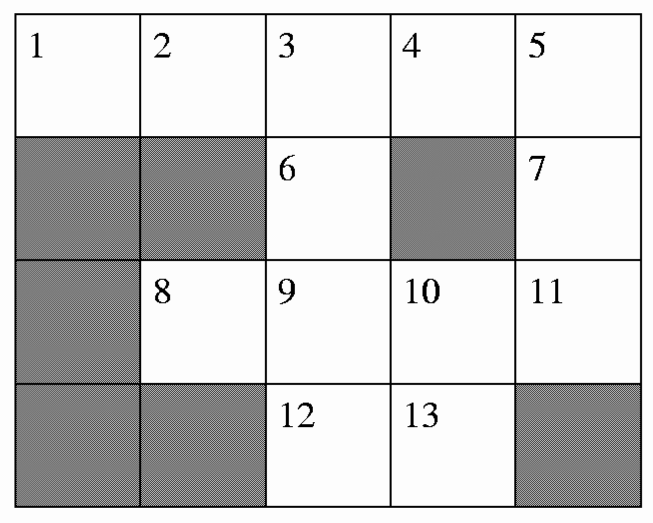
		Q	
Q			
			Q
	Q		

	Q		
			Q
Q			
		Q	

- Solution 1? Solution 2?

The crossword puzzle

You have
3 minutes!

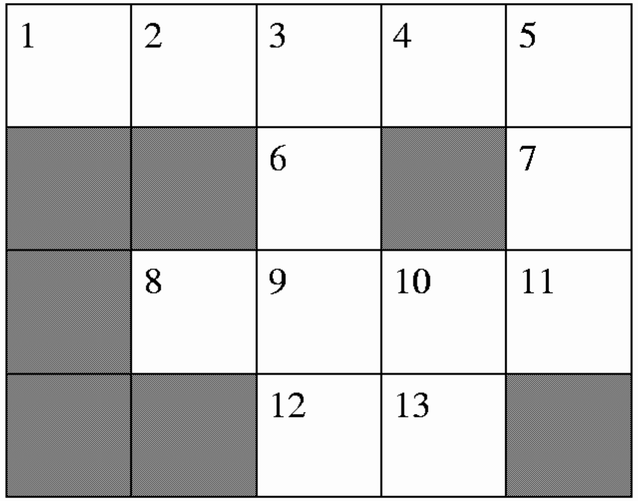


1	2	3	4	5
▨	▨	6	▨	7
▨	8	9	10	11
▨	▨	12	13	▨

{HOSES, LASER, SHEET, SNAIL, STEER, ALSO, EARN, HIKE, IRON, SAME, EAT, LET, RUN, SUN, TEN, YES, BE, IT, NO, US}

The crossword puzzle: constraints

- $R_{\{1,2,3,4,5\}}$ = {(H,O,S,E,S)(L,A,S,E,R)(S,H,E,E,T)(S,N,A,I,L)(S,T,E,E,R)}
- $R_{\{3,6,9,12\}}$ = {(A,L,S,O),(E,A,R,N),(H,I,K,E),(I,R,O,N),(S,A,M,E)}
- $R_{\{5,7,11\}}$ = {(E,A,T)(L,E,T)(R,U,N)(S,U,N)(T,E,N)(Y,E,S)}
- $R_{\{8,9,10,11\}} = R_{\{3,6,9,12\}}$
- $R_{\{10,13\}}$ = {(B,E),(I,T),(N,O),(U,S)}
- $R_{\{12,13\}} = R_{\{10,13\}}$



1	2	3	4	5
▨	▨	6	▨	7
▨	8	9	10	11
▨	▨	12	13	▨

Consistent partial assignments

- Over the set of variables $\{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{12}\}$

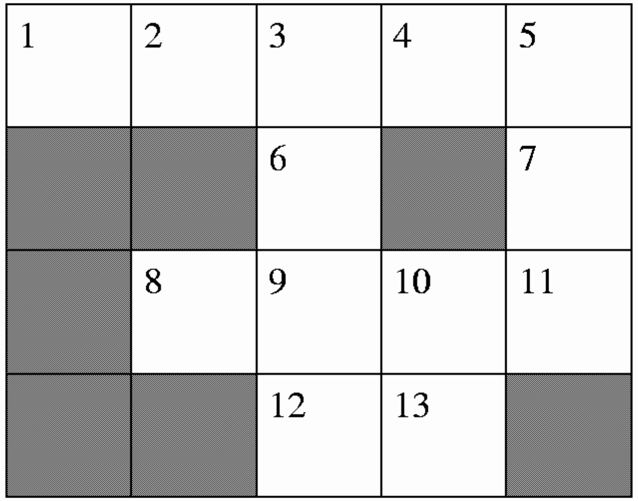
  {(H,O,S,E,S,A,M,E),

  (L,A,S,E,R,A,M,E),

  (S,H,E,E,T,A,R,N),

  (S,N,A,I,L,L,S,O),

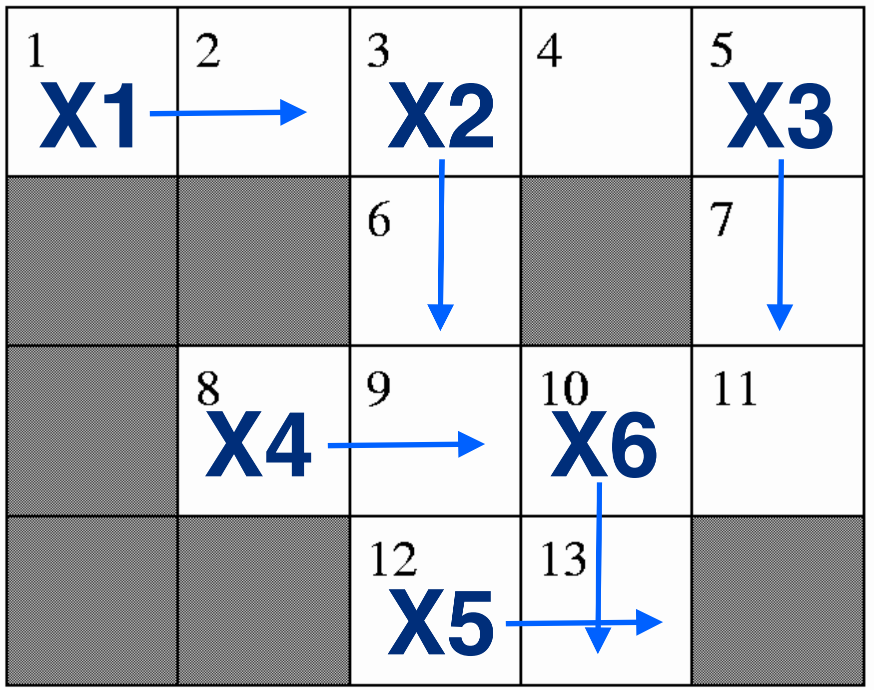
  (S,T,E,E,R,A,R,N)}



1	2	3	4	5
▨	▨	6	▨	7
▨	8	9	10	11
▨	▨	12	13	▨

Alternative formulation

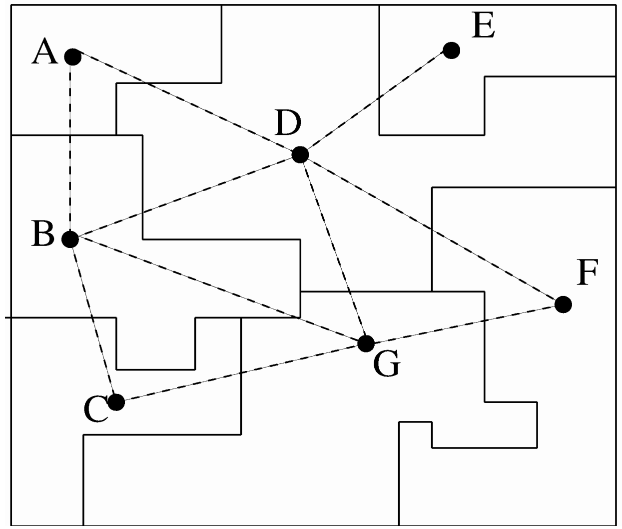
- x_1 (1, horizontal)
- x_2 (3, vertical)
- x_3 (5, vertical)
- x_4 (8, horizontal)
- x_5 (12, horizontal)
- x_6 (10, vertical)



- D_1 = {hoses,laser,sheet,snail,steer}

- R_{12} = {(hoses,same),(laser,same),(sheet,earn),(snail,also),(steer,earn)}

Constraint graph: map coloring

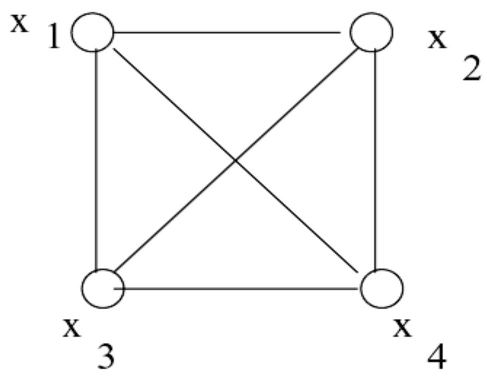


- Each node is a variable
- Arcs collect all nodes whose variables are included in a constraint scope

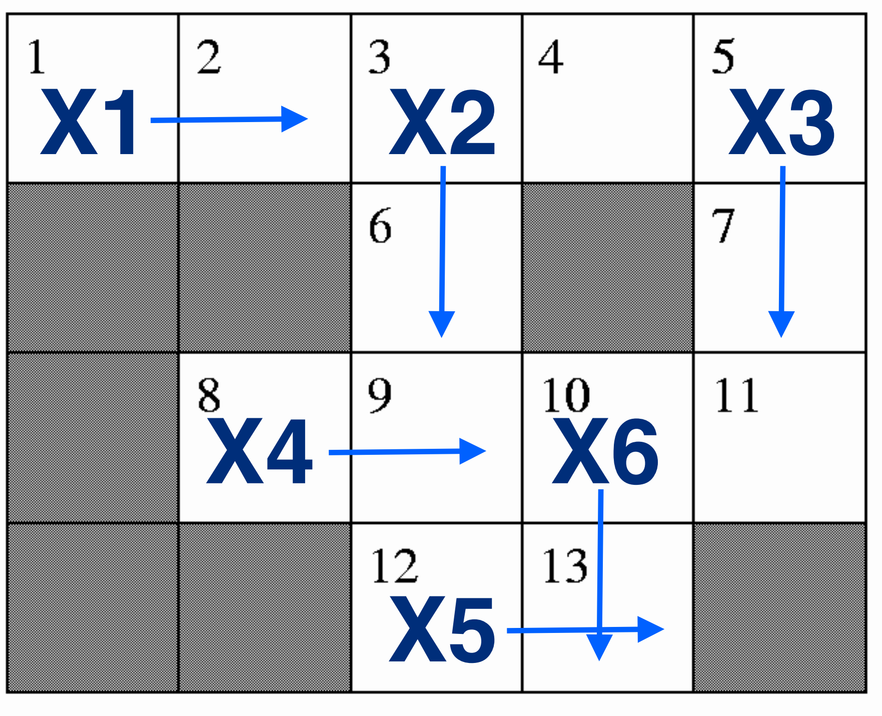
Constraint graph: 4 Queens

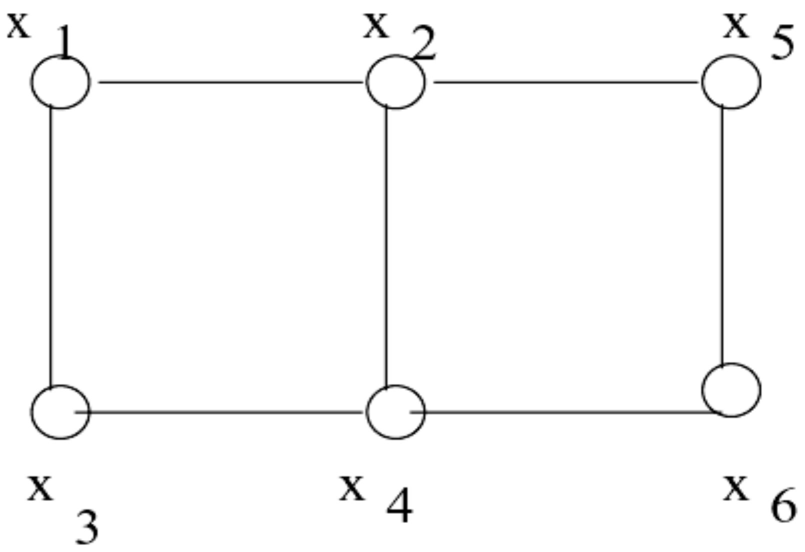
You have
2 minutes!

Constraint graph: 4 Queens



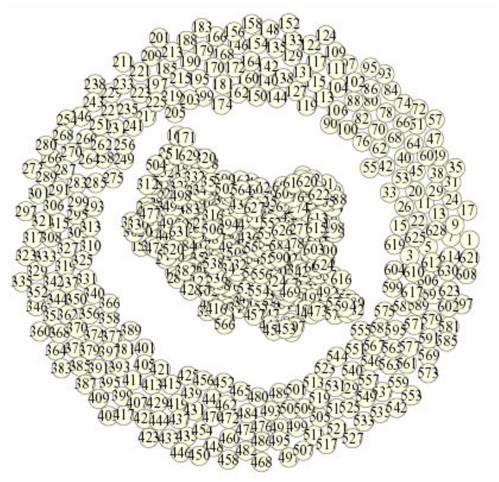
Constraint graph: crossword

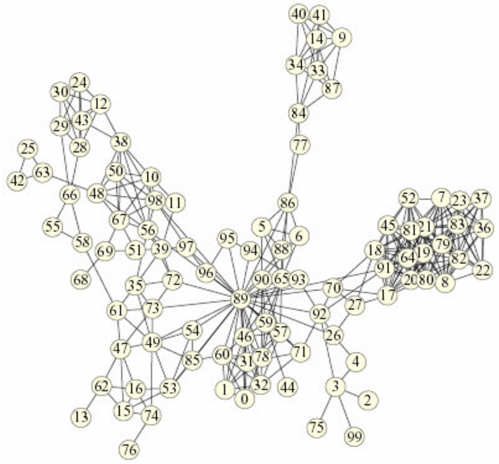


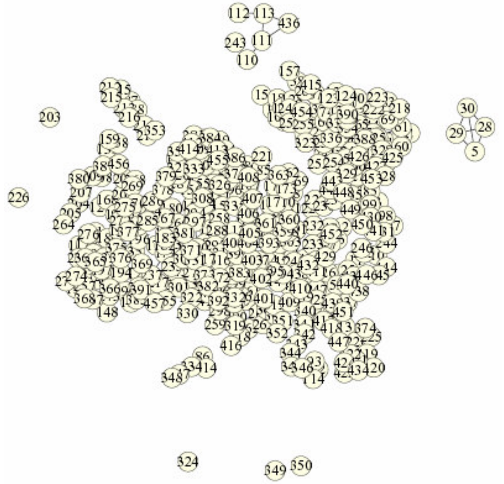


Radio link frequency assignment problem

- Real problems with +- 1000 variables and more than 5000 constraints





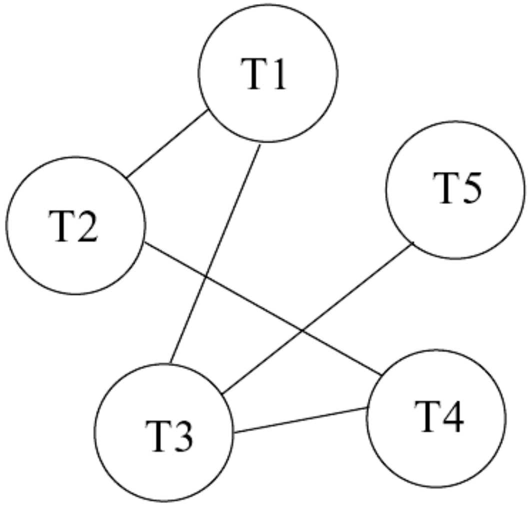


Scheduling example (I)

You have
5 minutes!

- Goal: schedule five tasks T1, …, T5 [variables]
- Each task takes 1h to complete
- Tasks may start at 1:00, 2:00 or 3:00 [domains]
- Tasks may be executed simultaneously, but… [constraints]
  - T1 must start after T3
  - T3 must start before T4 and after T5
  - T2 cannot execute at the same time as T1 or T4
  - T4 cannot start at 2:00

Scheduling example (II)



Unary constraint

$D_{T4} = \{1{:}00, 3{:}00\}$

Binary constraints

$R_{\{T1,T2\}}$: $\{(1{:}00, 2{:}00), (1{:}00, 3{:}00), (2{:}00, 1{:}00),$
$(2{:}00, 3{:}00), (3{:}00, 1{:}00), (3{:}00, 2{:}00)\}$

$R_{\{T1,T3\}}$: $\{(2{:}00, 1{:}00), (3{:}00, 1{:}00), (3{:}00, 2{:}00)\}$

$R_{\{T2,T4\}}$: $\{(1{:}00, 2{:}00), (1{:}00, 3{:}00), (2{:}00, 1{:}00)$
$(2{:}00, 3{:}00), (3{:}00, 1{:}00), (3{:}00, 2{:}00)\}$

$R_{\{T3,T4\}}$: $\{(1{:}00, 2{:}00), (1{:}00, 3{:}00), (2{:}00, 3{:}00)\}$

$R_{\{T3,T5\}}$: $\{(2{:}00, 1{:}00), (3{:}00, 1{:}00), (3{:}00, 2{:}00)\}$

Numeric constraints

- Express constraints by arithmetic expressions

$$(3x_i + 2x_j \le 3) \wedge (-4x_i + 5x_j < 1)$$

- 4-Queens

$$R_{ij} = \{(x_i, x_j) \mid x_i \in D_i, x_j \in D_j, x_i \neq x_j, \text{ and } |x_i - x_j| \neq |i - j|\}$$

- Crypto-arithmetic puzzles
  - SEND + MORE = MONEY

SEND + MORE = MONEY

You have
5 minutes!

Replace each letter by a different digit so that

$$\begin{array}{r} SEND \\ +\ \ MORE \\ \hline MONEY \end{array}$$

is a correct sum.

SEND + MORE = MONEY as a CSP

Variables: $S, E, N, D, M, O, R, Y,$

Domains:

$[1..9]$ for $S, M,$

$[0..9]$ for $E, N, D, O, R, Y.$

Constraints

$$\begin{aligned} & 1000 \cdot S + 100 \cdot E + 10 \cdot N + D \\ + \ & 1000 \cdot M + 100 \cdot O + 10 \cdot R + E \\ = 10000 \cdot M + \ & 1000 \cdot O + 100 \cdot N + 10 \cdot E + Y \end{aligned}$$

$$\texttt{all\_different}(S, E, N, D, M, O, R, Y).$$

SEND + MORE = MONEY: alternative

Use “carry” variables $C_1, \ldots, C_4 \in [0..1]$:

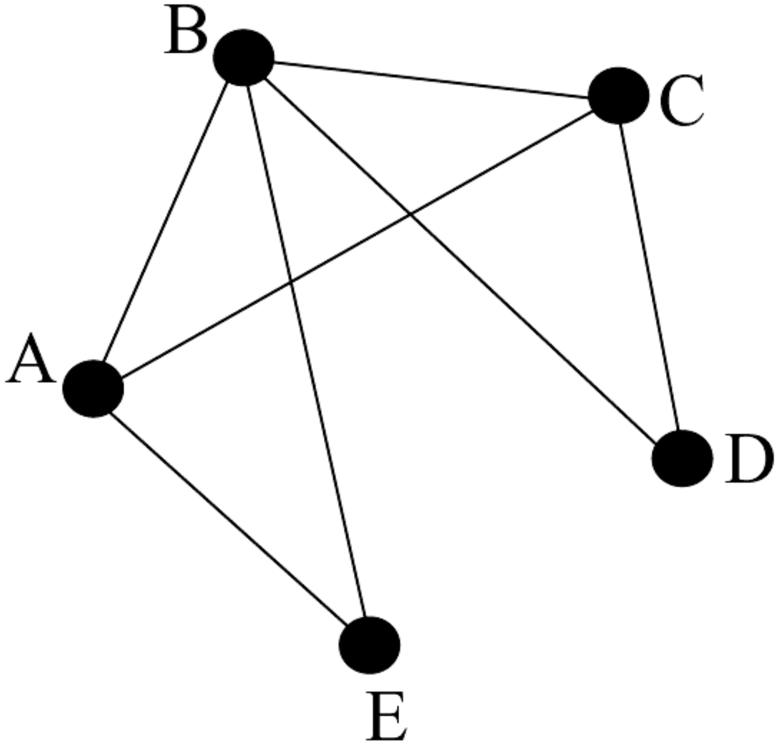
$$\begin{aligned}
&D + E = 10 \cdot C_1 + Y,\\
&C_1 + N + R = 10 \cdot C_2 + E,\\
&C_2 + E + O = 10 \cdot C_3 + N,\\
&C_3 + S + M = 10 \cdot C_4 + O,\\
&C_4 = M.
\end{aligned}$$

$$\texttt{all\_different}(S, E, N, D, M, O, R, Y).$$

Boolean constraints

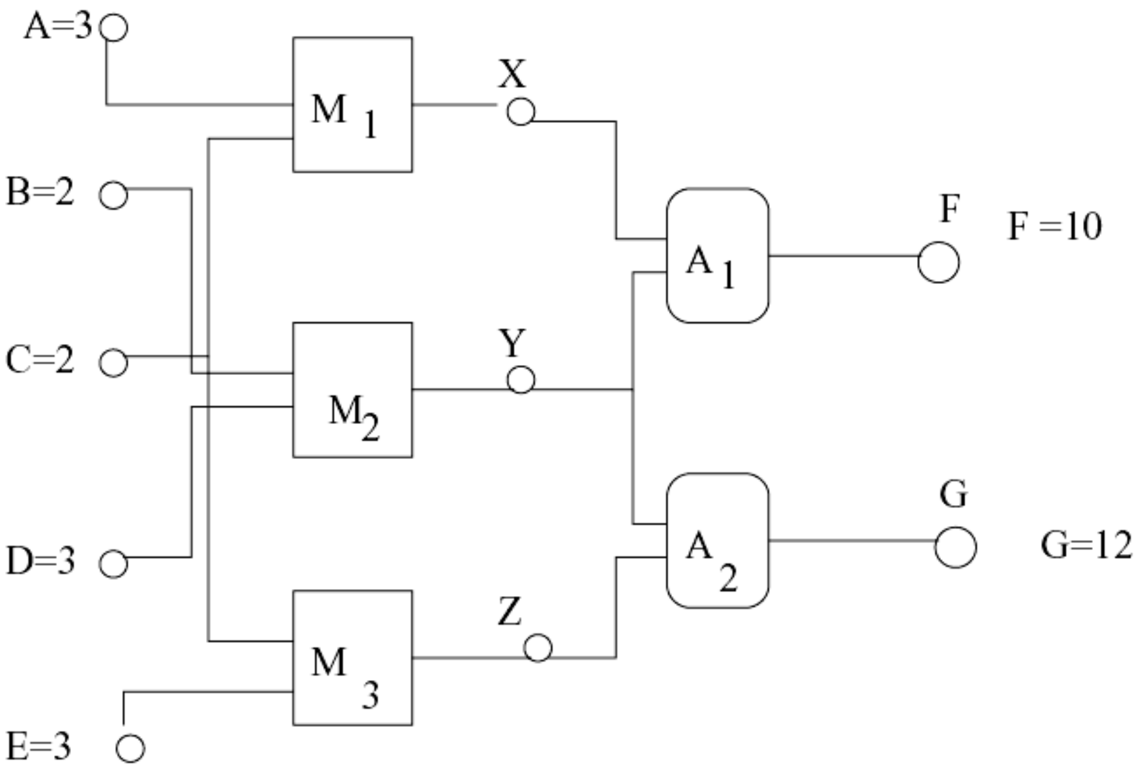
$\phi = \{(\neg C), (A \vee B \vee C), (\neg A \vee B \vee E), (\neg B \vee C \vee D)\}$

- One node for each Boolean variable
- One edge for each pair of nodes corresponding to variables appearing in the same clause



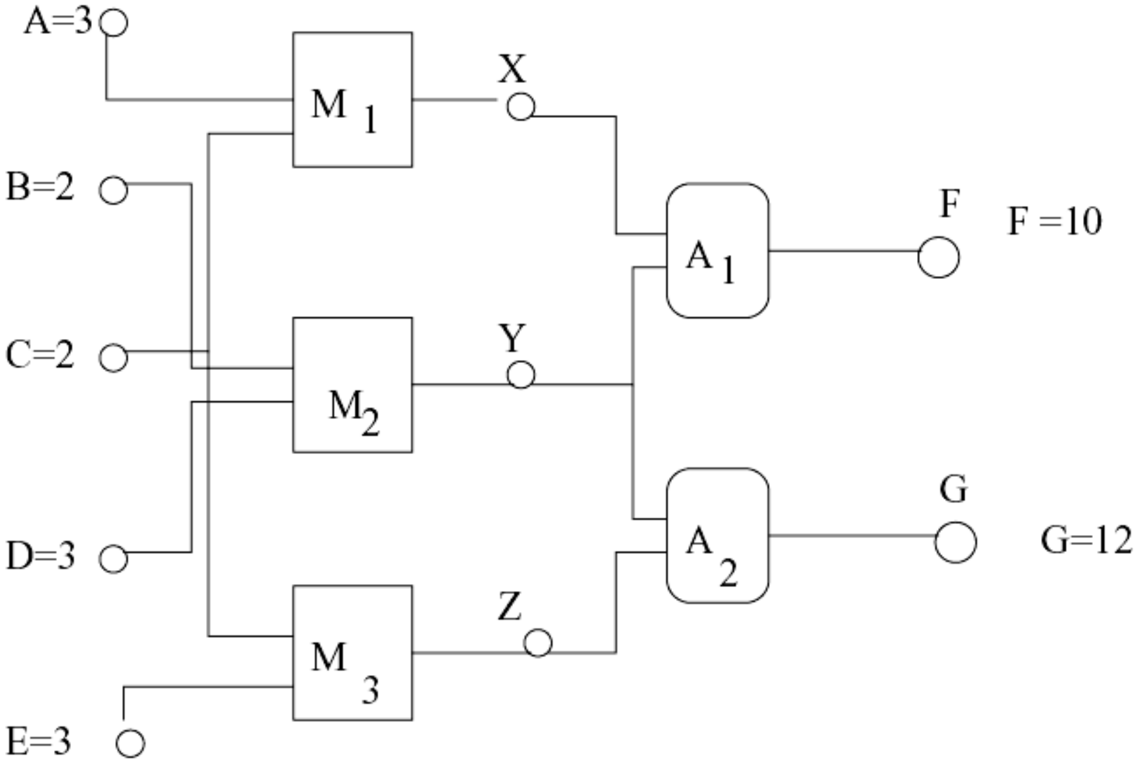
Combinatorial circuits diagnosis

- Identify a subset of the gates that explain the faulty observed output
- Variables are inputs (A-E), outputs (F,G), intermediate output (X,Y,Z) and components (M1-3,A1-2)
  - Inputs and outputs have integer domains
  - Components have Boolean domains



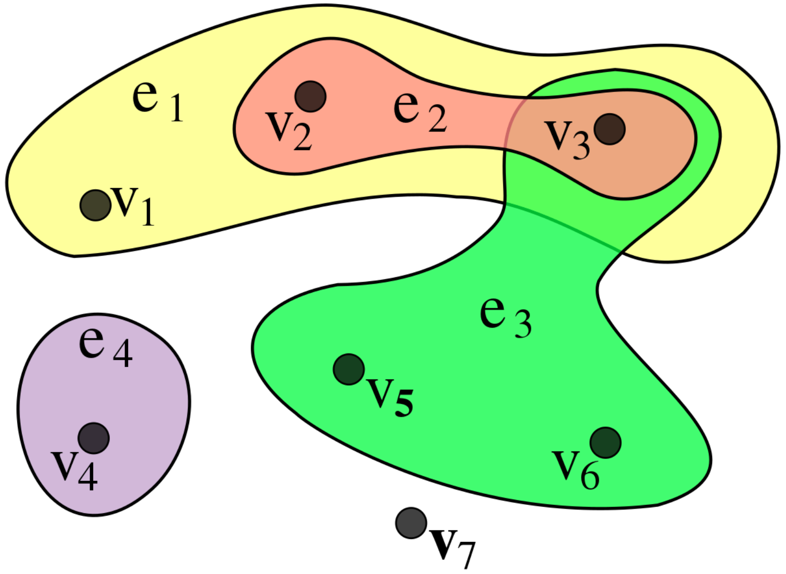
Combinatorial circuits diagnosis: constraints

- One constraint associating each component variable to the inputs and output
- Example of a constraint assuming Boolean domain and M_1 to be an AND gate
  - $M_1 \rightarrow (A \wedge C \rightarrow X)$



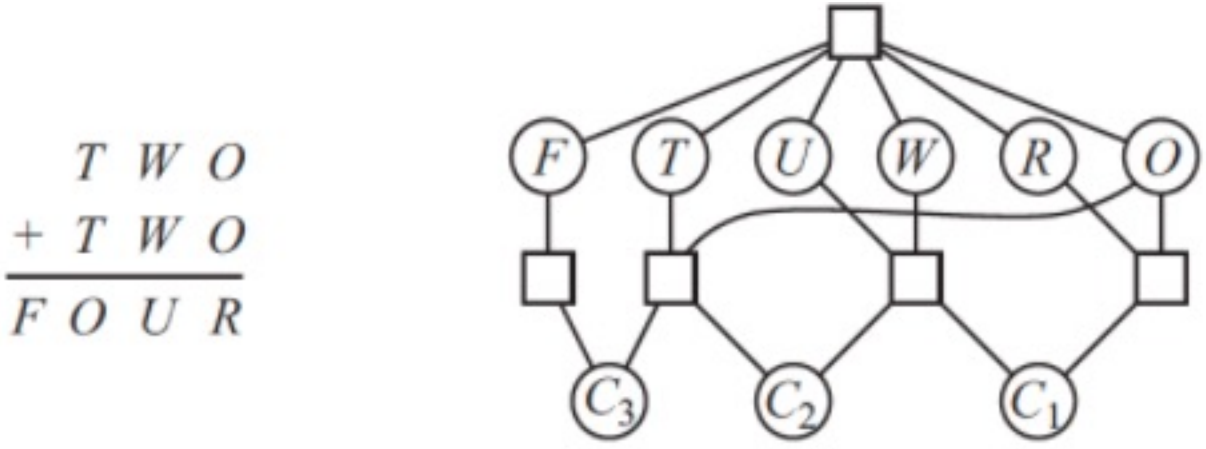
Primal vs dual graphs

- Dual graphs are related with hypergraphs
  - An hypergraph is a pair (V,S)
  - V is a set of vertices
  - S is a set of subsets of V, called hyperedges
  - Hyperedges may “connect” more than 1 or 2 vertices



Constraint hypergraphs

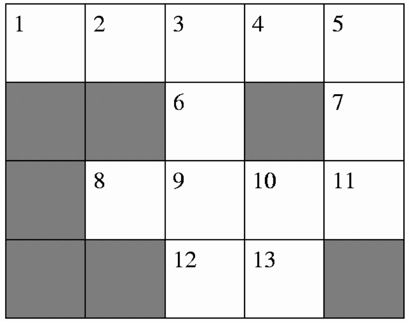
- Nodes represent the variables
- Hyperedges (drawn as regions/squares) are scopes of constraints



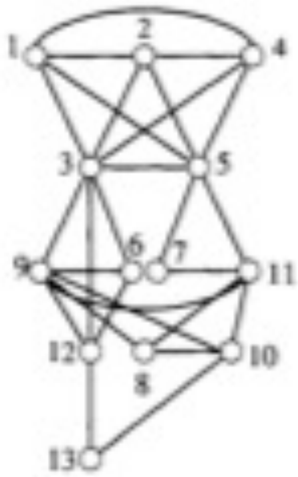
- Dual constraint graph is a related representation

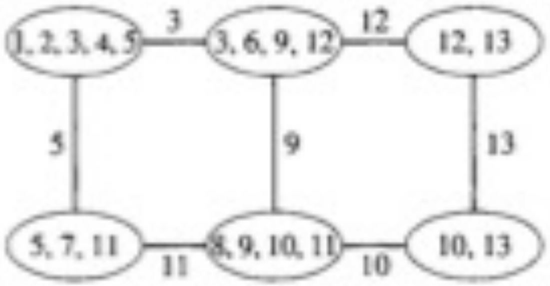
Primal vs dual graphs: crossword

- Nodes represent constraint scopes
- Arcs connect nodes with shared variables
  - Labels correspond to those variables



1	2	3	4	5
▨	▨	6	▨	7
▨	8	9	10	11
▨	▨	12	13	▨





Resembles something???

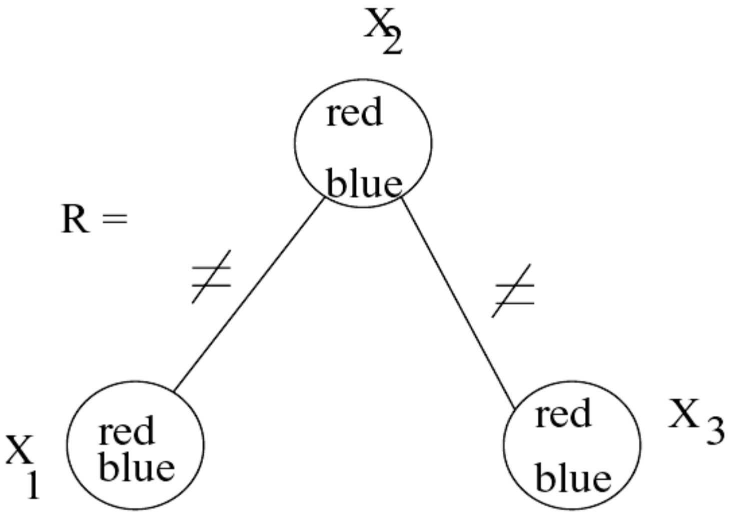
Equivalence and deduction

- New constraints can be inferred from an initial set of constraints
  - Deduction or constraint inference
- For example, from x → y and y → z we can infer x → z
- Adding inferred constraints yields an equivalent constraint network
  - Inferred constraints are redundant!

Example: 2-graph coloring

- Variables x_1, x_2, x_3
- Domain {red, blue}
- $R_{21} = R_{32}$ = {(blue,red),(red,blue)}
- Two solutions: (red,blue,red) and (blue,red,blue)

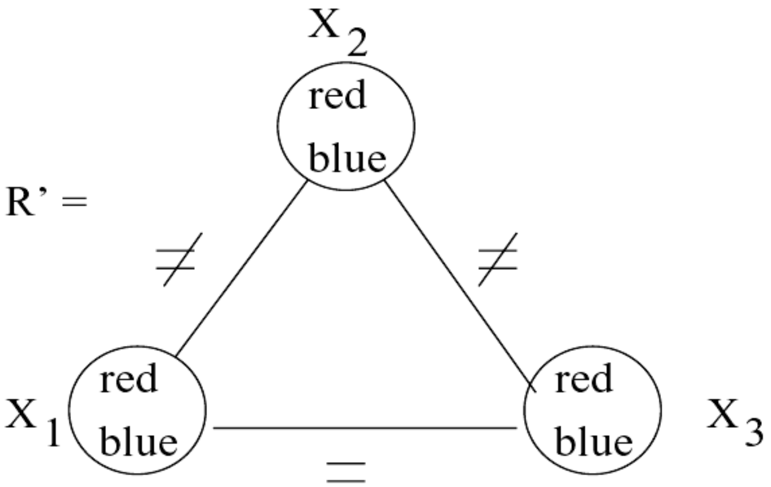
- Impact of disallowing the pair (x_1=red,x_3=blue)?



2-graph coloring: deduction

- Impact of adding the constraint R_{13}={(red,red),(blue,blue)}?
- The same two solutions: (red,blue,red) and (blue,red,blue)
- The new constraint could be inferred
- The two networks R and R' are equivalent!

- When is a constraint redundant?




That's all Folks!