VANIER COLLEGE
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# Cart, Pendulum and Spring System

By

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Differential Equations (203-HTK 01)

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## Abstract:

This paper studies the interaction between a cart, a spring, and a pendulum. The spring is attached to the cart and a wall, while the pendulum is attached to the bottom of the cart. The motion as well as the velocity of the pendulum and the cart have been modeled using Newtonian physics and differential equations. Those differential equations are then solved in a python program using the Runge-Kutta 45 numerical solver. The program also creates a graph of the motion of the cart vs the angle of the pendulum, a graph of the velocity of the cart vs the angular velocity of the pendulum, and an animated video of the system. In general, changing the mass of the cart will affect the kinetic energy of the system, while changing the mass of the pendulum will change the potential energy. Changing the mass of either part of the system also has an affect on the periodicity of the motion of both the cart and the pendulum. The spring affects the potential energy of the cart as well as the period of the cart. The length of the pendulum changes the potential energy of the pendulum as well as affecting the period of the pendulum. The friction between the cart and the pendulum causes the total energy and the amplitude of the pendulum to decrease over time, while the friction between the cart and the floor causes the total energy and the amplitude of the cart to decrease over time. Changing the initial position of either the cart or the pendulum will increase the potential energy, while changing the initial velocity will change the kinetic energy.

**Table of Contents:**

## Introduction:

The system is composed of three parts: the cart, the spring, and the pendulum. The spring is attached to a wall and the cart, while the pendulum is attached to the bottom of the cart. There are two separate oscillating systems, one being the spring and the mass, with the second being the pendulum. The oscillations from the cart affect the oscillations of the pendulum and vice-versa. This paper will only deal with two dimensions, as a three-dimensional system would be difficult to accurately model. The system is set off by either extending or compressing the spring, or by raising the pendulum above its equilibrium position. The system can also be set off by giving either the cart or the pendulum an initial velocity.

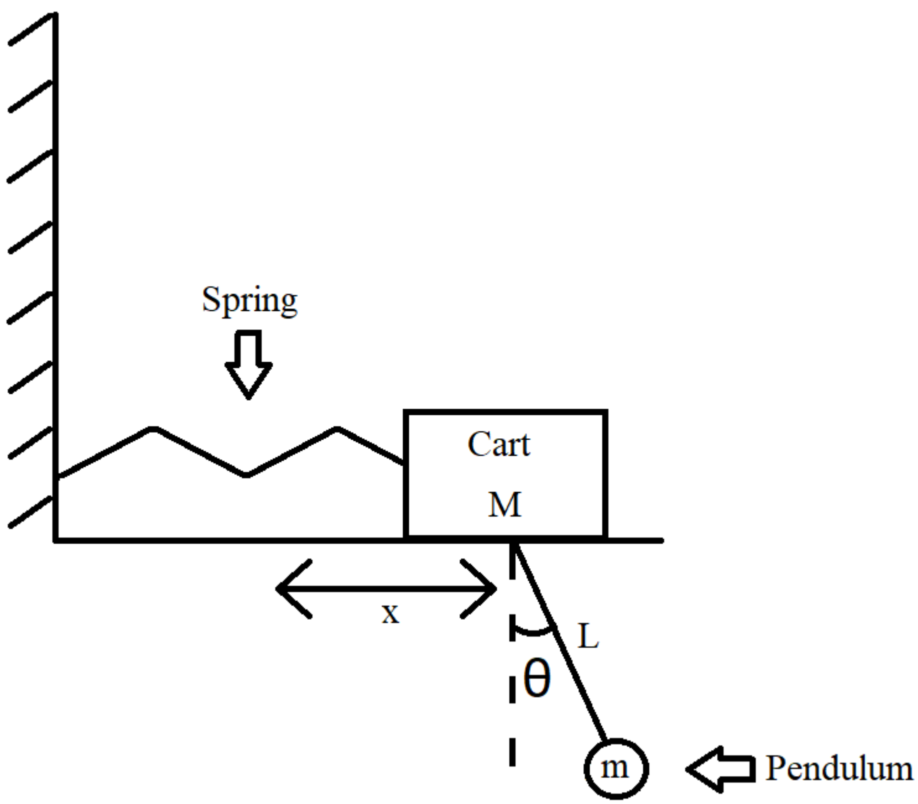


*Figure 1: A diagram showing the spring mass pendulum system.*

Figure one shows a diagram of the system, where:

M is the mass of the cart,

m is the mass of the bob of the pendulum,

x is the distance of the cart from the origin,

L is the length of the pendulum, and

$\theta$ is the angle of the pendulum compared to the neutral position.

As the system moves, the x position of the cart and the angle of the pendulum vary. When the system is at rest, the cart is at x=0 and the pendulum is at $\theta$=0.

## Analysing the System:

Instead of analysing the system as a whole, the system can be broken up into two different parts: the cart and the bob of the pendulum. Determining the forces acting on each part of the system will be crucial in modelling the system. A useful method to use when determining the forces on an object is a free body diagram.

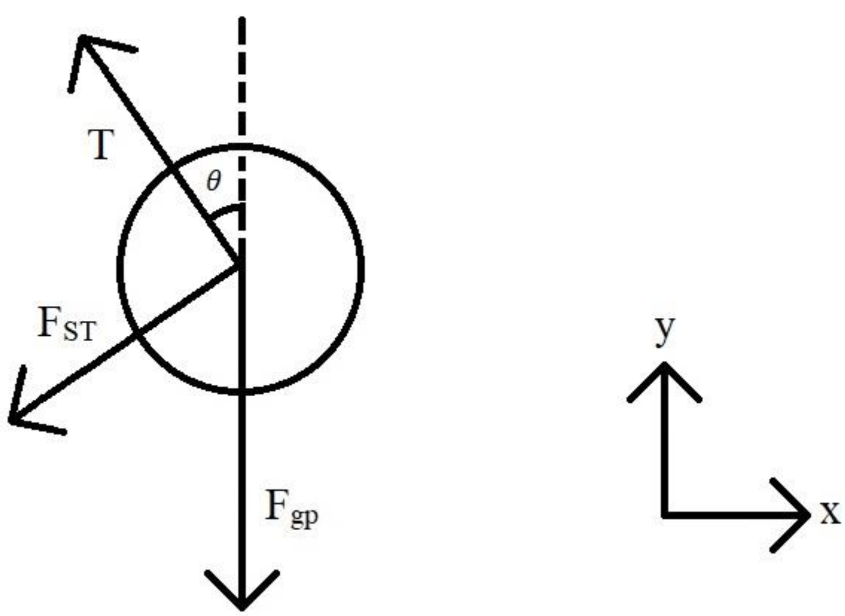


*Figure 2: Diagram showing the forces on the bob of the pendulum.*

Figure 2 shows the free body diagram for the bob of the pendulum, where:

T is the tension in the rod caused by the pendulum,

$F_{gp}$ is the force of gravity on the pendulum,

$F_{ST}$ is the force of friction between the pendulum and the cart.

The mass of the rod connecting the bob to the card is considered to be negligible, so the force of gravity on the bob is simply $F_{gp}$= mg. T, the tension, is a reactive force against the friction and the force of gravity. The tension also includes the force that the cart acts on the pendulum. The friction is due to the connection between the cart and the pendulum, and it opposes the motion of the pendulum. As the friction is a force and the rod has a length L, the friction produces torque which can be represented by the equation (Neumann, 2001):

$$\tau = \gamma\omega$$

Where $\omega$ is the angular velocity of the pendulum bob.

Since the torque produced by the friction is tangent to the direction of motion, the angle between the force of friction and the rod is 90°. In the above equation, the torque can be replaced by the equation for torque, $\tau = rFsin\theta$, where r is the distance of the force from the axis of rotation, F is the magnitude of the force, and $\theta$ is the angle between the force and the lever arm, giving the new equation (Neumann, 2001):

$$LF_{ST}\sin(90) = \gamma\omega$$

Which can be simplified to the following equation by noticing that $\omega=\theta'$, and that $sin(90)=1$ (Neumann, 2001):

$$F_{ST} = \frac{\gamma\theta'}{L}$$

Using the above free body diagram, newton's second law and the equations for friction, the following equations can be written that representing the forces on the pendulum (Neumann, 2001):

$$\sum F_x = ma_p = mx'' \quad = \frac{-\gamma\theta'}{L}cos\theta - Tsin\theta \tag{1}$$

$$\sum F_y = ma_p = mx'' \quad = \frac{-\gamma\theta'}{L}sin\theta - mg + Tcos\theta \tag{2}$$

These equations could be further simplified by deriving the equations for the position of the pendulum twice to find the equation of acceleration on the pendulum, as shown below (Neumann, 2001):

$$x_p = x + Lsin\theta$$

$$y_p = -Lcos\theta$$

Deriving once gives the equations for velocity along the x and y directions:

$$x'_p = x' + L\theta' cos\theta$$

$$y'_p = L\theta' sin\theta$$

Deriving a second time gives the equations for acceleration along the x and y directions:

$$x''_p = x'' + L\theta'' cos\theta - L\theta'^2 sin\theta$$

$$y''_p = L\theta''^2 sin\theta + L\theta' cos\theta$$

Finally, the equations for acceleration along the x and y directions can be placed into equations (1) and (2) respectively, giving (Neumann, 2001):

$$\sum F_x = m(x'' + L\theta'' cos\theta - L\theta'^2 sin\theta) = \frac{-\gamma\theta'}{L} cos\theta - Tsin\theta \tag{3}$$

$$\sum F_y = m(L\theta''^2 sin\theta + L\theta'^2 cos\theta) = \frac{-\gamma\theta'}{L} sin\theta - mg + Tcos\theta \tag{4}$$

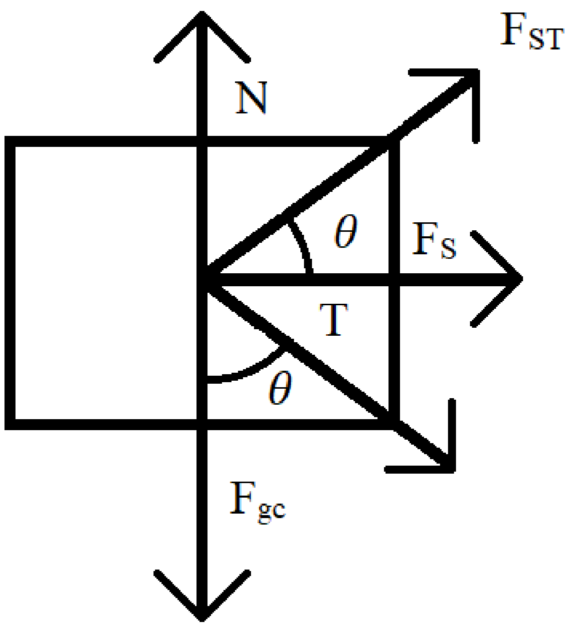


*Figure 3: Diagram showing the forces exerted on the cart.*

Figure 3 shows the free body diagram for the cart, where:

$F_S$ is the force that the spring exerts on the cart,

$F_{gc}$ is the force of gravity on the cart,

N is the normal force that the floor acts on the cart, and

$F_{SF}$ is the coefficient of friction between the floor and the cart.

The forces on the cart are the force of gravity, $F_{gc} = Mg$, the reactive normal force N, the tension in the rod caused by the pendulum T, the friction produced between the cart and the floor $F_{SF} = \mu x'$, the friction caused between the cart and the pendulum $F_{ST} = \frac{\gamma\theta\prime}{L}$, and the force from the spring on the cart $F_S = -kx$. These forces can be separated into two equations, one of the forces acting along the x-axis and another of the forces acting along the y-axis. It is important to note that since the cart experiences no vertical motion, the vertical acceleration is equal to 0. The two equations are (Neumann, 2001):

$$\sum F_x = -kx + Tsin\theta - (\mu x') + \frac{\gamma\theta'}{L}cos\theta = Ma_x = Mx'' \tag{5}$$

$$\sum F_y = N - Mg - Tcos\theta + \frac{\gamma\theta'}{L}sin\theta = Ma_y = 0 \tag{6}$$

While equations (3), (4), and (5) are useful as they are, since tension varies depending on the position of the pendulum, it would be beneficial to eliminate it from the equations. To eliminate T from all the equations, the first step is to add equations (3) and (5) together, which gives the following (Neumann, 2001):

$$x''(M + m) = mL\theta'^2 sin\theta - mL\theta'' cos\theta - kx - \mu x' \tag{7}$$

Next, multiplying equation (4) by sin$\theta$ and then replacing the resulting T sin $\theta$ by equation (3) will allow the trigonometric identity $\sin^2\theta + \cos^2\theta=1$ to be used. Upon completing the multiplication, equation (4) will turn into (Neumann, 2001):

$$mL\theta''^2 sin^2\theta + L\theta'^2 sin\theta cos\theta = \frac{-\gamma\theta'}{L} sin^2\theta - mgsin\theta + Tsin\theta cos\theta$$

Isolating T sin $\theta$ in equation (3) will result in the new equation:

$$Tsin\theta = -\frac{\gamma\theta'}{L}cos\theta - mx'' + mL\theta'' cos\theta - mL\theta'^2 sin\theta$$

Then replacing T sin $\theta$ in the previous equation and then simplifying with the trigonometric identity $\sin^2\theta + \cos^2\theta=1$ with what was just derived gives (Neumann, 2001):

$$mL\theta'' + \frac{\gamma\theta'}{L} + mgsin\theta + mx''cos\theta = 0 \tag{8}$$

Together, equation (7) and (8) describe the motion of the system without the force of tension. Observing the two equations shows that they are both nonlinear differential equations, as there are multiple sin$\theta$ and cos$\theta$ terms in both equations, and there is a $\theta^2$ term in equation (7). This makes the two differential equations difficult to solve analytically, so the Runge-Kutta method will be used to solve the equations numerically.

## Adapting the Equations for Runge-Kutta:

Runge-Kutta is a numerical method used to approximate solutions to ordinary differential equations. It involves computing weighted averages of function values at multiple points within a time step, which improves the accuracy of the approximation. The method works by inputting the equation of the slope of the system as well as the domain that the solver will be solving over. Therefore, to get equations (7) and (8) ready to be inputted into the Runge Kutta solver, they must be modified once again to isolate x'' and $\theta$''. The modifications by isolating $\theta$'' in equation (8):

$$\theta'' = \frac{\frac{-\gamma\theta'}{L} - mgsin\theta - mx''cos\theta}{mL} \tag{9}$$

Equation (9) could then be used to replace $\theta$'' in equation (7), which comes out to the following:

$$x''(M+m) = mL\theta'^2 sin\theta - mL\left(\frac{\frac{-\gamma\theta'}{L} - mgsin\theta - mx''cos\theta}{mL}\right)cos\theta - kx - \mu x'$$

Simplifying this gives the equation for the acceleration (x'') of the cart:

$$x'' = \frac{mL\theta'^2 sin\theta + \frac{\gamma\theta'}{L}cos\theta + mgsin\theta cos\theta - kx - \mu x'}{M + msin^2\theta} \tag{10}$$

Finding an equation for $\theta$'' that doesn't depend on x'' is more complicated process. One way to isolate it would be to multiply equation (7) by m cos$\theta$ and multiplying equation (8) by –(M+m). Adding the two new equations then cancels out x'', leaving the isolated equation for $\theta$''. Upon conduction the multiplications, the new equations are:

$$mx''(M+m)cos\theta = m^2L\theta'^2 sin\theta cos\theta - m^2L\theta''cos^2\theta - mcos\theta kx - mcos\theta\mu x'$$

$$-mL\theta''(M+m) - \frac{\gamma\theta'}{L}(M+m) - mgsin\theta(M+m) - mx''cos\theta(M+m) = 0$$

Adding these 2 equations together gives the following:

$$mx''(M+m)cos\theta - mL\theta''(M+m) - \frac{\gamma\theta'}{L}(M+m) - mgsin\theta(M+m) - mx''cos\theta(M+m) = m^2L\theta'^2 sin\theta cos\theta - m^2L\theta''cos^2\theta - mcos\theta kx - mcos\theta \mu x'$$

This can be simplified in the following way:

$$m^2L\theta''cos^2\theta - mL\theta''(M+m) = m^2L{\theta'}^2 sin\theta cos\theta - mcos\theta kx - mcos\theta\mu x' + \frac{\gamma\theta'}{L}(M+m) + mgsin\theta(M+m)$$

Then:

$$\theta''(m^2Lcos^2\theta - mL(M+m)) = m^2L{\theta'}^2 sin\theta cos\theta - mcos\theta kx - mcos\theta\mu x' + \frac{\gamma\theta'}{L}(M+m) + mgsin\theta(M+m)$$

Finally:

$$\theta'' = \frac{-mL{\theta'}^2 sin\theta cos\theta + kxcos\theta + \mu x' cos\theta - \frac{\gamma\theta'}{L}\left(\frac{M}{m}+1\right) - gsin\theta(M+m)}{L(M+msin^2\theta)} \quad (11)$$

Therefore, the two equations that represent the acceleration (x'') and angular acceleration ($\theta$'') are equations (10) and (11) (Neumann, 2001):

$$x'' = \frac{mL\theta'^2 sin\theta + \frac{\gamma\theta'}{L}cos\theta + mgsin\theta cos\theta - kx - \mu x'}{M + msin^2\theta} \quad (10)$$

$$\theta'' = \frac{-mL{\theta'}^2 sin\theta cos\theta + kxcos\theta + \mu x' cos\theta - \frac{\gamma\theta'}{L}\left(\frac{M}{m}+1\right) - gsin\theta(M+m)}{L(M+msin^2\theta)} \quad (11)$$

As Runge-Kutta only works with first order equations, equations (10) and (11) need to be modified to that they become first order. This can be done by changing the variables. The change of variables to bring equations (10) and (11) are:

$a_1 = x$

$a_2 = x'$

$a_3 = \theta$

$a_4 = \theta'$

Using these new variables, the equations (10) and (11) become:

$$a_1' = a_2$$

$$a_2{}' = \frac{mLa_4{}^2 sina_3 + \frac{\gamma a_4}{L} cosa_3 + mgsina_3 cosa_3 - ka_1 - \mu a_2}{M + msin^2 a_3}$$

$$a_3' = a_4$$

$$a_4{}' = \frac{-mLa_4{}^2 sina_3 cosa_3 + kxcosa_3 + \mu a_2 cosa_3 - \frac{\gamma a_4}{L}\left(\frac{M}{m} + 1\right) - gsina_3(M + m)}{L(M + msin^2 a_3)}$$

These four equations are what will be inputted into python to be solved with the Runge-Kutta 45 solver. Inputting values for the mass of the cart, the mass of the pendulum bob, the spring constant of the spring, the length of the rod of the pendulum, the coefficient of friction between the cart and the floor, the coefficient of friction between the cart and the pendulum, the initial x position of the cart, the initial velocity of the cart, the initial angle of the pendulum, and the initial angular velocity of the pendulum will allow graphs about the positions and velocities of the cart and the pendulum. The solved equations of the system will also be used to animate the system.

## **Examining Cases:**

Each case will begin with a proposition of values to be inputted into the simulation, followed by a prediction of how the system will react. Then, several graphs that depict how the system reacted will be shown, as well as an explanation for what occurred. It is important to note that the gravitational constant will always remain $g=9.8N/m^2$

Case 1:

Case 1 will set the following values:

$M=5kg$

$m=1kg$

$k=10N/m$

$L=0.5m$

$\mu=0.1$

$\gamma=0.01$

$x_0=0$

$v_0=0$

$\theta_0=\frac{\pi}{4}=45°$

$\omega_0=0$

In this example, only the pendulum starts with energy as it has potential energy from being raised by 45°. This means that the motion of the pendulum should drive the motion of the cart, while the pendulum should be experiencing damped harmonic motion. Upon placing the values into the Runge-Kutta solver, the following graphs were produced:

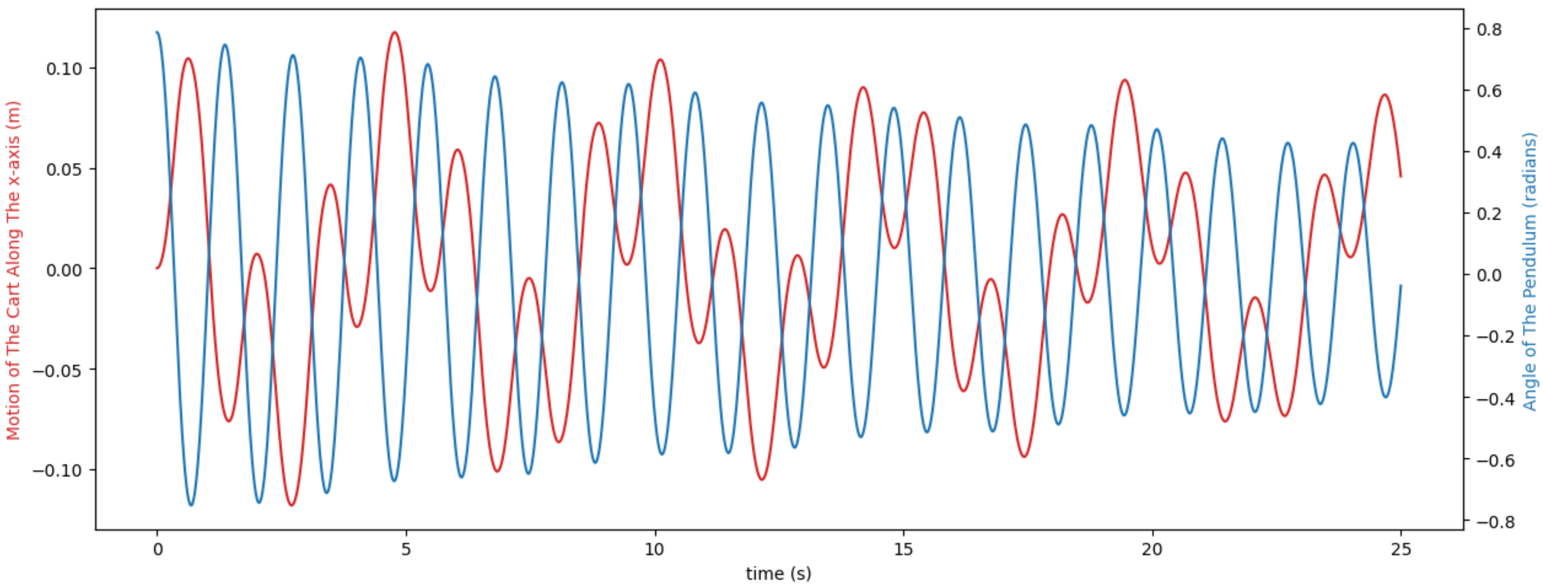


*Figure 4: A graph of Case 1 showing the motion of the cart along the x axis as well as the angle of the pendulum over 25 seconds.*

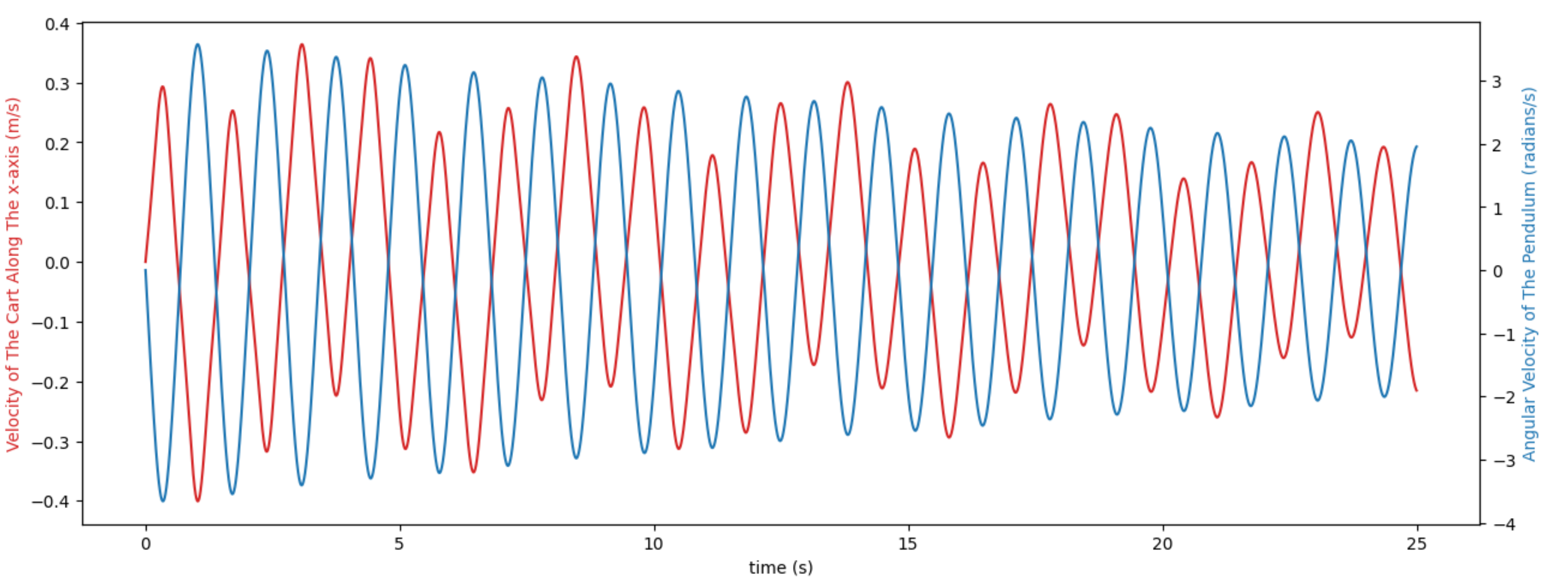


*Figure 5: A graph of Case 1 showing the velocity of the cart along the x axis as well as the angular velocity of the pendulum over 25 seconds.*

As expected, the pendulum seems to roughly exhibit damped harmonic motion. The affect of friction is clearly visible in the both graphs, as the maximum displacement and velocity of both the cart and the pendulum visibly decrease over time. This decay in the maximum angle of the pendulum is caused by the coefficient of friction between the pendulum and the cart. Over time, a small fraction of the total energy in the pendulum is lost to that friction. The decay in the cart is similarly due to the friction between the cart and the floor. The friction between the cart and the floor does not affect the motion of the pendulum as it does not act on the pendulum.

In this case, the motion of the cart is caused by pendulum, as well as the spring. As the force of gravity makes the pendulum swing from side to side, there must be a reactive force to counteract the side to side motion (Moebs, Ling, & Sanny, 2016). The resulting reactive force ends up acting on the cart through the tension in the rod. The acceleration of the cart and the angular acceleration of the pendulum are always opposite to each other, explaining why the cart and the pendulum tend to move in opposite directions of each other. As can be oberved in figure 4, the cart and the pendulum are in phase with each other. This can be observed on both graphs, where the peaks of each component seem to line up. This could be explained by the fact that only the pendulum started with initial energy, and since the cart and pendulum are directly connected they must move at the same time. While the motion of the cart is periodic like the motion of the pendulum, the cart seemes to jitter between absolute maximum and absolute minimum. This could be explained by how the cart interacts with the spring. As the cart moves, it loads and unloads the spring. Sometimes the force that the pendulums acts on the cart is in the same direction as the force that the spring acts on the cart, which can be seen just after t=0 seconds, where the cart quickly moves between 0.10 m and -0.075m. Both the spring and the pendulum are acting on the cart in the negative x direction, which allows the cart to reach its maximum velocity of as can be seen in figure 5. Conversely, sometimes the spring and pendulum work opposite to each other, as is the case around time= 2.5 seconds. The velocities around that time are relatively slower that the velocities at time=0 seconds, which suggest that the pendulum force and the spring force have opposite direction. Additionally, instead of going from maximum to maximum as was the case when the pendulum and the spring, the cart bounces between x= -0.075m and x=0m, showing that the displacement is much less. In this case, the spring is acting a force on the cart towards the origin, while the force that the pendulum acts on the cart is acting toward the negative x direction. Therefore, the jitter in the movement of the cart is caused by the variation in the direction of the forces that the spring and the pendulum act on the cart.

Case 2:

Case 2 will set the following values:

$M=5kg$

$m=1kg$

$k=10N/m$

$L=1m$

$\mu=1.0$

$\gamma=0.01$

$x_0=1.0m$

$v_0=2m/s$

$\theta_0=\frac{\pi}{3}=60°$

$\omega_0=-\frac{\pi}{2}$ rad/s

This case is more complex that the last as the system has potential energy in the form of the pendulum being at an angle, and the spring being compressed, while also having kinetic energy in the form of an initial velocity of the cart as well as the pendulum having an initial angular velocity. This would suggest that the system is likely to be less uniform as the last as there are higher chances of forces acting with against each other. For example, unlike in the last case, the cart has its own initial velocity, which means there may be more cases where the spring is acting with and against the pendulum. It is also important to note that the friction between the floor and the cart has increased, which suggests that the maximum displacement of the cart will have a steeper decrease over time. Running the values in the python program generates the following graphs:

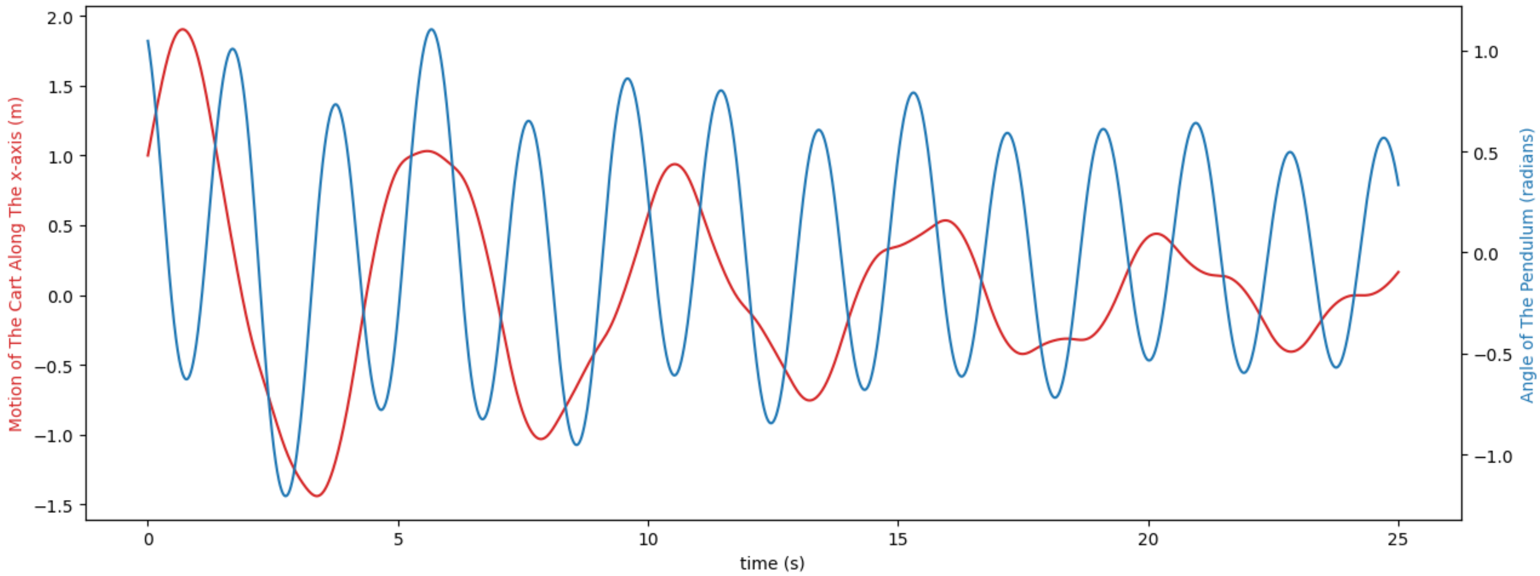


*Figure 6: A graph of Case 2 showing the motion of the cart along the x axis as well as the angle of the pendulum over 25 seconds.*

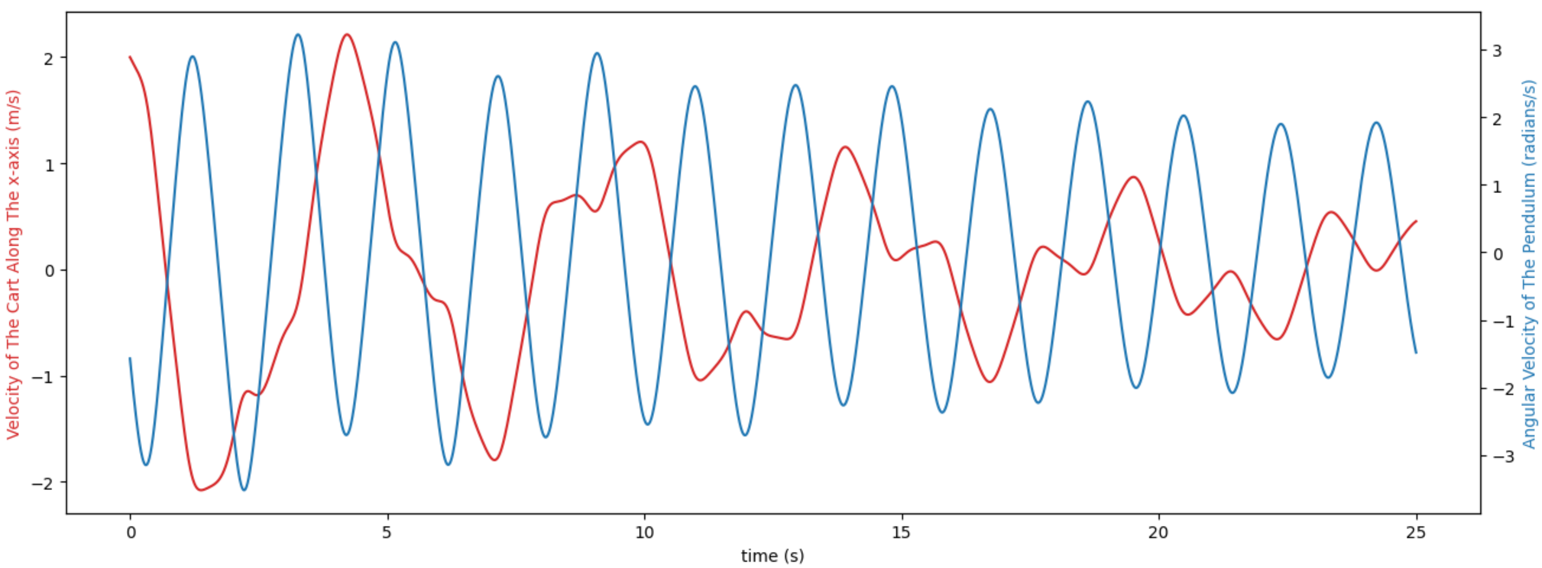


*Figure 7: A graph of Case 2 showing the velocity of the cart along the x axis as well as the angular velocity of the pendulum over 25 seconds.*

Observing the graphs shows that these values are indeed are less smooth and more unpredictable than the last case. The pendulum doesn't cleanly vary between its maximum and minimum as in the last time, the maximum seems to vary. The motion of the cart seems far more erratic than the last case. Instead of being periodic, the motion of the cart in this case starts off periodic but quickly devolves into decreasingly periodic motion. The opposite seems to be happening with the pendulum. As time increases, the pendulum seems to exhibit increasingly periodic motion. Regarding friction, the increased friction between the cart and the floor seems to have a greater affect the motion of the cart more as the maximum of the cart decreases over time. The friction between the cart and the floor also seems to affect the pendulum. As time goes on, the erratic motion of the pendulum seems to trend towards damped harmonic motion. This can be attributed to the fact that the total energy in the cart is getting dissipated due to the friction. Because the cart has less and less energy, it isn't able to affect the motion of the pendulum as much as it previously would have. This would mean that as time goes on, the cart has less affect on the pendulum, explaining why the pendulum eventually trends towards regular damped harmonic motion.

The increased erratic nature of the pendulum and the cart can be attributed to the initial conditions of the system. Unlike the last case, the cart does not start at rest, therefore it is not in phase with the pendulum. The phase difference means that the cart and the pendulum transfer energy to each other at different times, which affects the motion of both elements in the system. Observing figure 7, right before time =10, the maximum angle of the pendulum goes from roughly 0.5 radians to 1.25 radians. Due to friction, the maximum angle of the pendulum should not increase, so this increase has to come from the cart. Figure 8 shows that at this time the cart and the pendulum are moving in the same direction, so the increase in the angle of the pendulum can be attributed to the extra energy that the cart transferred to the pendulum. The sharp decrease in the velocity of the cart around this time further proves this point. The initial velocity as well as the initial position of the cart and the pendulum increases the total energy of the system. The initial velocities increase the kinetic energy of each part of the system, while the initial positions of the cart and the pendulum increases the potential energy.

For case 2, the length of the pendulum was increased to 1m from 0.5m. The affect on the system is that the longer pendulum has greater potential energy than the shorter pendulum. This is because the longer pendulum is effectively at a greater height. The height of the pendulum above the neutral position can be found using the formula (Moebs, Ling, & Sanny, 2016):

$$\Delta y = L - Lcos\theta$$

When the pendulums are both at 45°, the longer pendulum is 0.47m above the at rest position, while the shorter pendulum is at 0.24m above the neutral position. The longer pendulum is at twice the height of the shorter pendulum, meaning it has twice the kinetic energy according to the formula for potential energy (Moebs, Ling, & Sanny, 2016):

$$E_p = mg\Delta y$$

The higher potential energy means the pendulum exerts a larger force on the cart. Another effect the change in length has on the system is that the longer pendulum has a larger period compared to the pendulum with the shorter length. This is because the longer pendulum has a larger distance to cover due to the increased circumference caused by the larger radius. As the forces on the pendulum don't change, it takes longer for the pendulum to cover the increased distance. Comparing between case 1, where the rod is 0.5m long, and case 2, where the rod is 1m, a clear increase in the pendulums period can be observed. This increase in period is another explanation for the difference in period between the pendulum and the cart.

Case 3:

Case 3 will set the following values:

$M$=5kg

$m$=1kg

$k$=50N/m

$L$=1m

$\mu$=1.0

$\gamma$=0.01

$x_0$=-0.5m

$v_0$=2m/s

$\theta_0 = \frac{\pi}{3} = 60°$

$\omega_0 = \frac{\pi}{2}$ rad/s

Case 3 has is similar  to case 2, with the only changes the spring constant was changed from 10N/m to 50N/m, the initial x position of the cart was changed from 0.5m to -0.5m, and the initial angular velocity of the pendulum was changed to $\frac{\pi}{2}$ rad/s from - $\frac{\pi}{2}$ rad/s . The larger spring constant will increase the initial potential energy of the cart, in turn increasing the total energy of the cart. This increase will likely cause the motion of the cart to have a greater effect on the motion of the pendulum, while also causing the motion of the pendulum to have less of an affect on the cart. The new values produce the following results:

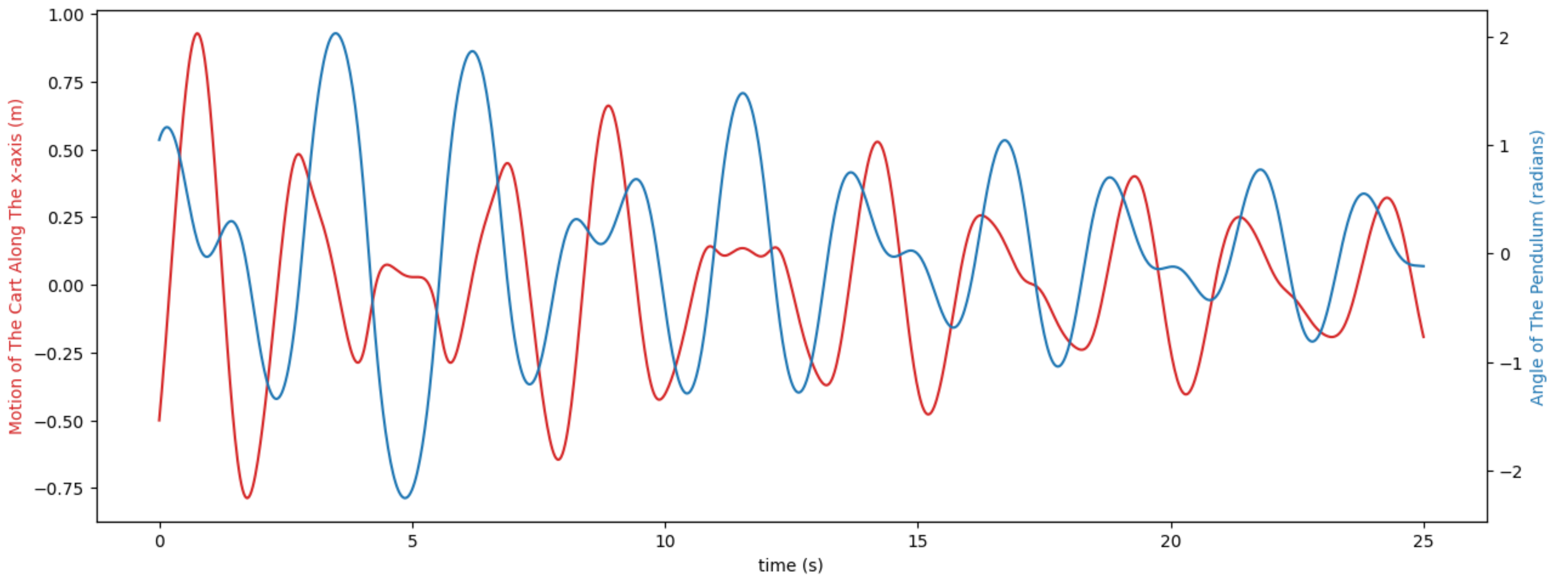


*Figure 8: A graph of Case 3 showing the motion of the cart along the x axis as well as the angle of the pendulum over 25 seconds.*

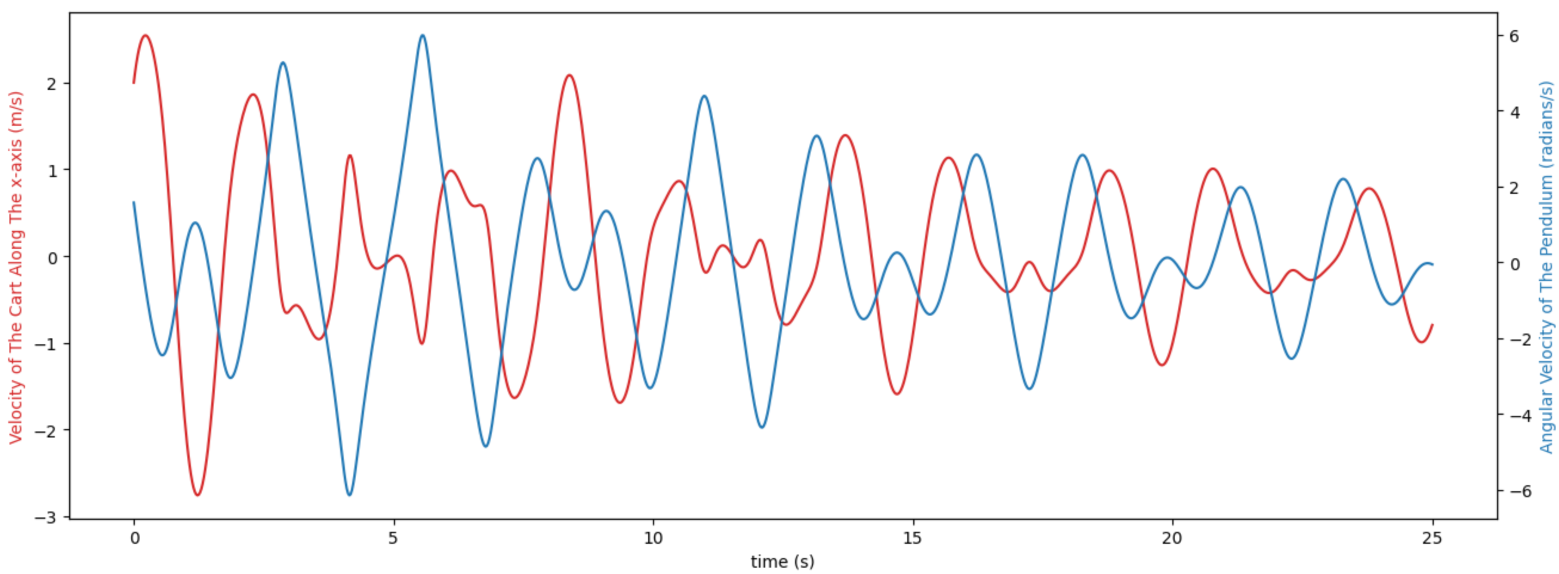


*Figure 9: A graph of Case 3 showing the velocity of the cart along the x axis as well as the angular velocity of the pendulum over 25 seconds.*

Analysing the two graphs shows that the cart does indeed experience an increase in total energy due to the larger spring constant. Interestingly however, the maximum displacement of the cart has decreased despite the increase in total energy. Observing the graph for the position of the cart shoes that compared to case 2, the periodicity of the cart seems remain roughly constant rather than decaying over time. Another observation that can be made in that the motion of the pendulum has become more chaotic as opposed to case 2. In case 2, the pendulum exhibited mostly periodic motion, however in case 3 the variation in maximum angle seems to vary more and the motion of the pendulum can sometimes look "bumpy", as is the case at time= 1 second, time= 9 seconds, time= 15 seconds, and time = 20 seconds.

The decrease in the maximum displacement compared to case 2 can be explained by the increase in the spring constant. The increased spring constant also increases the opposition the spring has to the motion of the cart, so since the initial velocity of the cart is the same as in case 2, the overall displacement of the cart decreases. The larger spring constant gives more potential energy to the cart, while having no effect on the kinetic energy of the cart, which in total gives more potential energy to the cart. The increase in the potential energy of the cart can be explained by observing the formula of potential energy due to a spring (Moebs, Ling, & Sanny, 2016):

$$P_{Es} = \frac{1}{2}kx^2$$

Where $P_{Es}$ is the elastic potential energy due spring. The spring constant is directly proportional to the spring constant, so any increase in the spring constant directly increases the potential energy of the cart, and vice-versa.

Since the cart has far more energy than case 2, it is able to transfer more energy to the pendulum. This increased energy causes a large increase in the maximum angle of the pendulum. In case 2, the pendulum had a maximum angle of just over 1 radian, while in case 3 the pendulum opens up to around 2 radians. Around the same time as that maximum angle, the pendulum also experiences its maximum angular velocity of 6 rad/s. This high velocity causes the pendulum to quickly go from -2 radians to 2 radians. At the same time as this massive shift in angle of the pendulum, the cart barely moves from the origin position. This suggests that the total energy of the cart was transferred to the pendulum, explaining why the pendulum had so much velocity. The increase in the energy transferred from the cart to the pendulum also causes the motion of the pendulum to appear "bumpier" compared to case 2. The increased energy being transferred to the pendulum is enough to overcome the inertia of the pendulum, causing the erratic motion.

Case 4:

Case 4 will set the following values:

$$M=50kg$$

$$m=1kg$$

$$k=50N/m$$

$$L=1m$$

$$\mu=5.0$$

$$\gamma=0.05$$

$$x_0= -0.5m$$

$$v_0=2m/s$$

$$\theta_0= \frac{\pi}{3} = 60°$$

$$\omega_0= \frac{\pi}{2} \text{ rad/s}$$

Case 4 will examine the effect that the mass of the pendulum has on the system. The changes are that the mass of the cart has changed to 50kg up from 5kg, the coefficient of friction between the cart and the floor has been increased to 5, and the coefficient of friction between the cart and the pendulum has increased to 0.05. These changes will increase the kinetic energy of the cart. This change in potential energy will likely have an effect on both the motion of the cart as well as the motion of the pendulum. The change in the friction is intended to counteract the increase in energy. The updated numbers for case 4 produce the following graphs:

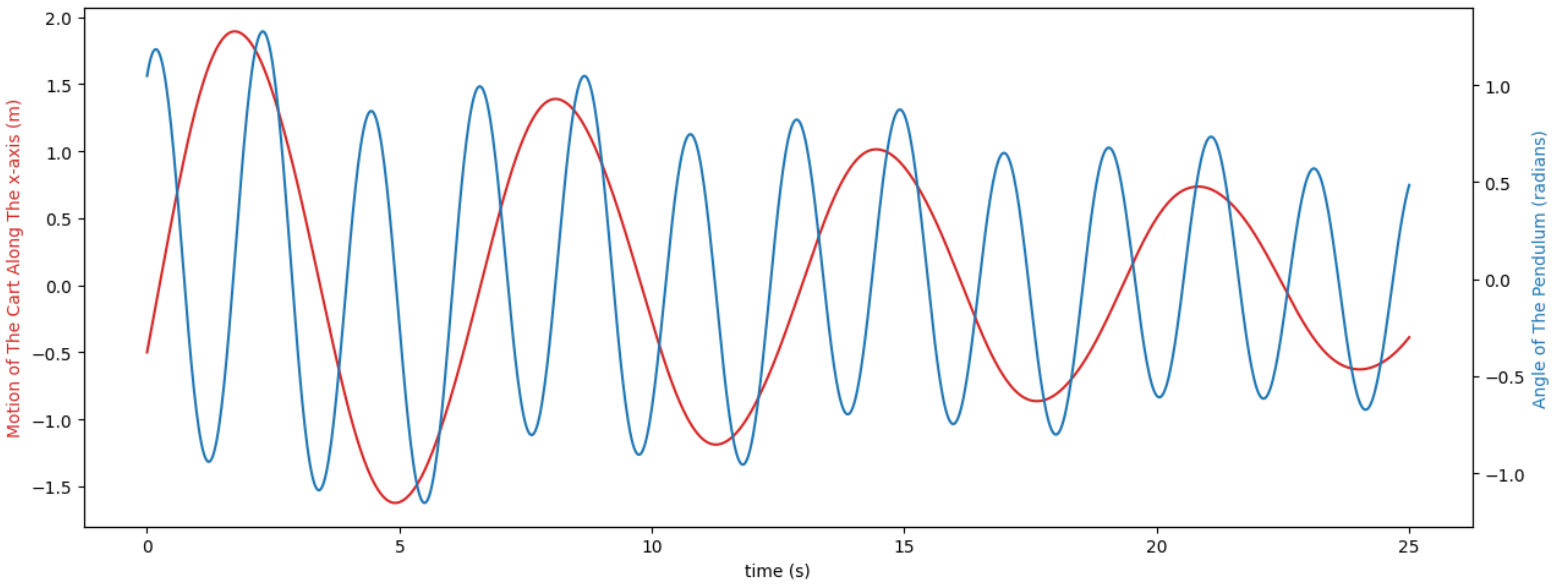


*Figure 10: A graph of Case 4 showing the motion of the cart along the x axis as well as the angle of the pendulum over 25 seconds.*

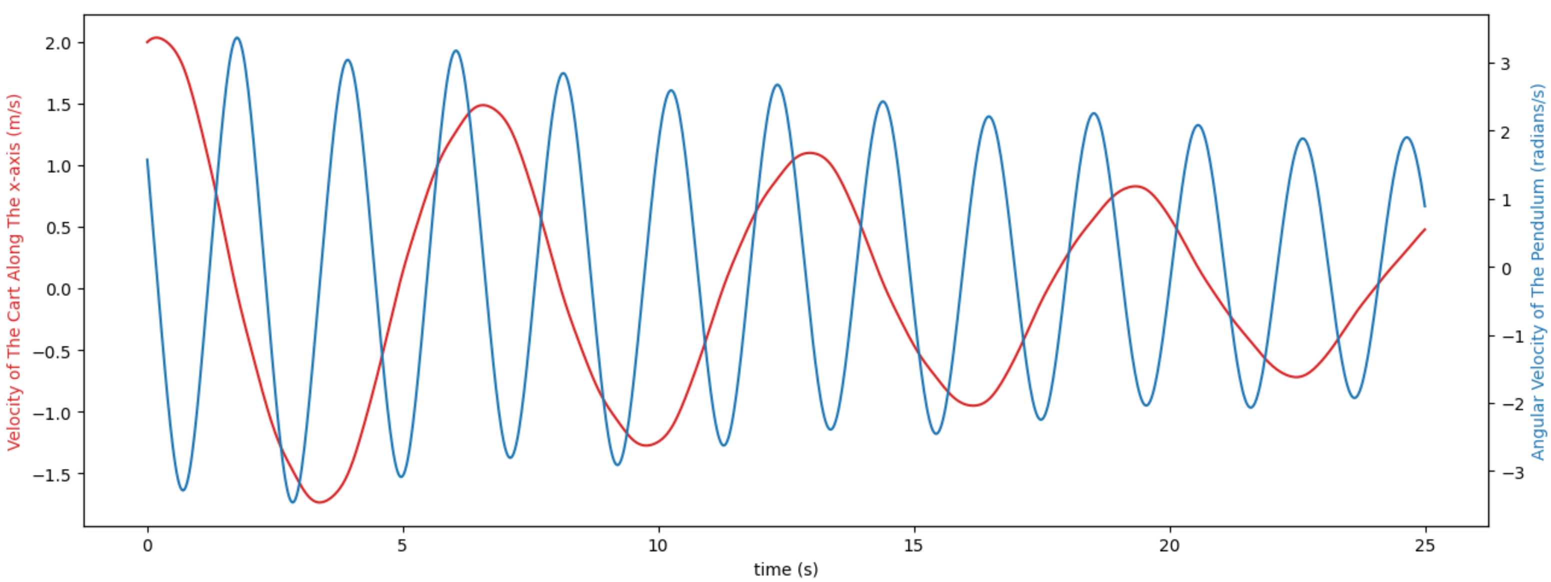


*Figure 11: A graph of Case 4 showing the velocity of the cart along the x axis as well as the angular velocity of the pendulum over 25 seconds.*

An immediate observation that can be made is that the motion of the both the pendulum and the cart seem more periodic and less erratic compared to case 2 and case 3. Another observation that can be made is that the period of the cart is far larger compared to case 2 and 3, while the period of the pendulums seems to have remained relatively constant. The changed friction seems to have had an affect as both the pendulum and the cart seem to have a larger decrease in their velocities over time.

As expected, the increase in the mass of the cart has caused an increase in the kinetic energy of the cart. This increase in energy can be seen in the increase in the displacement that the cart has compared to case 3. In case 4, the cart extends to a maximum of around 2 meters while in case 3 the cart on extends to around 1 meter. The increase in kinetic energy can be attributed to the additional mass, as mass is directly proportional to kinetic energy as shown by the formula for kinetic energy (Moebs, Ling, & Sanny, 2016):

$$P_K = \frac{1}{2}Mv^2$$

Where $P_K$ is the kinetic energy of the cart, M is the mass of the cart and v is the velocity of the cart.

Despite the cart and the pendulum not being in phase, the motion of both the cart and the pendulum still seems periodic and not erratic as was the case in case 2 and case 3. For the cart, this could be explained by the fact that the mass of the cart is so high that the pendulum doesn't have enough inertia to significantly affect the motion of the cart. The graphs of the velocity of the cart shows that the pendulum does have some effect cart, as it is not perfectly smooth, suggesting the pendulum is affecting the cart, but in a limited capacity. The smooth motion of the pendulum can be attributed to the fact that the acceleration of the cart smaller than case 2 and case 3. In case 4, as the cart has a higher mass, it has a higher inertia, therefore it has a higher resistance to changes in direction from the spring and the pendulum. This lower acceleration from then pendulum means that less energy is transferred to the pendulum, allowing the pendulum to exhibit periodic motion. There is still energy being transferred to the pendulum, which can be seen in the increase in the amplitude of the pendulum at $t = 2.5$ seconds and $t = 15$ seconds. Since the pendulum and the cart seem to have less of an effect on each other compared to other cases, their motion can be considered to be almost independent of each other.

The increase in the period of the cart can be attributed to the increased inertia of the cart. As the spring constant has not changed compared to case 3, the force that the spring exerts on the cart does not change. This means that the force has less of an affect on the increased inertia of the cart, explaining why the acceleration of the cart is lower than in the three other cases. This lower acceleration is the cause for the larger period.

## Conclusion:

The cart pendulum is a complex system that involves interactions between a car, a spring, and a pendulum. Changing certain parameters can have different effects on the system. The mass of the cart determines the inertia of the cart, which affects the period of the motion of the cart. The mass of the cart also affects how much initial kinetic energy the cart. As for the pendulum, the mass affects the period of the motion of the pendulum as well as the potential energy of the pendulum. The higher energy one part of the system has, the more it affects the motion of the other. Case 1 and case 4 demonstrate the affects of the mass of the pendulum and the cart on the system respectively.

The spring constant affects the period of motion of the cart and is directly proportional to the motion of the cart. The length of the pendulum has a similar effect on the pendulum as the spring constant has for the cart. The length affects the period of the pendulum, as well as the potential energy of the pendulum. The affects of changing spring constant can be observed in case 3, while the effects changing the length of the pendulum can be seen in case 3. Overall, similar to changing the mass, changing he period of motion and energy state of either the pendulum or the cart ends up affecting the system as a whole.

The final parameters that were discussed were the initial x position of the cart, the initial angle of the pendulum, the initial velocity of the cart and the initial angular velocity of the pendulum. The initial position and angle of the cart and the pendulum affects the potential energy of both parts of the system respectively. On the other hand, the initial velocity and angular velocity affect the kinetic energy of the cart and the pendulum respectively. Case 2 demonstrates the effects pf changing the initial positions and velocities can affect the system as a whole.

## References

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