

协方差 – 线性性质

- 资产 1 的回报是 X , 资产 2 的回报是 Y
- 组合的回报是 $Z = w_1X + w_2Y$
- 资产 1 和组合的协方差
 - $COV(X, Z) = COV(X, w_1X + w_2Y) = w_1COV(X, X) + w_2COV(X, Y)$
 - $COV(X, Z) = w_1\sigma_1^2 + w_2COV(X, Y) = w_1\sigma_1^2 + w_2\rho_{12}\sigma_1\sigma_2$

协方差 – 矩阵法

- 协方差矩阵
 - $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$
- 权值向量
 - $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$
- 新向量 $\vec{c} = \Sigma \times \vec{w}$ 的每个元素是每个资产和组合的协方差
 - $\vec{c} = \Sigma \times \vec{w} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1\sigma_1^2 + w_2\rho_{12}\sigma_1\sigma_2 \\ w_2\sigma_2^2 + w_1\rho_{12}\sigma_1\sigma_2 \end{bmatrix}$
- 矩阵法适合多个资产, 很灵活

协方差 – 组合方差偏导法

- 组合方差
 - $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1w_2\sigma_1\sigma_2$
- 方差对权值 w_1 的偏导
 - $\frac{\partial \sigma_p^2}{\partial w_1} = w_1\sigma_1^2 + \rho_{12}w_2\sigma_1\sigma_2 = cov_{1p}$
- $cov_{1p} = w_1\sigma_1^2 + \rho_{12}w_2\sigma_1\sigma_2 = w_1\sigma_1^2 + w_2cov_{12}$

组合标准差

- 定义
 - $\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho_{12}w_1w_2\sigma_1\sigma_2}$
- 矩阵法
 - $\sigma_p^2 = \vec{w}^T \times \Sigma \times \vec{w}$
- 标准差对权值 w_1 的偏导
 - $\frac{\partial \sigma_p}{\partial w_1} = \frac{1}{2\sigma_p} \frac{\partial \sigma_p^2}{\partial w_1} = \frac{w_1\sigma_1^2 + \rho_{12}w_2\sigma_1\sigma_2}{\sigma_p} = \frac{cov_{1p}}{\sigma_p}$

Marginal VaR 各个元素相同

- 协方差法
 - $MVaR_1 = MVaR_2 \rightarrow Z \times \frac{cov_{1p}}{\sigma_p} = Z \times \frac{cov_{2p}}{\sigma_p}$
 - $\rightarrow Cov_{1p} = Cov_{2p}$
 - $\rightarrow w_1\sigma_1^2 + w_2\rho_{12}\sigma_1\sigma_2 = w_2\sigma_2^2 + w_1\rho_{12}\sigma_1\sigma_2$

- $\rightarrow \frac{w_1}{w_2} = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}$
- $\rightarrow \frac{w_1}{w_2} = \frac{\sigma_2^2}{\sigma_1^2}$ 如果 2 个资产是独立的($\rho_{12} = 0$)
- $C \times \vec{w}$ 的每个元素相同, 可以写成常数向量 $C \times \vec{w} = k \times \vec{1}$

组合 Portfolio VaR

- 定义
 - $VaR_p = P \times Z \times \sigma_p$
- 矩阵法
 - $VaR_p^2 = P \times Z \times (\vec{w}^T \times \Sigma \times \vec{w})$
- Two-asset
 - $VaR_p = \sqrt{VaR_1^2 + VaR_2^2 + 2 \times VaR_1 \times VaR_2 \times \rho}$
- 假定独立 uncorrelated VaR
 - $VaR_p = \sqrt{VaR_1^2 + VaR_2^2}$
- 假定线性相关 undiversified VaR
 - $VaR_p = VaR_1 + VaR_2$
- Multiple-asset (equally weighted and each has the same std and correlation)
 - $VaR_p = \sigma \sqrt{\frac{1}{n} + (1 - \frac{1}{n}) \times \rho}$
- Diversified VaR
 - $DVaR_1 = VaR_1 + VaR_2 - VaR_p$

Marginal VaR 的不同计算方法

- Incremental VaR 和 Component VaR 需要利用 Marginal VaR。
- 组合 Portfolio VaR
 - $VaR_p = P \times Z \times \sigma_p$
- 个体 Individual VaR
 - $VaR_1 = P_1 \times Z \times \sigma_1$
- 定义
 - $MVaR_1 = \frac{\partial VaR_p}{\partial P_1}$
- Use Covariance
 - $MVaR_1 = \frac{\partial VaR_p}{\partial P_1} = Z \times \frac{\partial \sigma_p}{\partial w_1} = Z \times \frac{cov_{1p}}{\sigma_p}$
- Use Beta
 - $\beta_1 = \frac{cov_{1p}}{\sigma_p^2} \rightarrow MVaR_1 = Z \times \frac{cov_{1p}}{\sigma_p} = Z \times \sigma_p \times \beta_1$
- Use Portfolio VaR
 - $VaR_p = P \times Z \times \sigma_p \rightarrow MVaR_1 = Z \times \sigma_p \times \beta_1 = \frac{VaR_p}{P} \times \beta_1$
- Use Correlation
 - $\beta = \frac{cov_{1p}}{\sigma_p^2} = \rho_1 \frac{\sigma_1}{\sigma_p} \rightarrow MVaR_1 = Z \times \sigma_p \times \beta_1 = Z \times \sigma_1 \times \rho_1$
- Use Individual VaR

$$\circ \text{VaR}_1 = P_1 \times Z \times \sigma_1 \rightarrow \text{MVaR}_1 = Z \times \sigma_1 \times \rho_1 = \frac{\text{VaR}_1}{P_1} \times \rho_1$$

Applications

- Component VaR
 - $\text{CVaR}_1 = P_1 \times \text{MVaR}_1 = \text{VaR}_p \times w_1 \times \beta_1 = \text{VaR}_1 \times \rho_1$
 - $\sum_i w_i \times \beta_i = 1$
- Percentage contribution 百分比贡献
 - $\text{contribution}\% = \frac{\text{CVaR}_1}{\text{VaR}_p} = w_1 \times \beta_1$
- Incremental VaR
 - Full evaluation (time consuming, VaR complicated)
 - Incremental Steps
 - Map to factors and estimate the positions of them
 - Prepare the marginal MVaR vector
 - Cross product
 - $\text{IVaR}_1 = \text{Increment} \times \text{MVaR}_1 = \text{Incr} \times Z \times \frac{\text{cov}_{1p}}{\sigma_p}$
 - Both can be dollar valued
- Risk Management (Global Minimization)
 - Minimize $\text{VaR}_p \rightarrow \sigma_p$
 - Condition
 - $\text{MVaR}_i = \text{MVaR}_j$
 - **Allocate more to positions with lower MVaR**
 - Covariance equal
 - $\text{cov}_{1p} = \text{cov}_{2p}$
 - $w_1 \sigma_1^2 + w_2 \rho_{12} \sigma_1 \sigma_2 = w_2 \sigma_2^2 + w_1 \rho_{12} \sigma_1 \sigma_2$
 - $\rightarrow \frac{w_1}{w_2} = \frac{\sigma_2^2 - \rho_{12} \sigma_1 \sigma_2}{\sigma_1^2 - \rho_{12} \sigma_1 \sigma_2}$
 - Beta equal
 - $\beta_i = \beta_j \rightarrow \rho_1 \times \sigma_1 = \rho_2 \times \sigma_2$
- Portfolio Management
 - Maximize $\frac{\text{Return-risk-free}}{\text{VaR}_p}$
 - Condition
 - $\frac{R_i - R_f}{\text{MVaR}_i} = \frac{R_j - R_f}{\text{MVaR}_j}$
 - If distribution are elliptic (beta)
 - $\frac{R_i - R_f}{\beta_i} = \frac{R_j - R_f}{\beta_j}$
 - Allocate **more** weight to higher ratio (excess return to MVaR)

Question

26. Consider a portfolio consisting of USD 10 million of Intel shares and USD 5 million of GE shares. The returns on the two stocks have a bivariate normal distribution with a correlation of 0.3. The daily return volatility of Intel and GE is 2% and 1%, respectively. The standard deviation of daily changes in the value of the Intel position is USD 0.2 million, and the standard deviation of daily changes in the value of the GE position is USD 0.05 million. The daily VaR at the 99% confidence level is USD 0.5131 million. What is the incremental daily VaR of the portfolio for a small increase in the position on Intel shares over a one-day horizon at 99% confidence level?

- a. USD 0.0455 million
- b. USD 2.275 million
- c. USD 0.0453 million
- d. USD 0.0195 million

```
>>> var=0.5131
>>> s1=0.02
>>> s2=0.01
>>> p1=10
>>> p2=5
>>> r=0.3
>>> z=2.33
>>> pcov = p1*s1*s1 + p2*s1*s2*r
>>> mvar = z*z/var * pcov
>>> mvar
0.04549653089066459
>>>
```

$$\sigma_p = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_1 \sigma_2 \rho} \Rightarrow \frac{\partial \sigma_p}{\partial w_1} = \frac{w_1 \sigma_1^2 + \rho w_2 \sigma_1 \sigma_2}{\sigma_p}$$

$$\text{marginal VaR} = \frac{\partial \text{VaR}}{\partial p_1} = \frac{\partial (P \cdot z \cdot \sigma_p)}{\partial (P \cdot w_1)} = z \frac{\partial \sigma_p}{\partial w_1} = z \cdot \frac{\text{COV}}{\sigma_p} \quad \text{where } \text{COV} = w_1 \sigma_1^2 + \rho w_2 \sigma_1 \sigma_2$$

$$\text{Portfolio VaR} = P \cdot z \cdot \sigma_p$$

$$\text{mVaR} = \left[\frac{z^2}{\text{VaR}} \text{COV} \cdot P \right], \quad \text{where } P \cdot \text{COV} = p_1 \sigma_1^2 + p_2 \rho \sigma_1 \sigma_2$$

QUESTIONS 61 AND 62 REFER TO THE FOLLOWING INFORMATION

A financial risk consultant assumes that the joint distribution of returns is multivariate normal and calculates the following risk measures for a two-asset portfolio managed by a mid-sized insurance company:

Asset	Position (CNY)	Individual VaR (CNY)	Marginal VaR	VaR Contribution (CNY)
Financial	15,000,000	3,494,700	0.216	3,240,000
Energy	15,000,000	6,999,300	0.462	6,931,238
Portfolio	30,000,000	9,241,650		9,241,650

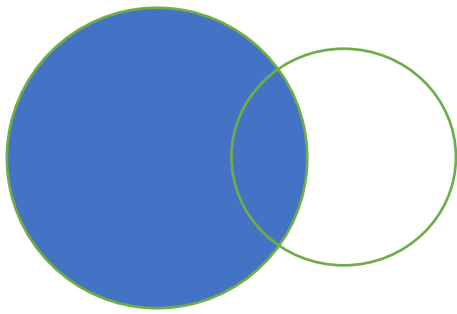
61. If the energy asset is dropped from the portfolio, what will be the reduction in portfolio VaR?

- A. CNY 2,242,350
- B. CNY 3,494,700
- C. CNY 5,746,950
- D. CNY 6,999,300

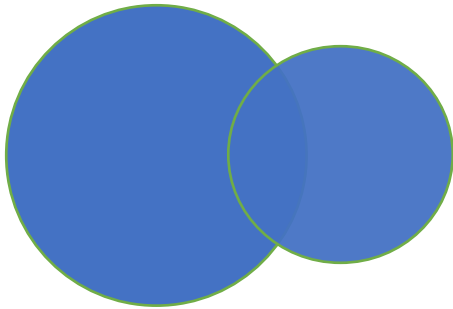
$$0.462 \times 15,000,000$$

~~D~~ C

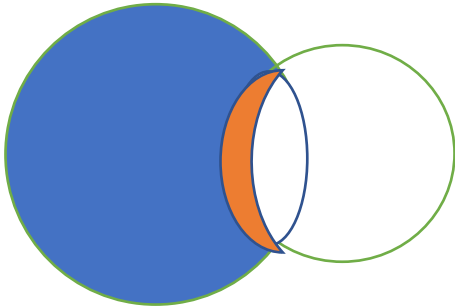
- Reduction when the second asset is removed, the reduction in VaR is $VaR_p - VaR_1$



阴影部分是 **Individual VaR₁**
单独集合 A

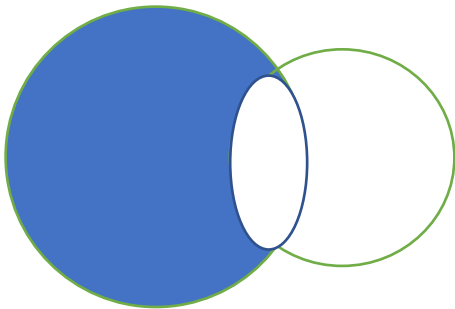


阴影部分是 **Portfolio VaR_p**
集合的并集 $A \cup B$

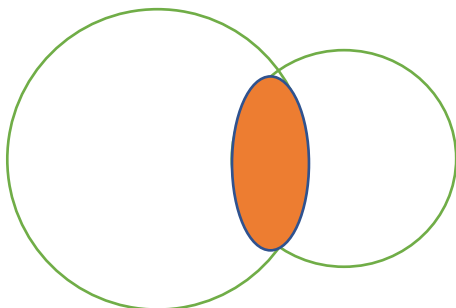


阴影部分是 **Component CVaR₁**
右边白色的是 **Component CVaR₂**

也可以理解成 **Marginal VaR**, 因为
 $Component VaR = P \times Marginal VaR$



阴影部分是去掉 1 后的 **Reduction**
 $VaR_p - VaR_2$
右边白色是余下的 VaR
集合的左对称差 $A \cup B - B$



阴影部分是投资组合的风险分散：**Diversified VaR**
 $Undiversified VaR - VaR_p = VaR_1 + VaR_2 - VaR_p$
集合的交集 $A \cap B$