Term-Structure Models (constant, time, interest)

- Time-dependent: exponential
- Interest-dependent: mean-reversion, squared

Name	Formula $\Delta w \sim N(0, \Delta t)$	Drift	Volatility	Interest	Note
Model 1	$\Delta r = \sigma \Delta w$	No	Constant		
Model 2	$\Delta r = \lambda \Delta t + \sigma \Delta w$	Constant	Constant		
Ho-Lee	$\Delta r = \lambda_t \Delta t + \sigma \Delta w$	Time	Constant		
Vasicek	$\Delta r = k(\theta - r) \Delta t + \sigma \Delta w$	Interest: mean-reversion	Constant		
General	$\Delta r = \lambda_t \Delta t + \sigma_t \Delta w$	Time	Time		**General version of Ho-Lee**
Model 3	$\Delta r = \lambda_t \Delta t + e^{-\alpha t} \sigma \Delta w$	Time	Time: exponential		Specific version
CIR	$\Delta r = k(\theta - r) \Delta t + \sqrt{r} \sigma \Delta w$	Interest: mean-reversion	Interest: squared		Improved version of **Basicek**
Model 4 Lognormal	$\Delta \ln r = \Delta r / r = \lambda \Delta t + \sigma \Delta w$	Constant	Constant	Log	Log version of **Model 2**
Lognormal Deterministic	$\Delta \ln r = \lambda_t \Delta t + \sigma \Delta w$	Time	Constant	Log	Log version of **Ho-Lee**
Lognormal Black-Karasinski	$\Delta \ln r = k_t(\ln \theta_t - \ln r) \Delta t + \sigma_t \Delta w$	Interest: Mean-reversion	Time	Log	Log version of **Vasicek**

- 升级：先调整 drift，再调整 volatility，再调整 interest 到 Log 空间
- Drift: model 1 (no drift) -> model 2 (constant) -> Ho-Lee (time dependent) -> Vasicek (interest-dependent)
- 记 lognomal 模型时，可以认为分别是 model2，Ho-lee，Vasicek 的 log 形式

- **Cox-Ingersoll-Ross (CIR)**
  - Basis-point volatility increase with short-term rate sqrt(t)
  - dr = k (theta - r) dt + sigma * sqrt(r) * dw
- **Lognormal - Deterministic Shift Model 4**
  - dln(r) = lambda(t) * dt + sigma * dw
    - dr = lambda* r *dt + sigma * r * dw
    - lnr_0 + lamba dt + sigma dw
    - r_0 => r_0 exp(lambda dt) exp(sigma dw), multiplicative
- **Lognormal with Mean Reversion ()**
  - d[ln(r)] = k(t) [ln(theta(t)) – ln(r)] dt + sigma(t) dw
  - not recombing: the time intervals between interest rate changes are recalibrated to force the nodes to recombine.

The Art of Term Structure Models: Drift

- **Short-Term Interest Rate Tree Construction**
- **Parameters**
  - dt small interval in year, 1 month=1/12 year
  - s: volatility
  - Normal distribution dw ~ N(0, sqrt(dt))
- **Model 1 – No Drift**
  - **dr = s* dw (expected rate change)**
  - Change to rate => parallel shift, a flat term structure of volatility
  - Limitations
    - Not flexible, only one factor, volatility is flat, parallel shift
- **Negative Interest Rate**
  - Problem is greater when interest rate is low or the time get longer
  - Solutions
    - Lognormal or chi-squared distribution
      - But introduce Skewness or inappropriate volatilities
    - Set to zero (preferred)
  - Bond less affected, but option depends on asymmetric payoff affected more
- **Model 2 – Constant Drift**
  - **dr = lambda * dt + s * dw**
  - Positive drift -> positive risk premium
  - Limitations
    - value of drift is high.
- **Ho-Lee Model – Time-dependent Drift**
  - **dr = lambda_t * dt + s * dw**
- **Arbitrage-Free and Equilibrium Models**
  - Arbitrage models
    - Used to quote the prices of securities that are illiquid or customized.
    - **Constructed** using on-the-run Treasury securities, **predict** off-the-run securities
    - Pricing **derivative** based on observable prices of underlying **securities**
    - Assumption: Prices are **accurate,** subject to suitability of model.
  - Equilibrium
    - Used for relative analysis
- **Vasicek Model – Mean-Reversion**
  - dr = k (theta - r) dt + s * dw
    - theta: **long-run equilibrium rate** value of short-term rate assuming risk neutrality
  - lambda = k (theta – r_l) = annual drift
    - **theta** = r_l + lambda / k, where r_l is the long-run **true rate** of interest
  - **non-recombine**
    - r_ud: take the average of the up-then-down and down-then-up rates
    - modify up p and down probability (1-p)
    - modify up-and-up r_uu and down-and-down probability r_dd
    - Equations for computing p and r_uu
      - Mean: p*r_uu + (1-p)*r_ud = r_0 + k(theta-r)dt
      - Variance: p*(r_uu - m)^2 + (1-p) (r_ud - mean)^2 = s * sqrt(dt)
  - **Exponentially Decay**
    - Difference decay exponentially exp(-k*t)

        - Interest Rate at time t
            - r_t = r_0 * w + theta* (1-w), where w = exp(-kt)
            - r_t = r_0 * exp(-kt) + theta * (1 – exp(-kt))
            - theta – r_t = (theta - r_0) exp(-kt)
        - half life
            - exp(kt) = 2 => t = ln2 /k
    - **Effectiveness**
        - It produces a term structure of volatility that is **declining**. The short-term volatility is **overstated** and long-term volatility is **understated**
        - **Nonparallel** shift: Upward shift in short term rate, short-term rate will be impacted more than long-term rate
        - Natural shock: larger (smaller) k, quicker (slower) the news is incorporated; smaller -> news is long-lived

The Art of Term Structure Models: Volatility and Distribution

- **Time-dependent volatility**
    - dr = lambda(t) dt + sigma(t) dw
- **Model 3**
    - dr = lambda(t) dt + sigma * exp(-alpha * t) dw
    - volatility decrease exponentially to 0 when alphan > 0
    - Effectiveness
        - Caps and floors
- **Model 3 and Vasicek**
    - **Same STD**: The same initial volatility and decay rate (alpha) =mean-reverting rate (k), the **standard deviations** of the terminal distributions are the same.
    - Same **Distribution**: If time-dependent drift = average interest rate path in Vasicek Model, terminal distributions are **identical**.
    - **Difference**
        - Model 3 parallel shift, Vasicek nonparallel shift
    - **Application**
        - Price options on fixed income instruments, model 3
        - Value or hedge fixed income or options, Vasicek (mean reverting)
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- **Lognormal with Mean Reversion (Black- Karasinski)**
    - d[ln(r)] = k(t) [ln(theta(t)) – ln(r)] dt + sigma(t) dw
    - not recombing: the time intervals between interest rate changes are recalibrated to force the nodes to recombine.