Quantile-critical value (single tail)

- **Single side**
  - **1% 5% 10%**
  - **2.326 1.645 1.28**
- **Two-side**
  - **1% 5% 10%**
  - **2.58 1.96 1.65**

VaR Define

- **Return**
  - Arithmetic: r = (P2-P1+D)/P1, no reinvestment, not suitable for long time
  - Geometric: R = ln ((P2+D)/P1), continuously reinvest
- **Parametric**
  - Normal (arithmetic return)
    - $r = \frac{P_2 - P_1}{P_1} \sim N(\mu, \sigma)$
    - $P_2 - P_1 = P_1 \times (u - z \times \sigma) < 0$
    - $\rightarrow \text{VaR} = \text{P1} - \text{P2} = \mathbf{P_1} \times (\boldsymbol{z} \times \boldsymbol{\sigma} - \boldsymbol{\mu}) > 0$
  - Lognormal (geometric return)
    - $R = \ln P_2 - \ln P_1 = \ln \frac{P_2}{P_1} \sim N(\mu, \sigma)$
    - $P_2 - P_1 = P_1 \times e^R - P_1 = P_1 \times (e^R - 1) = P_1 \times (e^{u - z \times \sigma} - 1) < 0$
    - $\rightarrow \text{VaR} = \text{P1} - \text{P2} = \boldsymbol{P_1} \times (\mathbf{1} - \boldsymbol{e^{u - z \times \sigma}}) > 0$

Non-parametric
  - Historical simulation and Variants (realized return)
- **Historical Simulation (Raw data)**
  - Given a confidence level c, order returns in ascending order, find the separating point: **alpha * n + 1**, whose cumulative probability > alpha=1 - confidence level 大于不是等于
  - Assumes that losses in the future will occur with the same frequency and magnitude as they have in the past.
  - Cannot adjust for changing economic conditions or abrupt shift
  - **Ghost effect**: a point remains after n periods
  - assume i.i.d
- **HS Variants**
  - HS + **Bootstrap** (Sample)
    - Sampling with Replacement
    - Apply to VaR and ES
    - More prices ES than raw data
  - HS + Smooth (**Density Estimation**)
    - Discrete -> continuous
    - Connect the middle point using line or connect curves
- **HS Weighting**
  - HS + Age-Weighted / Hybrid Approach
    - Weight recent data more and distant less, decay
    - W_i = lambda ^(i-1)*W_1
    - W_1 = 1- lambda / (1 – lambda ^n)
    - Decay inverse to lambda
    - Lambda large, decay slow
  - HS + Volatility-Weighted

        - r_t, s_t and r, s => r_t' = r_t/s_t*s (adjust using recent volatility)
        - s can be forecasted by GARCH and EWMA
        - near-term VaR are **sensible** to current market condition, sensitive to **changing** market conditions
        - VaR can be **higher** than estimates with historical data/can predict loss **outside** historical range
    - HS + Correlation-Weighted
        - Covariance matrix (historical -> updated/revised to new information)
    - HS + Filtered (Conditional Volatility + Bootstrap)
        - Most comprehensive, most complicated
        - Conditional Volatility model (GARCH or asymmetric GARCH)
        - Suitable: **longer** holding period, **multi**-asset portfolios, **large** portfolios
        - Flexible: can capture **conditional** volatility, volatility **clustering**, surprise factor that have an **asymmetric** effect on volatility.
- **Advantage**
    - Intuitive, simple
    - No underlying **assumption**
    - Not hindered by **skewness, fat-tails**
    - Avoids complex variance-covariance matrix and dimension problems
    - Data is readily available and does **not** require adjustment (financial statements adjustments)
    - Can do complex analysis
- **Disadvantages**
    - Rely on **historical** data
    - **insufficient** data
    - volatile periods -> high, quite period -> low
    - difficult to detect **structural shifts/regime** changes
    - difficult to estimate loss **larger** than historical (volatility-weighted, to some degree)
- **Non-parametric – MC (hypothesis return)**
- Stressed VaR
- Liquid VaR
- OpVaR
- Portfolio VaR

Log

- $\mathbf{Log_a^b = 1 / log_b^a}$

Other Risk Measures

- **Expected Shortfall (ES)**
    - E(loss | loss > VaR), Expected loss > threshold 大于
    - n increase -> ES increase
    - \sum P_i / (1-C) * Loss_i, where
        - P_i is the global probability 不是累计概率，是区间概率
        - C is the confidence level, 1-c=alpha
        - P_i / (1-C) is the local probability
    - Example
        - 2% - $30, 3% - $20, 5% - $10 => c=10%

        - ES = 2/10 * 30 + 3/10 * 20 + 5/10*10 = 17
- **Coherent Measure**
    - Generalize ES to all 0-100 quantile.
    - Given n, divide the tail into n-1 regions, then average the quantile weighted by a function.
- **QQ-plot**
    - Identify an empirical distribution to a theoretical distribution
    - Empirical vs theoretical

Backtesting VaR

- **Difficulties**
    - VaR is on static portfolio, but actual portfolio is dynamic
        - Use a relatively short time horizon
    - Test both actual and hypothetical returns
- **Failure rate**
    - **VaR Confidence** level c, one side, failure rate p = 1-c , VaR 是单边的
    - Binomial distribution B (p*T, sqrt(p*(1-p)*T))
    - Z-score= (x – u) / s = (x – n*T) / sqrt(p*(1-p)*T)
    - **Test** Confidence level, **two** side, 95% -> 1.96，双边检验
    - Confidence level small is better
        - Too large, a small number of samples, a long time to wait until exceptions happen，可以接受范围减少。
    - Sample size
        - Large, non-reject area is too small,
        - 样本多，拒绝区间增加，更容易拒绝
- **Type I and Type II errors**
    - Type I (bad luck): reject a model when it is true
    - Type II (faulty model): accept a model when it is false
    - Confidence level decrease, Type I and Type II increase
- **Unconditional Coverage (log-likelihood ratio)**
    - Kupiec test, LR = 2 log P(N/T) / P(p), where $P(x) = C_N^T * x^N (1-x)^{(T-N)}$
    - Concern with the total number of exceptions, not the independence or timing
    - LR > (1.96)^2=3.84 => **reject** the model
    - Sample size increase => **easily** to reject the model
    - Confidence level increase => **difficult** to backtest since the numbers of small
    - T large, chi-squared distribution with df=1
- **Conditional Coverage (serially independence)**
    - LR_cc = LR_uc + LR_ind, chi-squared
    - Assume exceptions are **equally** distributed and **serial** independence
    - LR_cc>5.99 => reject the model at 95%
    - LR_ind > 3.84 => reject the independence alone
    - When exceptions are **clustered**, should use this
- **Breach Cluster**
    - Solution: decrease horizon and decrease confidence
- **Basel Committee Rules**
    - More concerned about **type II** problem
    - 250 days, 99% confidence level,
    - Zone (#exceptions and multiplier)
        - Green: 0-4 => 3

        - Yellow: 5-> 3.4, 6-> 3.5, 7-> 3.65, 8->3.75, 9-> 3.85
        - Red: >=10 => 4
    - Category
        - Model lack basic integrity => should apply
        - Model accuracy needs improvement => should apply
        - Intraday trading activity => considered
        - Bad luck => no guidance is provided
- **Market Risk Charge**
    - MRC = max(VaR(99%,10-day), k* VaR(60 day average))

VaR Mapping

- **Advantages**
    - Aggregate risk exposure
    - Simplify risk exposure into primitive **risk factors**
    - Can Measure **changes** over time
    - When no historical data
- **Map (头寸->因子)**
    - map each position (market value) -> risk factors
    - Sum each risk factor exposures from all positions -> risk factor exposure distribution
- **General and specific risk factors**
    -
- **Mapping Fixed-income securities**
    - Principle mapping (average maturity)
        - M=\sum w_i * T_i, where w_i=P_i/P is principle%, T_i is the maturity
    - Duration mapping (average duration = delta_P / detal_r)
        - Decompose into cash flow
        - Sum cash flow for each time
        - Discount into PV
        - D =\sum PV * t / sum PV
    - Cash flow mapping
        - Decompose into cash flow, sum cash flow for each time,
        - Consider **inter-maturity** correlations
        - Undiversified VaR = \sum_t PV_t * VaR_t 直接求和
        - Diversified VaR = y^T C y, where y_t = PV_t * VaR_t
- **Stress Testing**
    - PV, VaR -> New PV = PV – PV * VaR% = PV(1-VaR%)
- **Benchmark**
    - Match duration
    - Tracking error: std of the return difference
    - Minimize abs VaR != minimize tracking error (cash flow match)
    - The tracking error VaR is smallest when it matches based on **cash flow**.
- **Mapping Linear Derivative**
    - Delta-normal linear
    - Foreign Currency Forward
        - Long forward contract = Long foreign currency spot + Long foreign currency bill + Short U.S. dollar bill
    - Forward rate agreement (FRA)
        - Long 6 x 12 FRA = Long 6-month bill + Short 12-month bill

        - Sell 6*12 FRA on P: Borrow P for a 6-month, invest in 12-month
    - Interest Rate Swap
        - Interest-rate swaps can be viewed in two different ways: as (1) a combined position in a fixed -rate bond and in a floating-rate bond or (2) a portfolio of forward contracts.
- **Mapping Nonlinear Derivative**
    - Delta-normal can be applied to short periods of time
    - Option
        - Long option = Long Δ*asset + Short (Δ*asset - c) bill
- **Delta**
    - Deep in the money: 1
    - At the money: 0.5
    - Deep out of money: 0
    - Forward: 1

Messages from the Academic Literature on Risk Measurement for the Trading Book

- **VaR Implementation**
    - Time-varying volatility
        - Underestimate risk, also need to consider time-varying correlations
        - Tend to use short time horizons, 10-day VaR
        - Longer period should be used for economic capital.
    - Backtesting not effective
        - The number of exceptions is small
        - Longer time horizons due to portfolio instability
- **Liquidity VaR**
    - Exogenous liquidity 外生性
        - market specific, average transaction costs, ask-bid spread
        - LVaR = VaR + V * spread/2
    - Endogenous liquidity 内生性
        - **price effect** of liquidating positions
        - most applicable to **exotic/complex** trading positions and high-stress market conditions
        - trade size, elasticity of price to trading volume
        - E = dp% / N%
- **Risk Measures**
    - VaR
        - Not consider the **severity** of loss in the tail of the return distribution
        - Not subadditive
    - ES
        - Complex and computationally intensive
        - Solve the two issues of VaR
        - mitigate the impact a specific confidence level choice
    - Spectral Risk
        - Consider investment manager's **risk** aversion
        - Better **smoothing** properties when weighting observations
        - Modify to reflect investor's specific risk aversion
        - Rarely used in practice
- **Stress Testing**

- Historical scenarios
  - Previous market data
- Predefined scenarios
  - Predetermined risk factors
- Mechanical-search stress tests
  - Automated routines to cover possible changes in risk factors

- **Stressed VaR**
  - VaR in Financial stressed period
- **Integrated VaR**
  - Compartmentalized
    - For each individual type, and summed
    - Basel capital requirements
  - Unified
    - Consider correlations of risk types
- **Risk Aggregation**
  - Top-down
    - Cleanly divided into market, credit, and operational risk measures
  - Bottom-up
    - Interaction (diversification or risk compounding)
- **Balance Sheet Management**
  - Actively managed -> Leverage become procyclical
  - **Cyclical Feedback**
    - Leverage **inversely** related to market value
    - Purchase assets when prices are rising
    - Sell assets when prices are declining
  - VaR / Economic Capital requirements
    - Amplify boom and bust cycles
    - Current regulation to limit risk-taking actual **increase** risk

Some Correlation Basics: Properties, Motivation, Terminology

- **Correlation Risk**
  - An increase in correlation is typical in a severe systemic crisis
  - **Structured** products are becoming an increasing area of concern regarding correlation risk.
- **CDS**
  - The **spread** is based on the default probability of the **reference asset** and the joint default correlation of CDS **seller**.
  - If there is **positive** correlation risk between Bank and France bonds, the investor has wrong-way risk (**WWR**).
  - The **higher** the correlation risk, the **lower** the CDS spread.
- **Quanto Option**
  - allows a **domestic** invest or to exchange his potential option payoff in a foreign currency back into his home currency at a fixed exchange rate.
  - US investor buy Japan **Nikkei** index and currency **USD/JPY**.
  - correlation between foreign index Nikkei and foreign currency USD/JPY.
    - The more **positive** the correlation coefficient, the **lower** the price for the quanto option. Favorable for the seller.
    - the **lower** the correlation, the more expensive the quanto option.
- **Correlation swap**
  - Buyer Pay fixed correlation, receive actual average pair-wise correlation

        - r_realized = sum_i>j p_ij / (n^2-n)/2
    - Buy index call option + sell individual call option
        - buy call options on an index such as the Dow Jones Industrial Average (the Dow) and **sell** call options on **individual** stocks of the Dow. Benefit from increasing correlation.
        - There is a positive relationship between correlation and volatility.
- **Variance Swap**
    - A further way to buy correlation is to pay fixed in a variance swap on an **index** and to receive fixed in variance swaps on **individual** components of the index.
- **Correlation Crisis**
    - The first correlation-related crisis occurred in May 2005.
    - Hedge funds had put on a strategy where they were **short** the equity tranche of CDO to receive **high** premium and **long** the mezzanine tranche of CDO to pay **low** premium.
    - When the correlations of the assets in the CDO **decreased**, the hedge funds lost on both position.
    - The equity tranche **premium increased** sharply. Hence the **fixed** premium that the hedge funds received in the original transaction was now significantly lower than the current market spread, resulting in a loss.
    - In addition, the hedge funds lost on their long mezzanine tranche positions. since a lower correlation lowers the mezzanine tranche **premium**. Hence the spread that the hedge funds paid in the original transactions was now **higher** than the market spread, resulting in a loss.
- 2007-2009 Crisis
    - From 2007 to 2009, **default correlations** of the mortgages in the CDOs **increased**.
    - If default correlations **increase**, the equity (mezzanine) tranche premium **decreases** (increases), leading to an **increase** (decrease) in the value of the equity (mezzanine) tranche.
    - Premium increase -> price decrease
- **Correlation Risk and Market Risk**
    - Market risk: interest rate, currency, equity, commodity
    - Covariance is an integral part of market risk VaR
- **Correlation Risk and Credit Risk**
    - default correlation **within** sectors is **higher** than between sectors.
    - For most investment grade bonds, the term structure of default probabilities **increases** in time.
    - For bonds in distress, however, the default term structure of default probabilities **decreases** in time.
- **Correlation Risk and Systemic Risk**
    - Systemic risk and correlation risk are highly dependent.
- **Correlation Risk and Concentration Risk**
    - Concentration risk is the risk of financial loss due to a concentrated exposure to a group of counterparties.
    - **Concentration ratio = 1/number of counterparties**
    - a lower concentration ratio and a lower correlation coefficient reduce the worst-case scenario for a creditor, the joint probability of default of his debtors.

Empirical Properties of Correlation: How Do Correlations Behave in the Real World?

- **Equity Correlation Study**
  - The correlation levels are **lowest** in strong economic **growth** times. The reason may be that in strong growth periods equity prices react primarily to **idiosyncratic**, not macroeconomic factors.
  - In **recessions**, correlation levels typically **increase**. **Macroeconomic** factors seem to dominate idiosyncratic factors, leading to a downturn of multiple stocks.
  - A **positive** relationship between **correlation level** and **correlation volatility**.
  - Correlation are **high** for **recessions**, and correlation **volatility** is highest for **normal** periods. Volatility: recession > expansion
  - 相关水平：经济好时低（百花齐放，互不相关），经济不好时（一切凋零，都很相关）
  - 相关波动性：经济正常时最高，不确定性高，可以变好，也可以变差。经济不好时
- **Mean Reversion** of Equity **Correlation**
  - Relationship: $D(S_t - S_{t-1})/DS_{t-1} < 0$
  - Formula
    - $S_t - S_{t-1} = a\ (u - S_{t-1})\ dt + sigma*e*sqrt(dt)$
    - Simplified: $S_t - S_{t-1} = a\ (u - S_{t-1})$
  - Regression
    - Y = alpha + beta X
      - $Y = S_t - S_{t-1}$, $X = S_{t-1}$
      - alpha = au, beta=-a
- **Autocorrelation**
  - **One-period auto correlation + mean reversion = 1**
  - Autocorrelation is the "reverse property" to mean reversion
  - Autocorrelation: ARCH and GARCH
  - Autocorrelation of correlation

    $$AC(\rho_t, \rho_{t-i}) = \frac{Cov(\rho_t, \rho_{t-i})}{\sigma(\rho_t)\sigma(\rho_{t-i})}$$

    ρ_t: Correlation values for time period t
    - ρ_{t-i}: Correlation values for time period t - i
  - It decays with longer time period lags
- **Best-fit distribution**
  - Equity - JSB
    - **Johnson SB** distribution (two shape, one location, one scale)
    - Poor: Normal, lognormal, beta
    - Mean reversion is high
  - Bond - GEV
    - **Generalized extreme value**
    - **Normal** is also good
  - Default probability - JSB
    - **Johnson SB**

Statistical Correlation Models: Can We Apply Them To Finance?

- **The Pearson correlation Limitations**
  - **Linear** relationship
  - Zero does not mean **independence**

- Correlation is not defined unless variances are **finite**.
- Correlation is a good measure of dependence when the measured variables are distributed as multivariate **elliptical**.
- Not meaningful for **transformed**.

- **The Spearman rank correlation - nonprametric**
  - Steps
    - Compute rank for X
    - Compute rank for Y
    - Compute rank difference squared: (Rx - Ry)^2
    - Metric
      - Sum (Rx - Ry)^2 / T
      - T = n(n^2-1)/6
- **The Kendall's τ - nonprametric**
  - Steps
    - Compute rank for X
    - Compute rank for Y
    - Classify each point 对每个点的分类
      - Positive: Rx < Ry 正的
      - Negative: Rx > Ry 负的
      - Zero: Rx = Ry 零（最后不会考虑的）
    - Sum by category 按照类别统计次数
      - p: number of positive points 正类的个数
      - n: number of negative points 负类的个数
      - z: number of zeros points 零类的个数
      - $N = p + n + z$
    - 逻辑
      - Positive 类别里互相是 concordant
      - Negative 类别里互相是 Concordant
      - Positive 和 Negative 之间的是 Discordant
    - Pairs 计算对数
      - Concordant: $C_n^2 + C_p^2$ 一致的对数
      - Discordant: $n \times p$ 不一致的对数
      - Total: C_N^2 总对数
    - Metric
      - $\frac{C_n^2 + C_p^2 - n \times p}{C_N^2}$

Pair Combination

	Positive	Negative	Zero
Positive	C_p^2		
Negative	$n \times p$	C_n^2	
Zero	$z \times p$	$z \times n$	C_z^2

组合只看下三角：红色是 concordant pairs，蓝色是 discordant pairs，黑色是被忽略的

需要证明下三角之后是所有的对数

$C_p^2 + C_n^2 + n \times p + z \times p + z \times n + C_z^2 = C_N^2$

$$=> C_p^2 + C_n^2 + n \times p + z \times (N - z) + C_z^2 = C_N^2$$
$$=> p \times (p-1) + n \times (n-1) + 2 \times p \times n + 2 \times z \times (N - z) + z \times (z-1) = N \times (N-1)$$
$$=> p^2 - p + n^2 - n + 2 \times p \times n + 2 \times z \times (N - z) + z^2 - z = N^2 - N$$
$$=> p^2 + 2 \times p \times n + n^2 + 2 \times z \times (N - z) + z^2 - n - p - z = N^2 - N$$
$$=> (p+n)^2 + 2 \times z \times (N - z) + z^2 = N^2$$
$$=> (N-z)^2 + 2 \times z \times (N - z) + z^2 = N^2$$
$$=> (N - z + z)^2 = N^2$$

- Weakness
  - Ordinal (有序的) : Spearman, Kendall
  - Good for credit rating
  - Should not be used for **cardinal** or numeric (基数)
  - Less sensitive to outliers
  - Under stress conditions, underestimate risk by ignoring outliers.
  - Kendall
    - A large number of pairs are neither concordant or discordant.
    - They are ignored

Financial Correlation Modeling Bottom-Up Approaches

- **Copula**
  - A copula creates a **joint** probability distribution between two or more variables while maintaining their individual **marginal** distributions.
  - Mapping multiple distributions to a **single multivariate** distribution
  - Copula enables the **structures** of **correlation** between variables to be calculated separately from their **marginal** distributions.
- **Gaussian Copula**
  - Maps the marginal distribution to the **standard normal** distribution N(0,1)
  - Mapping is done on **percentile-to-percentile** basis.
- **Gaussian default time copula**
  - **Marginal** distributions of cumulative **default** probabilities
- **Correlated Default Time (sample)**
  - When a Gaussian copula is used to derive the default time relationship for more than two assets, a **Cholesky decomposition** is used to derive a sample
  - Mn(x) = Q_i(t_i)

Empirical Approaches to Risk Metrics and Hedging

- **DV01-Neutral Hedge**
  - Assumes that the yield on a bond and the yield on a hedging instrument rise and fall by the same number of basis points.
  - The nominal yield adjusts by more than one basis point for every basis point adjustment in the real yield.
- **Regression Hedge - volatility**
  - Using a regression hedge examines the **volatility** of historical rate differences and adjusts the DV01 hedge accordingly, based on historical **volatility**.
  - It automatically gives an estimate of the hedged portfolio s **volatility**.
  - TIPS: real interest, T-Bond: nominal interest

- DV01 Neutral hedge
  - $\mathbf{F_r \times DV01_r = F_n \times DV01_n}$
- Regression: Hedge Adjustment Factor
  - $\Delta y_n = a + \beta \times \Delta y_r + \epsilon$
- Regression Hedge
  - $\mathbf{F_r \times DV01_r \times \Delta y_r = F_n \times DV01_n \times \Delta y_n}$
  - $F_r = F_n \times \frac{DV01_n}{DV01\_r} \times \frac{\Delta y_n}{\Delta y_r} = F_n \times \frac{DV01_n}{DV01\_r} \times \beta$

- **Two-Variable Regression Hedge**
  - Use 10 and 30 years to hedge 20 years
  - Equation: F10*DV10*Dy10+ F30*DV30*Dy30=F20*DV20*Dy20
  - Regression: Dy20 = a + beta10 * Dy10 + beta30*Dy30
  - Beta
    - Beta10 =Dy20/Dy10 = F10*DV10 / F20*DV20
    - Beta30 =Dy30/Dy10 = F30*DV30 / F20*DV20
- **Level and Change Regression**
  - Change-on-Change: Dy_t = alpha + beta Dx_t + de_t (somewhat correlated)
    - Dy_t = y_t – y_(t-1), Dx_t = x_t – x_(t-1)
  - Level-on-Level: y_t = alpha + beta * x_t + e_t (completely correlated)
  - Both are unbiased, correct, not efficient (error terms are **serially correlated**)
  - Error-on-Error
    - e_t = r * e_(t-1) + v_t
- **PCA**
  - Explain all factor exposures using a small number of uncorrelated exposures
  - Minimize the sum of variances

The Science of Term Structure Models

- **Interest Rate Tree (Binomial Model)**
  - The interest rates at each node in this interest rate tree are 1-period forward rates corresponding to the nodal period. Beyond the root of the tree,
- **Construction - Forward**
  - The values for **on-the-run issues** generated using an interest rate tree should match its **market price** to prohibit **arbitrage** opportunities.
  - It must maintain the interest rate **volatility** assumption of the underlying model.
- **Valuation – Backward Induction**
  - Node value: average of present values of two values from the next period. The interest rate is determined at the **beginning** of a year.
  - **(Average + coupon) then discount**.
- **Risk-Neutral Pricing Tree**
  - **Probabilities**
    - True probability: 0.5 up and 0.5 down
    - Risk-neutral probability: equate PV = market price
    - Interest rate drift: difference between true and risk-neutral probabilities
  - **Risk-Neutral Tree**
    - Adjust interest rate: Start with spot and forward rates, then adjust the **interest rate**. Use real-world probability.
    - Adjust risk-neutral probability, take the rates on the tree as given.
- **Recombining and Nonrecombining Tree**

- Recombining tree: Up-then-down probability = down-then-up probability
- Nonrecombining: does not equal
  - State-dependent volatility

- **Value Bond Derivative**
  - Steps
    - Compute the **value** of the bond at each node
    - Compute the **intrinsic** value of derivative at each node at maturity
    - Compute the expected discounted value by using backward induction
- **Value European Option**
  - Option is exercised at **maturity**.
  - Compute the bond price of nodes **at maturity**, no need to compute it for the former nodes, then work backward.
  - For a node, its **price** does **not include** its **coupon**. It includes the discounted expected coupon of its next nodes.
- **Constant Maturity Treasury Swap (CMT)**
  - Each node: price = cash flow + discounted expected value of next nodes
- **Option-Adjusted Spread (OAS)**
  - The interest **spread** added to **each node** to equal the current market price=PV
- **Fixed-Income Securities and BSM (does not apply)**
  - No upper **limit**. But bond has a **maximum** when interest=0
  - Assume **risk-free** rate is constant. But changes in short-term rate occur.
  - Assume price **volatility** is constant. But bond volatility **decreases** as bond approaches maturity.
- **Bond with Embedded Options**
  - Callable bond
    - Issuer has the right to buy it back at a fixed price.
    - **Less** price volatility.
    - At low yield: **negative** convexity, capital gains are **capped**, **reinvestment** risk rises
  - Puttable bond
    - Buyer has the right to sell it back at a fixed price

The Evolution of Short Rates and the Shape of the Term Structure

- **Interest Rate Expectation**
  - Node rate are forwards rates => spot rate (geometric mean)
  - Volatility creates convexity => **lower** spot rate
  - Flat, upward-sloping, downward-sloping
  - Can describe short-term not long-term **shape** but can describe the **level** of interest rates for long-term horizons.
- **Interest Rate Volatility**
  - There is uncertainty regarding expected rates, the **volatility** of expected rates causes the future spot rates to be **lower**. With the implied rate, we can compute the value of **convexity** for the 2-year zero-coupon bond as: 8%-7.9816% = 0.0184% or 1.84 basis points.
- **Convexity Effect**
  - Jensen's equity: E[1/(1+r)] > 1 / E[1+r]
    - f(x) = 1/x, convex, E[f(x)] > f(E[x]), let x = 1+r
  - convexity occurs due to **volatility**.
  - convexity increases with **volatility and maturity**.

    - convexity increases PV, lower **yields**, reduction in yield is the value of convexity
- **Risk Premium**
    - Convexity lower spot rate and use risk premium to increase spot rate
    - risk-averse investors require a risk premium for bearing this interest rate risk
    - There is only **uncertainty** in what the 1-year rate will be one and two years from today.
    - Two-year zero-coupon bond 30bps, three-year zero-coupon bond 60bnps

The Art of Term Structure Models: Drift

- **Short-Term Interest Rate Tree Construction**
- **Parameters**
    - dt small interval in year, 1 month=1/12 year
    - s: volatility
    - Normal distribution dw ~ N(0, sqrt(dt))
- **Model 1 – No Drift**
    - **dr = s* dw (expected rate change)**
    - Change to rate => parallel shift, a flat term structure of volatility
    - Limitations
        - Not flexible, only one factor, volatility is flat, parallel shift
- **Negative Interest Rate**
    - Problem is greater when interest rate is low or the time get longer
    - Solutions
        - Lognormal or chi-squared distribution
            - But introduce Skewness or inappropriate volatilities
        - Set to zero (preferred)
    - Bond less affected, but option depends on asymmetric payoff affected more
- **Model 2 – Constant Drift**
    - **dr = lambda * dt + s * dw**
    - Positive drift -> positive risk premium
    - Limitations
        - value of drift is high.
- **Ho-Lee Model – Time-dependent Drift**
    - **dr = lambda_t * dt + s * dw**
- **Arbitrage-Free and Equilibrium Models**
    - Arbitrage models
        - Used to quote the prices of securities that are illiquid or customized.
        - **Constructed** using on-the-run Treasury securities, **predict** off-the-run securities
        - Pricing **derivative** based on observable prices of underlying **securities**
        - Assumption: Prices are **accurate,** subject to suitability of model.
    - Equilibrium
        - Used for relative analysis
- **Vasicek Model – Mean-Reversion**
    - dr = k (theta - r) dt + s * dw
        - theta: **long-run equilibrium rate** value of short-term rate assuming risk neutrality
    - lambda = k (theta – r_l) = annual drift
        - **theta** = r_l + lambda / k, where r_l is the long-run **true rate** of interest

- **non-recombine**
  - r_ud: take the average of the up-then-down and down-then-up rates
  - modify up p and down probability (1-p)
  - modify up-and-up r_uu and down-and-down probability r_dd
  - Equations for computing p and r_uu
    - Mean: p*r_uu + (1-p)*r_ud = r_0 + k(theta-r)dt
    - Variance: p*(r_uu - m)^2 + (1-p) (r_ud - mean)^2 = s * sqrt(dt)
- **Exponentially Decay**
  - Difference decay exponentially exp(-k*t)
  - Interest Rate at time t
    - r_t = r_0 * w + theta* (1-w), where w = exp(-kt)
    - r_t = r_0 * exp(-kt) + theta * (1 – exp(-kt))
    - theta – r_t = (theta - r_0) exp(-kt)
  - half life
    - exp(kt) = 2 => t = ln2 /k
- **Effectiveness**
  - It produces a term structure of volatility that is **declining**. The short-term volatility is **overstated** and long-term volatility is **understated**
  - **Nonparallel** shift: Upward shift in short term rate, short-term rate will be impacted more than long-term rate
  - Natural shock: larger (smaller) k, quicker (slower) the news is incorporated; smaller -> news is long-lived

The Art of Term Structure Models: Volatility and Distribution

- **Time-dependent volatility**
  - dr = lambda(t) dt + sigma(t) dw
- **Model 3**
  - dr = lambda(t) dt + sigma * exp(-alpha * t) dw
  - volatility decrease exponentially to 0 when alphan > 0
  - Effectiveness
    - Caps and floors
- **Model 3 and Vasicek**
  - **Same STD**: The same initial volatility and decay rate (alpha) =mean-reverting rate (k), the **standard deviations** of the terminal distributions are the same.
  - Same **Distribution**: If time-dependent drift = average interest rate path in Vasicek Model, terminal distributions are **identical**.
  - **Difference**
    - Model 3 parallel shift, Vasicek nonparallel shift
  - **Application**
    - Price options on fixed income instruments, model 3
    - Value or hedge fixed income or options, Vasicek (mean reverting)
- **Cox-Ingersoll-Ross (CIR)**
  - Basis-point volatility increase with short-term rate sqrt(t)
  - dr = k (theta - r) dt + sigma * sqrt(r) * dw
- **Lognormal - Deterministic Shift Model 4**
  - dln(r) = lambda(t) * dt + sigma * dw
    - dr = lambda* r *dt + sigma * r * dw
    - lnr_0 + lamba dt + sigma dw
    - r_0 => r_0 exp(lambda dt) exp(sigma dw), multiplicative

- **Lognormal with Mean Reversion (Black- Karasinski)**
  - d[ln(r)] = k(t) [ln(theta(t)) – ln(r)] dt + sigma(t) dw
  - not recombing: the time intervals between interest rate changes are recalibrated to force the nodes to recombine.

Volatility Smile (strike price)

- **Call-Put Parity** (no-arbitrage equilibrium)
  - c – p = S – X exp(-rT) (market)
  - c_bsm – p_bsm = S – X exp(-rT) (BSM)
  - c_mkt – p_mkt = c_bsm – p_bsm
- **Volatility Smiles - Foreign Currency Options**
  - volatility depend on **strike** price, volatility smiles
  - Higher for deep in-the-money and deep out-of-the-money (away-from-money)
  - **Greater** chance of **extreme** price movements than predicted by a lognormal distribution
- **Volatility Smirk/Skew (half-smile) - Equity Options**
  - High **implied** volatility for **low** strike price options
    - In-the-money call and out-of-the-money puts
  - **Left-Skewed Distribution (asymmetric)**
    - Large **down** movements in price than large **up** movements in price, compared with a **lognormal** distribution
  - **Leverage (inverse relation between volatility and asset value)**
    - Equity value decrease -> leverage increase -> increase volatility asset
    - Equity value increase -> leverage decrease -> decrease volatility asset
  - **Crashophobia**
    - Used since U.S. stock market crisis of 1987. Afraid of another crash, place a premium on the probability of stock prices falling precipitously
    - Deep-out-of-money puts have high premium since they provide protection against drop in equity prices.
    - Implied volatilities are higher for low strike price because traders want to protect themselves against another substantial drop in the market.
- **Alternative Volatility Smile**
  - Stock price, X -> X/S_0 => more stable volatility smile.
  - Forward price of asset, X -> X/F_0 => better gauge of at the money option
  - Option's delta, X -> Delta => other than European and American options
- **Volatility term structure (TTM)**
  - A function of time to expiration for at-the-money options.
  - Similar to mean-reverting characteristic
- **Volatility Surface (TTM * Strike)**
  - Combination of volatility term structure with volatility smiles
- **Option Greeks**
  - Sticky strike rule: assume implied volatility is the same over short time period
  - Sticky delta rule: delta will be larger than that given by BSM
  - Both assume volatility smile is **flat** for all option **maturities**.
- **Price Jump / Volatility frowns**
  - News cause the price to move up or down by a large amount.
  - jumps occur in asset prices.
  - Two lognormal model

只看左边的 2 个图，不需要去记住那个分布图。
equity 就是下面的 foreign currency 就是上面的图
考试时
1 先画图（equity, currency)
2 画一个水平线，在中间的那个虚线和曲线相交的地方，横着画一个线（可以认为这个线就是对应的 lognormal）
3 根据 in/out-of-money call/put 画点

根据这个关系图推理：值高，波动高，fat tail

Extreme Value Theory (EVT)
- Focuses on data that is generally considered outliers.
- For low probability, high impact events; not everyday occurrences.
- A Cluster analysis is appropriate for financial data with time dependency.
- Distributions
    - Weilbull distribution
    - Frechet distribution
    - Generalized Pareto distribution

peaks-over-threshold (POT)
- A Fewer estimated parameters than the GEV approach and shares one parameter with the GEV
- Determine the cut-off between typical and extreme values.

Overnight Indexed Swap (OIS)
- Is a stable **proxy** in **stressed** market conditions
- Does not lead to an incorrect no-default value
- Does not result in **double** counting for credit risk.
- Generally reflects **low** credit risk
- The rate reflects a lack of credit risk because it is a function of the federal funds overnight rate, which bears minimal default risk and the adjustment to the rate in a transaction with a counterparty is typically **small**.
-

LIBOR
- as the discount rate to value a non-collateralized portfolio may result in **double counting** for risk.
- LIBOR is more volatile than the OIS rate and, therefore, not reflective of a true risk-free rate.
- LIBOR is a rate on **unsecured** borrowing.

Delta-Normal & Option
- The delta-normal VaR method cannot be expected to provide an accurate estimate of true VaR over ranges where deltas are unstable. That would occur when options are **at-the-money**.
- Deep-out-of-the-money and deep-in-the-money options have relatively **stable** deltas.

Price Value of a basis point (PVBP)

- PVBP: Change in Portfolio value for a 1bps change in rates
- At x% probability level change in interest rates is y% or higher.
- VaR at (100-x)% is PVBP * y * 100