**协方差 – 线性性质**

- 资产 1 的回报是 X，资产 2 的回报是 Y
- 组合的回报是 $Z = w_1 X + w_2 Y$
- 资产 1 和组合的协方差
  - $COV(X,Z) = COV(X, w_1 X + w_2 Y) = w_1 COV(X,X) + w_2 COV(X,Y)$
  - $COV(X,Z) = w_1\sigma_1^2 + w_2 COV(X,Y) = w_1\sigma_1^2 + w_2\rho_{12}\sigma_1\sigma_2$

**协方差 – 矩阵法**

- 协方差矩阵
  - $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$
- 权值向量
  - $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$
- 新向量 $\vec{c} = \Sigma \times \vec{w}$ 的每个元素是每个资产和组合的协方差
  - $\vec{c} = \Sigma \times \vec{w} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1\sigma_1^2 + w_2\rho_{12}\sigma_1\sigma_2 \\ w_2\sigma_2^2 + w_1\rho_{12}\sigma_1\sigma_2 \end{bmatrix}$
- 矩阵法适合多个资产，很灵活

**协方差 – 组合方差偏导法**

- 组合方差
  - $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1\sigma_2$
- 方差对权值$w_1$的偏导
  - $\frac{\partial\, \sigma_p^2}{\partial w_1} = w_1\sigma_1^2 + \rho_{12} w_2 \sigma_1\sigma_2 = cov_{1p}$
- $\mathrm{cov}_{1p} = w_1\sigma_1^2 + \rho_{12} w_2\sigma_1\sigma_2 = w_1\sigma_1^2 + w_2 cov_{12}$

**组合标准差**

- 定义
  - $\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho_{12} w_1 w_2 \sigma_1\sigma_2}$
- 矩阵法
  - $\sigma_p^2 = \vec{w}^T \times \Sigma \times \vec{w}$
- 标准差对权值$w_1$的偏导
  - $\frac{\partial\, \sigma_p}{\partial w_1} = \frac{1}{2\,\sigma_p}\frac{\partial \sigma_p^2}{\partial w_1} = \frac{w_1\sigma_1^2 + \rho_{12} w_2\sigma_1\sigma_2}{\sigma_p} = \frac{cov_{1p}}{\sigma_p}$

**Marginal VaR 各个元素相同**

- 协方差法
  - $MVaR_1 = MVaR_2 \rightarrow Z \times \frac{cov_{1p}}{\sigma_p} = Z \times \frac{cov_{2p}}{\sigma_p}$
  - $\rightarrow \mathrm{Cov}_{1p} = Cov_{2p}$
  - $\rightarrow w_1\sigma_1^2 + w_2\rho_{12}\sigma_1\sigma_2 = w_2\sigma_2^2 + w_1\rho_{12}\sigma_1\sigma_2$

- $\rightarrow \frac{w_1}{w_2} = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}$
- $\rightarrow \frac{w_1}{w_2} = \frac{\sigma_2^2}{\sigma_1^2}$ 如果 2 个资产是独立的($\rho_{12} = 0$)

- $C \times \vec{w}$的每个元素相同，可以写成常数向量$C \times \vec{w} = k \times \vec{1}$

**组合 Portfolio VaR**

- 定义
  - $\mathrm{VaR_p} = \mathrm{P} \times \mathrm{Z} \times \sigma_\mathrm{p}$
- 矩阵法
  - $\mathrm{VaR_p^2} = \mathrm{P} \times \mathrm{Z} \times (\vec{w}^\mathrm{T} \times \Sigma \times \vec{w})$
- Two-asset
  - $VaR_p = \sqrt{VaR_1 + VaR_2 + 2 \times VaR_1 \times VaR_2 \times \rho}$
- 假定独立 uncorrelated VaR
  - $VaR_p = \sqrt{VaR_1 + VaR_2}$
- 假定线性相关 undiversified VaR
  - $VaR_p = VaR_1 + VaR_2$
- Multiple-asset (equally weighted and each has the same std and correlation)
  - $VaR_p = \sigma\sqrt{\frac{1}{n} + (1 - \frac{1}{n}) \times \rho}$
- **Diversified VaR Reduction**
  - $\mathrm{DVaR_1} = \mathrm{VaR_1} + VaR_2 - VaR_p$

## Marginal VaR 的不同计算方法

- Incremental VaR 和 Component VaR 需要利用 Marginal VaR。
- 组合 Portfolio VaR
  - $\mathrm{VaR_p} = \mathrm{P} \times \mathrm{Z} \times \sigma_\mathrm{p}$
- 个体 Individual VaR
  - $\mathrm{VaR_1} = \mathrm{P_1} \times Z \times \sigma_1$
- 定义
  - $MVaR_1 = \frac{\partial VaR_p}{\partial P_1}$
- Use Covariance
  - $MVaR_1 = \frac{\partial VaR_p}{\partial P_1} = \mathrm{Z} \times \frac{\partial \sigma_p}{\partial w_1} = \mathrm{Z} \times \frac{cov_{1p}}{\sigma_p}$
- Use Beta
  - $\beta_1 = \frac{cov_{1p}}{\sigma_p^2} \rightarrow MVaR_1 = \mathrm{Z} \times \frac{cov_{1p}}{\sigma_p} = \mathrm{Z} \times \sigma_\mathrm{p} \times \beta_1$
- Use Portfolio VaR
  - $\mathrm{VaR_p} = \mathrm{P} \times \mathrm{Z} \times \sigma_\mathrm{p} \rightarrow \mathrm{MVaR_1} = \mathrm{Z} \times \sigma_\mathrm{p} \times \beta_1 = \frac{VaR_p}{P} \times \beta_1$
- Use Correlation
  - $\beta = \frac{cov_{1p}}{\sigma_p^2} = \rho_1 \frac{\sigma_1}{\sigma_p} \rightarrow \mathrm{MVaR_1} = \mathrm{Z} \times \sigma_\mathrm{p} \times \beta_1 = Z \times \sigma_1 \times \rho_1$
- Use Individual VaR

- $\mathrm{VaR}_1 = \mathrm{P}_1 \times Z \times \sigma_1 \rightarrow \mathrm{MVaR}_1 = Z \times \sigma_1 \times \rho_1 = \frac{VaR_1}{P_1} \times \rho_1$

**Applications**

- Component VaR
  - $\mathrm{CVaR}_1 = \mathrm{P}_1 \times \mathrm{MVaR}_1 = VaR_p \times \mathrm{w}_1 \times \beta_1 = VaR_1 \times \rho_1$
  - $\sum_{\mathrm{i}} w_i \times \beta_i = 1$
- **Percentage contribution 百分比贡献**
  - $\mathrm{contribution\%} = \frac{\mathrm{CVaR}_1}{VaR_p} = \mathrm{w}_1 \times \beta_1$
- Incremental VaR
  - Full evaluation (time consuming, VaR complicated)
  - Incremental Steps
    - Map to factors and estimate the positions of them
    - Prepare the marginal MVaR vector
    - Cross product
  - $\mathrm{IVaR}_1 = Increment \times \mathrm{MVaR}_1 = Incr \times \mathrm{Z} \times \frac{cov_{1p}}{\sigma_p}$
    - Both can be dollar valued
- Risk Management (Global Minimization)
  - Minimize $\mathrm{VaR}_\mathrm{p} \rightarrow \sigma_p$
  - Condition
    - $\mathrm{MVaR}_\mathrm{i} = \mathrm{MVaR}_\mathrm{j}$
    - **Allocate more to positions with lower MVaR**
  - Covariance equal
    - $\mathrm{cov}_{1\mathrm{p}} = cov_{2p}$
    - $w_1\sigma_1^2 + w_2\rho_{12}\sigma_1\sigma_2 = w_2\sigma_2^2 + w_1\rho_{12}\sigma_1\sigma_2$
    - $\rightarrow \frac{w_1}{w_2} = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}$
  - Beta equal
    - $\beta_\mathrm{i} = \beta_\mathrm{j} \rightarrow \rho_1 \times \sigma_1 = \rho_2 \times \sigma_2$
- Portfolio Management
  - Maximize $\frac{\mathrm{Return-risk-free}}{\mathrm{VaR_p}}$
  - Condition
    - $\frac{\mathrm{R_i - R_f}}{\mathrm{MVaR_i}} = \frac{R_j - \mathrm{R_f}}{\mathrm{MVaR_j}}$
  - If distribution are elliptic (beta)
    - $\frac{\mathrm{R_i - R_f}}{\beta_\mathrm{i}} = \frac{R_j - \mathrm{R_f}}{\beta_\mathrm{j}}$
  - Allocate **more** weight to higher ratio (excess return to MVaR)
- Expected Return on Capital
  - Expected Return $\mathrm{R}_1 = \mathrm{R_f} + \beta_1 \times (\mathrm{R_M} - \mathrm{R_F})$
    - $MVaR_1 = \frac{VaR_p}{P} \times \beta_1$
  - ROC (individual value and VaR)
    - $\mathrm{ROC}_1 = \frac{P_1 \times \mathrm{R}_1}{VaR_1}$

Question

26. Consider a portfolio consisting of USD 10 million of Intel shares and USD 5 million of GE shares. The returns on the two stocks have a bivariate normal distribution with a correlation of 0.3. The daily return volatility of Intel and GE is 2% and 1%, respectively. The standard deviation of daily changes in the value of the Intel position is USD 0.2 million, and the standard deviation of daily changes in the value of the GE position is USD 0.05 million. The daily VaR at the 99% confidence level is USD 0.5131 million. What is the incremental daily VaR of the portfolio for a small increase in the position on Intel shares over a one-day horizon at 99% confidence level?

- **a.** USD 0.0455 million
- **b.** USD 2.275 million
- **c.** USD 0.0453 million
- **d.** USD 0.0195 million

```
>>> var=0.5131
>>> s1=0.02
>>> s2=0.01
>>> p1=10
>>> p2=5
>>> r=0.3
>>> z=2.33
>>> pcov = p1*s1*s1 + p2*s1*s2*r
>>> mvar = z*z/var * pcov
>>> mvar
0.04549653089066459
>>>
```

$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho} \Rightarrow \frac{\partial \sigma_p}{\partial w_1} = \frac{w_1\sigma_1^2 + \rho w_2\sigma_1\sigma_2}{\sigma_p}$

marginal VaR $= \frac{\partial VaR}{\partial P_1} = \frac{\partial P \cdot z \cdot \sigma_p}{\partial P \cdot w_1} = z\frac{\partial \sigma_p}{\partial w_1} = z \cdot \frac{cov}{\sigma_p}$，其中 $\boxed{cov = w_1\sigma_1^2 + \rho w_2\sigma_1\sigma_2}$

Portofio VaR $= P \cdot z \cdot \sigma_p$

mVaR $= \boxed{\frac{z^2}{VaR} cov \cdot P}$，其中 $P \cdot cov = P_1\sigma_1^2 + P_2 \cdot \rho\sigma_1\sigma_2$

**QUESTIONS 61 AND 62 REFER TO THE FOLLOWING INFORMATION**

*A financial risk consultant assumes that the joint distribution of returns is multivariate normal and calculates the following risk measures for a two-asset portfolio managed by a mid-sized insurance company:*

| Asset | Position (CNY) | Individual VaR (CNY) | Marginal VaR | VaR Contribution (CNY) |
|---|---|---|---|---|
| Financial | 15,000,000 | 3,494,700 | 0.216 | 3,240,000 |
| Energy | 15,000,000 | 6,999,300 | 0.462 | 6,931,238 |
| Portfolio | 30,000,000 | 9,241,650 | | 9,241,650 |

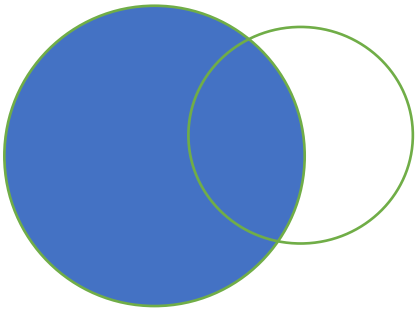
61. *If the energy asset is dropped from the portfolio, what will be the reduction in portfolio VaR?*

A. CNY 2,242,350
B. CNY 3,494,700
C. CNY 5,746,950
D. CNY 6,999,300

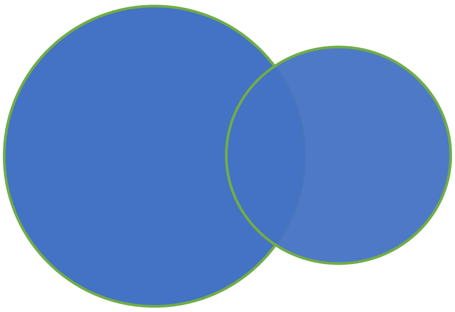
0.462×15000000

~~D~~ C

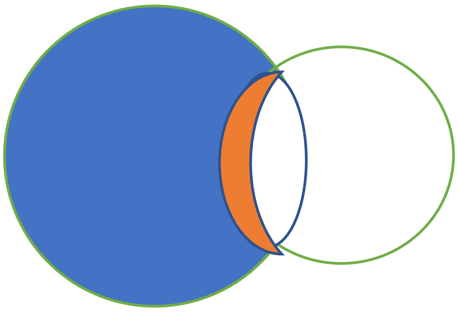
- Reduction when the second asset is removed, the reduction in VaR is $\mathrm{VaR_p} - VaR_1$



阴影部分是 **Individual** $\mathrm{VaR}_1$
单独集合 A

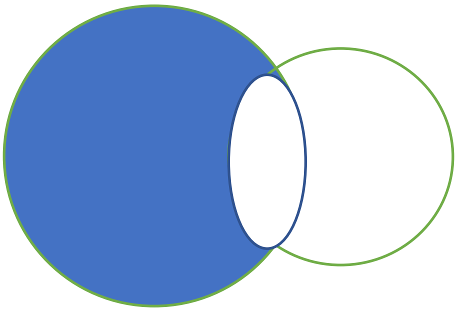


阴影部分是 **Portfolio** $\mathrm{VaR}_\mathrm{p}$
集合的并集 $\mathrm{A} \cup \mathrm{B}$

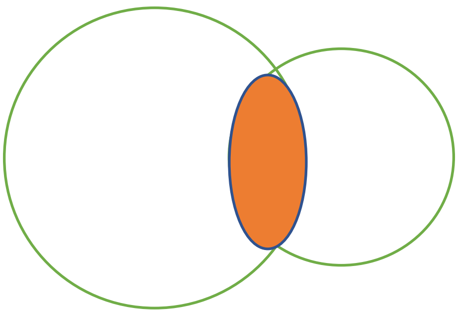


阴影部分是 Component $\mathrm{CVaR}_1$
右边白色的是 Component $\mathrm{CVaR}_2$

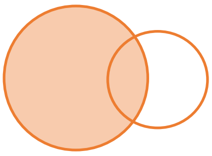
也可以理解成 Marginal VaR，因为
$Component\ VaR = P\ \times Margianl\ VaR$



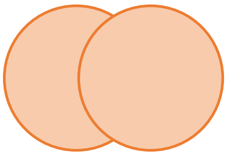
阴影部分是去掉 1 后的 Reduction
$\mathbf{VaR_p} -\ \mathrm{VaR}_2$
右边白色是余下的 VaR
集合的左对称差 $\mathrm{A} \cup \mathrm{B} -\ \mathrm{B}$



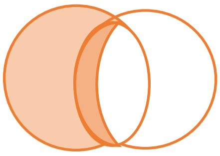
阴影部分是投资组合的风险分散： VaR
Undiversified VaR − $\mathrm{VaR}_\mathrm{p} = \mathrm{VaR}_1 + VaR_2 -\ \mathrm{VaR}_\mathrm{p}$
集合的交集 $\mathrm{A} \cap \mathrm{B}$



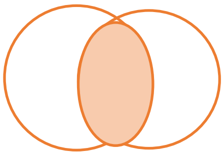
阴影部分是 **Individual** $\mathrm{VaR}_1$
单独集合 A



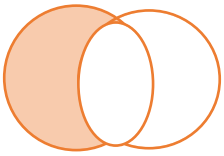
阴影部分是 **Portfolio** $\mathrm{VaR}_\mathrm{p}$
集合的并集 $\mathrm{A} \cup \mathrm{B}$



阴影部分是 **Component** $\mathrm{CVaR}_1$
右边白色是 Component $\mathrm{CVaR}_2$
也可理解成 MVaR，因为
$CVaR = P \times MVaR$



阴影是投资组合的风险分散
Undiversified VaR $-\ \mathrm{VaR}_\mathrm{p}$
$= \mathrm{VaR}_1 + VaR_2 - \mathrm{VaR}_\mathrm{p}$
集合的交集 $\mathrm{A} \cap \mathrm{B}$



阴影是去掉 1 后的风险降低
$\mathbf{VaR_p} -\ \mathrm{VaR}_2$
右边白色是余下的 VaR
集合的左对称差 $\mathrm{A} \cup \mathrm{B} -\ \mathrm{B}$