Error :

Exact value − Approximate value

Sources of Error :

- Round off /up Error
- Truncation Error
- Machine Error.

Propagation of Error

$$X = X' + \epsilon_x$$
$$Y = Y' + \epsilon_y$$

1) Addition

$$\epsilon_{x+y} = \epsilon_x + \epsilon_y$$

$$|\epsilon_{x+y}| \leq |\epsilon_x| + |\epsilon_y|$$

2) Subtraction:

$$\epsilon_{x-y} = \epsilon_x - \epsilon_y$$

$$|\epsilon_{x-y}| \leq |\epsilon_x| + |\epsilon_y|$$

3) Multiplication:

$$\epsilon_{xy} = x'\epsilon_y + y'\epsilon_x + \epsilon_x\epsilon_y$$

$$|\epsilon_{xy}| \leq |x'||\epsilon_y| + |y'||\epsilon_x| + |\epsilon_x||\epsilon_y|$$

4) Division:

$$X/Y = \left(\frac{x'}{y'} + \frac{\epsilon_x}{y'}\right)\left(1 + \frac{\epsilon_y}{y'}\right)^{-1}$$

This binomial expression can be expanded to find the error.

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \ldots$$

Representation of Error:

Absolute Error:

|Exact - Approximate|

Relative Error:

$$\frac{\text{Absolute Error}}{\text{Reference Value}} = \frac{|\text{Exact} - \text{Approx}|}{\text{Ref}}$$

usually reference value is taken as exact

Percentage Error

Relative Error $\times 100$

Error in Function:

$$z = f(x, y)$$

$$\epsilon x = x - x'$$

$$\epsilon_y = y - y'$$

$$\Delta z = \epsilon x \frac{\partial z}{\partial x} + \epsilon y \frac{\partial z}{\partial y}$$

Derivative is constant → Linear equation
Derivative has variable → Non-linear equation.

Roots of a Non-Linear Equation:

$x_R \in D_f$ such that

$$f(x_R) = 0$$

Bisection Method: (Interval Halving Method)

$a, b \in D_f$

$f(a)f(b) < 0$

$$c = \frac{a+b}{2}$$

Newton-Raphson Method: (Tangent Method)

$x_0 \in D_f$

$$f'(x_0) \neq 0$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$\forall \ i = 0, 1, 2, \ldots$

If $f'(x_i) = 0$, method fails to converge.

Secant Method:

$x_0, x_1 \in D_f$

$f(x_0) \neq f(x_1)$

$$x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)}$$

If $f(x_i) = f(x_{i+1})$, method fails to converge.

False Position: (Regula-Falsi Method):

$x_0, x_1 \in D_f$

$f(x_0) f(x_1) < 0$

$f(x_{i+1}) f(x_i) < 0$

$$x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)}$$

$$c = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$f(a) f(b) < 0$

Numerical Integration:

Trapezoidal Rule:

$c_1 x + c_2$

$$\int_a^b f(x)\,dx \cong \left(\frac{f(a)+f(b)}{2}\right)(b-a)$$

Composite:

$$\int_a^b f(x)\,dx \cong \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots y_{N-1}) + y_N\right]$$

N - Number of intervals

$h = \dfrac{b-a}{N}$ → width of interval.

Simpson's $1/3^{rd}$ Rule:

$$c_1 x^2 + c_2 x + c_3$$

$$\int_a^b f(x)\,dx \cong \frac{h}{3}\left[f(a) + 4f(c) + f(b)\right]$$

$$h = \frac{b-a}{2}$$

$$c = \frac{a+b}{2}$$

Composite:

$$\int_a^b f(x)\,dx \cong \frac{h}{3}\Big[y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{N-1})$$
$$+ 2(y_2 + y_4 + \dots y_{N-2})$$
$$+ y_N\Big]$$

N should be even.

$$h = \frac{b-a}{N}$$

Simpson's 3/8th rule:

$$c_1 x^3 + c_2 x^2 + c_3 x + c_4$$

$$\int_a^b f(x)\,dx \cong \frac{3}{8}h\left[f(a) + 3f(c) + 3f(d) + f(b)\right]$$

$$h = \frac{b-a}{3}$$

$$c = a + h$$

$$d = a + 2h$$

Composite:

$$\int_a^b f(x)\,dx \cong \frac{3h}{8}\Big[y_0 + 3(y_1 + y_4 + y_7 + \dots + y_{N-2})$$

$$+ 3(y_2 + y_5 + y_8 + \dots + y_{N-1})$$

$$+ 2(y_3 + y_6 + y_9 + \dots + y_{N-3})$$

$$+ y_N\Big]$$

N should be multiple of 3.

$$h = \frac{b-a}{N}$$

Numerical Solution of Ordinary Differential Equation:

$$\frac{dy}{\partial x} = f(x, y)$$

$$y(x_0) = y_0$$

Euler's Method:

$$y_{i+1} = y_i + h f(x_i, y_i)$$

$$\forall \; i = 0, 1, 2, 3, \dots$$

Modified Euler Method:

$$y^* = y_i + h f(x_i, y_i) \quad \text{at } x_{i+1}$$

$$y_{i+1} = y_i + \frac{h\,[f(x_i, y_i) + f(x_{i+1}, y^*)]}{2}$$

at x_{i+1}

$$\forall \; i = 0, 1, 2, 3, \dots$$

Runge-Kutta Method of Order 4.
(R-K)

$$y_{i+1} = y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf(x_i + h/2, y_i + k_1/2)$$

$$k_3 = hf(x_i + h/2, y_i + k_2/2)$$

$$k_4 = hf(x_i + h, y_i + k_3)$$

$\forall\ i = 0, 1, 2, 3, 4, \ldots$

Numerical solution of a system of linear equations:

LU - Decomposition Method
(Matrix Factorization Technique)

→ Avoid Inverse

→ Direct Method.

$$A\underline{x} = \underline{b} \quad - (1)$$

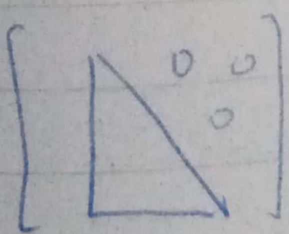
Let $$A = LU \quad - (2)$$

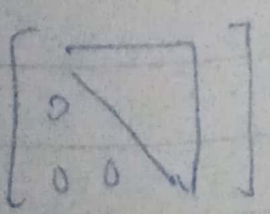
$$LU\underline{x} = \underline{b}$$

$$U\underline{x} = \underline{y} \quad - (3)$$ ↑ Backward substitution

$$L\underline{y} = \underline{b} \quad - (4)$$ ↓ Forward substitution.

where

$L =$  lower triangular matrix

$R =$  upper triangular matrix

Jacobi- Iterative Method:

$$A \underline{x} = \underline{b}$$

coefficient matrix A should be diagonally dominant.

$$|a_{11}| \geq |a_{12}| + |a_{13}| + |a_{14}| + \dots$$

$$|a_{22}| \geq |a_{21}| + |a_{23}| + |a_{24}| + \dots$$

$$\vdots$$

$$x_1^{i+1} = \frac{b_1}{a_{11}} - \frac{1}{a_{11}}\left(a_{12} x_2^{i} + a_{13} x_3^{i} + \dots + a_{1n} x_n^{i}\right)$$

$$x_2^{i+1} = \frac{b_2}{a_{22}} - \frac{1}{a_{22}}\left(a_{21} x_1^{i} + a_{23} x_3^{i} + \dots + a_{2n} x_n^{i}\right)$$

$$\vdots$$

$$x_n^{i+1} = \frac{b_n}{a_{nn}} - \frac{\left(a_{n1} x_1^{i} + a_{n2} x_2^{i} + \dots + a_{n,n-1} x_{n-1}^{i}\right)}{a_{nn}}$$

Gauss - Seidel Iterative Method :

$$A\underline{x} = \underline{b}$$

coefficient matrix A should be diagonally dominant.

$$|a_{11}| \geq |a_{12}| + |a_{13}| + |a_{14}| + \cdots$$

$$|a_{22}| \geq |a_{21}| + |a_{23}| + |a_{24}| + \cdots$$

$$\vdots$$

$$x_1^{i+1} = \frac{b_1}{a_{11}} + \frac{1}{a_{11}} (a_{12} x_2^{i} + a_{13} x_3^{i} + \cdots + a_{1n} x_n^{i})$$

$$x_2^{i+1} = \frac{b_2}{a_{22}} + \frac{1}{a_{22}} (a_{21} x_1^{i+1} + a_{23} x_3^{i} + \cdots + a_{2n} x_n^{i})$$

$$\vdots$$

$$x_n^{i+1} = \frac{b_n}{a_{nn}} + \frac{1}{a_{nn}} (a_{n1} x_1^{i+1} + a_{n2} x_2^{i+1} + \cdots + a_{n,n-1} x_{n-1}^{i+1})$$

Interpolation:

Forward Difference (Δ)

$n+1$ equally spaced data set.

$$\Delta f(x) = f(x+h) - f(x)$$

Backward Difference (∇)

$$\nabla f(x) = f(x) - f(x-h)$$

$$\Delta^n y_0 = \nabla^n y_n$$

If there are $n+1$ data set, then the maximum degree that can fit the polynomial is n.

If $\Delta^i y$ becomes constant, i^{th} order polynomial will fit.

$\Delta^2 y \rightarrow$ constant

quadratic polynomial.

Newton's Forward Difference Formula:

$n+1$ data set $\rightarrow$ equally spaced.

$h \rightarrow$ width.

$$f(x) = y_0 + \frac{\Delta y_0}{1!}\frac{(x-x_0)}{h} + \frac{\Delta^2 y_0}{2!}\frac{(x-x_0)(x-x_1)}{h^2}$$

$$+ \frac{\Delta^3 y_0}{3!}\frac{(x-x_0)(x-x_1)(x-x_2)}{h^3}$$

$$\vdots$$

$$+ \frac{\Delta^n y_0}{n!}\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{h^n}$$

Newton's Backward Difference Formula:

$$f(x) = y_n + \frac{\nabla y_n}{1!}\frac{(x-x_n)}{h} + \frac{\nabla^2 y_n}{2!}\frac{(x-x_n)(x-x_{n-1})}{h^2}$$

$$+ \frac{\nabla^3 y_n}{3!}\frac{(x-x_n)(x-x_{n-1})(x-x_{n-2})}{h^3}$$

$$\vdots$$

$$+ \frac{\nabla^n y_n}{n!}\frac{(x-x_n)(x-x_{n-1})\cdots(x-x_1)}{h^n}$$

Lagrange Interpolation Formula:

unequally spaced data

n - data set

→ local error = 0

→ Maximum n-1 degree polynomial:

(x_0, y_0) , (x_1, y_1)

$$f(x) = L_0(x) y_0 + L_1(x) y_1$$

$$\boxed{f(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(y_1 - y_0)} y_1}$$

(x_0, y_0) , (x_1, y_1) , (x_2, y_2)

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

for n+1 points.

$$f(x) = \sum_{i=0}^{n} L_j(x)\, y_i$$

$$L_i(x) = \sum_{\substack{j=0 \\ j\neq 0}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

Curve Fitting:

n - dataset

Linear Fit

$$\hat{y} = ax + b$$

$$a = \frac{n \Sigma xy - \Sigma x \Sigma y}{n \Sigma x^2 - (\Sigma x)^2}$$

$$b = \bar{y} - a\bar{x}$$

$$= \frac{\Sigma y}{n} - a \frac{\Sigma x}{n}$$

Quadratic Fit:

$$\hat{y} = ax^2 + bx + c$$

$$a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2 = \Sigma yx^2$$

$$a\Sigma x^3 + b\Sigma x^2 + c\Sigma x = \Sigma yx$$

$$a\Sigma x^2 + b\Sigma x + nc = \Sigma y$$

or

$$\begin{bmatrix} \Sigma x^4 & \Sigma x^3 & \Sigma x^2 \\ \Sigma x^3 & \Sigma x^2 & \Sigma x \\ \Sigma x^2 & \Sigma x & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \Sigma x^2 y \\ \Sigma xy \\ \Sigma y \end{bmatrix}$$

Exponential Fit :

$$\hat{y} = me^{px}$$

$$p = \frac{n\Sigma xy' - \Sigma x \Sigma y'}{n\Sigma x^2 - (\Sigma x)^2}$$

$$m = e^{\bar{y'} - p\bar{x}}$$

where

$$y' = \ln y$$

Gradient Descent Method

$$x_{i+1} = x_i - t \frac{\partial z}{\partial x}\bigg|_{(x_i, y_i)}$$

$$y_{i+1} = y_i - t \frac{\partial z}{\partial x}\bigg|_{(x_i, y_i)}$$

$$\forall \; i = 0, 1, 2, 3, \ldots$$

LU Decomposition:

3x3

$$\begin{bmatrix} L_{11}U_{11} & L_{11}U_{12} & L_{11}U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + L_{22}U_{22} & L_{21}U_{13} + L_{22}U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} \end{bmatrix}$$

4x4

$$\begin{bmatrix} L_{11}U_{11} & L_{11}U_{12} & L_{11}U_{13} & L_{11}U_{14} \\ L_{21}U_{11} & L_{21}U_{12} + L_{22}U_{22} & L_{21}U_{13} + L_{22}U_{23} & L_{21}U_{14} + L_{22}U_{24} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} & L_{31}U_{14} + L_{32}U_{24} + L_{33}U_{34} \\ L_{41}U_{11} & L_{41}U_{12} + L_{42}U_{23} & L_{41}U_{13} + L_{42}U_{23} + L_{43}U_{33} & L_{41}U_{14} + L_{42}U_{24} + L_{43}U_{34} + L_{44}U_{44} \end{bmatrix}$$

Stopping Criteria for

- Jacobi Iterative Method
- Gauss-Seidle Iterative Method

Residue vector

$$\underline{r} = A\underline{x}^{i+1} - \underline{b}$$

Norm of residue vector:

$|\underline{r}| = 0$ ← Exact solution.

$|\underline{r}| \leq$ tolerance (e.g. 10^{-2}) ← Stopping criteria.

Use skipping technique as it is complex.

$$|\underline{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2 \ldots}$$

Curve Fitting.

For comparing multiple curves:

$$R^2 = 1 - \frac{SSE}{SST}$$

SSE = Sum of square error.

SST = Sum of square total

$$SSE = \Sigma (\hat{y_i} - y_i)^2$$

$$SST = \Sigma (y_i - \bar{y})^2$$