ERROR ANALYSIS:

$$\boxed{\text{Error} = \text{Exact Solution} - \text{Approximate}}$$

Sources Of Error: Best Method => Error = 0

1) Round off /up

Exact = 7.18293148

Approx = 7.183.

2) Truncation Error

$\pi = 3.142$ irrational numbers.

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$ (Taylor Series)

3) Machine Error

Automatic truncation of decimal numbers due to less hardware architecture.

32-bit, 64-bit architecture

Propagation Of Error In Calculations:

$$\bar{x} = x' + \epsilon_x$$
$$y = y' + \epsilon_y$$

$x \to$ Exact Solution
$x' \to$ Approximate
$\epsilon_x \to$ Error

1) Addition:

$$x + y = (x' + \epsilon_x) + (y' + \epsilon_y)$$

$$\boxed{x + y = x' + y' + \epsilon_x + \epsilon_y}$$

$$\epsilon_{x+y} = \epsilon_x + \epsilon_y$$

$\epsilon_x = 0.1$ $\quad$ $\epsilon_{x+y} = 0.1 - 0.1 = 0$

$\epsilon_y = -0.1$

Algebraically cancellation of errors is Catastrophic Cancellation.

If we are not interested in the sign of Error, only in magnitude:

$$|\epsilon_{x+y}| = |\epsilon_x + \epsilon_y| \le |\epsilon_x| + |\epsilon_y|$$

$\downarrow$

Absolute Error / Quantity

$\downarrow$

we only take Magnitude

2) Subtraction:

$$x - y = (x' + \epsilon_x) - (y' + \epsilon_y)$$

$$\boxed{x - y = x' - y' + \epsilon_x - \epsilon_y}$$

$$\epsilon_{x-y} = \epsilon_x - \epsilon_y$$

Absolute Error:

$$|\epsilon_{x-y}| = |\epsilon_x - \epsilon_y| = |\epsilon_x + (-\epsilon_y)|$$

$$|\epsilon_{x-y}| = |\epsilon_x + (-\epsilon_y)| \le |\epsilon_x| + |\epsilon_y|$$

3) Multiplication:

$$xy = (x' + \epsilon_x)(y' + \epsilon_y)$$

$$\boxed{xy = x'y' + x'\epsilon_y + y'\epsilon_x + \epsilon_x\epsilon_y}$$

$$\epsilon_{xy} = x'\epsilon_y + y'\epsilon_x + \epsilon_x\epsilon_y$$

Absolute Error:

$$|\epsilon_{xy}| = |x'\epsilon_y + y'\epsilon_x + \epsilon_x\epsilon_y| \le$$

$$|x'\epsilon_y| + |y'\epsilon_x| + |\epsilon_x\epsilon_y| \le$$

$$|x'||\epsilon_y| + |y'||\epsilon_x| + |\epsilon_x||\epsilon_y|$$

4) Divisi

$$\frac{x}{y} =$$

$$\frac{x}{y} =$$

$$\frac{x}{y} =$$

4) Division:

$$\frac{x}{y} = \frac{(x' + \epsilon x)}{(y' + \epsilon y)}$$

$$\frac{x}{y} = \frac{x' + \epsilon x}{y'\left(1 + \frac{\epsilon y}{y'}\right)}$$

$$\frac{x}{y} = \left(\frac{x'}{y'} + \frac{\epsilon x}{y'}\right)\underbrace{\left(1 + \frac{\epsilon y}{y'}\right)^{-1}}$$

binomial expression which can be expanded.

REPRESENTATION OF AN ERROR:

$$\text{Error} = \text{Exact} - \text{Approx}$$

1) ABSOLUTE ERROR: (Jb sign se mtlb na ho sirf magnitude chahiye ho)

$$\text{Abs. Error} = |\text{Error}|$$

2) RELATIVE ERROR: (Comparison)
(Error ko kisi dusri quantity se relate kr k express krna)

- Relative error has no unit.
- Tells how many times absolute error is relative to exact error? → Reference value ↓ should be fixed or constant.

$$\text{Rel. Error} = \frac{\text{Abs. Error}}{\text{Exact}}$$

3) PERCENTAGE ERROR: Percentage Relative Error.

%age error = relative error x 100 (100 pieces)

Per thousand = relative error x 1000 (1000 pieces)

Per million = relative error X 1M (1M pieces)

$y'(x) =$ constant (linear)
variable (non-linear)

— Domain ki aisi value jisse $\boxed{\text{range ki value zero hoye}}$ $y=0$

ROOTS OF A NON-LINEAR EQUATIONS:

Consider a non-linear function $y=f(x)$.

Let $x_R \in D_f$, such that $f(x_R)=0$.

D_f → Domain of function

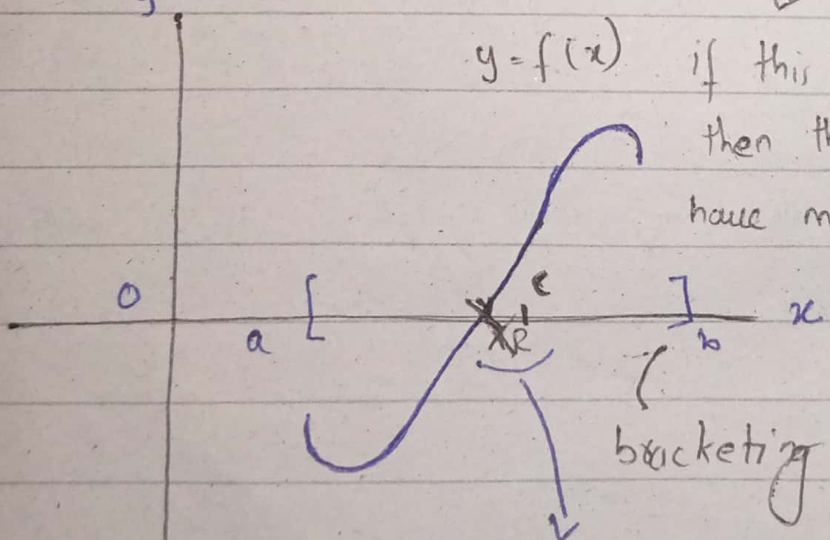
Assume equation has min one root.

1- BISECTION METHOD (INTERVAL HALVING METHOD) bounded/bracketing

Root bracketing

Let $a, b \in D_f$, such that $f(a)f(b) \leq 0$

if this satisfies then the interval have minimum one root in that interval



$[a, c]$ Root is in $a-c$

$[c, b]$ x discard

- Aik interval nikalna hai jise root bracket hjye

2- Bi-sect the interval (divide in two)

$$c = \frac{a+b}{2}$$

STEPS:

-> Select Interval

-> Select Midpoint

-> Repeat

disadv

-> Iteration wise slow

-> Convergence wise easy

adv

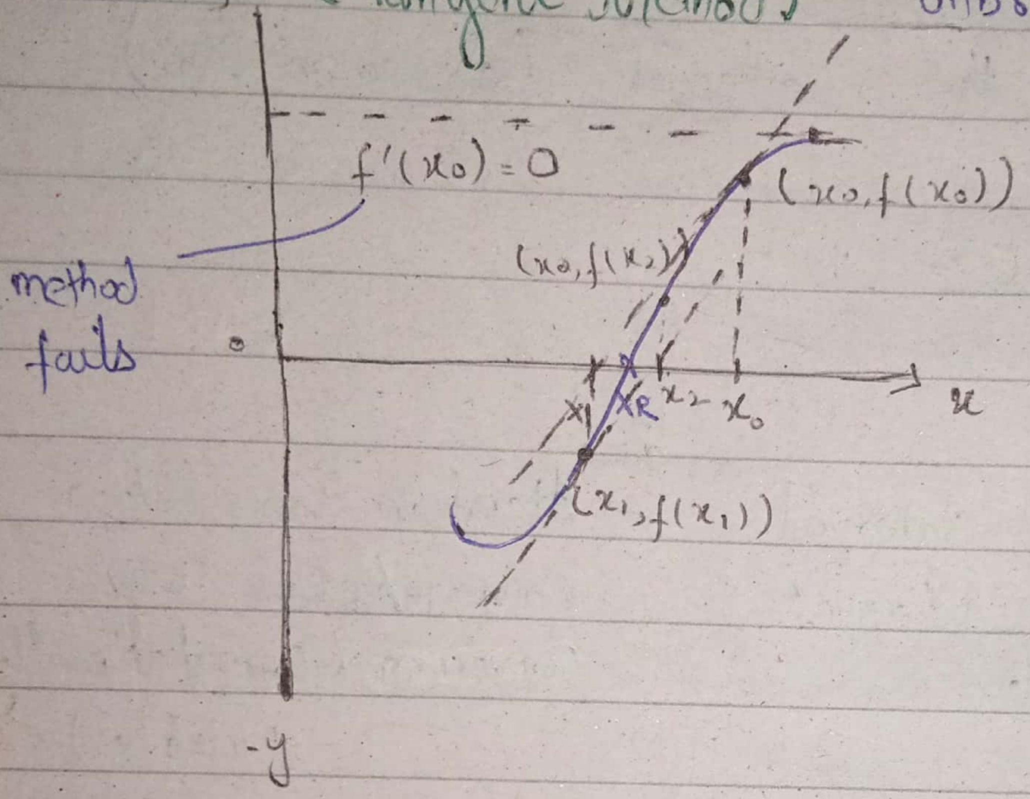
conversion (computationally suitable)

Above method is called Bisection method for finding the roots of non-linear equations. This method is slowest in terms of no. of iterations. This method is sensitive towards the choice of initial interval.

Adv -> It will never breakdown.

2- NEWTON RAPHSON METHOD:

(Slope-based search) unbounded/
(Tangent Method) unbracketing



Let $\bar{x}_0 \in D_f$, such that $f'(x_0) \neq 0$

$x_0 \longrightarrow (x_0, f(x_0))$

Equation Tangent To : $f'(x_0) =$ slope $\longrightarrow \dfrac{y_2 - y_1}{x_2 - x_1}$

$$f'(x_0) = \frac{\overset{0}{y} - f(x_0)}{x_{①} - x_0}$$

$\longrightarrow$ If $x = x_1$ then $y = 0$ intercept

$$x_1 = x_0 - \left(\frac{f(x_0)}{f'(x_0)} \right)$$

Numerator $= 0$ → Root agya equation ka

Denominator $= 0$ → Method fails to converge

Both N & D $= 0$ → Indeterminant form

$$\boxed{x_{i+1} = x_i - \left(\frac{f(x_i)}{f'(x_i)}\right)}$$

$\forall i = 0, 1, 2, \ldots$

→ It is faster in terms of iteration.

→ Restart if failed to converge.

↓

If $f'(x_i) = 0$, Method failed to converge.

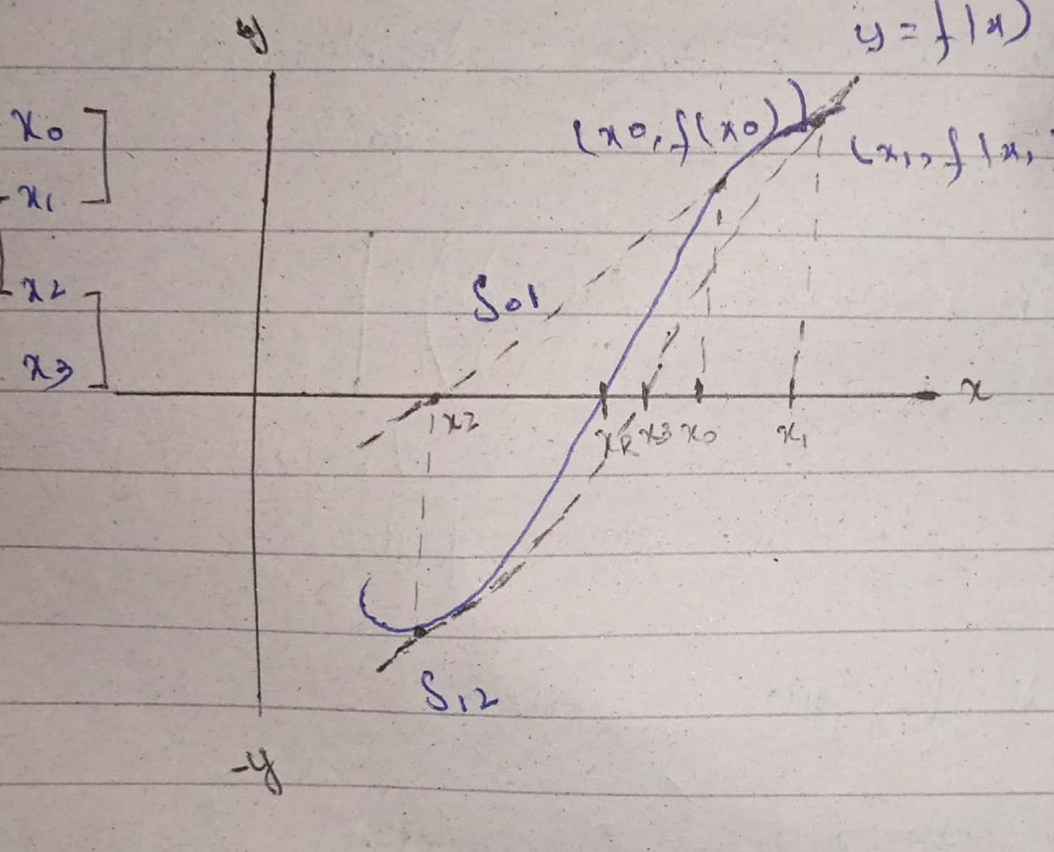
Above method is called Newton-Raphson's method.

- fastests in terms of iteration.
- fails to converge if tangent is parallel to x-axis i.e $f'(x_i) = 0$
- If $f(x_i) = 0$, the value becomes constant & we get the answer (root).

3- Secant Method: unbounded

Consider a non-linear function $y = f(x)$

let $x_R \in D_f$ such that $f(x_R) = 0$



Let $x_0, x_1 \in D_f$ such that $f(x_0) \neq f(x_1)$

$x_0, x_1 \longrightarrow (x_0, f(x_0)), (x_1, f(x_1))$

If $f(x_1) - f(x_0) = 0$,

- Secant become parallel to x-axis
- failed to converge.

Eq of Sec (Two-point form)

$$\frac{\overset{0}{y} - f(x_0)}{x_2 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

if $x = x_2$ then $y = 0$

intercept

$$\frac{x_2 - x_0}{-f(x_0)} = \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$x_2 = x_0 + \frac{x_0 f(x_0) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{x_0 f(x_1) - \cancel{x_0 f(x_0)} + \cancel{x_0 f(x_0)} - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$\boxed{x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)}}$$

$\forall\ i = 0, 1, 2, \ldots$

→ faster than bisection.
→ Slower than Newton Raphson.

4) FALSE POSITION METHOD (Regula Falsi)

combo of Secant & bisection
bounded

$$\boxed{c = \frac{a f(b) - b f(a)}{f(b) - f(a)}}$$

when $f(a) f(b) < 0$

$a = c$ or $b = c$
for next iteration

$- \ x_0$
$+ \ x_1$
x_2 (if $-ve$)
then x_1, x_2
(if $+ve$)
the x_0, x_2

In False Position method, root must be bounded. This method will not fail to converge as compared to secant method.
Slower than newton.
Faster than bisection.

Minimum 4 decimal places

Solve the following equation using
1) Bisection Method
2) Newton Raphson Method
3) Secant Method
4) False Position Method

Use stopping criteria/condition

$$|f(x_i)| \leq 10^{-2}$$

0.01

$f(x) = 3x^3 + 2x^2 - x + 10$ (Type 1)

1) BISECTION METHOD:

Tip: Interval choty se chota

-2, 0 ✗

-2, -1 ✓ preferable

-ve replaces -ve

+ve " +ve

in the next iteration

$f(0) = 10$

$f(-1) = 10$ ✓

$f(-2) = -4$ ✓

Dualside Conversions or Pendulum behaviour } root k left side right side move kr rha h..

Iteration #	a (-ve)	f(a)	b (+ve)	f(b)	$c = \frac{a+b}{2}$	f(
1	-2	-ve	-1	+ve	-1.5	+5.875
2	-2	-ve	-1.5	+ve	-1.75	+1.796
3	-2	-ve	-1.75	+ve	-1.875	-0.869
4	-1.875	-ve	-1.75	+ve	-1.8125	-0.519
5	-1.875	-ve	-1.8125	+ve	-1.8437	-0.15
6	-1.8437	-ve	-1.8125	+ve	-1.8281	0.1837
7	-1.8437	-ve	-1.8281	+ve	-1.8359	0.0130
8	-1.8437	-ve	-1.8359	+ve	-1.8398	-0.072
9	-1.8398	-ve	-1.8359	+ve	-1.8378	-0.02
10	-1.8378	-ve	-1.8359	+ve	-1.8368	-0.0

Bisection = 10 Iterations

2) NEWTON - RAPHSON METHOD:

$$\boxed{x_{i+1} = x_i - \left(\frac{f(x_i)}{f'(x_i)}\right)}$$

$\forall\ i = 0, 1, 2, \dots$

$f'(x_i) \neq 0$

$\boxed{x_0 = -1.5}$ → midpoint of interval , $\frac{-1-2}{2} = -1.5$

$f'(x) = 9x^2 + 4x - 1$

$f'(-1.5) = 13.25$

$i = 0$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = (-1.5) - \frac{f(-1.5)}{f'(-1.5)}$$

$x_1 = -1.9434$

$f(x_1) = -2.5225$

$i = 1:$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = (-1.9434) - \frac{f(-1.9434)}{f'(-1.9434)}$$

$$x_2 = -1.8434$$

$$f(x_2) = -0.15$$

$i = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = -1.8365$$

$$f(x_3) = -1.03 \times 10^{-4}$$

Newton - Raphson = 3 Iterations

3) SECANT METHOD :

$$x_{i+2} = \frac{x_i\, f(x_{i+1}) - x_{i+1}\, f(x_i)}{f(x_{i+1}) - f(x_i)}$$

$\forall\ i = 0, 1, 2, \ldots$

$f(x_i) \neq f(x_{i+1})$

$x_0 = -1.2 \qquad x_1 = -1.3$

$i = 0 : \qquad x_0 = -1.2 \qquad x_1 = -1.3$

Check: $f(x_0) \neq f(x_1)$

$$x_2 = \frac{x_0\, f(x_1) - x_1\, f(x_0)}{f(x_1) - f(x_0)}$$

$x_2 = -2.3023$

$i = 1 \qquad x_1 = -1.3 \qquad x_2 = -2.303$

Check $f(x_1) \neq f(x_2)$

$$x_3 = \frac{x_1\, f(x_2) - x_2\, f(x_1)}{f(x_2) - f(x_1)}$$

$x_3 = -1.6719$

$i = 2$ $\quad x_2 = -2.303$ $\quad x_3 = -1.6719$

$f(x_2) \neq f(x_3)$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$x_4 = -1.7924$

$i = 3$ $\quad x_3 = -1.6719$ $\quad x_4 = -1.7924$

$f(x_3) \neq f(x_4)$

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$f(x_3) = 3.242265751$

$f(x_4) = 0.9424771249$

$x_5 = -1.841782144$

$i = 4$ $\quad x_4 = -1.7924$ $\quad x_5 = -1.8418$

$$x_6 = \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)}$$

$f(x_5) = -0.1172$

1. False Position Method (Regula Falsi)

$$c = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$f(a) f(b) < 0$$

Let $a = -2$ $\quad b = -1$

$f(-2) = -4$ $\quad f(-1) = 10$

$x_6 = -1.8363$

$i = 5$ $\quad x_5 = -1.8418$ $\quad x_6 = 1.1616$

$$x_7 = \frac{x_5 f(x_6) - x_6 f(x_5)}{f(x_6) - f(x_5)}$$

$f(x_6) = 16.2391$

$f(x_6) = 0.004297$

C ki value a or b -2 -1
se bahir nhi jaegi

Iter	a (-ve)	f(a)	b (+ve)	f(b)	$c = \frac{af(b) - bf(a)}{f(b) - f(a)}$	f(c)
1	-2	-ve	-1	+ve	-1.7142	2.4781
2	-2	-ve	-1.7142	+ve	-1.8235	0.2835
3	-2	-ve	-1.8235	+ve	-1.8351	0.0288
4	-2	-ve	-1.8351	+ve	-1.8363	0.00472

Answer: -1.8363

store A
store B
$x = 3F^3 + 2F^2 - F + 10$
Give F = A
Change x → y
$y = 3F^3 + 2F^2 - F + 10$
Give F = B
$C = \frac{Ay - Bx}{y - x}$
Change y → E
$E = 3F^3 + 2F^2 - F + 10$
Give F = C

Computational Complexity of an algorithm represents, how many arithmetic operations are required to perform certain tasks. If 'n' represent computational complexity of a certain algorithm for which total complexity can be represented by the following equations:

$$C(n) = n^3 - n^2 + n - 20$$

Find the no. of operations required to have maximum total complexity equal to 150.

$$\Rightarrow n^3 - n^2 + n - 20 = 150$$

$$\Rightarrow n^3 - n^2 + n - 170 = 0$$

$$f(n) = 0$$

Curve cannot be represented by straight lines

NUMERICAL INTEGRATION:

$$\int_{a}^{b} \boxed{f(x)}\, dx \cong \int_{a}^{b} P_n(x)\, dx$$

Integrand

Limits

$$f(x) \cong P_n(x)$$

$$a \le x \le b$$

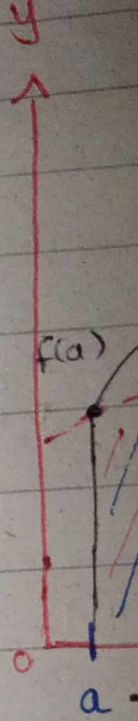
Replace with equivalent Polynomial.

METHODS FOR EQUIVALENT POLYNOMIAL:

① TREPEZOIDAL RULE OF INTEGRATION:

Degree 1 Polynomial

$$\int_{a}^{b} f(x)\, dx \cong \int_{a}^{b} P_1(x)\, dx = \int_{a}^{b} (c_1 x + c_2)\, dx$$



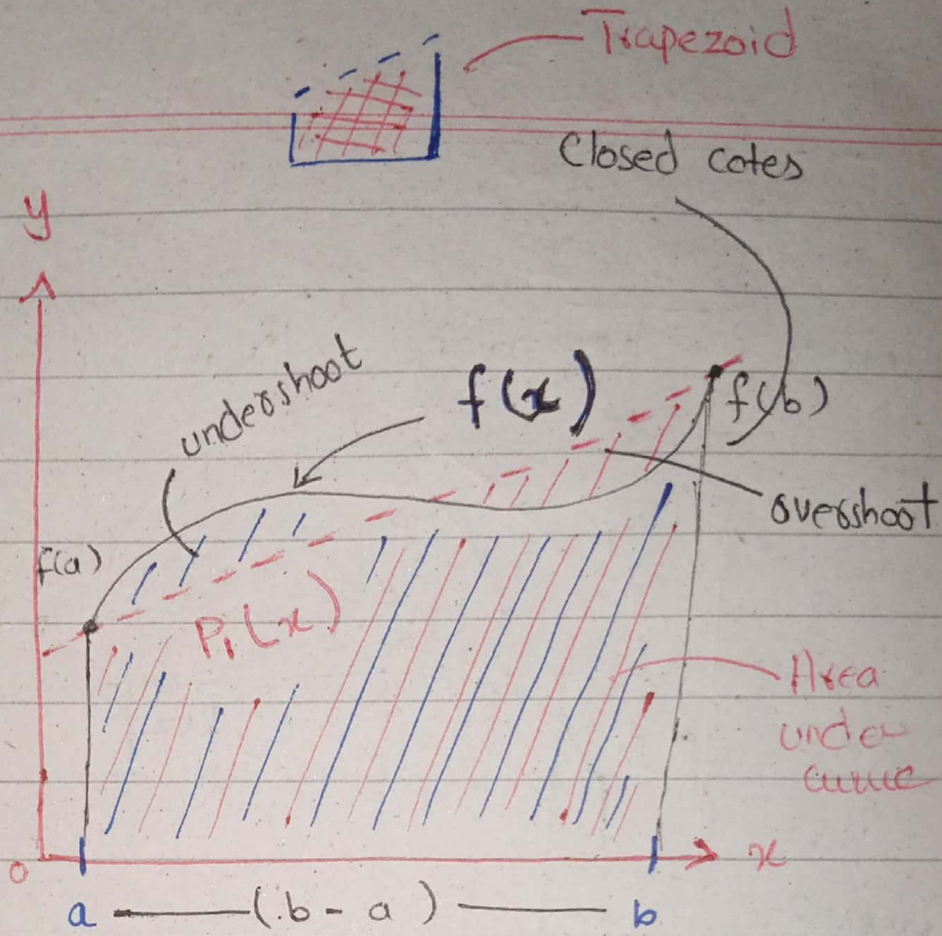
Closed Cotes:

Open Cotes:

Area of Trap

Problem in

overshoots on



Closed Cotes: Ending points se polynomial line pass hgi.

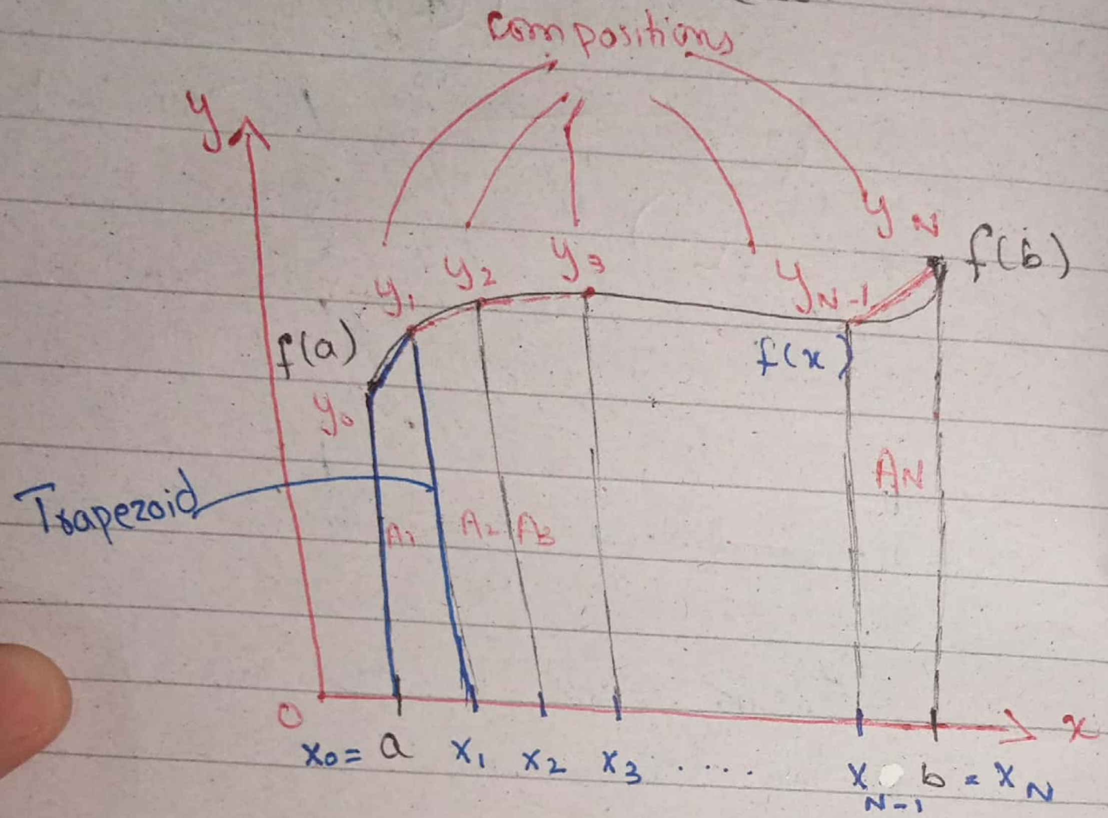
Open Cotes: " " " pass nhi hgi.

Area of Trapezoid $= \left(\frac{f(a)+f(b)}{2}\right)(b-a)$ — 1

Problem in this rule is there are overshoots and undershoots

To enhance the accuracy of Trapezoidal Rule of integration.

Composite Trapezoidal Rule:



Break into N Intervals

$$h = \frac{b-a}{N} \qquad \begin{matrix} N \to \infty \\ h \to 0 \end{matrix}$$

$$\int_a^b f(x)\,dx \cong A_1 + A_2 + \cdots + A_N \cong \left(\frac{y_0 + y_1}{2}\right)h + \left(\frac{y_1 + y_2}{2}\right)h + \cdots + \left(\frac{y_{N-1} + y_N}{2}\right)h$$

Every y is appearing 2 times except y_0 & y_N that's why we are using y_N.

$$\int_a^b f(x)\,dx \cong \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots y_{N-1}) + y_N\right] \quad — (2)$$

Disadv ; any non-linear integral cannot be solved in trapezoid

improvement -> composite (intervals -> sub intervals) hp-adaptivity → h km se km accuracy zyada se zyada

width degree

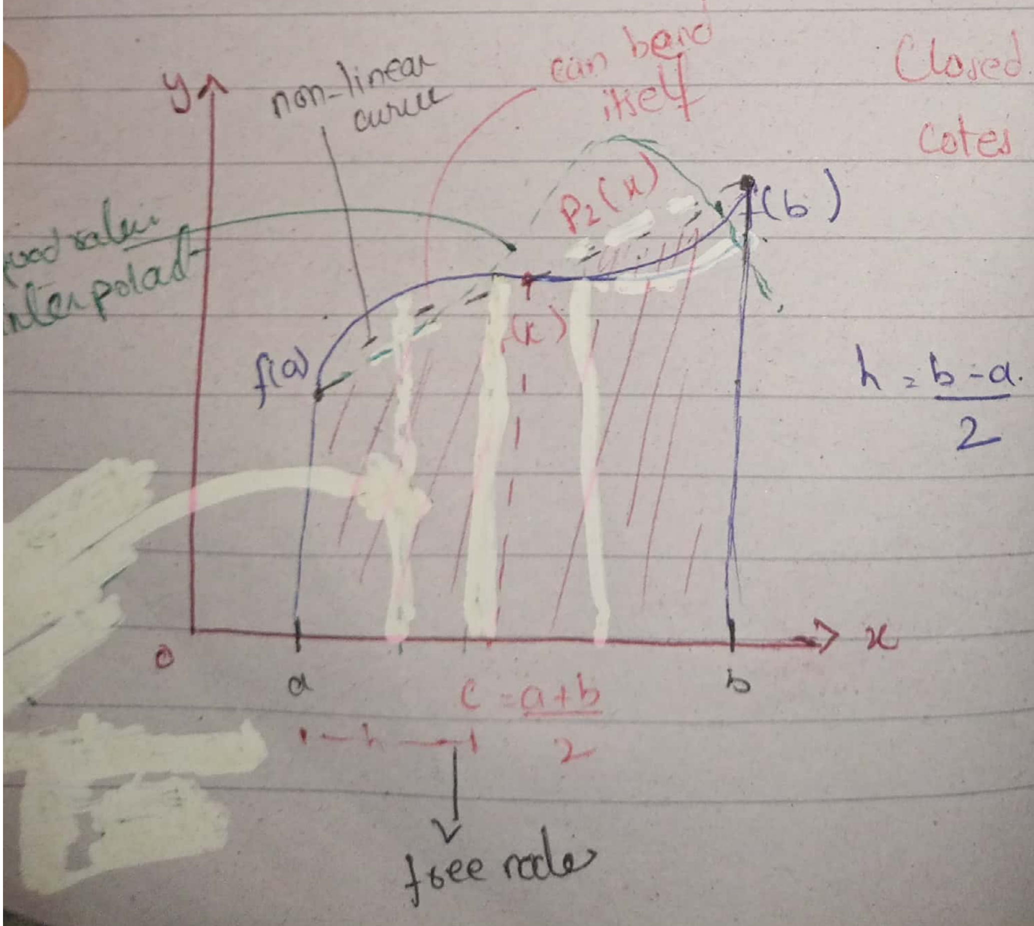
degree 2 possibility

② SIMPSON'S $\frac{1}{3}$rd RULE:

$$\int_a^b f(x)\,dx \cong \int_a^b P_2(x)\,dx = \int_a^b (c_1x^2 + c_2x + c_3)\,dx$$

width of interval + degree of polynomial →
balance b/w two. → hp-adaptivity

Consider above integral.



$$c_1 a^2 + c_2 a + c_3 = f(a) \quad \text{— (1)}$$
$$c_1 b^2 + c_2 b + c_3 = f(b) \quad \text{— (2)}$$

• Free node should be the mid point of interval.

$$c_1 c^2 + c_2 c + c_3 + f(c) \quad \text{— (3)}$$

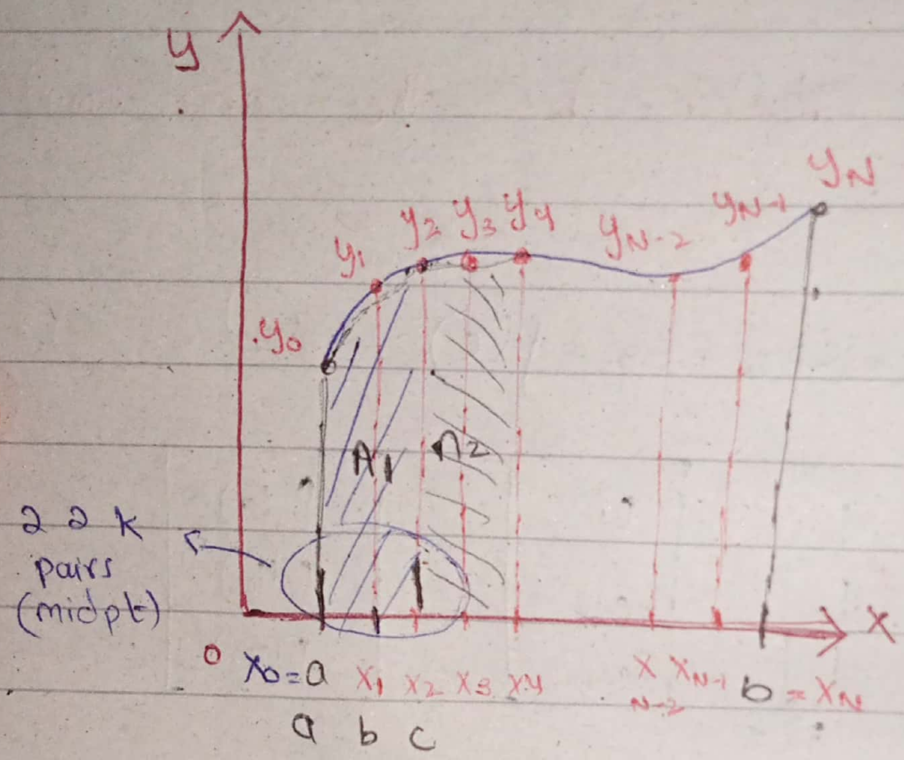
$$\underset{3\times 3}{\begin{bmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} f(a) \\ f(b) \\ f(c) \end{bmatrix}$$

$$\int_a^b f(x)\,dx \cong \underbrace{\frac{h}{3}\left[f(a) + 4f(c) + f(b) \right]}_{\text{Area of } P_2(x)} \quad \text{— (3)}$$

Still there are over & undershoots.

To enhance accuracy:

Composite Simpson's $\frac{1}{3}$rd Rule:



N = no. of intervals.

$h = \frac{b-a}{N}$ $\quad N \to \infty$

$h \to 0$

reason

Restriction on N:

$\to$ N must be even no / multiple of 2..

odd hoga tw koi aik interval reh jyega. (puri domain krni h cover that's why)

$$\int_a^b f(x)\,dx \cong A_1 + A_2 + A_3 + \cdots + A_{N/2}$$

$$\int_a^b f(x)\,dx \cong \frac{h}{3}[y_0 + 4y_1 + y_2] +$$

$\times 2$

$$\frac{h}{3}[y_2 + 4y_3 + y_4] +$$

$$\frac{h}{3}[y_4 + 4y_5 + y_6] + \cdots +$$

$$\frac{h}{3}[y_{N-2} + 4y_{N-1} + y_N]$$

$$\int_a^b f(x)\,dx \cong \frac{h}{3}\Big[y_0 + 4(y_1 + y_3 + y_5 + \cdots y_{N-1})$$

$$2(y_2 + y_4 + y_6 + \cdots + y_{N-2}) + y_N)\Big]$$

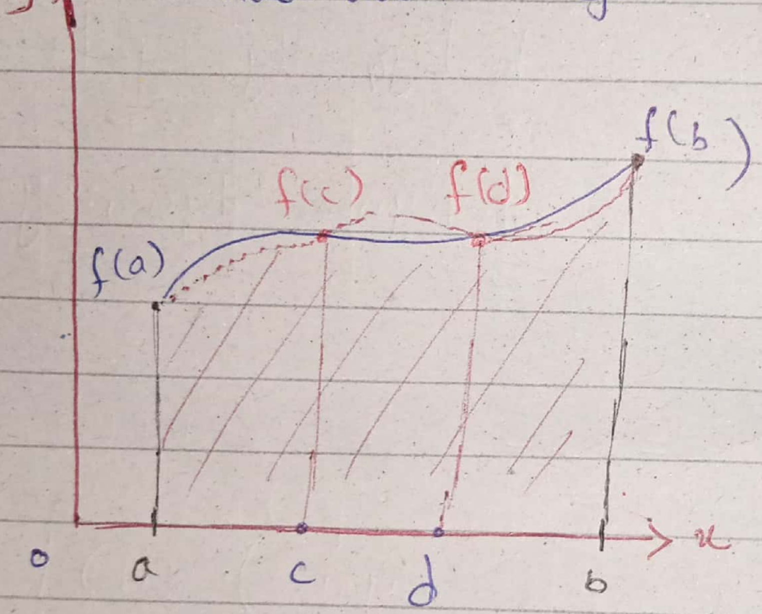
—— (4)

③ Simpson's $\frac{3}{8}$th Rule:

$$\int_a^b f(x)\,dx \simeq \int_a^b P_3(x)\,dx =$$

$$\int_a^b (c_1x^3 + c_2x^2 + c_3x + c_4)\,dx$$

Consider above integral

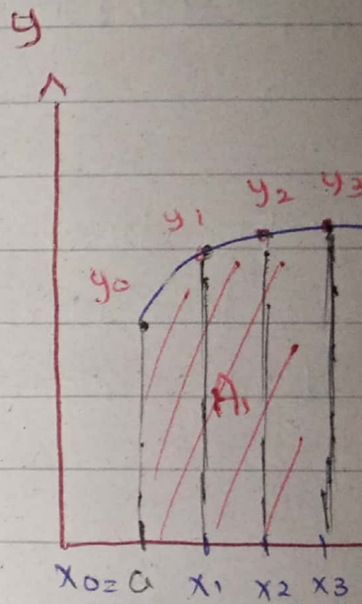


$$h = \frac{b-a}{3}$$

$$\int_a^b f(x)\,dx \simeq \frac{3h}{8}\left[f(a) + 3f(c) + 3f(d) + f(b)\right]$$

Still overshoo

Composite Simps



N must

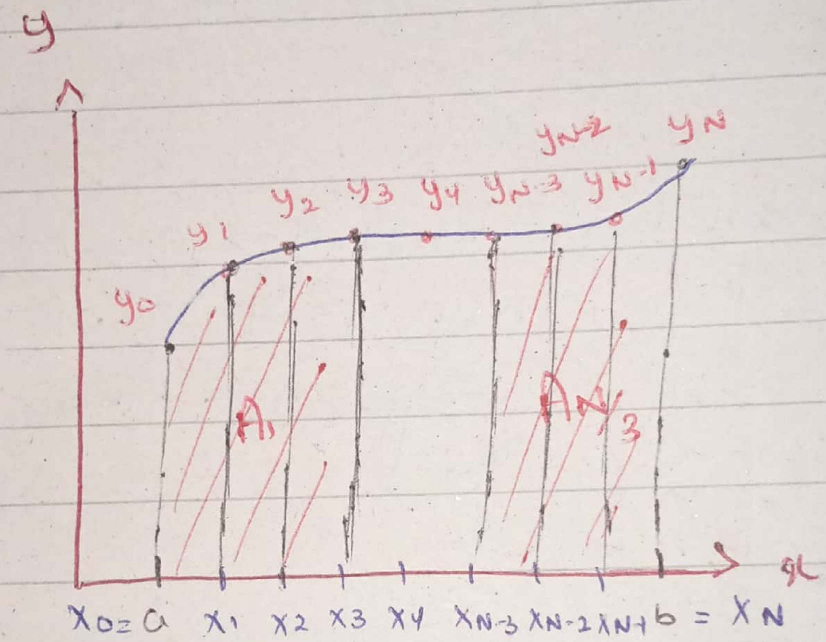
$$h = \frac{b-a}{N}$$

$$\int_a^b f(x)\,dx \simeq$$

$$\int_a^b f(x)\,dx \simeq 3$$

Still overshoots undershoots.

Composite Simpson's $\frac{3}{8}$th Rule:



N must be multiple of 3.

$$h = \frac{b-a}{N}$$

$$\int_a^b f(x)\,dx \cong A_1 + A_2 + A_3 \cdots A_{N/3}$$

$$\int_a^b f(x)\,dx \cong \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right] +$$

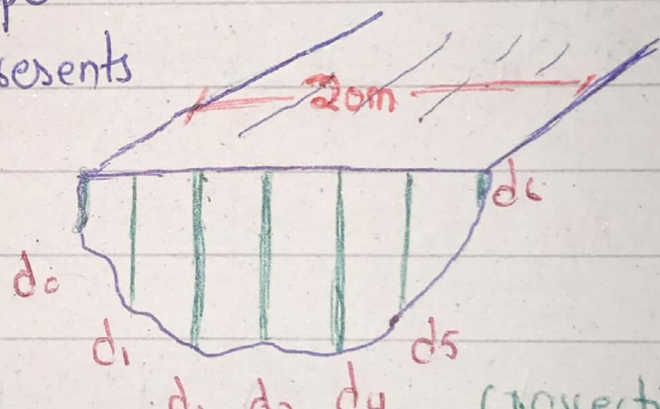
$$\frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right] + \cdots +$$

$$\frac{3h}{8}\left[y_{N-3} + 3y_{N-2} + 3y_{N-1} + y_N\right]$$

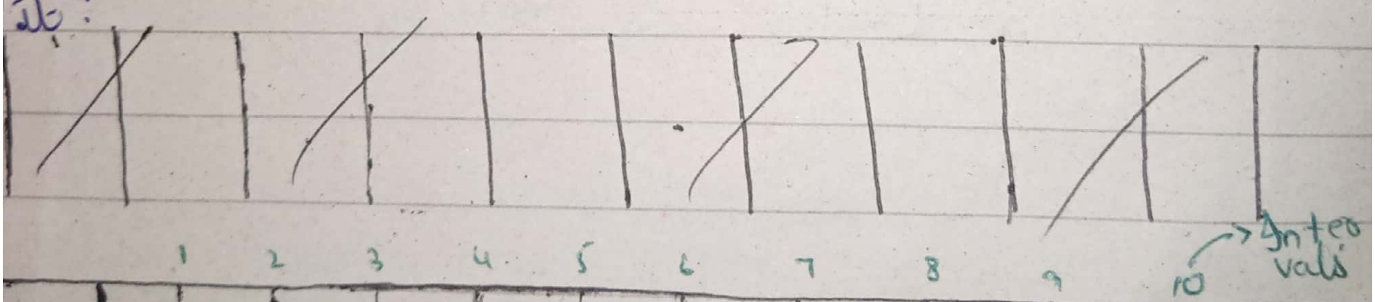
$$\int_a^b f(x)\,dx \cong \frac{3h}{8}\left[y_0 + 3(y_1 + y_4 + y_7 + \cdots y_{N-2}) + 3(y_2 + y_5 + y_8 + \cdots y_{N-1}) + 2(y_3 + y_6 + y_9 + \cdots y_{N-3}) + y_N\right] \quad (6)$$

Consider the following data which represents the depth of river bed (d) obtained at different locations along the crossectional areas. Find the crossectional area of the riverbed and estimate the volume of the river with the breath of the crossection is 20m and length of the river is 30m

It is crossection shape which somewhat represents parabolic functions but to find accuracy 3/8 will be used, so that if it is 3 degree it can also compute it.



Crossection of river



x(m)	0	2	4	6	8	10	12	14	16	18	20
d(m)	1	3	5	7	10	14	17	16	14	12	5
	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}

$3/8$ cannot be applied here as no. of intervals are not multiples of 3.
(i.e, 10)

Approach ①:

but We want to use $\frac{3}{8}$, we will achieve it by hybrid approach.

Apply trapezoid from 0-2 (1 interval)
9 are left (which will be solved by $\frac{3}{8}$)

Approach ②: ✓✓

$\frac{1}{3}$ on (0 - 4) 4 intervals

$\frac{3}{8}$ on (rest) 6 intervals

Using $\frac{1}{3rd}$ rule of integration.

$h = 2m$ unit

Crossectional Area $= \frac{h}{3}[d_0 + 4(d_1 + d_3 + d_5 + d_7 +$

$d_9) + 2(d_2 + d_4 + d_6 + d_8) + d_{10}]$

Crossectional Area $= 204\ m^2$

Volume $= 204\ m^2 \times 30m$

Volume $= 6120\ m^3$

Q2 (a) pg# 7 $\int_0^{10}$ or $\int_0^t$ // correction

(b) $t \in [0, 10]$

Q3 (b) no. of interval = 7
break into
4 and 3
solve 1/3 3/8

can also be solved by trapezo

Q4. Independent (V)
Dependent (P)

* Verify the lengths of intervals are same

Limit starting from 0.25

$$\underset{\text{independent}}{\frac{dy}{dx}} = \overset{\text{dependent}}{xy}$$

Numerical Solution of O.D.E:

Single variable dependency

$\frac{dy}{dx} = f(x,y)$ {Initial value problem} will be given

$$\boxed{\frac{dy}{dx} = e^{xy}}$$

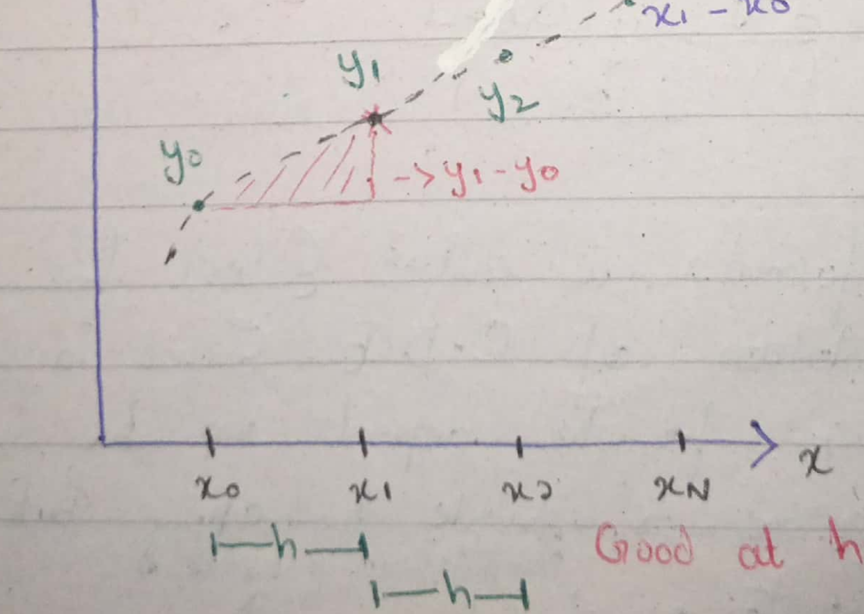
No analytical method to solve this problem.

Order & Degree = 1

Consider a differential equation of the following form $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$. i.e. Initial value Problem (I.V.P)

Method # 1: "Euler's Method"

$$y_N \quad \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h} = \text{slope}$$



Good at $h \to 0$

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h} = \text{Slope}$$

$$\text{Slope} = f(x_0, y_0)$$

$$\frac{y_1 - y_0}{h} = f(x_0, y_0)$$

$$\boxed{y_1 = y_0 + h f(x_0, y_0)}$$

Better at $h \to 0$ but will take longer time

$$\boxed{y_{i+1} = y_i + h f(x_i, y_i)}$$

$$\forall i = 0, 1, 2, \ldots$$

$$\frac{dy}{dx} = \underbrace{e^{xy}}_{f(x,y)} \quad y(\overset{x_0}{0}) = \overset{y_0}{1} \quad x_N = 1$$

Example

Above formula is called Euler's for the solution of O.D.E. Error in the above formula depends on h.

This method is simple to apply but not accurate enough. computationally suitable

$h \downarrow$ Accuracy $\uparrow$

also known as 2-stage R-K Method.

new = old + h Slope

METHOD # 2:
MODIFIED EULER'S METHOD: → 2-Stage Method calculates Slope 2 times.

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

→ Intermediate value

$$y^* = y_0 + h f(x_0, y_0) \text{ at } x_1$$

Predicted value

$$\boxed{y_1 = y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y^*)}{2}\right]}$$

at x_1

Average of old and new slopes → predicted & last value

For $x_0 \to x_1$:

$y_0 \to y^* \to y_1$

Intermediate value

→ Euler's se hi niklega

$$\boxed{y^* = y_i + h f(x_i, y_i) \text{ at } x_{i+1}}$$

$$\boxed{y_{i+1} = y_i + h\left[\frac{f(x_i, y_i) + f(x_{i+1}, y^*)}{2}\right]}$$

$i = 0, 1, 2, 3 \ldots$

Error is less as slope is taken two times & taken average so the deviation because of h is not so large.

We can take larger value of 'h', it will still give close to accurate reading.

h fixed $\rightarrow$ R.K

new = old + h slope → framework

Method #3
Range - Kutta Method of Stage 4
(R-K Method)

4 slopes

$$\boxed{y_{i+1} = y_i + \underbrace{\frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}}_{\text{Weighted average}}}$$

where,

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_i + h, y_i + k_3\right)$$

$\forall i = 0, 1, 2, 3, \ldots$

There are two extreme (k_1 and k_4), 2 in middle (k_2 and k_3)

* Interdependency / Sequential dependency

* Accuracy depends on h

h↓ Computation↑

h↓ Error↓

Initial value problem

Solve the following I.V.P → starting value

$$\frac{dy}{dx} = x^2 dy, \quad y(0) = 1$$

Use $N = 2$ and obtained solution at $x = 0.5$

① Euler's Method
② Modified Euler's Method
③ Runge-Kutta Method of order 4

① EULER'S METHOD:

$$y_{i+1} = y_i + h f(x_i, y_i) \quad \forall\, i = 0$$

$$h = \frac{0.5 - 0}{2} = 0.25$$

$i = 0$:

$$y_1 = y_0 + h f(x_0, y_0)$$

$$y_1 = 1 + (0.25)(x_0^2 + y_0)$$

$y_1 = 1 + (0.25)(0+1)$

$y_1 = 1.25$ at $x_1 = 0.25$

$i = 1$:

$y_2 = y_1 + hf(x_1, y_1)$

$y_2 = 1.25 + (0.25)(x_1^2 + y_1)$

$y_2 = 1.25 + (0.25)(0.25^2 + 1.25)$

$y_2 = 1.578$ at $x = 0.5$

② MODIFIED EULER'S METHOD:

$y^* = y_i + hf(x_i, y_i)$ at x_{i+1}

$$y_{i+1} = y_i + h\left[\frac{f(x_i, y_i) + f(x_{i+1}, y^*)}{2}\right]$$

$i = 0$:

$y^* = y_0 + hf(x_0, y_0)$

$y^* = 1 + (0.25)(x_0^2 + y_0)$

$y^* = 1 + (0.25)(0 + 1)$

$y^* = 1.25$ at $x_1 = 0.25$

$y_1 = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y^*))$

$y_1 = 1 + \frac{0.25}{2}(f(0,1) + f(0.25, 1.25))$

$y_1 = 1.289$ at $x_1 = 0.25$

$i = 1$

$y^* = y_1 + h f(x_1, y_1)$

$y^* = 1.289 + (0.25)(x_1^2 + y_1)$

$y^* = 1.289 + (0.25)(0.25^2 + 1.289)$

$y^* = 1.6268$ at $x_2 = 0.5$

$$y_2 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y^*)\right)$$

$$y_2 = 1.289 + \frac{0.25}{2}\left(f(0.25, 1.289) + f(0.5, 1.6268)\right)$$

$$y_2 = 1.289 + (0.125)(1.3515 + 1.8768)$$

$$y_2 = 1.6925 \quad \text{at} \quad x_2 = 0.5$$

3 RUNGE KUTTA'S METHOD:

$$y_{i+1} = y_i + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}$$

where,

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_i + h, y_i + k_3)$$

$i = 0$:

$$y_1 = y_0 + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}$$

$$k_1 = hf(x_0, y_0)$$

$$k_1 = (0.25) f(0, 1)$$

$$k_1 = (0.25)(0^2 + 1)$$

$$k_1 = 0.25$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_2 = (0.25) f\left(0 + \frac{0.25}{2}, 1 + \frac{0.25}{2}\right)$$

$$k_2 = (0.25) f(0.125, 1.125)$$

$$k_2 = (0.25)(0.125^2 + 1.125)$$

$k_2 = 0.2851$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_3 = (0.25)\, f\left(0 + \frac{0.25}{2}, 1 + \frac{0.2851}{2}\right)$$

$$k_3 = (0.25)\, f(0.125, 1.1425)$$

$$k_3 = (0.25)(0.125^2 + 1.1425)$$

$$k_3 = 0.2895$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k_4 = (0.25)\, f(0 + 0.25, 1 + 0.2895)$$

$$k_4 = (0.25)\, f(0.25; 1.2895)$$

$$k_4 = (0.25)(0.25^2 + 1.2895)$$

Numerical Solution Of A System Of Linear Equations:

Direct Method:

① LU - Decomposition Method:
(Matrix Factorization Technique)

$\underline{x}$ x-bar ($\underline{x}$)

$\underline{b}$ b-bar ($\underline{b}$)

$$A\underline{x} = \underline{b} \quad \text{— ①}$$

in form of $A = \underbrace{BC}$ Factors Matrix

Let,

$$A = LU \quad \text{— ②}$$

LUb known

x unknown

Put in ①

$$LU\underline{x} = \underline{b} \quad \text{— ③}$$

let,

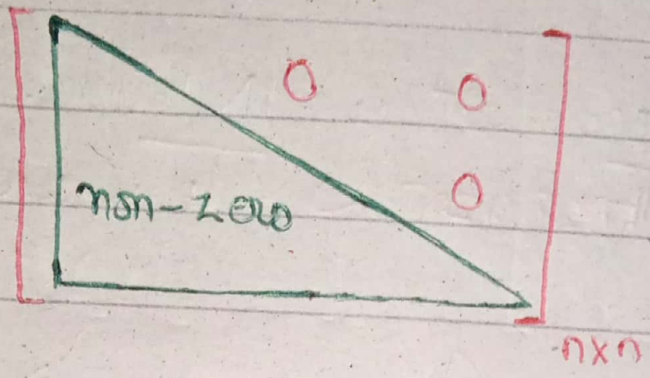
y-bar ($\bar{y}$)

$$U\underline{x} = \underline{y} \quad \text{— ④}$$

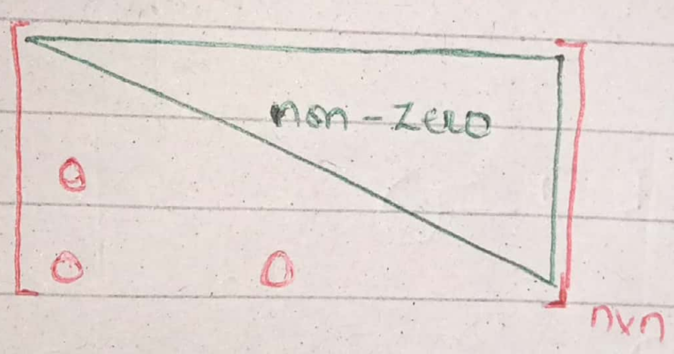
$$L\underline{y} = \underline{b} \quad \text{— ⑤}$$

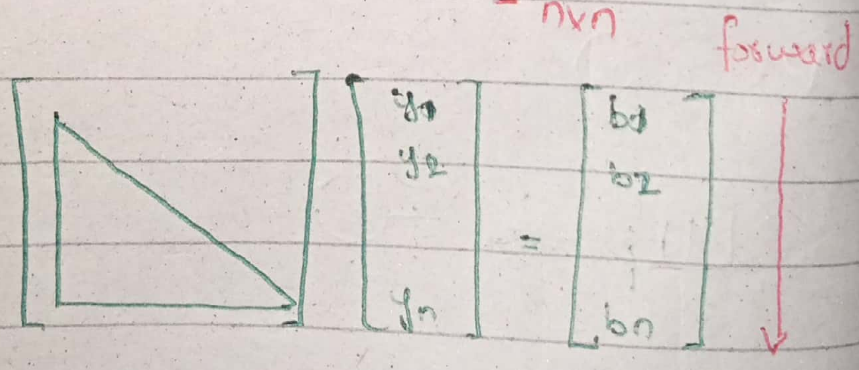
where 'L' is lower triangular matrix.

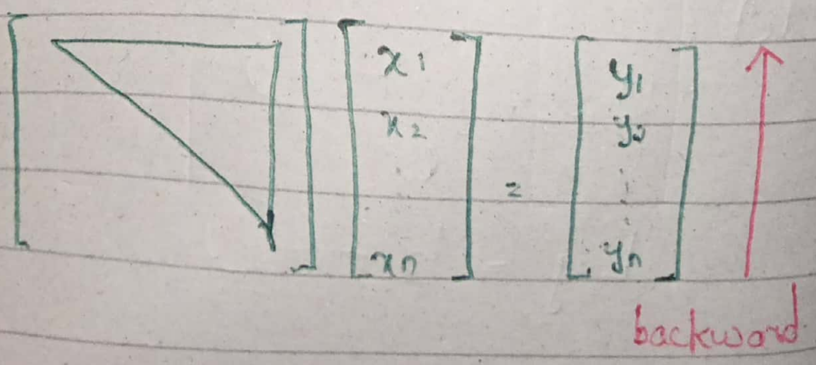
Lower Triangular e.g:

L = 

Upper Triangular e.g

U = 

Ly = b => 

Ux = y => 

$A = LU$

$|A| = |L| \cdot |U| = 0$ either $|L| = 0$ or $|U| = 0$

Diagonal m. matrix by zero ajye tu determinant zero. determinant zero means system does not exist.

$n = 4$

$un = 20$

$eq = 16$

Consider 3×3

Eq #

$A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$Eq = 9$

$Unknown = 12$

$n = 3$

$Eq > Unknown$

Assume krege 3.

Put diagonal elements same of one of the matrix.

Crout's method

Lower Matrix main diagonal main 1

Upper Matrix main " " "

Do-little's method

Solve the following system of equations using LU decomposition technique.

$$\begin{bmatrix} 9 & 8 & 1 & 7 \\ 3 & 9 & 8 & 1 \\ 4 & 2 & 2 & 7 \\ 8 & 3 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 6 \\ 20 \end{bmatrix}$$

SOLUTION:

Let,

$$A = \begin{bmatrix} 9 & 8 & 1 & 7 \\ 3 & 9 & 8 & 1 \\ 4 & 2 & 2 & 7 \\ 8 & 3 & 8 & 9 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \& \quad \underline{b} = \begin{bmatrix} 9 \\ 8 \\ 6 \\ 20 \end{bmatrix}$$

$$\Rightarrow \quad A\underline{x} = \underline{b} \quad \text{— (1)}$$

Let,

$$A = LU$$

$$\Rightarrow LU\underline{x} = \underline{b}$$

Let,

$$U\underline{x} = \underline{y} \quad \text{— (2)}$$

$$\Rightarrow L\underline{y} = \underline{b} \quad \text{— (3)}$$

where, 'L' and 'U' are lower and upper triangular matrices.

LU can be computed using the following factorization:

$$A = LU$$

$$\begin{bmatrix} 9 & 8 & 1 & 7 \\ 3 & 9 & 8 & 1 \\ 4 & 2 & 2 & 7 \\ 8 & 3 & 8 & 9 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} L_{11}U_{11} & L_{11}U_{12} & L_{11}U_{13} & L_{11}U_{14} \\ L_{21}U_{11} & L_{21}U_{12} + L_{22}+U_{22} & L_{31}U_{13}+L_{32} & \\ & & & \end{bmatrix}}$$

$$= \begin{bmatrix} L_{11}U_{11} & L_{11}U_{12} & L_{11}U_{13} & L_{11}U_{14} \\ L_{21}U_{11} & L_{21}U_{12}+L_{22}U_{22} & L_{21}U_{13}+L_{22}U_{23} & L_{21}U_{14}+L_{22}U_{24} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+L_{33}U & L_{31}U_{14}+L_{32}U_{24}+L_{33}U_{34} \\ L_{41}U_{11} & L_{41}U_{12}+L_{42}U_{22} & L_{41}U_{13}+L_{42}U_{23}+L_{43}U_{33} & L_{41}U_{14}+L_{42}U_{24}+L_{43}U_{34}+L_{44}U_{44} \end{bmatrix}$$

Let

$L_{11} = L_{22} = L_{33} = L_{44} = 1$

$L_{11} U_{11} = 9$

$(1) U_{11} = 9 \quad => \quad U_{11} = 9$

$L_{11} U_{12} = 8$

$(1) U_{12} = 8 \quad => \quad U_{12} = 8$

$L_{11} U_{13} = 1$

$(1) U_{13} = 1 \quad => \quad U_{13} = 1$

$L_{11} U_{14} = 7$

$(1) U_{14} = 7 \quad => \quad U_{14} = 7$

$L_{21} U_{11} = 3$

$L_{21} (9) = 3 \quad => \quad L_{21} = \frac{1}{3}$

$L_{21} U_{12} + L_{22} U_{22} = 9$

$\left(\frac{1}{3}\right)(8) + (1) U_{22} = 9$

$8 + 3 U_{22} = 27 \quad => \quad U_{22} = \frac{19}{3}$

$L_{21}U_{13} + L_{22}U_{23} = 8$

$\left(\frac{1}{3}\right)(1) + (1)U_{23} = 8 \Rightarrow U_{23} = 8 - \frac{1}{3}$

$U_{23} = \frac{23}{3}$

$L_{21}U_{14} + L_{22}U_{24} = 1$

$\left(\frac{1}{3}\right)(7) + (1)U_{24} = 1 \Rightarrow U_{24} = 1 - \frac{7}{3}$

$U_{24} = -\frac{4}{3}$

$L_{31}U_{11} = 4$

$L_{31}(9) = \frac{4}{\cancel{4}} \Rightarrow L_{31} = \frac{4}{9}$

$L_{31}U_{12} + L_{32}U_{22} = 2$

$\left(\frac{4}{9}\right)(8) + L_{32}\left(\frac{19}{3}\right) = 2$

$L_{32} = \left(2 - \frac{32}{9}\right)\left(\frac{3}{19}\right) = -\frac{14}{\cancel{9}\,3}\left(\frac{\cancel{3}}{19}\right)$

$L_{32} = \cancel{-\frac{90}{19}} \quad \cancel{-\frac{42}{}} \quad -\frac{14}{57}$

$A = LU$

$\det(A) = \det(L) + \det(U)$

If A is singular

$\det(A) = 0$ $\qquad$ ~~$\det(L)\det(U) = 0$~~ one of them

check diagonal of U or V, if it is 0 then singular.

$$L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} = 2$$

Not Recommended!

1.- JACOBI ITERATIVE METHOD: indirect

$$A_{n\times m} X = \underline{b} \quad -① $$

Consider a diagonally dominant system of equations i.e.

Matrix A is diagonally dominant

Diagonally Dominant: (to control the growth of coefficients)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{matrix} |a_{11}| \geq |a_{12}| + |a_{13}| \\ |a_{22}| \geq |a_{21}| + |a_{23}| \\ |a_{33}| \geq |a_{31}| + |a_{32}| \end{matrix}$$

Diagonally Dominant

Diagonal m zero nh askta.. agya zero agr tu rectangular matrix bnjyeg $\geq$ (equals) k case may. Yeh algo square matrix k liye bna h.

Rows interchange kr k bh check krna h, Diagonally Dominant tu nhe hai system.

e.g

$$\begin{bmatrix} 4 & 2 & 9 \\ 3 & -7 & -1 \\ 8 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 1 & 2 \\ 3 & -7 & -1 \\ 4 & 2 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_1 \\ b_2 \end{bmatrix}$$

System is diagonally dominant after interchanging R_1 & R_3.

We have following form of systems of equation:

$$x_1 = \frac{b_1}{a_{11}} - \frac{(a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n)}{a_{11}}$$

$$x_2 = \frac{b_2}{a_{22}} - \frac{(a_{21}x_1 + a_{23}x_3 + \cdot + a_{2n}x_n)}{a_{22}}$$

$\vdots$

$$x_n = \frac{b_n}{a_{nn}} - \frac{(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn-1}x_{n-1})}{a_{nn}}$$

initial vector $x^0 = \begin{bmatrix} x_0^0 \\ x_1^0 \\ x_2^0 \\ x_3^0 \end{bmatrix}$

$x^0 \to x^1 \to x^2 \to$ exact

x_1 better than x_0

x^2 better than x_0, x_1

Solve the following system of equations using Jacobbi Iterative Method and Iterative Method. Use error tolerance = 10^{-2}.

$$\begin{bmatrix} 2 & 2 & 1 & 3 \\ 4 & 1 & 1 & 1 \\ 2 & 8 & 3 & 1 \\ 1 & 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 9 \\ 8 \end{bmatrix}$$

Solution:

) Jacobbi - Iterative Method:

The Diagonally Dominant form of the system:

$$\begin{bmatrix} 4 & 1 & 1 & 1 \\ 2 & 8 & 3 & 1 \\ 1 & 2 & 7 & 1 \\ 2 & 1 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \\ 4 \end{bmatrix}$$

$$x_1^{i+1} = \frac{1}{4} - \frac{(x_2^i + x_3^i + x_4^i)}{4}$$

$$x_2^{i+1} = \frac{9}{8} - \frac{(2x_1^i + 3x_3^i + x_4^i)}{8}$$

$$x_3^{i+1} = \frac{8}{7} - \frac{(x_1^i + 2x_2^i + x_4^i)}{7}$$

$$x_4^{i+1} = \frac{4}{8} - \frac{(2x_1^i + x_2^i + x_3^i)}{8}$$

$\forall i = 0, 1, 2, \ldots$

Starting vector $\quad \underline{x^0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$i = 0$:

$$x_1^1 = \frac{1}{4} - \frac{(0+0+0)}{4}$$

$$x'_1 = \frac{1}{4}$$

$\cancel{x = x}$

$$x'_2 =$$

$$x'_2 = \frac{9}{8}$$

$$x'_3 =$$

$$x'_3 = \frac{8}{7}$$

$$x'_4 = 8$$

$$x'_4 = \frac{4}{8}$$

$\sqrt{(x)^2 + (y)^2 + (z)^2}$

$\underline{r} = \underline{0} = A\underline{x}' - \underline{b}$ = residue

$|\underline{r}| = |A\underline{x}' - \underline{b}| = 0$

$\leq$ tolerance

~~Itr~~ $i = 1$

$$\underline{x}' = \begin{bmatrix} 1/4 \\ 9/8 \\ 8/7 \\ 4/8 \end{bmatrix}$$

slope/gradient

Conjugate Gradient Method main se aik method steepest gradient method.

STEEPEST GRADIENT METHOD:

or Steepest Decent Method

$z = x^2 + y^2 + 10$

$x = ?$ $y = ?$ so that z would be maximum. or minimum

$y = f(x)$ $z = f(x, y)$ (x_0, y_0)

gradient/slope jis direction may change hota hga, us direction may move krenge tw maxima aiga ya minima

$\nabla z = \frac{\partial z}{\partial x} i + \frac{\partial z}{\partial y} k \longrightarrow$ gradient {is a changed vector}

$\Rightarrow \nabla z = 2x\hat{i} + 2y\hat{k}$

$\frac{\partial z}{\partial x} = (x - x_0)$

starting point

direction

$x - x_0 = \Delta x \frac{\partial z}{\partial x}$

$= y_0 + \Delta y \frac{\partial z}{\partial y}$

$x_1 = x_0 + \Delta x \frac{\partial z}{\partial x}$

steps

When two consecutive points are same, we are reached to maxima or minima point

$$\boxed{y_1 = y_0 + \Delta y \frac{\partial z}{\partial y}}$$

slope = 0

Let $\Delta x = \Delta y = h$

$$\boxed{x_{i+1} = x_i + h \left.\frac{\partial z}{\partial x}\right|_{(x_i, y_i)}}$$

$$\boxed{y_{i+1} = y_i + h \left.\frac{\partial z}{\partial y}\right|_{(x_i, y_i)}}$$

$\forall i = 0, 1, 2$

For maximize, just use the gradient

For minimize, use $-h$

h k sath -ve

model decided on the basis of data trend & set

model decided (quadratic) etc on our system ignoring Data trend & set.

INTERPOLATION & CURVE FITTING:

Interpolation

$x = 1.5$ $y = ?$

x	y
1	9
2	10
3	12
4	13
5	14

equally spaced

$x = 5.5$ $y = ?$

Dataset se bahir value is extrapolation

INTERPOLATION:

Interpolation
- Equally Spaced
- Unequally spaced

• EQUALLY SPACED:

1- Newton's forward difference method

2- Newton's backward difference method

• UNEQUALLY SPACED:

3- Lagrange Interpolation formula

Δ^2 -> Second order forward diff / Forward diff square

Δ -> Forward difference

① FORWARD DIFFERENCE (Δ)

(x_i, y_i) $\forall$ $i = 0, 1, 2, \cdots n$

$n+1$ dataset

$n+1$ equally spaced data set,

$x_{i+1} - x_i = h$

Equally & Unequally will be decided only on the basis of independent Variable (x).

$\boxed{\Delta y_i = y_{i+1} - y_i}$

x	y	Δ	Δ^2	$\Delta^3 \ldots \Delta^n$
x_0	y_0	$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
x_1	y_1	$\Delta y_1 = y_2 - y_1$		
x_2	y_2			
$\vdots$	$\vdots$			
x_{n-1}	y_{n-1}	$\Delta y_{n-1} = y_n - y_{n-1}$		
x_n	y_n			

Δ -> no Δy_n (last)

Δ^2 -> 2nd last won't be assigned $\Delta^2 y_{n-1}$

Δ^3 -> 3rd " "

$\nabla \rightarrow$ nabla

last value (Δy_n) ka forward nhi nkle paiga in fwd diff (Δ)

Sbse zyada first value k forward difference niklte hain (y_0) jisko hm anchor point bnaty hain.

② BACKWARD DIFFERENCE (∇)

$$\nabla y_i = y_i - y_{i-1}$$

First value y_0 ka backward nhi nkle paiga

FORMULAE:

NEWTON'S FORWARD DIFFERENCE FORMULA: (NFDF)

n points $\rightarrow$ $(n-1)$ difference

Let us consider $(n+1)$ data set of equally spaced.

2 points

3 points

$$f(x) = y_0 + \frac{\Delta y_0}{1!}\frac{(x-x_0)}{h} + \frac{\Delta^2 y_0}{2!}\frac{(x-x_0)}{h}\frac{(x-x_1)}{h} + \frac{\Delta^3 y_0}{3!}\frac{(x-x_0)}{h}\frac{(x-x_1)}{h}\frac{(x-x_2)}{h}$$

4 points

$$\ldots + \frac{\Delta^n y_0}{n!}\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{h^n}$$

local error = 0

$$f(x) = y_0 + \frac{\Delta y_0}{1!} \frac{(x-x_0)}{h} + \frac{\Delta^2 y_0}{2!} \frac{(x-x_0)(x-x_1)}{h^2} +$$

$$\frac{\Delta^3 y_0}{3!} \frac{(x-x_0)(x-x_1)(x-x_2)}{h^3} + \dots +$$

$$\frac{\Delta^n y_0}{n!} \frac{(x-x_0)}{h}\frac{(x-x_1)}{h} \cdots \frac{(x-x_{n-1})}{h}$$

(x_1, y_1)

$$f(x_1) = y_0 + \frac{(y_1 - y_0)}{1} \frac{(x_1 - x_0)}{h}$$

$$= \cancel{y_0} + (y_1 - \cancel{y_0}) \frac{\cancel{h}}{\cancel{h}}$$

$$f(x_1) = y_1$$

→ Above formula is Newton's forward difference formula for interpolation of equally spaced data.

NEWTON'S BACKWARD DIFFERENCE FORMULA:

$$f(x) = y_n + \frac{\nabla y_n}{1!}\frac{(x-x_n)}{h} + \frac{\nabla^2 y_n}{2!}\frac{(x-x_n)(x-x_{n-1})}{h^2}$$

$$+ \frac{\nabla^3 y_n}{3!}\frac{(x-x_n)(x-x_{n-1})(x-x_{n-2})}{h^3} + \cdots$$

$$\cdots \frac{\nabla^n y_n}{n!}\frac{((x-x_n)(x-x_{n-1})(x-x_{n-2})\cdots(x-x_1))}{h^n}$$

Above formula is Newton's backward difference formula for interpolation of equally spaced data. and used for the ~~Interformation~~ ~~of~~ interpolation (interpolating point) lying near the end of data.

Trend same ho to koi bh lagyen back forward fiq nhi pata below example have same trent.

Consider the following data and interpolate the result at $x = 1.5$ and $x = 4.5$.

x	y
1	1
2	8
3	27
4	64
5	125
6	216
7	343

$h = 2 - 1 = 1$

check all for verfical

① Check whether data is equaly spaced or not.

~~Solution~~:

② Take midpt ie 3.5

$1.5 < 3.5$ forward

$4.5 > 3.5$ backward

no need to show

Difference Table:

forward →

(3) degree ka trend mojood hai

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^n y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	y_0 1					0	0	0
		Δy_0 7				0	0	
2	8		$\Delta^2 y_0$ 12			0		
		19		$\Delta^3 y_0$ 6				
3	27		18					
		37		6 ← constant				
4	64		24					
		61		6				
	125		30					
		91		6 $\nabla^3 y_6$				
	216		36 $\nabla^2 y_6$					
		127 ∇y_6						
	343 y_6							

→ backward

exactly data $y=x^3$ ko follow kr rha

six] trend btata hai equation nhi btata

Jo column constant hjye data us column ki degree ko follow kr rha hga

Data may trend jitny points hn us se km degree ka hna chahiye. 7 ~~points~~ wrna interpolation error aega.

(1) $x = 1.5$

$$f(x) = y_0 + \frac{(x-x_0)}{h}\frac{\Delta y_0}{1!} + \frac{\Delta^2 y_0}{2!}\frac{(x-x_0)}{h}\frac{(x-x_1)}{h} + \frac{\Delta^3 y_0}{3!}\frac{(x-x_0)(x-x_1)(x-x_2)}{h^3}$$

$0 + 0 + 0$

$$f(1.5) = 3.375 = \frac{27}{8}$$

(2) $x = 4.5$

$$f(x) = y_6 + \frac{\nabla y_6}{1!}\frac{(x-x_6)}{h} + \frac{\nabla^2 y_6}{2!}\frac{(x-x_6)}{h}\frac{(x-x_5)}{h} + \frac{\nabla^3 y_6}{3!}\frac{(x-x_6)}{h}\frac{(x-x_5)}{h}\frac{(x-x_4)}{h}$$

$+ 0 + 0 + 0$

$f(4.5) =$

Works on equally spaced as well

Local error = 0 hna chahyiye   Max degree = 1

→ Not computationally suitable

1) LAGRANGE INTERPOLATION FORMULA:

(Unequally Spaced)

For 2-points: $(x_0, y_0), (x_1, y_1)$

→ Possible degree → Less than 2

Associated with

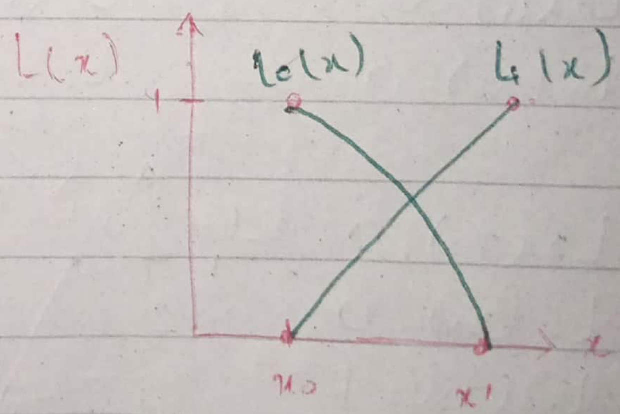
$$f(x) = L_0(x)y_0 + L_1(x)y_1$$

→ Lagrange Interpolating Polynomial

Lagrange Basses

$L_0(x_0) = 1$
$L_0(x_1) = 0$ → Apni node k ilawa jitni bh nodes hn un sb pr off (0) hoga

$L_1(x_0) = 0$
$L_1(x_1) = 1$



$$L_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$\boxed{f(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right) y_1}$$

$x_0 \leq x \leq x_1$

Local Error = 0

Maximum Degree = 1

For 3-points:

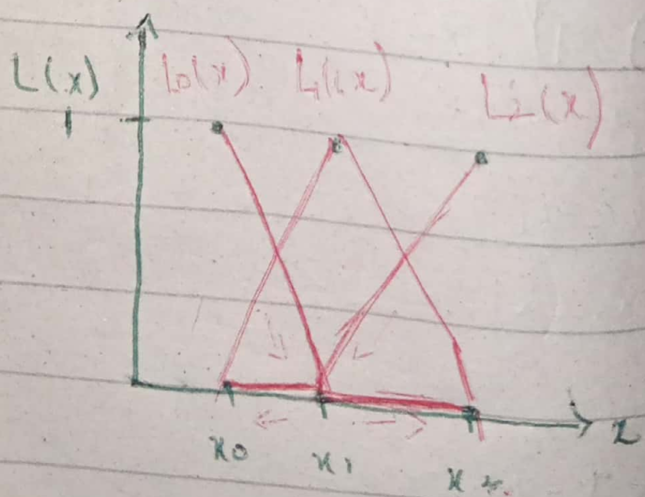
$(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$f(x) = L_0(x) y_0 + L_1(x) y_1 + L_2(x) y_2$

$L_0(x_0) = 1$
$L_0(x_1) = 0$
$L_0(x_2) = 0$

$L_1(x_0) = 0$
$L_1(x_1) = 1$
$L_1(x_2) = 0$

$L_2(x_0) = 0$
$L_2(x_1) = 0$
$L_2(x_2) = 1$



$$L_0(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$L_1(x) = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$L_2(x) = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)$$

For $(n+1)$ - points:

$$f(x) = \sum_{i=0}^{n} L_i(x) y_i$$

like summation $\pi = (\ )(\ )(\ )$

where,

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

This is called Lagrange interpolation formula of nth degree.

Data zyada hotw piecewise
Strategy

Linear Model	Quadratic Model

Q. Consider the following data and interpolate the result at $x = 4.3$

			x_0	x_1		
x	0	3	4	9	13	15
y	10	2	5	8	6	3

Using Lagrange interpolating formula
Use 2 points & 3 points interpolation.
(Piecewise Strategy)

Solution:

Using 2-point Lagrange formula
(Linear fitting)

$(x_0, y_0) = (4, 5)$
$(x_1, y_1) = (9, 8)$

$$f(x) = \frac{(x-9)}{(4-9)}(5) + \frac{(x-4)}{(9-4)}(8)$$

Fit a Poly nomi khenge x ki term m order

Pat

f(x

f(

Using

f(x)

f(x)

$\left(\frac{x-x}{x_1-}\right.$

f(x) =

Put $x = 4.3$

$$f(x) = \frac{(4.3-9)}{(4-9)}(5) + \frac{(4.3-4)}{(9-4)}(8)$$

$f(4.5) = 5.18$ → $\overset{y_0}{5.08}$ $\overset{y_1}{8}$ k bich may ani chahiye.

Using 3 point Lagrange formula:

$$(x_0, y_0) = (4, 5)$$
$$(x_1, y_1) = (9, 8)$$
$$(x_2, y_2) = (13, 6)$$

~~$f(x) = L_0(x)$~~

$$f(x) = \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)(y_0) +$$
$$\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)(y_1) + \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)(y_2)$$

$$f(x) = \left(\frac{x-9}{4-9}\right)\left(\frac{x-13}{4-13}\right)(5) + \left(\frac{x-4}{9-4}\right)\left(\frac{x-13}{9-13}\right)(8)$$

$$+\left(\frac{x-4}{13-4}\right)\left(\frac{x-9}{13-9}\right)(6)$$

Put $x = 4.3$

$$f(4.3) = \left(\frac{4.3-9}{4-9}\right)\left(\frac{4.3-13}{4-13}\right)(5) +$$

$$\left(\frac{4.3-4}{9-4}\right)\left(\frac{4.3-13}{9-13}\right)(8) +$$

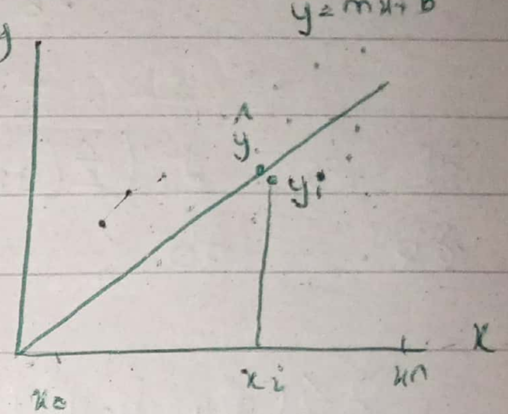
$$\left(\frac{4.3-4}{13-4}\right)\left(\frac{4.3-9}{13-9}\right)(6)$$

$$f(4.3) = 5.3523$$

yhn local error zero nhi
zaruri nhi
Total error ko minimum
krege

Curve Fitting

(4) Linear Fitting: Scattered Data
(Line of Best Fit)



Consider n-data set
(x_i, y_i) $\forall = i = 1, 2, 3, \ldots n$
We want to fit a
model $\hat{y} = ax + b$ — (0)
Slope y-intercept

Local Error = $(\hat{y}_i - y_i)$

Method Of Least Square Error (M.L.E):

error = $S.S.E = \sum_{i=1}^{n} (\hat{y}_i - y_i)^2$ -> Total Error

$$E(a,b) = E = \sum_{i=1}^{n} (ax_i + b - y_i)^2 \text{ — (1)}$$

must minimize

$$\frac{\partial E}{\partial a} = 0 = \frac{\partial}{\partial a} \sum_{i=1}^{n} (ax_i + b - y_i)^2$$

$$= 2\sum_i (ax_i + b - y_i)(x_i) = 0$$

$$a\sum_i x_i^2 + b\sum_i x_i - \sum_i x_i y_i = 0$$

$$a\sum_i x_i^2 + b\sum_i x_i = \sum_i x_i y_i \text{ — (2)}$$

Consider $\frac{\partial E}{\partial b} = 0$

$$\frac{\partial E}{\partial b} = 0 = \frac{\partial}{\partial b}\left(\sum (ax_i + b - y_i)^2\right)$$

$$2\sum (ax_i + b - y_i)(1) = 0$$

$$a\sum_i x_i + nb - \sum y_i = 0$$

$$a\sum_i x_i + nb = \sum y_i \quad \text{--- } (3)$$

Solving (2) and (3) gives following

$$a = \frac{n\sum_i x_i y_i - \sum_i x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$b = \bar{y} - a\bar{x} = \frac{\sum y}{n} - a\frac{\sum_i x_i}{n}$$

② Quadratic fit (2nd Degree Polynomial)

$$E(a,b,c) = \sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)^2 \quad —①$$

$$\frac{\partial E}{\partial a} = 0 = \frac{\partial}{\partial a} \sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)^2$$

$$2 \sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)(x_i^2) = 0$$

$$a \sum_{i=1}^{n} x_i^4 + b \sum_{i=1}^{n} x_i^3 + c \sum_{i=1}^{n} x_i^2 - \sum x_i^2 y_i = 0$$

$$a \sum_{i=1}^{n} x_i^4 + b \sum_{i=1}^{n} x_i^3 + c \sum_{i=1}^{n} x_i^2 = \sum x_i^2 y_i \quad —②$$

Consider $\frac{\partial E}{\partial b} = 0$

$$\frac{\partial E}{\partial b} = 0 = \frac{\partial}{\partial b} \sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)^2$$

$$2 \sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)(x_i) = 0$$

$$a \sum_{i=1}^{n} x_i^3 + b \sum_{i=1}^{n} x_i^2 + c \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i y_i = 0$$

$$a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i \quad -(3)$$

Consider $\frac{\partial E}{\partial c} = 0$

$$a\sum x_i^2 + b\sum x_i + nc = \sum y_i \quad -(4)$$

Solving 2, 3 & 4.

$$\begin{bmatrix} \Sigma x_i^4 & \Sigma x_i^3 & \Sigma x_i^2 \\ \Sigma x_i^3 & \Sigma x_i^2 & \Sigma x_i \\ \Sigma x_i^2 & \Sigma x_i & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \Sigma x_i^2 y_i \\ \Sigma x_i y_i \\ \Sigma y_i \end{bmatrix}$$

Above system will give the value of Parameters a, b & c such that some of square errors will be minimum.

~~Not included.~~

③ Exponential Fit ($\hat{y} = me^{nx}$)

$(x_i, y_i)\ \forall i = 1, 2, \cdots, n$

$$E(m,n) = \sum (me^{nx_i} - y_i)^2$$

$$= 2\sum (me^{nx_i} - y_i)e^{nx_i}$$

Linear equation nae bnsi

Non-linear equation ka system bnsa.

Assumption $\hat{y} > 0$

Taking Natural log both sides

$$\ln(y) = \ln(me^{nx})$$

$$\ln y = \ln m + \ln e^{nx}$$

$$\ln y = \ln m + nx$$

Let, $y' = \ln y$

$$\boxed{y' = nx + b}$$

Where,

$b = \ln m$

$m = e^b$

From
Above equation we will the value of n and by using linear fit b/w x and y'. Where $y' = \log y$.

$$n = \frac{N \sum xy' - \sum x \sum y'}{N \sum x^2 - (\sum x)^2}$$

Where N represent total no. of dataset

$b = \bar{y} - n\bar{x}$

$$\boxed{m = e^b}$$

EXAMPLE:

Consider the following data & fit, linear, quadratic & exponential mathematical model on the following data.

x	y	x^4	x^3	x^2	xy	x^2y	$y' = \ln y$	xy'
0	01	0	0	0	0	0	0	0
1	2	1	1	1	2	4	0.6931	0.693
3	10	81	27	9	30	300	2.30	6.9
5	26	625	125	25	130	3380	3.2580	16.29
8	65	4096	512	64	520	33800	4.174 ~~[illegible]~~	33.392 ~~[illegible]~~
10	101	10000	1000	100	1010	100010	4.6151	46.151
Σ = 27	205	14803	1665	199	1692	134,474 15002	15.0102	103.426

① For Linear Fit:

$$\hat{y} = ax + b$$

where,

$$a = \frac{n\sum_i xy - \sum_i x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$b = \bar{y} - a\bar{x}$$

$$a = \frac{6(16.92) - (27)(205)}{6(199) - (20)^2}$$

$$a = 9.928$$

$$b = \left(\frac{205}{6}\right) - (9.929)\left(\frac{27}{6}\right)$$

$$b = 10.573$$

$$\hat{y} = 9.293x - 10.573$$

For Quadratic Fit:

$$\hat{y} = ax^2 + bx + c$$

where,

a, b, c can be computed as follows.

$$\begin{bmatrix} \sum x^4 & \sum x^3 & \sum x^2 \\ \sum x^3 & \sum x^2 & \sum x \\ \sum x^2 & \sum x & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum x^2 y \\ \sum xy \\ \sum y \end{bmatrix}$$

$$\begin{bmatrix} 14803 & 1665 & 199 \\ 1665 & 199 & 27 \\ 199 & 27 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15002 \\ 1692 \\ 205 \end{bmatrix}$$

Calculation $\begin{cases} \to \text{Inverse} \\ \to \text{How to Multiply Matrix with vector?} \end{cases}$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{y} = x^2 + 1$$

Which one is best

$n = 6$

$R^2 = 1 - \frac{SSE}{SST}$

– variance of y

3) Exponential Fit:

$S.S.T = \sum_i (y_i - \bar{y})^2$

$$\hat{y} = m e^{px}$$

$$p = \frac{n\sum xy' - \sum x \sum y'}{n\sum x^2 - (\sum x)^2}$$

$$m = e^{(\bar{y} - p\bar{x})}$$

…traints: production may units (1) 0 nhi hng…
(2) -ve " "
(3) decimal " "

So graph will be in 1st quadrant only

Unconstraint Optimization but variable applies constraint

A company produces two products A and B. if x represents no. of units of product A & y represents no of units of product B and the cost function for the production of both products is given as follows:

$$C(x,y) = 4x^2 - 13y^2 - 10$$

Find the no. of units of product A and B such that the cost of production (C) is minimum.

$x > 0, y > 0$

gradient optimization
constaint optimization

Solution:

Using Gradient descent method

works for problem must have { parabolic / or quadratic

maxima or minima
eith inc or dec

Local maxima :

e.g company has constraint that it can produd 5 units of A & 10 units of B.

$$x_{i+1} = x_i - t \left.\frac{\partial c}{\partial x}\right|_{(x_i, y_i)}$$

x_i represents quantity of product A

y_i " " " " B

$$y_{i+1} = y_i - t \left.\frac{\partial c}{\partial y}\right|_{(x_i, y_i)} \quad \forall_i = 0, 1, 2, \ldots$$

Choti se choti rkhni h but not 0

Let,

$$(x_0, y_0) = (1, 1)$$

kum se km ek unit tu bnega

Let,

$$t = 0.1$$

Final answer intgr mai hna chahiye round off ho ke likhiye. Process ko darmiyan interrupt nahi karna.

$$\frac{\partial c}{\partial x} = 8x \qquad \frac{\partial c}{\partial y} = 6y$$

$i = 0$

$$x_{i+1} = x_i - (t)(8x_i)$$
$$y_{i+1} = x_i - (t)(6y_i)$$

$$x_1 = x_0 - (t)(8x_0)$$

$$x_1 = 1 - 8t$$

$$y_1 = y_0 - (t)(6y_0)$$

$$y_1 = 1 - 6t$$

put $t = 0.1$

$x_1 = 1 - 8(0.1)$ $\qquad$ $y_1 = 1 - 6(0.1)$

$x_1 = 0.2$ $\qquad$ $y_1 = 0.4$

$(x_1, y_1) = (0.2, 0.4)$

$i = 1$

consumer rather

$x_2 = x_1 - (t)8x_1$

$x_2 = 0.2 - t\,8\,(0.2)$

$x_2 = 0.2 - (0.1)(8)(0.2)$

$x_2 = 0.04$

$y_2 = y_1 - (t)6y_1$

$y_2 = 0.4 - (0.1)(6)(0.4)$

$y_2 = 0.16$

$0 < 0.04 < 1$ $\qquad$ $0 < 0.16 < 1$

put 0, 1 in eq and check which one is close.

$i = 2 \quad (x_2, y_2) = (0.04, 0.16)$

$x_3 = x_2 - (t)\,\delta x_2$

$x_3 =$