E pluribus unum (Out of many, one)

— Official motto of the United States of America

John: *Who's your daddy? C'mon, you know who your daddy is! Who's your daddy? D'Argo, tell him who his daddy is!*"
D'Argo: *I'm your daddy.*

— *Farscape*, "Thanks for Sharing" (June 15, 2001)

What rolls down stairs, alone or in pairs, rolls over your neighbor's dog?
What's great for a snack, and fits on your back? It's Log, Log, Log!

It's Log! It's Log! It's big, it's heavy, it's wood!
It's Log! It's Log! It's better than bad, it's good!

— *Ren & Stimpy*, "Stimpy's Big Day/The Big Shot" (August 11, 1991)
lyrics by John Kricfalusi

The thing's hollow - it goes on forever - and - oh my God! - it's full of stars!

— Capt. David Bowman's last words(?)
2001: A Space Odyssey by Arthur C. Clarke (1968)

17 Data Structures for Disjoint Sets

In this lecture, we describe some methods for maintaining a collection of disjoint sets. Each set is represented as a pointer-based data structure, with one node per element. We will refer to the elements as either 'objects' or 'nodes', depending on whether we want to emphasize the set abstraction or the actual data structure. Each set has a unique 'leader' element, which identifies the set. (Since the sets are always disjoint, the same object cannot be the leader of more than one set.) We want to support the following operations.

- MAKESET(x): Create a new set $\{x\}$ containing the single element x. The object x must not appear in any other set in our collection. The leader of the new set is obviously x.
- FIND(x): Find (the leader of) the set containing x.
- UNION(A, B): Replace two sets A and B in our collection with their union $A \cup B$. For example, UNION(A, MAKESET(x)) adds a new element x to an existing set A. The sets A and B are specified by arbitrary elements, so UNION(x, y) has exactly the same behavior as UNION(FIND(x), FIND(y)).

Disjoint set data structures have lots of applications. For instance, Kruskal's minimum spanning tree algorithm relies on such a data structure to maintain the components of the intermediate spanning forest. Another application is maintaining the connected components of a graph as new vertices and edges are added. In both these applications, we can use a disjoint-set data structure, where we maintain a set for each connected component, containing that component's vertices.

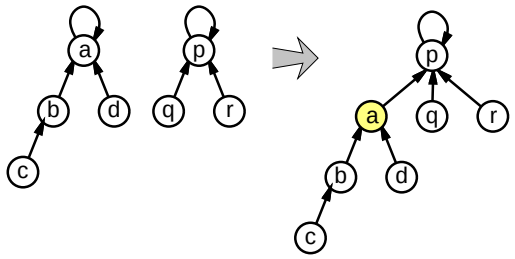
17.1 Reversed Trees

One of the easiest ways to store sets is using trees, in which each node represents a single element of the set. Each node points to another node, called its *parent*, except for the leader of each set, which points to itself and thus is the root of the tree. MAKESET is trivial. FIND traverses parent pointers up to the leader. UNION just redirects the parent pointer of one leader to the other. Unlike most tree data structures, nodes do *not* have pointers down to their children.

```
MakeSet(x):
  parent(x) ← x
```

```
Find(x):
  while x ≠ parent(x)
      x ← parent(x)
  return x
```

```
Union(x, y):
  x̄ ← Find(x)
  ȳ ← Find(y)
  parent(ȳ) ← x̄
```



Merging two sets stored as trees. Arrows point to parents. The shaded node has a new parent.

MAKE-SET clearly takes $\Theta(1)$ time, and UNION requires only $O(1)$ time in addition to the two FINDS. The running time of FIND(x) is proportional to the depth of x in the tree. It is not hard to come up with a sequence of operations that results in a tree that is a long chain of nodes, so that FIND takes $\boldsymbol{\Theta(n)}$ ***time*** in the worst case.

However, there is an easy change we can make to our UNION algorithm, called *union by depth*, so that the trees always have logarithmic depth. Whenever we need to merge two trees, we always make the root of the *shallower* tree a child of the *deeper* one. This requires us to also maintain the depth of each tree, but this is quite easy.

```
MakeSet(x):
  parent(x) ← x
  depth(x) ← 0
```

```
Find(x):
  while x ≠ parent(x)
      x ← parent(x)
  return x
```

```
Union(x, y)
  x̄ ← Find(x)
  ȳ ← Find(y)
  if depth(x̄) > depth(ȳ)
      parent(ȳ) ← x̄
  else
      parent(x̄) ← ȳ
      if depth(x̄) = depth(ȳ)
          depth(ȳ) ← depth(ȳ) + 1
```

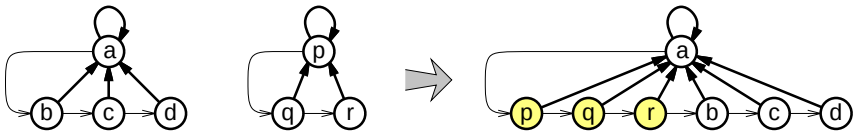
With this new rule in place, it's not hard to prove by induction that for any set leader $\overline{x}$, the size of $\overline{x}$'s set is at least $2^{depth(\overline{x})}$, as follows. If $depth(\overline{x}) = 0$, then $\overline{x}$ is the leader of a singleton set. For any $d > 0$, when $depth(\overline{x})$ becomes d for the first time, $\overline{x}$ is becoming the leader of the union of two sets, both of whose leaders had depth $d-1$. By the inductive hypothesis, both component sets had at least 2^{d-1} elements, so the new set has at least 2^d elements. Later UNION operations might add elements to $\overline{x}$'s set without changing its depth, but that only helps us.

Since there are only n elements altogether, the maximum depth of any set is $\lg n$. We conclude that if we use union by depth, both FIND and UNION run in $\boldsymbol{\Theta(\log n)}$ time in the worst case.

17.2 Shallow Threaded Trees

Alternately, we could just have every object keep a pointer to the leader of its set. Thus, each set is represented by a shallow tree, where the leader is the root and all the other elements are its children. With this representation, MAKESET and FIND are completely trivial. Both operations clearly run in constant time. UNION is a little more difficult, but not much. Our algorithm sets all the leader pointers in one set to point to the leader of the other set. To do this, we need a method to visit every element

in a set; we will 'thread' a linked list through each set, starting at the set's leader. The two threads are merged in the UNION algorithm in constant time.



Merging two sets stored as threaded trees.
Bold arrows point to leaders; lighter arrows form the threads. Shaded nodes have a new leader.

```
MAKESET(x):
  leader(x) ← x
  next(x) ← x
```

```
FIND(x):
  return leader(x)
```

```
UNION(x, y):
  x̄ ← FIND(x)
  ȳ ← FIND(y)

  y ← ȳ
  leader(y) ← x̄
  while (next(y) ≠ NULL)
      y ← next(y)
      leader(y) ← x̄

  next(y) ← next(x̄)
  next(x̄) ← ȳ
```

The worst-case running time of UNION is a constant times the size of the *larger* set. Thus, if we merge a one-element set with another n-element set, the running time can be $\Theta(n)$. Generalizing this idea, it is quite easy to come up with a sequence of n MAKESET and $n-1$ UNION operations that requires $\Theta(n^2)$ time to create the set $\{1, 2, \ldots, n\}$ from scratch.

```
WORSTCASESEQUENCE(n):
  MAKESET(1)
  for i ← 2 to n
      MAKESET(i)
      UNION(1, i)
```

We are being stupid in two different ways here. One is the order of operations in WORSTCASESEQUENCE. Obviously, it would be more efficient to merge the sets in the other order, or to use some sort of divide and conquer approach. Unfortunately, we can't fix this; we don't get to decide how our data structures are used! The other is that we always update the leader pointers in the larger set. To fix this, we add a comparison inside the UNION algorithm to determine which set is smaller. This requires us to maintain the size of each set, but that's easy.

```
MAKEWEIGHTEDSET(x):
  leader(x) ← x
  next(x) ← x
  size(x) ← 1
```

```
WEIGHTEDUNION(x, y)
  x̄ ← FIND(x)
  ȳ ← FIND(y)
  if size(x̄) > size(ȳ)
      UNION(x̄, ȳ)
      size(x̄) ← size(x̄) + size(ȳ)
  else
      UNION(ȳ, x̄)
      size(ȳ) ← size(x̄) + size(ȳ)
```

The new WEIGHTEDUNION algorithm still takes $\Theta(n)$ time to merge two n-element sets. However, in an amortized sense, this algorithm is much more efficient. Intuitively, before we can merge two large sets, we have to perform a large number of MAKEWEIGHTEDSET operations.

Theorem 1. *A sequence of m* MAKEWEIGHTEDSET *operations and n* WEIGHTEDUNION *operations takes $O(m + n \log n)$ time in the worst case.*

Proof: Whenever the leader of an object x is changed by a WEIGHTEDUNION, the size of the set containing x increases by at least a factor of two. By induction, if the leader of x has changed k times, the set containing x has at least 2^k members. After the sequence ends, the largest set contains at most n members. (Why?) Thus, the leader of any object x has changed at most $\lfloor \lg n \rfloor$ times.

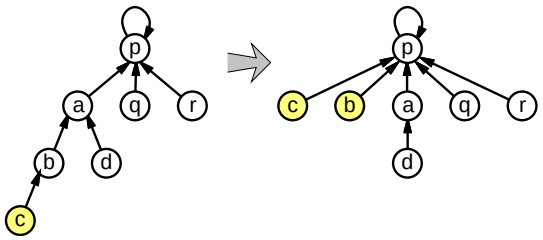
Since each WEIGHTEDUNION reduces the number of sets by one, there are $m - n$ sets at the end of the sequence, and at most n objects are *not* in singleton sets. Since each of the non-singleton objects had $O(\log n)$ leader changes, the total amount of work done in updating the leader pointers is $O(n \log n)$. □

The aggregate method now implies that each WEIGHTEDUNION has ***amortized cost $O(\log n)$***.

17.3 Path Compression

Using unthreaded tress, FIND takes logarithmic time and everything else is constant; using threaded trees, UNION takes logarithmic amortized time and everything else is constant. A third method allows us to get both of these operations to have *almost* constant running time.

We start with the original unthreaded tree representation, where every object points to a parent. The key observation is that in any FIND operation, once we determine the leader of an object x, we can speed up future FINDs by redirecting x's parent pointer directly to that leader. In fact, we can change the parent pointers of all the ancestors of x all the way up to the root; this is easiest if we use recursion for the initial traversal up the tree. This modification to FIND is called *path compression*.



Path compression during Find(c). Shaded nodes have a new parent.

```
FIND(x)
  if x ≠ parent(x)
      parent(x) ← FIND(parent(x))
  return parent(x)
```

If we use path compression, the 'depth' field we used earlier to keep the trees shallow is no longer correct, and correcting it would take way too long. But this information still ensures that FIND runs in $\Theta(\log n)$ time in the worst case, so we'll just give it another name: *rank*. The following algorithm is usually called *union by rank*:

```
MAKESET(x):
  parent(x) ← x
  rank(x) ← 0
```

```
UNION(x, y)
  x̄ ← FIND(x)
  ȳ ← FIND(y)
  if rank(x̄) > rank(ȳ)
      parent(ȳ) ← x̄
  else
      parent(x̄) ← ȳ
      if rank(x̄) = rank(ȳ)
          rank(ȳ) ← rank(ȳ) + 1
```

FIND still runs in $O(\log n)$ time in the worst case; path compression increases the cost by only most a constant factor. But we have good reason to suspect that this upper bound is no longer tight. Our new algorithm memoizes the results of each FIND, so if we are asked to FIND the same item twice in a row, the second call returns in constant time. Splay trees used a similar strategy to achieve their optimal amortized cost, but our up-trees have fewer constraints on their structure than binary search trees, so we should get even better performance.

This intuition is exactly correct, but it takes a bit of work to define precisely *how* much better the performance is. As a first approximation, we will prove below that the amortized cost of a FIND operation is bounded by the *iterated logarithm* of n, denoted $\log^* n$, which is the number of times one must take the logarithm of n before the value is less than 1:

$$\lg^* n = \begin{cases} 1 & \text{if } n \le 2, \\ 1 + \lg^*(\lg n) & \text{otherwise.} \end{cases}$$

Our proof relies on several useful properties of ranks, which follow directly from the UNION and FIND algorithms.

- If a node x is not a set leader, then the rank of x is smaller than the rank of its parent.
- Whenever $parent(x)$ changes, the new parent has larger rank than the old parent.
- Whenever the leader of x's set changes, the new leader has larger rank than the old leader.
- The size of any set is exponential in the rank of its leader: $size(\overline{x}) \ge 2^{rank(\overline{x})}$. (This is easy to prove by induction, hint, hint.)
- In particular, since there are only n objects, the highest possible rank is $\lfloor \lg n \rfloor$.
- For any integer r, there are at most $n/2^r$ objects of rank r.

Only the last property requires a clever argument to prove. Fix your favorite integer r. Observe that only set leaders can change their rank. Whenever the rank of any set leader $\overline{x}$ changes from $r-1$ to r, mark all the objects in $\overline{x}$'s set. Since leader ranks can only increase over time, each object is marked at most once. There are n objects altogether, and any object with rank r marks at least 2^r objects. It follows that there are at most $n/2^r$ objects with rank r, as claimed.

*17.4 $O(\log^* n)$ Amortized Time

The following analysis of path compression was discovered just a few years ago by Raimund Seidel and Micha Sharir.[1] Previous proofs[2] relied on complicated charging schemes or potential-function arguments; Seidel and Sharir's analysis relies on a comparatively simple recursive decomposition. (Of course, simple is in the eye of the beholder.)

Seidel and Sharir phrase their analysis in terms of two more general operations on set forests. Their more general COMPRESS operation compresses *any* directed path, not just paths that lead to the root. The new SHATTER operation makes every node on a root-to-leaf path into its own parent.

```
COMPRESS(x, y):
⟨⟨y must be an ancestor of x⟩⟩
  if x ≠ y
      COMPRESS(parent(x), y)
      parent(x) ← parent(y)
```

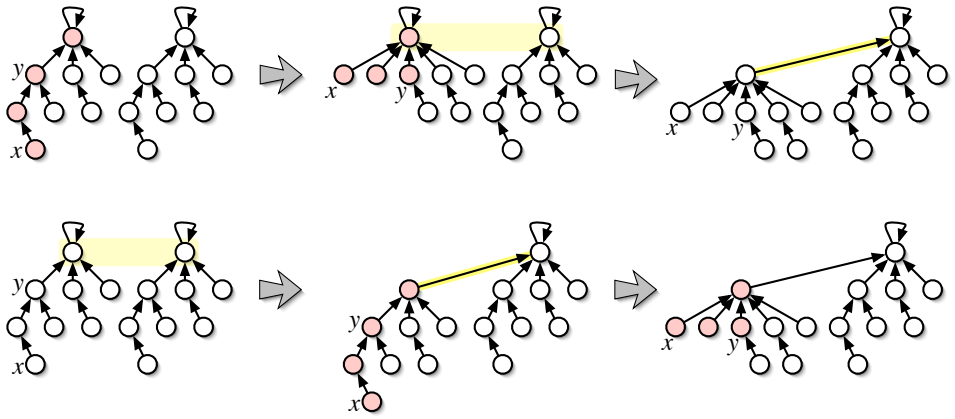
```
SHATTER(x):
  if parent(x) ≠ x
      SHATTER(parent(x))
      parent(x) ← x
```

[1] Raimund Seidel and Micha Sharir. Top-down analysis of path compression. *SIAM J. Computing* 34(3):515–525, 2005.

[2] Robert E. Tarjan. Efficiency of a good but not linear set union algorithm. *J. Assoc. Comput. Mach.* 22:215–225, 1975.

Clearly, the running time of FIND(x) operation is dominated by the running time of COMPRESS(x, y), where y is the leader of the set containing x. Thus, we can prove the upper bound by analyzing an *arbitrary* sequence of UNION and COMPRESS operations. Moreover, we can assume that the arguments of every UNION operation are set leaders, so that each UNION takes only constant worst-case time.

Finally, since each call to COMPRESS specifies the top node in the path to be compressed, we can reorder the sequence of operations, so that every UNION occurs before any COMPRESS, without changing the number of pointer assignments.



Top row: A COMPRESS followed by a UNION. Bottom row: The same operations in the opposite order.

Each UNION requires only constant time, so we only need to analyze the amortized cost of COMPRESS. The running time of COMPRESS is proportional to the number of parent pointer assignments, plus $O(1)$ overhead, so we will phrase our analysis in terms of pointer assignments. Let $\boldsymbol{T(m,n,r)}$ denote the worst case number of pointer assignments in any sequence of at most m COMPRESS operations, executed on a forest of at most n nodes, in which each node has rank at most r.

The following trivial upper bound will be the base case for our recursive argument.

Theorem 2. $T(m, n, r) \leq nr$

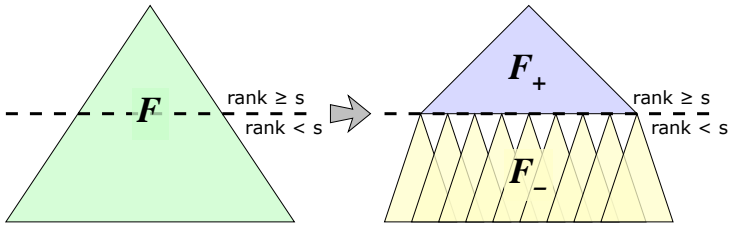
Proof: Each node can change parents at most r times, because each new parent has higher rank than the previous parent. □

Fix a forest F of n nodes with maximum rank r, and a sequence C of m COMPRESS operations on F, and let $\boldsymbol{T(F,C)}$ denote the total number of pointer assignments executed by this sequence.

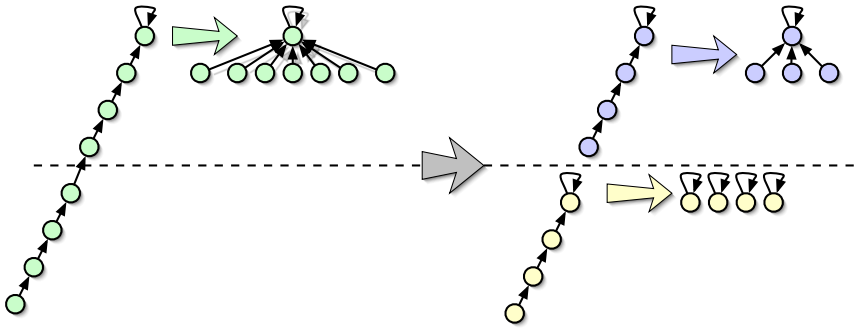
Let s be an arbitrary positive rank. Partition F into two sub-forests: a 'low' forest F_- containing all nodes with rank at most s, and a 'high' forest F_+ containing all nodes with rank greater than s. Since ranks increase as we follow parent pointers, every ancestor of a high node is another high node. Let n_- and n_+ denote the number of nodes in F_- and F_+, respectively. Finally, let m_+ denote the number of COMPRESS operations that involve any node in F_+, and let $m_- = m - m_+$.

Any sequence of COMPRESS operations on F can be decomposed into a sequence of COMPRESS operations on F_+, plus a sequence of COMPRESS and SHATTER operations on F_-, with the same total cost. This requires only one small modification to the code: We forbid any low node from having a high parent. Specifically, if x is a low node and y is a high node, we replace any assignment $parent(x) \leftarrow y$ with $parent(x) \leftarrow x$.

This modification is equivalent to the following reduction:



Splitting the forest F (in this case, a single tree) into sub-forests F_+ and F_- at rank s.



A COMPRESS operation in F splits into a COMPRESS operation in F_+ and a SHATTER operation in F_-

```
COMPRESS(x, y, F):          ⟨⟨y is an ancestor of x⟩⟩
  if rank(x) > s
      COMPRESS(x, y, F+)                ⟨⟨in C+⟩⟩
  else if rank(y) ≤ s
      COMPRESS(x, y, F−)                ⟨⟨in C−⟩⟩
  else
      z ← x
      while rank(parent_F(z)) ≤ s
          z ← parent_F(z)
      COMPRESS(parent_F(z), y, F+)      ⟨⟨in C+⟩⟩
      SHATTER(x, z, F−)
      parent(z) ← z                     (?)
```

The pointer assignment in the last line (?) looks redundant, but it is actually necessary for the analysis. Each execution of that line mirrors an assignment of the form $parent(z) \leftarrow w$, where z is a low node, w is a high node, and the previous parent of z was also a high node. Each of these 'redundant' assignments happens immediately after a COMPRESS in the top forest, so we perform at most m_+ redundant assignments.

Each node x is touched by at most one SHATTER operation, so the total number of pointer reassignments in all the SHATTER operations is at most n.

Thus, by partitioning the forest F into F_+ and F_-, we have also partitioned the sequence C of COMPRESS operations into subsequences C_+ and C_-, with respective lengths m_+ and m_-, such that the following inequality holds:

$$\boxed{T(F,C) \le T(F_+,C_+) + T(F_-,C_-) + m_+ + n}$$

Since there are only $n/2^i$ nodes of any rank i, we have $n_+ \le \sum_{i>s} n/2^i = n/2^s$. The number of different ranks in F_+ is $r - s < r$. Thus, Theorem 2 implies the upper bound

$$T(F_+,C_+) < rn/2^s.$$

Let us fix $\boldsymbol{s = \lg r}$, so that $T(F_+, C_+) \le n$. We can now simplify our earlier recurrence to

$$T(F, C) \le T(F_-, C_-) + m_+ + 2n,$$

or equivalently,

$$T(F, C) - m \le T(F_-, C_-) - m_- + 2n.$$

Since this argument applies to *any* forest F and *any* sequence C, we have just proved that

$$T'(m, n, r) \le T'(m, n, \lfloor \lg r \rfloor) + 2n,$$

where $T'(m, n, r) = T(m, n, r) - m$. The solution to this recurrence is $\boldsymbol{T'(n, m, r) \le 2n \lg^* r}$. Voilá!

Theorem 3. $T(m, n, r) \le m + 2n \lg^* r$

*17.5 Turning the Crank

There is one place in the preceding analysis where we have significant room for improvement. Recall that we bounded the total cost of the operations on F_+ using the trivial upper bound from Theorem 2. But we just proved a better upper bound in Theorem 3! We can apply precisely the same strategy, using Theorem 3 recursively instead of Theorem 2, to improve the bound even more.

Suppose we fix $s = \lg^* r$, so that $n_+ = n/2^{\lg^* r}$. Theorem 3 implies that

$$T(F_+, C_+) \le m_+ + 2n \frac{\lg^* r}{2^{\lg^* r}} \le m_+ + 2n.$$

This implies the recurrence

$$T(F, C) \le T(F_-, C_-) + 2m_+ + 3n,$$

which in turn implies that

$$T''(m, n, r) \le T''(m, n, \lg^* r) + 3n,$$

where $T''(m, n, r) = T(m, n, r) - 2m$. The solution to this equation is $\boldsymbol{T(m, n, r) \le 2m + 3n \lg^{**} r}$, where $\lg^{**} r$ is the *iterated* iterated logarithm of r:

$$\lg^{**} r = \begin{cases} 1 & \text{if } r \le 2, \\ 1 + \lg^{**}(\lg^* r) & \text{otherwise.} \end{cases}$$

Naturally we can apply the same improvement strategy again, and again, as many times as we like, each time producing a tighter upper bound. Applying the reduction c times, for any positive integer c, gives us $\boldsymbol{T(m, n, r) \le cm + (c+1)n \lg^{*^c} r}$, where

$$\lg^{*^c} r = \begin{cases} \lg r & \text{if } c = 0, \\ 1 & \text{if } r \le 2, \\ 1 + \lg^{*^c}(\lg^{*^{c-1}} r) & \text{otherwise.} \end{cases}$$

Each time we 'turn the crank', the dependence on m increases, while the dependence on n and r decreases. For sufficiently large values of c, the cm term dominates the time bound, and further iterations only make things worse. The point of diminishing returns can be estimated by *the minimum number of stars* such that $\lg^{**\cdots*} r$ is smaller than a constant:

$$\alpha(r) = \min\left\{ c \ge 1 \;\middle|\; \lg^{*^c} n \le 3 \right\}.$$

(The threshold value 3 is used here because $\lg^{*^c} 5 \ge 2$ for all c.) By setting $c = \alpha(r)$, we obtain our final upper bound.

Theorem 4. $T(m,n,r) \le m\alpha(r) + 3n(\alpha(r)+1)$

We can assume without loss of generality that $m \ge n$ by ignoring any singleton sets, so this upper bound can be further simplified to $T(m,n,r) = O(m\alpha(r)) = O(m\alpha(n))$. It follows that if we use union by rank, FIND with path compression runs in ***O(α(n)) amortized time***.

Even this upper bound is somewhat conservative if m is larger than n. A closer estimate is given by the function

$$\alpha(m,n) = \min\left\{ c \ge 1 \;\middle|\; \lg^{*^c}(\lg n) \le m/n \right\}.$$

It's not hard to prove that if $m = \Theta(n)$, then $\alpha(m,n) = \Theta(\alpha(n))$. On the other hand, if $m \ge n \lg^{*****} n$, for any constant number of stars, then $\alpha(m,n) = O(1)$. So even if the number of FIND operations is only *slightly* larger than the number of nodes, the amortized cost of each FIND is *constant*.

$O(\alpha(m,n))$ is actually a *tight* upper bound for the amortized cost of path compression; there are no more tricks that will improve the analysis further. More surprisingly, this is the best amortized bound we obtain for *any* pointer-based data structure for maintaining disjoint sets; the amortized cost of *every* FIND algorithm is at least $\Omega(\alpha(m,n))$. The proof of the matching lower bound is, unfortunately, far beyond the scope of this class.[3]

17.6 The Ackermann Function and its Inverse

The iterated logarithms that fell out of our analysis of path compression are the inverses of a hierarchy of recursive functions defined by Wilhelm Ackermann in 1928.[4]

$$\mathbf{2} \uparrow^{\boldsymbol{c}} \boldsymbol{n} := \begin{cases} 2 & \text{if } n = 1 \\ 2n & \text{if } c = 0 \\ 2 \uparrow^{c-1} (2 \uparrow^{c} (n-1)) & \text{otherwise} \end{cases}$$

For each fixed integer c, the function $2 \uparrow^c n$ is monotonically increasing in n, and these functions grow *incredibly* faster as the index c increases. $\mathbf{2} \uparrow \boldsymbol{n}$ is the familiar power function 2^n. $\mathbf{2} \uparrow\uparrow \boldsymbol{n}$ is the *tower* function:

$$2 \uparrow\uparrow n = \underbrace{2 \uparrow 2 \uparrow \ldots \uparrow 2}_{n} = 2^{2^{\cdot^{\cdot^{2}}}\Big\}n}$$

John Conway named $2 \uparrow\uparrow\uparrow n$ the *wower* function:

$$2 \uparrow\uparrow\uparrow n = \underbrace{2 \uparrow\uparrow 2 \uparrow\uparrow \cdots \uparrow\uparrow 2}_{n}.$$

And so on, *et cetera, ad infinitum*.

For any fixed c, the function $\log^{*^c} n$ is the inverse of the function $2 \uparrow^{c+1} n$, the $(c+1)$th row in the Ackerman hierarchy. Thus, for any remotely reasonable values of n, say $n \le 2^{256}$, we have $\log^* n \le 5$, $\log^{**} n \le 4$, and $\log^{*^c} n \le 3$ for any $c \ge 3$.

[3] Robert E. Tarjan. A class of algorithms which require non-linear time to maintain disjoint sets. *J. Comput. Syst. Sci.* 19:110–127, 1979.

[4] Ackermann didn't define his functions this way—I'm actually describing a slightly cleaner hierarchy defined 35 years later by R. Creighton Buck—but the exact details of the definition are surprisingly irrelevant! The mnemonic up-arrow notation for these functions was introduced by Don Knuth in the 1970s.

The function $\alpha(n)$ is usually called the *inverse Ackerman function*.[5] Our earlier definition is equivalent to $\alpha(n) = \min\{c \geq 1 \mid 2 \uparrow^{c+2} 3 \geq n\}$; in other words, $\alpha(n)+2$ is the inverse of the third column in the Ackermann hierarchy. The function $\alpha(n)$ grows *much* more slowly than $\log^{*^c} n$ for any fixed c; we have $\alpha(n) \leq 3$ for all even *remotely imaginable* values of n. Nevertheless, the function $\alpha(n)$ is eventually larger than any constant, so it is *not* $O(1)$.

$2 \uparrow^c n$	$n=1$	2	$n=3$	$n=4$	$n=5$
$2n$	2	4	6	8	10
$2 \uparrow n$	2	4	8	16	32
$2 \uparrow\uparrow n$	2	4	16	65536	2^{65536}
$2 \uparrow\uparrow\uparrow n$	2	4	65536	$2^{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}\Big\}65536$	$2^{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}\Big\}65536$
$2 \uparrow\uparrow\uparrow\uparrow n$	2	4	$2^{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}\Big\}65536$	$2^{2^{\cdot^{\cdot^{\cdot^2}}}\Big\}2^{\cdot^{\cdot^{\cdot^2}}}\Big\}65536}\Big\}\Big\}2^{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}\Big\}65536$	⟨⟨*Yeah, right.*⟩⟩
$2 \uparrow\uparrow\uparrow\uparrow\uparrow n$	2	4	$2^{2^{\cdot^{\cdot^{\cdot^2}}}\Big\}2^{\cdot^{\cdot^{\cdot^2}}}\Big\}65536}\Big\}\Big\}2^{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}\Big\}65536$	⟨⟨*Very funny.*⟩⟩	⟨⟨*Argh! My eyes!*⟩⟩

Small (!!) values of Ackermann's functions.

17.7 To infinity... and beyond!

Of course, one can generalize the inverse Ackermann function to functions that grow arbitrarily more slowly, starting with the *iterated* inverse Ackermann function

$$\alpha^*(n) = \begin{cases} 1 & \text{if } n \leq 4, \\ 1 + \alpha^*(\alpha(n)) & \text{otherwise,} \end{cases}$$

then the *iterated* iterated iterated inverse Ackermann function

$$\alpha^{*^c}(n) = \begin{cases} \alpha(n) & \text{if } c = 0, \\ 1 & \text{if } n \leq 4, \\ 1 + \alpha^{*^c}(\alpha^{*^{c-1}}(n)) & \text{otherwise,} \end{cases}$$

and then the diagonalized inverse Ackermann function

$$\textit{Head-asplode}(n) = \min\{c \geq 1 \mid \alpha^{*^c} n \leq 4\},$$

and so on ad nauseam. Fortunately(?), such functions appear extremely rarely in algorithm analysis. Perhaps the only naturally-occurring example of a super-constant sub-inverse-Ackermann function is a recent result of Seth Pettie[6], who proved that if a splay tree is used as a double-ended queue — insertions and deletions of only smallest or largest elements — then the amortized cost of any operation is $O(\alpha^*(n))$!

[5] Strictly speaking, the name 'inverse Ackerman function' is inaccurate. One good formal definition of the true inverse Ackerman function is $\tilde{\alpha}(n) = \min\left\{c \geq 1 \;\middle|\; \lg^{*^c} n \leq c\right\} = \min\left\{c \geq 1 \;\middle|\; 2 \uparrow^{c+2} c \geq n\right\}$. However, it's not hard to prove that $\tilde{\alpha}(n) \leq \alpha(n) \leq \tilde{\alpha}(n)+1$ for all sufficiently large n, so the inaccuracy is completely forgivable. As I said in the previous footnote, the exact details of the definition are surprisingly irrelevant!

[6] Splay trees, Davenport-Schinzel sequences, and the deque conjecture. *Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms*, 1115–1124, 2008.

Exercises

1. Consider the following solution for the union-find problem, called *union-by-weight*. Each set leader $\overline{x}$ stores the number of elements of its set in the field $weight(\overline{x})$. Whenever we UNION two sets, the leader of the *smaller* set becomes a new child of the leader of the *larger* set (breaking ties arbitrarily).

```
MAKESET(x):
  parent(x) ← x
  weight(x) ← 1
```

```
FIND(x):
  while x ≠ parent(x)
      x ← parent(x)
  return x
```

```
UNION(x, y)
  x̄ ← FIND(x)
  ȳ ← FIND(y)
  if weight(x̄) > weight(ȳ)
      parent(ȳ) ← x̄
      weight(x̄) ← weight(x̄) + weight(ȳ)
  else
      parent(x̄) ← ȳ
      weight(x̄) ← weight(x̄) + weight(ȳ)
```

Prove that if we use union-by-weight, the *worst-case* running time of FIND(x) is $O(\log n)$, where n is the cardinality of the set containing x.

2. Consider a union-find data structure that uses union by depth (or equivalently union by rank) *without* path compression. For all integers m and n such that $m \geq 2n$, prove that there is a sequence of n *MakeSet* operations, followed by m UNION and FIND operations, that require $\Omega(m \log n)$ time to execute.

3. Suppose you are given a collection of up-trees representing a partition of the set $\{1, 2, \ldots, n\}$ into disjoint subsets. **You have no idea how these trees were constructed.** You are also given an array $node[1..n]$, where $node[i]$ is a pointer to the up-tree node containing element i. Your task is to create a new array $label[1..n]$ using the following algorithm:

```
LABELEVERYTHING:
  for i ← 1 to n
      label[i] ← FIND(node[i])
```

   (a) What is the worst-case running time of LABELEVERYTHING if we implement FIND *without* path compression?

   (b) ***Prove*** that if we implement FIND using path compression, LABELEVERYTHING runs in $O(n)$ time in the worst case.

4. Consider an arbitrary sequence of m MAKESET operations, followed by u UNION operations, followed by f FIND operations, and let $n = m + u + f$. Prove that if we use union by rank and FIND with path compression, all n operations are executed in $O(n)$ time.

5. Suppose we want to maintain an array $X[1..n]$ of bits, which are all initially zero, subject to the following operations.

   - LOOKUP(i): Given an index i, return $X[i]$.

- BLACKEN(i): Given an index $i < n$, set $X[i] \leftarrow 1$.
- NEXTWHITE(i): Given an index i, return the smallest index $j \geq i$ such that $X[j] = 0$. (Because we never change $X[n]$, such an index always exists.)

If we use the array $X[1..n]$ itself as the only data structure, it is trivial to implement LOOKUP and BLACKEN in $O(1)$ time and NEXTWHITE in $O(n)$ time. But you can do better! Describe data structures that support LOOKUP in $O(1)$ worst-case time and the other two operations in the following time bounds. (We want a different data structure for each set of time bounds, not one data structure that satisfies all bounds simultaneously!)

(a) The worst-case time for both BLACKEN and NEXTWHITE is $O(\log n)$.

(b) The amortized time for both BLACKEN and NEXTWHITE is $O(\log n)$. In addition, the *worst-case* time for BLACKEN is $O(1)$.

(c) The amortized time for BLACKEN is $O(\log n)$, and the *worst-case* time for NEXTWHITE is $O(1)$.

(d) The worst-case time for BLACKEN is $O(1)$, and the amortized time for NEXTWHITE is $O(\alpha(n))$. *[Hint: There is no* WHITEN*.]*

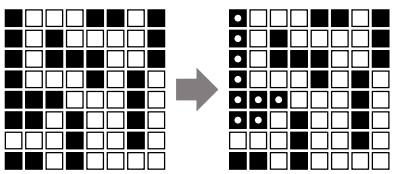
6. Suppose we want to maintain a collection of strings (sequences of characters) under the following operations:

- NEWSTRING(a) creates a new string of length 1 containing only the character a and returns a pointer to that string.
- CONCAT(S, T) removes the strings S and T (given by pointers) from the data structure, adds the concatenated string ST to the data structure, and returns a pointer to the new string.
- REVERSE(S) removes the string S (given by a pointer) from the data structure, adds the reversal of S to the data structure, and returns a pointer to the new string.
- LOOKUP(S, k) returns the kth character in string S (given by a pointer), or NULL if the length of the S is less than k.

Describe and analyze a *simple* data structure that supports CONCAT in $O(\log n)$ *amortized* time, supports every other operation in $O(1)$ *worst-case* time, and uses $O(n)$ space, where n is the sum of the *current* string lengths. Unlike the similar problem in the previous lecture note, there is no SPLIT operation. *[Hint: Why is this problem here?]*

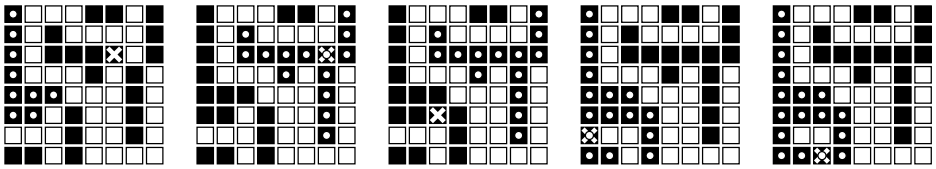
7. (a) Describe and analyze an algorithm to compute the size of the largest connected component of black pixels in an $n \times n$ bitmap $B[1..n, 1..n]$.

For example, given the bitmap below as input, your algorithm should return the number 9, because the largest conected black component (marked with white dots on the right) contains nine pixels.



(b) Design and analyze an algorithm BLACKEN(i, j) that colors the pixel $B[i, j]$ black and returns the size of the largest black component in the bitmap. For full credit, the *amortized* running time of your algorithm (starting with an all-white bitmap) must be as small as possible.

For example, at each step in the sequence below, we blacken the pixel marked with an X. The largest black component is marked with white dots; the number underneath shows the correct output of the BLACKEN algorithm.



(c) What is the *worst-case* running time of your BLACKEN algorithm?

*8. Consider the following game. I choose a positive integer n and keep it secret; your goal is to discover this integer. We play the game in rounds. In each round, you write a list of *at most* n integers on the blackboard. If you write more than n numbers in a single round, you lose. (Thus, in the first round, you must write only the number 1; do you see why?) If n is one of the numbers you wrote, you win the game; otherwise, I announce which of the numbers you wrote is smaller or larger than n, and we proceed to the next round. For example:

You	Me
1	It's bigger than 1.
4, 42	It's between 4 and 42.
8, 15, 16, 23, 30	It's between 8 and 15.
9, 10, 11, 12, 13, 14	It's 11; you win!

Describe a strategy that allows you to win in $O(\alpha(n))$ rounds!