Homework 5 Solutions

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Question 3

Consider two graphs G_1 and G_2 and let us denote the question 'is G_1 isomorphic to some subgraph of G_2' by $SI(G_1, G_2)$. We wish to prove that subgraph isomorphism (SI)is NP-complete:

1. Is SI in NP? This asks whether we can check a supposed solution in polynomial time. You should ask yourself, 'what does a solution look like?' of 'what is the form of a solution to SI?' An answer will be an isomorphism between the G_1 and some subgraph H of G_2. Hence, we are dealing with a bijective function f that maps the vertices of G_1 to the vertices of H. Said another way, the the solution could have $v \rightarrow f(v)$ for all v in G_1 and all $f(v)$ in H.

How do we check such a solution? In order to verify that this mapping is actually a valid isomorphism (ie. a solution), we check to make sure that if there is an edge (u, v) in G_1 then there is a corresponding edge $(f(u), f(v))$ in H (usually, you have to check the other way, but since we are dealing with a subgraph this doesn't matter).

Can we do this edge checking in polynomial time? Certainly. All we have to do is for every vertex pair of vertices u, v in G_1 that have an edge between them, we have to check that there is a corresponding edge between $f(u)$ and $f(v)$ in H. This can be done in $O(|V|^2|E|)$ time where V is the vertex set of G_1 and E is the vertex set of G_1 ($|V|^2$ pairs of vertices, each time we run through the edge set $|E|$).

2. Now we need to do the reduction part of such a proof. That means we need to select a known NP-complete problem; in this case we choose CLIQUE(G,k).

Then we need a function g that maps instances of $CLIQUE(G, k)$ to instances of $SI(G_1, G_2)$. There is also an added criteria that we want g to fulfill two things (remember that $CLIQUE(G, k)$ and $SI(G_1, G_2)$ are decision problems; they have 'yes' or 'no' answers):

A) We want to be able to say that if we have a solution to $SI(G_1, G_2)$ then we have a solution to $CLIQUE(G, k)$ under the mapping g.
B) We want to be able to say that if we have a solution to $CLIQUE(G, k)$ then we have a solution to $SI(G_1, G_2)$ under the mapping g.

Define g to take the problem instance $CLIQUE(G_2, k)$ to the problem instance $SI(K_k, G_2)$ where K_k is a clique (complete graph) on k vertices. Can this mapping be done in polynomial time? Yes, we are mapping a problem instance to a problem instance; that is all, there is no big construction like you saw in class with 3-SAT to CLIQUE. It might help to think about mapping as simply constructing a clique of k vertices (clearly doable in polynomial time) and feed that as input to SI.

Now, we need to check the added criteria of g:
A) Assume that I have an answer to $SI(K_k, G_2)$, do I have an answer to $CLIQUE(G_2, k)$? Yes. Assume, that the answer to $SI(K_k, G_2)$ is 'yes'. This means that a clique on k vertices is isomorphic to some subgraph of G_2. But then I know that G_2 itself contains a subgraph on k vertices that is a clique; hence, I have an answer to $CLIQUE(G_2, k)$. The same thing holds if the answer to $SI(K_k, G_2)$ is no.

B) Assume that I have an answer to $CLIQUE(G_2, k)$, do I have an answer to $SI(K_k, G_2)$? Yes. Knowing that G_2 contains a clique of size k guarantees us that a clique of size k will be isomorphic to some subgraph (namely that clique in G_2) in G_2. Again, an answer of 'no' to $CLIQUE(G_2, k)$ tells us that K_k is not isomorphic to any subgraph of G_2.

One may feel some confusion at this stage. We could ask 'how come we're not considering *other* instances of SI where the first graph is not necessarily a clique?'. The most important answer is that to show NP-completeness, we need only show the 'iff answer portion' under **under the mapping** g. Hence, we need only consider that an answer to $CLIQUE(G_2, k)$ gives us an answer to SI **under the mapping of** g which in turn means that is gives us an answer to $SI(K_k, G_2)$. A less important, but interesting, answer is to note (as we did in the review session) that $CLIQUE$ can be viewed as a special case of SI where the first graph is a clique. Hence, SI is larger and at least as hard as $CLIQUE$; so our mapping makes sense.

Having shown that SI is in NP and having provided a reduction from $CLIQUE$ to SI, we have proven that SI is NP-complete.

Question 4

We wish to show that Independent Set (IS) is NP-complete. Remember the steps for an NP-completeness proof.

1. Prove that IS is in NP. Remember to ask yourself, 'what does a solution look like?' A solution will be a set of vertices that form the independent set. Can we check that this solution provides a valid independent set? Yes. We simply take the first vertex in the given solution and make sure that it shares no edges with any of the other vertices in the solution. We repeat this for all vertices in the solution. Can we perform this verification in polynomial time? Certainly. We can do such a check in $O(|V|^2|E|)$ time (we have to check $|V|^2$ pairs of vertices, looking through the edge set each time).

2. Now we need to do the reduction portion of the NP-completeness proof. This means we need a function g that maps problem instances from some known NP-complete problem to problem instances of IS. The hint tells us to use CLIQUE as our known NP-complete problem.

Define g to map the problem instance $CLIQUE(\bar{G}, k)$ to the problem instance $IS(G, k)$ where $\bar{G}$ refers to the complement of G. The complement of $G = (V, E)$ is a graph $\bar{G} = (V, E')$; that is they have the same vertices, but the edges are not the same. Where there is an edge (u, v) in G, there is no edge (u, v) in $\bar{G}$. Conversely, if there is no edge (s, t) in G, then there is an edge (s, t) in $\bar{G}$.

Can g be done in polynomial time? This is asking 'Can we construct the complement of a graph in polynomial time?'. The answer is 'yes'. For every pair of vertices in, what will become, $\bar{G}$ we look at the same pair in G. If there is an edge in G we don't draw an edge in $\bar{G}$, otherwise we do. Hence, we can do this construction in at most $O(|V|^2|E|)$ time.

Now the answer part of the reduction:

A) Assume that we have an answer to $CLIQUE(\bar{G}, k)$. In $\bar{\bar{G}} = G$, the $k-$clique becomes an independent set. Hence, if we know that we have a clique of size k in $\bar{G}$, then we know that we have an independent set of size k in G (and similarly, for a 'no' answer).

B) Assume that we have an answer to $IS(G, k)$. In the process of taking the complement of G to get $\bar{G}$, the independent set of size k becomes a clique of size k in $\bar{G}$. Hence, if we know we know an independent set of size k in G, we then know that we have a clique of size k in $\bar{G}$ (similarly, for a 'no' answer).

Hence, we have completed all the steps we need to provide a proof that IS is NP-complete.