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Augmented Aztec bipyramid and dicube tilings

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ABSTRACT

We consider the enumeration of dicube tilings, where each tiling represents a three-dimensional tessellation of a polycube using dicubes. While the enumeration of domino tilings of polycubes like the Aztec diamond and the augmented Aztec diamond is well studied, we focus on the three-dimensional analogue, the augmented Aztec bipyramid. This polycube consists of unit cubes and resembles a Platonic octahedron. In this paper, we find a bijection between dicube tilings of the augmented Aztec bipyramid and three-dimensional Delannoy paths, and use this correspondence to determine the number of dicube tilings of the augmented Aztec bipyramid.

1. Introduction

In combinatorics, an *Aztec diamond* of order n is a quadrilaterally symmetric region composed of $2n$ rows of unit squares, where the rows have lengths $2, 4, \ldots, 2n-2, 2n, 2n, 2n-2, \ldots, 4, 2$. A domino is a 2×1 rectangle consisting of two unit squares. The Aztec diamond theorem states that the number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$. This theorem was proved firstly by Elkies, Kuperberg, Larsen and Propp [5], and many other proofs were presented in [2–4,6,7,11].

The *augmented Aztec diamond* is obtained from the Aztec diamond by replacing the two long columns in the middle with three columns. See Fig. 1. The enumeration for domino tilings of a region is known to be very sensitive to its boundary shape [12]. Sachs and Zernitz [16] found the number of domino tilings of the augmented Aztec diamond of order n, which is known as the Delannoy number $D(n) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 2^k$. In a survey paper [15], Propp listed dozens of enumeration problems related to domino tilings of the Aztec diamond and its variants with holes.

The authors recently obtained several results on enumeration for domino tilings of the following variations of the Aztec diamond in a series of papers:

- augmented Aztec rectangles and their variants which are generalized regions of augmented Aztec diamonds [8];
- Aztec octagons obtained from a rectangular region by deleting four triangular corners (not necessarily congruent) [9,13,14]; and

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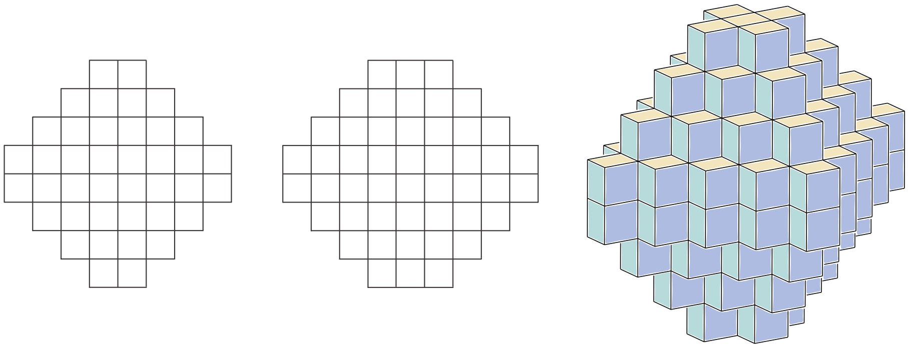

Fig. 1. Aztec diamond, augmented Aztec diamond, and three-dimensional augmented Aztec bipyramid of order 4.

- the Aztec diamond revisited with a simpler proof of the Aztec diamond theorem and domino-deficient augmented Aztec rectangles [10].

The main idea in [8–10] can be summarized as follows. After choosing a checkerboard coloring of the squares of each of these regions, we associate the region with a dual lattice graph connecting center points of the black squares in a specific way. Then we construct a bijection between the set of domino tilings of the region and the set of Delannoy path systems in the dual lattice graph, and use the bijection to count the number of the Delannoy path systems.

Tilings of two-dimensional regions have been widely studied. The analogous enumeration of tilings of three-dimensional regions is much more difficult to deal with and has not yet been studied much. A *polycube* is a connected solid figure formed by combining unit cubes face to face. A *dicube tiling* of a polycube is a tessellation of the polycube by dicubes, where a dicube consists of two unit cubes attached along a face like a $2\times1\times1$ cuboid. Bodini and Jamet [1] found a necessary and sufficient condition for a so-called pyramidal polycube to have a dicube tiling.

As a three-dimensional extension of an augmented Aztec diamond, an *augmented Aztec bipyramid* $\mathcal{P}_n$ of order n is a polycube satisfying the following properties (as drawn in the right figure in Fig. 1):

- all unit cubes in $\mathcal{P}_n$ have centers at integer lattice points in $\mathbb{R}^3$;
- both cross sections of $\mathcal{P}_n$ cut off by the planes $x=0$ and $y=0$ are the augmented Aztec diamond of order n; and
- each cross section of $\mathcal{P}_n$ cut off by the plane $x=\pm m$ or $y=\pm m$ for $m=1,2,\ldots,n$ is the augmented Aztec diamond of order $n-m$ with one more square attached on each of the top and the bottom.

It resembles a platonic octahedron. We state the main result of this paper.

Theorem 1. *The number of dicube tilings of the augmented Aztec bipyramid $\mathcal{P}_n$ of order n is given by*

$$\sum_{k=0}^{n}\binom{n+k}{n-k}\binom{2k}{k}^2.$$

Remark that if we attach one cube above the top-center and another cube below the bottom-center, then the resulting polycube is more like an octahedron, but has only one dicube tiling.

2. Replacing a dicube tiling by Delannoy paths

In this section, we consider dicube tilings of a general polycube, which is a union of a finite number of unit cubes in $\mathbb{R}^3$ and has a connected interior. A polycube is said to be *dicube-tilable* if it admits a dicube tiling. We find a simple necessary condition for a given polycube to be dicube-tilable. We also introduce three-dimensional Delannoy paths and then show that dicube tilings of a given polycube correspond bijectively to systems of disjoint Delannoy paths.

From now on, we place a polycube $\mathcal{C}$ in $\mathbb{R}^3$ so that the unit cubes in $\mathcal{C}$ have their centers located at points in $\mathbb{Z}^3$ and their faces are parallel to the xy-, yz- or zx-plane. The cubes in $\mathcal{C}$ are alternately colored black and white like a three-dimensional version of checkerboard coloring.

Given a colored polycube $\mathcal{C}$, a black cube in $\mathcal{C}$ is called a *pivot cube* if it has its top face in $\partial\mathcal{C}$. A unit cube in $\mathbb{R}^3$ which is not contained in $\mathcal{C}$ but adjacent to a white cube in $\mathcal{C}$ along its top face is called a *ghost cube* of $\mathcal{C}$. For our convenience, we color the ghost cube black. By exchanging black and white colors, if necessary, we may assume that there is always a pivot cube in $\mathcal{C}$.

Now we associate $\mathcal{C}$ with a graph $\Gamma_{\mathcal{C}}=(V_{\mathcal{C}},E_{\mathcal{C}})$, called the *dual lattice graph* of $\mathcal{C}$, and it is constructed as follows:

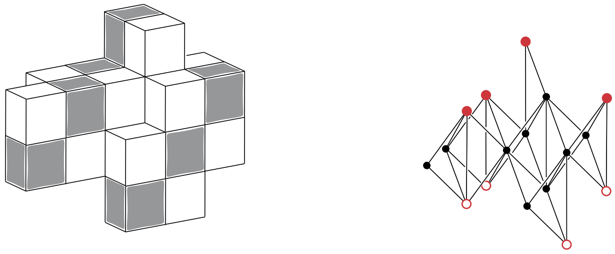

Fig. 2. Polycube $\mathcal{C}$ and its dual lattice graph $\Gamma_{\mathcal{C}}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

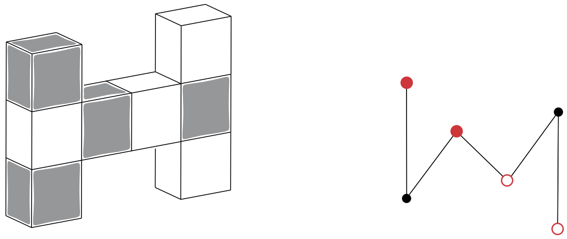

Fig. 3. A non-dicube-tilable polycube with the same number of pivot and ghost cubes.

- the center points of black cubes in $\mathcal{C}$ and the center points of ghost cubes of $\mathcal{C}$ form the vertex set $V_{\mathcal{C}}$ (especially the center points of pivot/ghost cubes are called the *pivot/ghost vertices*); and
- for each non-ghost vertex (x, y, z) in $V_{\mathcal{C}}$, if $\mathcal{C}$ contains a white cube centered at $(x\pm1, y, z)$ (resp. $(x, y\pm1, z)$ and $(x, y, z-1)$), then the two vertices (x, y, z) and $(x\pm1, y, z-1)$ (resp. $(x, y\pm1, z-1)$ and $(x, y, z-2)$) are joined by an edge in $E_{\mathcal{C}}$.

An example of $\mathcal{C}$ and its dual lattice graph $\Gamma_{\mathcal{C}}$ are depicted in Fig. 2. In the figure, the red bold dots indicate pivot vertices and the red circular dots indicate ghost vertices.

Before we count the number of dicube tilings of a polycube, it is natural to check whether the polycube is dicube-tilable or not. The following proposition gives a necessary condition for a polycube to be dicube-tilable.

Proposition 2. *If a polycube $\mathcal{C}$ is dicube-tilable, then $\mathcal{C}$ must have the same number of pivot cubes and ghost cubes (hence the dual lattice graph $\Gamma_{\mathcal{C}}$ has the same number of pivot vertices and ghost vertices).*

Proof. Choose a dicube-tilable polycube $\mathcal{C}$ which is a union of $2n$ unit cubes. More precisely, each number of black cubes and white cubes equals n. Let p and g be the number of the pivot and ghost cubes, respectively. We extend $\mathcal{C}$ to another polycube $\mathcal{C}'$ by attaching

- a white cube to each pivot cube along the top face; and
- all the ghost cubes to $\mathcal{C}$.

Then $\mathcal{C}'$ has p more white cubes and g more black cubes than $\mathcal{C}$.

We will show that $\mathcal{C}'$ is also dicube-tilable. By a column of $\mathcal{C}'$, we mean a set of all unit cubes in $\mathcal{C}'$ which are projected onto the same unit square in the xy-plane. Notice that a column possibly has more than one connected component. Each connected component of a column has a white cube at the top and a black cube at the bottom by its construction, and so it is dicube-tilable. This implies that $\mathcal{C}'$ is also dicube-tilable.

Since it has $n+p$ white cubes and $n+g$ black cubes, we conclude that $n+p=n+g$ or equivalently $p=g$. $\square$

The polycube in Fig. 3 shows that the converse of Proposition 2 does not hold in general.

We define a three-dimensional extension of Delannoy paths. A three-dimensional *Delannoy path* is a lattice path in $\mathbb{Z}^3$ of finite length with steps in

$$\{(0,1,-1),(0,-1,-1),(1,0,-1),(-1,0,-1),(0,0,-2)\}.$$

Given a dicube-tilable colored polycube $\mathcal{C}$, we also define a *Delannoy path system* in $\Gamma_{\mathcal{C}}$ to be a non-intersecting family of Delannoy paths with the properties that

- each path in the system runs from a pivot vertex to a ghost vertex in $\Gamma_{\mathcal{C}}$; and

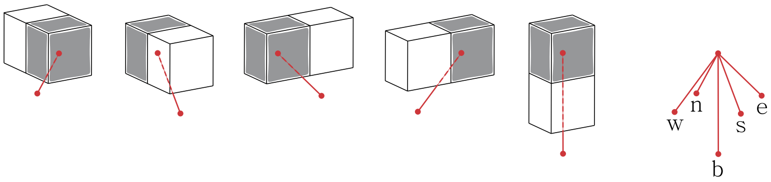

Fig. 4. Delannoy path replacement of dicubes.

- every pivot vertex of $\Gamma_{\mathcal{C}}$ is joined to a ghost vertex of $\Gamma_{\mathcal{C}}$ by a path in the system.

Note that the second property combined with Proposition 2 guarantees that every ghost vertex is an endpoint of a Delannoy path.

Now we construct a bijection between dicube tilings of a dicube-tilable polycube $\mathcal{C}$ and Delannoy path systems in $\Gamma_{\mathcal{C}}$. Any dicube consists of one black cube and one white cube, and according to the position of the white cube with respect to the black cube, we call it an n-, s-, e-, w-, b-, or t-*dicube* if the white cube lies to the north $(+y)$, south $(-y)$, east $(+x)$, west $(-x)$, bottom $(-z)$, or top $(+z)$, respectively, of the black cube. A dicube is called a D-*dicube* if it is not a t-dicube.

Theorem 3. *If a polycube $\mathcal{C}$ is dicube-tilable, then there is a bijection between the set of dicube tilings of $\mathcal{C}$ and the set of Delannoy path systems in $\Gamma_{\mathcal{C}}$.*

Proof. The proof is analogous to the proof of [8, Theorem 1]. We construct a map $\Phi\colon \mathbb{T}_{\mathcal{C}} \to \mathbb{S}_{\mathcal{C}}$, where $\mathbb{T}_{\mathcal{C}}$ is the set of dicube tilings of $\mathcal{C}$ and $\mathbb{S}_{\mathcal{C}}$ is the set of Delannoy path systems in $\Gamma_{\mathcal{C}}$. We define the map Φ as follows: Φ replaces each n-, s-, e-, w- or b-dicube in a dicube tiling $T \in \mathbb{T}_{\mathcal{C}}$ with an edge of $\Gamma_{\mathcal{C}}$ corresponding to the step $(0,1,-1)$, $(0,-1,-1)$, $(1,0,-1)$, $(-1,0,-1)$, or $(0,0,-2)$, respectively, starting at the center of the black cube of the dicube as shown in Fig. 4. We ignore all t-dicubes.

Now we show that Φ is really a function into $\mathbb{S}_{\mathcal{C}}$.

Claim 1. *For each pivot vertex, $\Phi(T)$ contains a Delannoy path joining it to some ghost vertex of $\Gamma_{\mathcal{C}}$.*

Proof. Let (x_1, y_1, z_1) be a pivot vertex. Let D_1 denote the D-dicube in T covering the pivot cube corresponding to (x_1, y_1, z_1). Such a D-dicube indeed exists by the fact that no white cube in $\mathcal{C}$ is adjacent to the top face of the pivot cube.

By the definition of Φ, we get a single-step path P_1 in $\Gamma_{\mathcal{C}}$, jumping from (x_1, y_1, z_1) to the vertex (x_2, y_2, z_2) by using one of the steps $(0,1,-1)$, $(0,-1,-1)$, $(1,0,-1)$, $(-1,0,-1)$ or $(0,0,-2)$ according to whether D_1 is an n-, s-, e-, w- or b-dicube. Moreover, notice that the cube centered at (x_2, y_2, z_2+1) is the white cube in D_1.

Next, consider the black cube centering at (x_2, y_2, z_2). If it is not in $\mathcal{C}$, then it must be a ghost cube of $\mathcal{C}$ so we are done. Otherwise, T contains another D-dicube D_2 covering the black cube. We extend the path P_1 to P_2, which ends at the vertex (x_3, y_3, z_3), by adjoining a new step in the same way as above. We continue this process to obtain a sequence of paths $\{P_n\}$. The process stops in finite steps since the number of unit cubes in $\mathcal{C}$ is finite, and the final path reaches some ghost vertex. □

Claim 2. *Any two Delannoy paths in $\Phi(T)$ do not intersect.*

Proof. Assume for a contradiction that two Delannoy paths P and P' intersect. Consider the vertices at which P and P' intersect. Among them, we choose the vertex (x, y, z) which has the highest z-coordinate. Clearly, the vertex (x, y, z) is not a pivot vertex. Let B be the black cube centered at (x, y, z) and W be the white cube centered at $(x, y, z+1)$. Since B is covered by a D-dicube D in T, which is not a t-dicube, W cannot be covered by D. Therefore, T contains another D-dicube D' that covers W. This implies that the edge joining two vertices centered at the black cubes in D and D' is uniquely determined by the definition of Φ. Then the edge is contained in both P and P', which contradicts the choice of (x, y, z). □

By the two claims above, we know that $\Phi(T)$ contains a Delannoy path system S in $\Gamma_{\mathcal{C}}$. We now show that $\Phi(T) = S$. Assume for a contradiction that there is a D-dicube D in T whose image under Φ is not contained in S. Let the black cube of D be centered at (x, y, z). Then we can construct a Delannoy path starting at (x, y, z) to some ghost vertex v_g of $\Gamma_{\mathcal{C}}$ as in the proof of Claim 1. A Delannoy path in S also terminates at v_g, which contradicts Claim 2. Therefore, $\Phi(T) \in \mathbb{S}_{\mathcal{C}}$, implying that Φ is a function into $\mathbb{S}_{\mathcal{C}}$.

The injectivity immediately follows from the definition of Φ. In the remainder of the proof we will show that Φ is surjective. Given a Delannoy path system in $\mathbb{S}_{\mathcal{C}}$, each step in the system is replaced with a D-dicube by reversing the

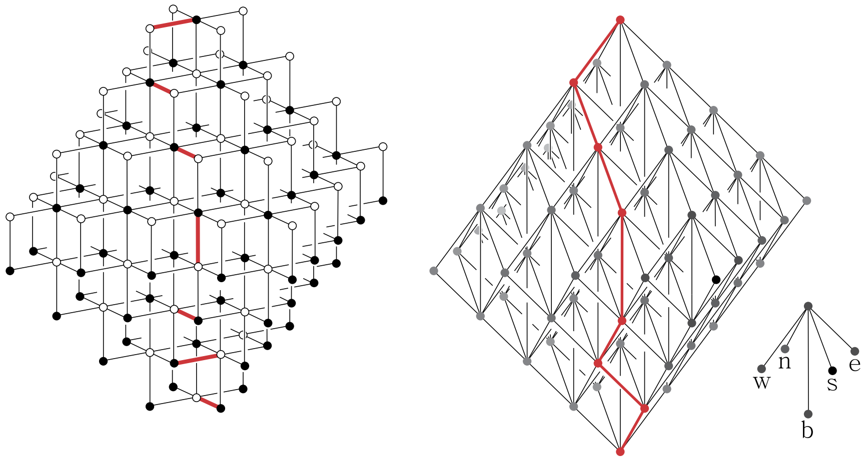

Fig. 5. The graphs $\mathcal{G}_n$ and Γ_n.

construction of Φ described in the first paragraph of this proof. Any two D-dicubes do not share a black (resp. white) cube, since otherwise the corresponding steps have the same starting (resp. ending) point.

Let Q be the sub-polycube of $\mathcal{C}$ which is the union of the D-dicubes obtained above. If Q equals $\mathcal{C}$, then we are done. Suppose not. Then we claim that Q^c can be tiled only by t-dicubes, where Q^c is the complementary polycube of Q in $\mathcal{C}$. Choose a white cube W in Q^c and let B denote the black cube adjacent to the bottom face of W. Clearly B is not in Q, otherwise the reverse construction of Φ implies that W is in Q. If B is not in $\mathcal{C}$, then it should be a ghost cube of $\mathcal{C}$. However it is a contradiction because a white cube which lies to the top of any ghost cube lies in Q. Therefore, if Q^c has a white cube W, then the black cube B just below W is contained in Q^c. Furthermore $\mathcal{C}$ has the same number of white and black cubes, and so does Q^c. It follows that Q^c can be tiled by t-dicubes only, as desired. Therefore we obtain a dicube tiling of $\mathcal{C}$ from the Delannoy path system, which ensures the surjectivity of Φ. $\square$

3. Proof of Theorem 1

In this section we prove Theorem 1. Recall that an augmented Aztec bipyramid $\mathcal{P}_n$ of order n is a polycube given as the union of unit cubes which have centers at integer lattice points (x, y, z) satisfying $|x| + |y| + |z| \le n+1$ with $z = 1, \dots, n$ or $|x| + |y| + |z| \le n$ with $z = 0, -1 \dots, -(n-1)$. We color the unit cubes of $\mathcal{P}_n$ so that the unit cube with the center point $(0, 0, n)$ is the only one pivot cube and the unit cube with the center point $(0, 0, -n)$ is the ghost cube.

We construct a graph $\mathcal{G}_n$ by taking a vertex at the center of each unit cube contained in $\mathcal{P}_n$ and joining by an edge any two vertices of distance 1 apart. By coloring each vertex of $\mathcal{G}_n$ with the color of the unit cube containing the vertex, we may consider $\mathcal{G}_n$ as a bipartite graph as shown on the left of Fig. 5.

For simplicity, we use Γ_n to denote the dual lattice graph $\Gamma_{\mathcal{P}_n}$ of $\mathcal{P}_n$, defined in Section 2. As shown on the right of Fig. 5, the vertex set of Γ_n consists of all the black vertices of $\mathcal{G}_n$ together with the ghost vertex at the point $(0, 0, -n)$, and two vertices of Γ_n are joined by an edge if they are located at points (x, y, z) and (x', y', z') such that $(x, y, z) + s = (x', y', z')$, where $s = (0, 1, -1), (0, -1, -1), (1, 0, -1), (-1, 0, -1)$ or $(0, 0, -2)$. One can see that a vertex of Γ_n is located at an integer lattice point (x, y, z) if and only if the following two conditions hold:

$$\begin{cases} \text{(i)} \ \ |x| + |y| + |z| \le n, \\ \text{(ii)} \ \ \text{the integer } x + y + z \text{ has the same parity as the integer } n. \end{cases} \tag{$*$}$$

Actually, from the two pictures in Fig. 5, we obtain a dicube tiling of $\mathcal{P}_4$ as follows. Consider the seven red edges $e_1, \dots, e_7$ of the graph $\mathcal{G}_4$, numbered from top to bottom, on the left of Fig. 5. Each edge e_i has one endpoint at a black vertex and the other at a white vertex. The endpoint of e_i at a white vertex and the endpoint of e_{i+1} at a black vertex are joined by a vertical edge of $\mathcal{G}_4$ $(i = 1, \dots, 6)$. Each edge e_i is a spine of a D-dicube, say, D_i. The union $Q = D_1 \cup \dots \cup D_7$ is the connected sub-polycube of $\mathcal{P}_4$ covered by D-dicubes, in the proof of Theorem 3, and the complementary sub-polycube Q^c can be covered only by t-dicubes. Combining the dicube tilings of the sub-polycubes Q and Q^c, we obtain a dicube tiling of $\mathcal{P}_4$. Also, the sequence $e_1, \dots, e_7$ corresponds to the Delannoy path in Γ_4 on the right of Fig. 5.

Generally, $\mathcal{P}_n$ is dicube-tilable for any positive integer n: $\mathcal{P}_n$ has a dicube tiling such that the central column is covered by b-dicubes and the others are covered by t-dicubes. By Theorem 3, dicube tilings of $\mathcal{P}_n$ correspond bijectively to Delannoy path systems in Γ_n. Since Γ_n has exactly one pivot vertex and exactly one ghost vertex, each Delannoy path system consists of exactly one Delannoy path. It follows that the number of dicube tilings of $\mathcal{P}_n$ is equal to that of Delannoy paths in Γ_n. We count the number of Delannoy paths in Γ_n. Recall that a Delannoy path in Γ_n is a lattice path with a finite number of steps in

$$\{(0,1,-1),(0,-1,-1),(1,0,-1),(-1,0,-1),(0,0,-2)\},$$

starting at the point $(0,0,n)$ and terminating at the point $(0,0,-n)$.

We claim that a Delannoy path in Γ_n determines three nonnegative integers a,b,c such that $a+b+c=n$.

Claim 3. *Let P be a Delannoy path in Γ_n consisting of m steps $s_1,\dots,s_m$. Define $a,b^{\pm},c^{\pm}$ as the following numbers:*

- *a: the number of steps $s_i=(0,0,-2)$ in P,*
- *$b^{\pm}$: the number of steps $s_j=(\pm1,0,-1)$ in P, and*
- *$c^{\pm}$: the number of steps $s_k=(0,\pm1,-1)$ in P.*

Then $b^+=b^-$, $c^+=c^-$, and $a+b+c=n$, where $b=b^+$, $c=c^+$.

Proof. The path P starts at $(0,0,n)$ and terminates at $(0,0,-n)$. By the definition of the numbers $a,b^{\pm},c^{\pm}$, we have

$$(0,0,n)+(b^+-b^-,c^+-c^-,-2a-b^+-b^--c^+-c^-)=(0,0,-n).$$

Hence $b^+=b^-$, $c^+=c^-$, and if we let $b=b^+$ and $c=c^+$ then $n-2(a+b+c)=n-2a-b^+-b^--c^+-c^-=-n$ and hence $a+b+c=n$. □

Conversely, we construct Delannoy paths in Γ_n from a triple (a,b,c) of nonnegative integers such that $a+b+c=n$.

Claim 4. *Let a,b,c be nonnegative integers such that $a+b+c=n$. Suppose that P is a Delannoy path in $\mathbb{Z}^3$, starting at the point $(0,0,n)$, with steps $s_1,\dots,s_m$. We further suppose that*

- *a is the number of steps $s_i=(0,0,-2)$,*
- *b is the number of steps $s_j=(\varepsilon,0,-1)$ for each $\varepsilon=\pm1$, and*
- *c is the number of steps $s_k=(0,\varepsilon,-1)$ for each $\varepsilon=\pm1$.*

Then P is a path in Γ_n.

Proof. To prove that P is a path in Γ_n, it is enough to show that the vertices of P belong to the vertex set of Γ_n. We actually verify the two conditions in ($*$). Let $v_0,v_1,\dots,v_m$ be the vertices of P, numbered in succession, where v_0 is located at $(0,0,n)$. For each ℓ ($1\le\ell\le m$), we define the numbers $a_\ell,b_\ell^{\pm},c_\ell^{\pm}$ as follows:

- a_ℓ is the number of steps $s_i=(0,0,-2)$ between v_0 and v_ℓ,
- $b_\ell^{\pm}$ is the number of steps $s_j=(\pm1,0,-1)$ between v_0 and v_ℓ, and
- $c_\ell^{\pm}$ is the number of steps $s_k=(0,\pm1,-1)$ between v_0 and v_ℓ.

Then $0\le a_\ell\le a$, $0\le b_\ell^{\pm}\le b$, $0\le c_\ell^{\pm}\le c$ and $a_m=a$, $b_m^+=b_m^-=b$, $c_m^+=c_m^-=c$. Suppose that the vertex v_ℓ is located at (x_ℓ,y_ℓ,z_ℓ). Then we have

$$\begin{aligned}
x_\ell&=b_\ell^+-b_\ell^-,\\
y_\ell&=c_\ell^+-c_\ell^-,\\
z_\ell&=n-2a_\ell-b_\ell^+-b_\ell^--c_\ell^+-c_\ell^-.
\end{aligned}$$

If $z_\ell>0$, then

$$\begin{aligned}
|x_\ell|+|y_\ell|+|z_\ell|&=|b_\ell^+-b_\ell^-|+|c_\ell^+-c_\ell^-|+n-2a_\ell-b_\ell^+-b_\ell^--c_\ell^+-c_\ell^-\\
&\le b_\ell^++b_\ell^-+c_\ell^++c_\ell^-+n-2a_\ell-b_\ell^+-b_\ell^--c_\ell^+-c_\ell^-\\
&=n-2a_\ell\le n.
\end{aligned}$$

Suppose $z_\ell\le0$. One easily sees that $|b_\ell^+-b_\ell^-|+b_\ell^++b_\ell^-=2\max(b_\ell^+,b_\ell^-)\le 2b$ and that $|c_\ell^+-c_\ell^-|+c_\ell^++c_\ell^-=2\max(c_\ell^+,c_\ell^-)\le 2c$. Thus

$$\begin{aligned}
|x_\ell|+|y_\ell|+|z_\ell|&=|b_\ell^+-b_\ell^-|+|c_\ell^+-c_\ell^-|-n+2a_\ell+b_\ell^++b_\ell^-+c_\ell^++c_\ell^-\\
&=-n+2a_\ell+2\max(b_\ell^+,b_\ell^-)+2\max(c_\ell^+,c_\ell^-)\\
&\le -n+2(a+b+c)=n.
\end{aligned}$$

Also, since the sum of x-, y-, z-coordinates of every step is either 0 or -2, the integers $x_\ell+y_\ell+z_\ell$ have the same parity with the integer n. Therefore each v_ℓ is a vertex in Γ_n. Finally,

$$\begin{aligned}
x_m &= b_m^+ - b_m^- = b - b = 0 \\
y_m &= c_m^+ - c_m^- = c - c = 0 \\
z_m &= n - 2a_m - b_m^+ - b_m^- - c_m^+ - c_m^- = n - 2(a + b + c) = -n.
\end{aligned}$$

Thus P terminates at $(0, 0, -n)$, so it is a Delannoy path in Γ_n. $\square$

By Claim 4, whenever we have a triple (a, b, c) of nonnegative integers with $a + b + c = n$, we obtain a Delannoy path in Γ_n by arranging, in a row, a steps $s_i = (0, 0, -2)$, b steps $s_j = (1, 0, -1)$, b steps $s_{j'} = (-1, 0, -1)$, c steps $s_k = (0, 1, -1)$ and c steps $s_{k'} = (0, -1, -1)$. The number of such arrangements is equal to the multinomial combination $\binom{a+2b+2c}{a,b,b,c,c} = \binom{b+c+n}{a,b,b,c,c}$. Thus we conclude that the number of Delannoy paths in Γ_n is the sum

$$\sum_{a+b+c=n} \binom{b+c+n}{a, b, b, c, c} = \sum_{a+b+c=n} \frac{(b+c+n)!}{a!\,(b!\,c!)^2}.$$

This is equal to the number of dicube tilings of $\mathcal{P}_n$ by Theorem 3. Letting $b + c = k$, we get

$$\begin{aligned}
\sum_{a+b+c=n} \frac{(b+c+n)!}{a!\,(b!\,c!)^2} &= \sum_{k=0}^{n} \sum_{b=0}^{k} \frac{(n+k)!}{(n-k)!\,(b!\,(k-b)!)^2} \\
&= \sum_{k=0}^{n} \sum_{b=0}^{k} \frac{(n+k)!}{(n-k)!\,(2k)!} \frac{(2k)!}{k!\,k!} \frac{(k!)^2}{(b!\,(k-b)!)^2} \\
&= \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k} \sum_{b=0}^{k} \binom{k}{b} \binom{k}{k-b} \\
&= \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k}^2,
\end{aligned}$$

where the last equality follows from Vandermonde's identity $\sum_{b=0}^{k} \binom{m}{b}\binom{n}{k-b} = \binom{m+n}{k}$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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