Some remarks on n-fibrations

Lab Lunch, Birmingham

Sina Hazratpour
sinhp@github.io

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- In the first section I will review essentials of theory of Grothendieck and Street fibrations of categories. They will be the base case of inductive definition of n-fibration in the last section.

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- The third section will be about internalization of Grothendieck and Street fibrations in 2-categories. Here, I will talk about some very recent work Steve and I did on relating internal fibrations to 2-fibrations.
- In the last section, I will present state of the art regarding theory of n-fibrations of n-categories.

A notation guide

I will stick to these more or less!

- For categories: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, . . ., $\mathcal{X}$
- Objects of categories, 2-categories, bicategories: A, B, . . ., Z
- For functors: $\mathfrak{p}$, $\mathfrak{q}$, . . .
- For bicategories: $\mathbb{C}$, $\mathbb{K}$, $\mathbb{L}$, . . ., $\mathbb{X}$
- For 2-functors and bifunctors: $\mathbb{P}$, $\mathbb{C}\mathbf{od}$, . . .
- Comma categories: $\mathcal{C}^{\rightarrow}$, $\mathcal{C}^{[1]}$, . . .

Fibrations of groupoids

Definition

A functor $\pi : \mathcal{G} \to \mathcal{H}$ of groupoids is a **fibration** whenever for every arrow $f : V \to U$ in $\mathcal{H}$ and every object X in $\mathcal{G}$ sitting above U, there is an arrow $F : Y \to X$ with $\pi(F) = f$. We call F *lift* of f ending at X.

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EXAMPLE

for a covering map $p : E \to B$ of topological spaces the fundamental groupoid functor $\Pi(p) : \Pi(E) \to \Pi(B)$ is a discrete fibration.

Cartesian morphisms

Definition

Suppose $\mathcal{X}$ and $\mathcal{C}$ are categories and $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a functor. A morphism $u : Y \to X$ in $\mathcal{X}$ is called $\mathfrak{p}$-**precartesian** whenever for any $\mathcal{X}$-morphism $v : Z \to X$ with $\mathfrak{p}(v) = \mathfrak{p}(u)$, there exists a unique vertical morphism w such that $u \circ w = v$. Morphism $u : X \to Y$ is said to be $\mathfrak{p}$-**cartesian** whenever for any $\mathcal{X}$-morphism $v : Z \to X$ and any $h : \mathfrak{p}(Z) \to \mathfrak{p}(X)$ with $\mathfrak{p}(u) \circ h = \mathfrak{p}(v)$, there exists a unique lift w of h such that $u \circ w = v$.

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Precartesian:

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists ! w"'] \arrow[dr, bend left, "v"] & \\ Y \arrow[r, "u"] & X \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d, mapsto, "\mathfrak{p}"'] \\ {} \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{p}Z \arrow[d, equal] \arrow[dr, bend left, "Pv"] & \\ \mathfrak{p}Y \arrow[r, "\mathfrak{p}u"'] & \mathfrak{p}X \end{tikzcd}$$

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Basic well-known facts about cartesian morphisms

which should not be questioned about in this talk!

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- Let $v\colon Y \to W$ be a $\mathfrak{p}$-cartesian morphism in $\mathcal{X}$. Any morphism $u\colon X \to Y$ is $\mathfrak{p}$-cartesian if and only if $v \circ u\colon X \to W$ is $\mathfrak{p}$-cartesian.

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- An $\mathcal{X}$-morphism $u \colon X \to Y$ is $\mathfrak{p}$-cartesian if and only if the following commuting square is a pullback diagram in **Set** for each object Z in $\mathcal{X}$:

$$\begin{tikzcd}
\mathcal{X}(Z,X) \arrow[r, "u\circ -"] \arrow[d, "\mathfrak{p}_{Z,X}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{X}(Z,Y) \arrow[d, "\mathfrak{p}_{Z,Y}"] \\
\mathcal{C}(\mathfrak{p}Z,\mathfrak{p}X) \arrow[r, "\mathfrak{p}(u)\circ -"'] & \mathcal{C}(\mathfrak{p}Z,\mathfrak{p}Y)
\end{tikzcd}$$

Typical example

- For any category $\mathcal{C}$, there is a codomain functor $\mathrm{cod} : \mathcal{C}^{[1]} \to \mathcal{C}$ which sends an object $f : J \to I$ of $\mathcal{C}^{[1]}$ to its codomain C and sends a morphism $\langle v, u \rangle : g \to f$ of $\mathcal{C}^{[1]}$, i.e. a commuting square, to f.

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- Cartesian morphisms in $\mathcal{C}^{[1]}$ are precisely pullback squares in $\mathcal{C}$.

$$
\begin{tikzcd}
Y \arrow[r, dashed, "u^* f"] \arrow[d, dashed, "f^* u"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "u"] \\
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Fibrations of categories

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A functor $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a *Grothendieck prefibration* (resp. *Grothendieck fibration*) whenever for each $X \in \mathcal{X}$, every morphism $A \xrightarrow{f} \mathfrak{p}X$ in $\mathcal{C}$ has a precartesian (resp. cartesian) lift in $\mathcal{X}$.

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Example

$\mathrm{cod} : \mathcal{C}^{[1]} \to \mathcal{C}$ is a fibration iff $\mathcal{C}$ has all pullbacks.

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Non-example

A simple functor which fails to be a fibration: consider category $\mathcal{C}$ consisting of two objects x_0 and x_1 with their identities and an arrow θ_x between them. Let $\mathcal{X}$ be the category extended by a fresh arrow $e\colon x_0 \to x_0$ with $e \circ e = e$ and $\theta_x \circ e = \theta_x$. The functor $\mathcal{X} \to \mathcal{C}$ which sends θ_x to itself and e to id_{x_0} is not a fibration since θ_x in $\mathcal{C}$ does not have a cartesian lift.

More examples of fibrations

- For a category $\mathcal{B}$, the category of families in $\mathcal{B}$ can be constructed as a comma object in $\mathfrak{Cat}$. The projection functor $\pi_1 : \mathbf{Fam}(\mathcal{B}) \to \mathbf{Set}$ is a Grothendieck fibration.

$$\begin{tikzcd}
\mathbf{Fam}(\mathcal{B}) \arrow[r, "!"] \arrow[d, "\pi_1"'] & 1 \arrow[d, "\mathcal{B}"] \\
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- Wait for a proof of this at the end of this section!

Cloven fibrations

Definition

A **cleavage** for a (pre)fibration $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a choice for each $X \in \mathcal{X}$ and $f : B \to \mathfrak{p}X$ in $\mathcal{C}$, a (pre)cartesian lift $\rho(f, X) : \rho_f X \to X$ of f in $\mathcal{X}$. More formally, the data of a cleavage is a term ρ of the following dependent type:

$$\rho : \prod_{B,A:\ \mathbf{Ob}(\mathcal{C})} \ \prod_{f:\ \mathcal{C}(B,A)} \ \prod_{X:\ \mathcal{X}_A} \ \sum_{Y:\ \mathcal{X}_B} \mathcal{C}art_{\mathcal{X}}(Y, X)$$

where the type $\mathcal{C}art_{\mathcal{X}}(Y, X)$ is type of all cartesian morphisms from Y to X. If the fibration $\mathfrak{p}$ is equipped with a cleavage ρ, then $(\mathfrak{p}, \rho)$ is called a **cloven fibration**. The cleavage ρ is said to be *splitting* if for any composable pair of morphisms f, g:

$$\rho(g \circ f, X) = \rho(g, X) \circ \rho(f, \rho_g X)$$

And *normalized* whenever for every object X in $\mathcal{X}$:

$$\rho(id_{\mathfrak{p}X}, X) = id_X$$

Example of cloven fibration

A cloven fibration $(\mathrm{cod}, \rho) : \mathcal{C}^{[1]} \to \mathcal{C}$ is precisely a category $\mathcal{C}$ with a choice of pullbacks.

Basic well-known properties of Grothendieck fibrations
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- A (cloven) prefibration $\mathfrak{p}$ is a (cloven) fibration iff $\mathfrak{p}$-precartesian morphisms are closed under composition.

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- $\langle \mathfrak{p}, \rho \rangle \colon \mathcal{X} \to \mathcal{C}$ is a cloven Grothendieck fibration if and only if for each object $X \in \mathcal{X}$, the induced functor $\mathfrak{p}_X \colon \mathcal{X}/X \to \mathcal{C}/\mathfrak{p}X$ has a right adjoint right inverse.

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- (Cloven) Grothendieck fibrations are closed under composition.
- (Cloven) Grothendieck fibrations are closed under pullback along other functor.
- A Pseudo-pullback of functors is equivalent to their strict pullback if either of the functors is a Grothendieck fibration.

A 2-category of Grothendieck fibrations

Definition

A **(pre)fibration map** between two (pre)fibrations $\mathfrak{q} : \mathcal{Y} \to \mathcal{D}$ and $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ consists of two functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{Y} \to \mathcal{X}$ such that

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "G"] \arrow[d, "\mathfrak{q}"'] & \mathcal{X} \arrow[d, "\mathfrak{p}"] \\ \mathcal{D} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

commutes, and moreover, G is cartesian, that is it carries $\mathfrak{q}$-cartesian (resp.precartesian) morphisms to $\mathfrak{p}$- cartesian (resp.precartesian) morphisms. A **(pre) fibration transformation** is a pair of natural transformations $(\alpha\colon F_0 \to F_1, \beta\colon G_0 \to G_1)$ such that $\mathfrak{p} \centerdot \beta = \alpha \centerdot \mathfrak{q}$.

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(Pre)fibrations, (pre)fibration maps, and (pre)fibration transformations form a 2-category.

A 2-category of Grothendieck fibrations

By looking at fibres of a cloven fibration (prefibration) we get a pseudofunctor (resp. lax functor) to 2-category of categories. Grothendieck construction makes this into a biequivalence of 2-categories:

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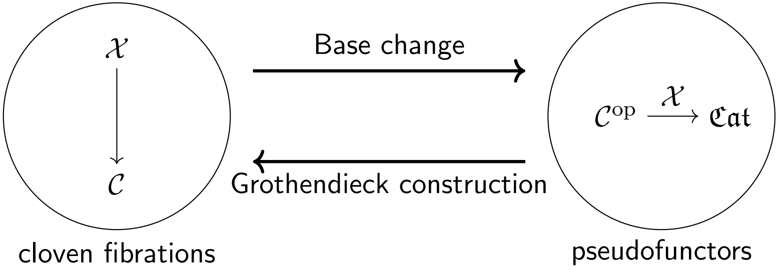

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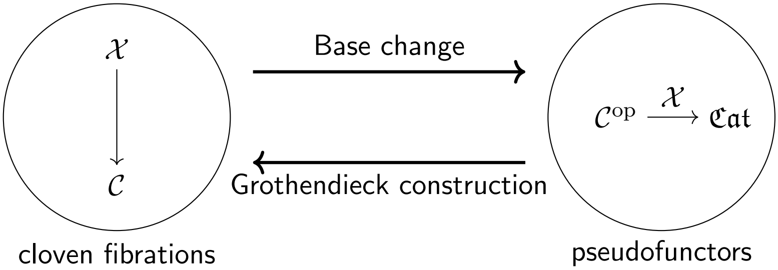
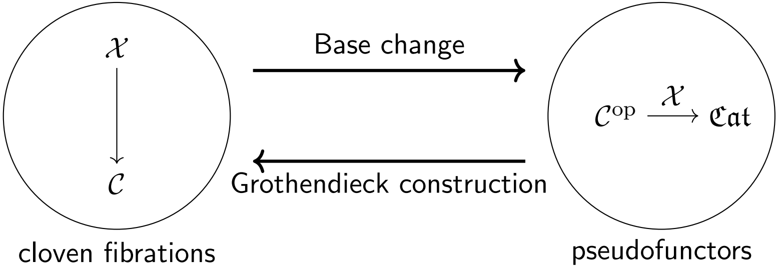


$$\mathbf{clPreFib}(\mathcal{C}) \simeq \mathbf{LaxFun}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Cat})$$

$$\mathbf{clFib}(\mathcal{C}) \simeq \mathbf{PsFun}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Cat})$$

where **clFib** is the 2-category of cloven fibrations.

A 2-category of Grothendieck fibrations



Remark

An interesting feature of the Grothendieck construction is that it reduces category level. That is it turns a 2-functor of 2-categories into a single morphisms in 2-category of categories. Other than a change in viewpoint it makes a world of difference when we work in higher levels; An n-stack in algebraic geometry can be conceived as a category fibred in spaces instead of an ∞-functor to the ∞-category $\mathscr{S}$ of spaces.

Fam as example of Grothendieck construction

For a category $\mathcal{B}$, the Grothendieck fibration $\mathbf{Fam}(\mathcal{B}) \to \mathbf{Set}$ is the Grothendieck construction of 2-functor

$$\mathbf{Fun}(-, \mathcal{B}) \colon \mathbf{Set}^{\mathrm{op}} \to \mathfrak{Cat}$$

where for an (indexing) set I, $\mathbf{Fun}(I, \mathcal{B})$ is the category of functors from discrete category I to category $\mathcal{B}$.

How forgetful are fibrations?

Homotopy type of homotopy fibre (aka bipullback) over a point tells us how forgetful the fibration is. For instance

$$\begin{tikzcd} U^*G \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathbf{Ab} \arrow[d, "U"] \\ 1 \arrow[r, "G"'] & \mathbf{Grp} \end{tikzcd}$$

And one can easily see that the fibre

$$U^*G \cong \left(\sum_{A \colon Ab} A \cong G\right)$$

is either empty or a contractible groupoid and in terminology of (Baez and Shulman, 2010), U forgets property of being abelian. However in the case of forgetful fibration $\mathbf{Ab} \to \mathbf{Set}$ the homotopy fibres are 0-groupoids (aka sets), and the forgetful functor forgets structure of abelian group.

Going one dimension higher

2-cartesian 1-cells

Suppose $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a 2-functor. Inspired by the case of 1-functors we define 2-cartesian 1-cells as follows.

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Definition

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$$\begin{tikzcd}
\mathbb{X}(W,X) \arrow[r, "u_*"] \arrow[d, "\mathbb{P}_{W,X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{X}(W,Y) \arrow[d, "\mathbb{P}_{W,Y}"] \\
\mathbb{C}(\mathbb{P}W,\mathbb{P}X) \arrow[r, "\mathbb{P}(u)_*"'] & \mathbb{C}(\mathbb{P}W,\mathbb{P}Y)
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Remark

By considering object component of pullback diagram above we observe that every 2-cartesian 1-cell is automatically 1-cartesian in the usual sense.

2-cartesian 1-cells in elementary terms

This definition gives us two layers of cartesian properties of 1-cells w.r.t. $\mathbb{P}$ in $\mathbb{X}$. First of all, u is 1-cartesian as usual. Second, every 2-cell $\alpha\colon v \Rightarrow w\colon W \to Y$ and every 2-cell $\sigma\colon g \Rightarrow h\colon \mathbb{P}W \to \mathbb{P}X$ with $\mathbb{P}(\alpha) = \mathbb{P}(u) \centerdot \sigma$ there is a unique lift ϕ of σ such that $u \centerdot \phi = \alpha$.

$$\begin{tikzcd}
W \arrow[r, dotted, bend left, "\overline{g}"] \arrow[r, dotted, bend right, "\overline{h}"'] \arrow[r, phantom, "\phi\Downarrow"] \arrow[rr, bend left=60, "v"] \arrow[rr, bend right=60, "w"'] & X \arrow[r, "u"] \arrow[l, phantom, "\Downarrow\alpha"] & Y
\end{tikzcd}$$

$$\big\downarrow \mathbb{P}$$

$$\begin{tikzcd}
\mathbb{P}W \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\sigma\Downarrow"] \arrow[rr, bend left=60, "\mathbb{P}v"] \arrow[rr, bend right=60, "\mathbb{P}w"'] & \mathbb{P}X \arrow[r, "\mathbb{P}u"] & \mathbb{P}Y
\end{tikzcd}$$

Strict 2-fibrations

Definition

A 2-functor $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a **2-fibration** if

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Remark

The second condition is equivalent to say that for any morphism g in $\mathbb{X}$ and a 2-cell $\alpha\colon f \Rightarrow \mathbb{P}g$, there is a cartesian 2-cell $\sigma\colon f \Rightarrow g$ with $\mathbb{P}\sigma = \alpha$.

An archetypal example of strict 2-fibration

Example

The 2-category **Fib** of Grothendieck fibrations is a 2-fibred over 2-category of categories via the codomain functor $\mathrm{cod}\colon \mathbf{Fib} \to \mathfrak{Cat}$

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Notice that every 2-fibration gives rise to a base change 2-functor. In this case, given a functor $H\colon \mathcal{C} \to \mathcal{D}$, we get a 2-functor $H^*\colon \mathbf{Fib}(\mathcal{D}) \to \mathbf{Fib}(\mathcal{C})$ which takes a fired category over $\mathcal{D}$ to its strict pullback along H.

Weak 2-cartesian 1-cells

Definition

Suppose $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a 2-functor. A 1-cell $f : X \to Y$ in $\mathbb{X}$ is **weakly cartesian** with respect to $\mathbb{P}$ whenever for each 0-cell W in $\mathbb{X}$ the following commuting square is a bipullback diagram in 2-category $\mathfrak{Cat}$ of categories.

$$\begin{tikzcd}
\mathbb{X}(W, X) \arrow[r, "f_*"] \arrow[d, "\mathbb{P}_{W,X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{X}(W, Y) \arrow[d, "\mathbb{P}_{W,Y}"] \\
\mathbb{C}(\mathbb{P}W, \mathbb{P}X) \arrow[r, "\mathbb{P}(f)_*"'] & \mathbb{C}(\mathbb{P}W, \mathbb{P}Y)
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Igor Bakovic (2012). "Fibrations in tricategories". In: *93rd Peripatetic Seminar on Sheaves and Logic, University of Cambridge* and
Mitchell Buckley (2014). "Fibred 2-categories and bicategories". In: vol. 218, pp. 1034–1074

Weak 2-cartesian 1-cells in elementary terms

Only a bit more complicated than last one-lifts up to iso

$$
\begin{tikzcd}
W \arrow[dr, dashed, "\hat{h}"'] \arrow[drr, bend left, "g"] & & \\
 & X \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\hat{\alpha}"] & Y
\end{tikzcd}
$$

$$
\Big\downarrow \mathbb{P}
$$

$$
\begin{tikzcd}
\mathbb{P}W \arrow[dr, "h"] \arrow[dr, bend right=60, "\mathbb{P}\hat{h}"'] \arrow[drr, bend left, "\mathbb{P}g"] & & \\
 & \mathbb{P}X \arrow[r, "\mathbb{P}f"'] \arrow[ur, phantom, "\Uparrow\alpha"] \arrow[ul, phantom, "\Uparrow\hat{\beta}"] & \mathbb{P}Y
\end{tikzcd}
$$

Input data:

1. $g\colon W \to Y$
2. $h\colon \mathbb{P}W \to \mathbb{P}X$
3. iso 2-cell $\alpha\colon \mathbb{P}(f) \circ h \Rightarrow \mathbb{P}(g)$

Output data:

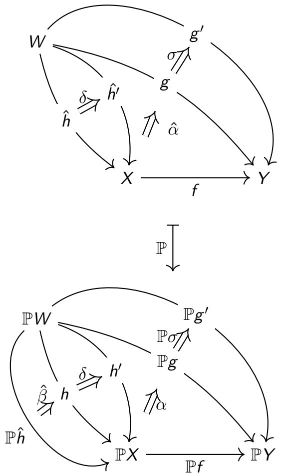
(not necc. unique)

1. $\hat{h}\colon W \to X$
2. iso 2-cell $\hat{\alpha}\colon f\hat{h} \Rightarrow g$
3. iso 2-cell $\hat{\beta}\colon \mathbb{P}(\hat{h}) \Rightarrow h$
4. an equality of 2-cells

$\alpha \circ (\mathbb{P}(f) \centerdot \hat{\beta}) = \mathbb{P}(\hat{\alpha}) \circ \Phi_{h,f}$

Weak 2-cartesian 1-cells in elementary terms

Continued



Input data:

1. $\sigma \colon g \Rightarrow g' \colon W \rightrightarrows Y$
2. $\delta \colon \mathbb{P}h \Rightarrow \mathbb{P}h' \colon \mathbb{P}W \rightrightarrows \mathbb{P}X$
3. iso 2-cells

$\alpha \colon \mathbb{P}(f) \circ h \Rightarrow \mathbb{P}(g)$

$\alpha' \colon \mathbb{P}(f) \circ h' \Rightarrow \mathbb{P}(g)$

4. an equality of 2-cells

$\alpha' \circ (\mathbb{P}f \,.\, \delta) = \mathbb{P}(\sigma) \circ \alpha$

Output data:

1. unique $\hat{\delta} \colon \hat{h} \Rightarrow \hat{h}'$
2. an equality $\hat{\alpha}' \circ (f \,.\, \hat{\delta}) = \sigma \circ \hat{\alpha}$
3. an equality $\delta \,.\, (\hat{\beta}) = \hat{\beta}' \circ \mathbb{P}\hat{\delta}$

Cartesian 2-cells

Definition

A 2-cell $\alpha\colon f \Rightarrow g\colon x \to y$ in $\mathbb{X}$ is **cartesian** if it is cartesian as a 1-cell for the functor $\mathbb{P}_{xy}\colon \mathbb{X}(x,y) \to \mathbb{C}(\mathbb{P}x, \mathbb{P}y)$.

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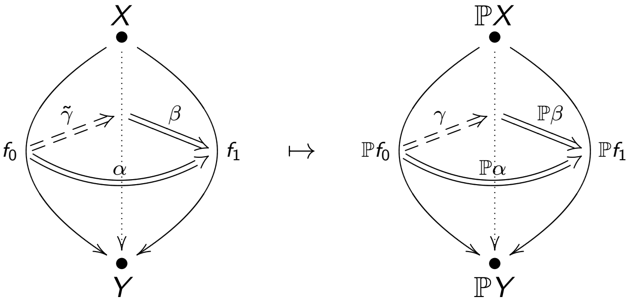
In elementary terms it means a 2-cell $\alpha\colon f_0 \Rightarrow f_1\colon X \rightrightarrows Y$ is cartesian if for any given 1-cell $e\colon X \to Y$ and 2-cell $\beta\colon e \to f_1$ with $\mathbb{P}\alpha = \mathbb{P}\beta \circ \gamma$ for some 2-cell γ, then there is a unique 2-cell $\tilde{\gamma}$ over γ such that $\alpha = \beta \circ \tilde{\gamma}$.



Weak 2-fibrations

As in the case of strict 2-fibrations, we say that $\mathbb{P}$ is *locally fibred* when $\mathbb{P}_{XY} \colon \mathbb{X}(x, y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a fibration for all X, Y in $\mathbb{X}$.

Definition

Let $\mathbb{P} \colon \mathbb{X} \to \mathbb{C}$ be a 2-functor. We say that $\mathbb{P}$ is a **fibration** whenever

1. for any $X \in \mathbb{X}$ and $f \colon B \to \mathbb{P}X$ in $\mathbb{C}$, there is a 2-cartesian 1-cell $\widetilde{f} \colon \widetilde{B} \to X$ with $\mathbb{P}\widetilde{f} = f$;

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Mitchell Buckley (2014). “Fibred 2-categories and bicategories”. In: vol. 218, pp. 1034–1074

Internal fibrations to 2-categories

The first and the most obvious way to internalize definition of Grothendieck fibration in a 2-category is the representable approach. The second approach was developed by Street in (Street, 1974) who introduced two-sided fibrations in 2-categories, followed by two-sided fibrations in bicategories. These fibrations are defined as algebras over certain 2-monads on 2-categories, or hyperdoctrines on bicategories respectively, and Chevalleys internal characterization of fibrations was obtained as a theorem.

Internal fibrations to 2-categories

The third approach was developed by Johnstone in (Johnstone, 1993) which is closer than Streets definition to the spirit of Grothendiecks original definition. For instance, the base change functors is part of data of definition. Johnstone also established the equivalence of his definition with the representable one.

Internal fibrations to 2-categories

Unlike Street's definition, Johnstone's definition does not require strictness of the 2-category nor the existence of the structure of strict pullbacks and comma objects. Indeed, this definition is most suitable for weak 2-categories such as 2-category of toposes where we do not expect diagrams of 1-cells to commute strictly. This definition is also very flexible in terms of existence of bipullbacks: one only needs existence of bipullbacks of the class of 1-cells one would like to define as (op)fibrations. We will call these 1-cells *carrable*.

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Ross Street (1974). "Fibrations and Yoneda's lemma in a 2-category". In: *Lecture Notes in Math., Springer, Berlin* Vol.420, pp. 104–133

Peter Johnstone (1993). "Fibrations and partial products in a 2-category". In: *Applied Categorical Structures* vol.1

Turning iso 2-cells of $\mathbb{K}$ into 1-cells

Suppose $\mathbb{K}$ is a 2-category and $\mathbb{I}$ is the interval category. We can form a new 2-category $\mathbb{K}^{\mathbb{I}} := \mathbf{Fun}_{ps}(\mathbb{I}, \mathbb{K})$ consisting of (strict) 2-functors, pseudo-natural transformations and modifications between them.

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- 0-cells are of the form

$$\begin{tikzcd} E \arrow[d, "p"'] \\ S \end{tikzcd}$$

  where $p \in \mathbb{K}_1$.

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- 1-cells from q to p are of the form $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f} \rangle$

$$\begin{tikzcd} F \arrow[r, "\overline{f}"] \arrow[d, "q"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & E \arrow[d, "p"] \\ A \arrow[r, "\underline{f}"'] & S \end{tikzcd}$$

  where $\overset{\blacktriangledown}{f} : p\overline{f} \Rightarrow \underline{f}q$ is an iso 2-cell in $\mathbb{K}$.

- 2-cells between 1-cells f and g are of the form $\alpha = \langle \overline{\alpha}, \underline{\alpha} \rangle$ where $\overline{\alpha} : \overline{f} \Rightarrow \overline{g}$ and $\underline{\alpha} : \underline{f} \Rightarrow \underline{g}$ are 2-cells in $\mathbb{K}$

$$
\begin{tikzcd}
F \arrow[rr, bend left, "\overline{g}"] \arrow[rr, bend right, "\overline{f}"'] \arrow[rr, phantom, "\Uparrow\overline{\alpha}"] \arrow[dd, "q"'] & & E \arrow[dd, "p"] \\
& \overset{\blacktriangledown}{f} \Downarrow \quad \overset{\blacktriangledown}{g} \Downarrow & \\
A \arrow[rr, bend left, dotted, "\underline{g}"] \arrow[rr, bend right, "\underline{f}"'] \arrow[rr, phantom, "\Uparrow\underline{\alpha}"] & & S
\end{tikzcd}
$$

in such a way that the obvious diagram of 2-cells commutes.

Remark

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Recall that a triple category is an internal category in the category $\mathcal{D}bl$ of strict double categories and strict double functors. Also recall that a double category is an internal category in the category $\mathcal{C}at$ of small categories and functors.

Fibration 0-cells in 2-category $\mathbb{K}^{\mathbb{I}}$

Let $p\colon E \to S$ be a 0-cell in $\mathbb{K}$. We call p a **fibration** 0-cell in 2-category $\mathbb{K}^{\mathbb{I}}$ whenever for any 2-cell $\underline{\alpha} : \underline{f} \Rightarrow \underline{g} : A \to S$ in $\mathbb{K}$, we have

- a 1-cell $\langle \overline{r(\alpha)}, \overset{\blacktriangledown}{r_\alpha}, 1_A \rangle \colon \underline{g}^* p \to \underline{f}^* p$
- and a 2-cell $(\overline{\alpha}, \underline{\alpha})\colon f \circ r(\alpha) \Rightarrow g$

Unpack

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Unpack

in $\mathbb{K}^{\mathbb{I}}$, and moreover the following axioms are satisfied:

1. There is an isomorphism $(\overline{\tau_f}, id_{1_A}) : id_{\underline{f}^\star p} \Rightarrow r_{\underline{f}}$ such that $(\overline{id_f}, id_{\underline{f}}) \circ (\overline{f}\,\overline{\tau_0}, id_{\underline{f}}) = (id_{\overline{f}}, id_{\underline{f}})$.

Unpack

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Unpack

2. If $\underline{\beta} : \underline{g} \Rightarrow \underline{h}$ is another 2-cell in $\mathbb{K}$, then there exists an iso 2-cell $\tau_{\alpha,\beta} : r(\alpha) \circ r(\beta) \Rightarrow r(\beta\alpha)$ such that the following diagram of 2-cells in $\mathbb{K}^{\mathbb{I}}$ commutes:

$$\begin{array}{ccc} f \circ r(\alpha) \circ r(\beta) & \xRightarrow{\alpha \centerdot r(\beta)} & g \circ r(\beta) \\ {\scriptstyle f \centerdot \tau_{\alpha,\beta}}\Big\Downarrow & = & \Big\Downarrow{\scriptstyle \beta} \\ f \circ r(\beta\alpha) & \xRightarrow[\beta\alpha]{} & h \end{array}$$

3. Lifting of α is compatible with left whiskering; That is, given any 1-cell $\underline{k} : B \to A$ in $\mathbb{K}$, we require $r(\alpha \centerdot k)$ to fit into the following bipullback square in $\mathbb{K}^{\mathbb{I}}$:

$$\begin{tikzcd}
(\underline{g}\,\underline{k})^* p \arrow[r, "k_g"] \arrow[d, "r(\alpha.k)"'] \arrow[dr, phantom, "\cong_\kappa"] & \underline{g}^* p \arrow[d, "r(\alpha)"] \\
(\underline{f}\,\underline{k})^* p \arrow[r, "k_f"'] & \underline{f}^* p
\end{tikzcd}$$

where k_f and k_g are pullback 1-cells over $\underline{k}$.
We also require pasting of 2-cells α and κ to be equal to 2-cell $\alpha \centerdot k$.

Unpack

4. For any 1-cells $y = \langle \overline{y}, id, 1_A \rangle$ where $\overline{y} \colon D \to \underline{g}^* E$, and $x = \langle \overline{x}, \overset{\blacktriangledown}{x}, 1_A \rangle \colon \underline{g}^* p \circ \overline{y} \to \underline{f}^* p$ where $\overline{x} \colon D \to \underline{f}^* E$, and , any 2-cell $\beta = \langle \overline{\beta}, \underline{\alpha} \rangle \colon f \circ x \Rightarrow g \circ y$ in $\mathbb{K}^{\mathbb{I}}$ is uniquely factored through α, that is there is a unique 2-cell μ in $\mathbb{K}^{\mathbb{I}}$ with property $(\alpha \centerdot y) \circ (f \centerdot \mu) = \beta$, that is to say the two pasting diagrams in below are equal:

$$
\begin{tikzcd}
\underline{g}^* p \circ \overline{y} \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\mu"] & \underline{f}^* p \arrow[d, "f"] \\
\underline{g}^* p \arrow[ur, "r(\alpha)"'] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\alpha"] & p
\end{tikzcd}
=
\begin{tikzcd}
\underline{g}^* p \circ \overline{y} \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\beta"] & \underline{f}^* p \arrow[d, "f"] \\
\underline{g}^* p \arrow[r, "g"'] & p
\end{tikzcd}
$$

Unpack

Fibration 0-cells in $\mathbb{K}^{\mathbb{I}}$ are Johnstone fibrations in $\mathbb{K}$

Proposition

A 0-cell in $\mathbb{K}^{\mathbb{I}}$ is a fibration iff it is a fibration as a 1-cell in $\mathbb{K}$ in the sense of Johnstone.

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Example

A 0-cell fibration in 2-category $\mathfrak{Cat}^{\mathbb{I}}_{st}$ = A Johnstone fibration in 2-category $\mathfrak{Cat}_{st}$ = A Grothendieck fibration of categories

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A 0-cell fibration in 2-category $\mathfrak{Cat}^{\mathbb{I}}_{ps}$ = A Johnstone fibration in 2-category $\mathfrak{Cat}_{ps}$ = A weak fibration (aka Street fibration) of categories

Fibrations in 2-cats vs. fibrations between 2-cats

Here is what recently Steve and I realized recently:

PROPOSITION

a 1-cell in $\mathbb{K}^{\mathbb{I}}$ is cartesian iff it is a bipullback square in $\mathbb{K}$.

$$
\begin{tikzcd}
F \arrow[r, dashed, "\overline{f}"] \arrow[d, dashed, "\underline{f}^* u"'] \arrow[dr, phantom, "\Downarrow r_{\underline{f}}"] & E \arrow[d, "p"] \\
A \arrow[r, "\underline{f}"] & S
\end{tikzcd}
$$

$$
\Big\downarrow \mathbb{C}\mathbf{od}
$$

$$
A \xrightarrow[\underline{f}]{} S
$$

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a 1-cell in $\mathbb{K}^{\mathbb{I}}$ is cartesian iff it is a bipullback square in $\mathbb{K}$.

$$
\begin{tikzcd}
F \arrow[r, dashed, "\overline{\underline{f}}"] \arrow[d, dashed, "\underline{f}^* u"'] \arrow[dr, phantom, "\lrcorner" very near start] \arrow[dr, phantom, "\Downarrow r_{\underline{f}}"] & E \arrow[d, "p"] \\
A \arrow[r, "\underline{f}"] & S
\end{tikzcd}
$$

$$\Big\downarrow \mathbb{C}\mathbf{od}$$

$$A \xrightarrow[\underline{f}]{} S$$

PROPOSITION

The 2-functor $\mathbb{C}\mathbf{od} \colon \mathbb{K}^{\mathbb{I}} \to \mathbb{K}$ is a weak 2-fibration of 2-categories iff every p in $\mathbb{K}^{\mathbb{I}}$ is a fibration and furthermore,

A few questions to think about

- Weak double categories and more generally n-fold categories are one of the main examples of Segal-type higher categories (Paoli, 2017). Can we generalize the notion of 2-fibration between 2-categories/bicategories to a notion of fibration between weak double categories? Or further to a notion of fibration between n-fold categories?

A few questions to think about

- Weak double categories and more generally n-fold categories are one of the main examples of Segal-type higher categories (Paoli, 2017). Can we generalize the notion of 2-fibration between 2-categories/bicategories to a notion of fibration between weak double categories? Or further to a notion of fibration between n-fold categories?
- Following the first question, since double categories have a well-established notion of weak equivalence, we can ask whether the class of fibrations and weak equivalences form a weak factorization system and whether the fibrant objects are framed bicategories in the sense of (Shulman, 2009)

n-fibrations: a tentative definition

In order to define fibration of (weak) n-categories first we have to say what we mean by a (weak) n-category. There has been a two decades of work on this and it is still not a conclusive matter what an n-category should be. A few of them:

- Batanin Leinster: n-globular set with an action of a suitable globular operad.
- Street: simplicial set satisfying certain horn-filling conditions $\simeq$ truncation of a definition of ω-category
- Tamsamani Simpson: a simplicial object in $(n-1)$-categories satisfying object-discreteness and the Segal condition aka a Segal n-category
- Baez- Dolan: an opetopic set having enough n-universal fillers, ...

n-fibrations: a tentative definition

The idea of following tentative definition was put forward initially by Baez and Shulman in John C. Baez and Michael Shulman (2010). "Lectures on n-Categories and Cohomology". In: *Baez J., May J. (eds) Towards Higher Categories, Springer, New York, NY*. URL: arXiv:math/0608420 and later more explicitly on nLab by Shulman:

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Definition

Let $P \colon \mathbb{X} \to \mathbb{C}$ be a functor between (weak) n-categories. A morphism $f \colon X \to Y$ in $\mathbb{X}$ is **cartesian** w.r.t P if for any $W \in \mathbb{X}_0$, the following square:

$$\begin{tikzcd}
\mathbb{X}(W,X) \arrow[r, "f_*"] \arrow[d, "P_{W,X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{X}(W,Y) \arrow[d, "P_{W,Y}"] \\
\mathbb{C}(PW,PX) \arrow[r, "P(f)_*"'] & \mathbb{C}(PW,PY)
\end{tikzcd}$$

is a (weak) pullback of $(n-1)$-categories.

n-fibrations: a tentative definition

Definition

We say that $P\colon \mathbb{X} \to \mathbb{C}$ is a **weak n-fibration** if

1. For any object $X \in \mathbb{X}_0$ and morphism $f\colon X \to PA$ in $\mathbb{C}$, there exists a cartesian morphism $\tilde{f}\colon \tilde{X} \to A$ and an equivalence $P(\widetilde{f}) \simeq f$ in the slice n-category $\mathbb{C}/PA$,
2. For any objects $X, Y \in \mathbb{X}_0$, the functor $P_{X,Y} : \mathbb{X}(X,Y) \to \mathbb{C}(PX, PY)$ is an $(n-1)$-fibration,
3. For any $W, X, Y \in \mathbb{X}_0$, the square

$$\begin{tikzcd}
\mathbb{X}(X,Y) \times \mathbb{X}(W,X) \arrow[r, "\circ_{\mathbb{X}}"] \arrow[d, "P_{X,Y} \times P_{W,X}"'] & \mathbb{X}(W,Y) \arrow[d, "P_{W,Y}"] \\
\mathbb{C}(PX,PY) \times \mathbb{C}(PW,PX) \arrow[r, "\circ_{\mathbb{C}}"'] & \mathbb{C}(PW,PY)
\end{tikzcd}$$

is a morphism of $(n-1)$-fibrations.

n-fibrations: a tentative definition

Remark

An n-fibration is **strict** if in the first condition of definition above, equivalence is replaced with equality. The idea is that a strict 1-fibration should correspond to a Grothendieck fibration of categories while a weak 1-fibration corresponds to a Street fibration of categories.

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End!

THANK YOU FOR YOUR ATTENTION!

Bi-pullback: review

A bi-pullback of an opspan $A \xrightarrow{f} C \xleftarrow{g} B$ in a 2-category $\mathbb{K}$ is given by a 0-cell P together with 1-cells d_0, d_1 and an iso 2-cell $\delta\colon fd_0 \Rightarrow gd_1$ satisfying a universal property which states that given another iso cone $(l_0, l_1, \lambda\colon fl_0 \cong gl_1)$ over f, g (with vertex X) there exists a 1-cell u with two iso 2-cells γ_0 and γ_1 such that the pasting diagrams below are equal

Bi-pullback: review

$$
\begin{tikzcd}
X \arrow[dr, dashed, "u"] \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] \arrow[r, phantom, "\cong_{\gamma_1}"] \arrow[dd, phantom, "\cong_{\gamma_0}"] & {} & & & X \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] \arrow[ddrr, phantom, "\cong_{\lambda}"] & & \\
& P \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\cong_{\delta}"] & B \arrow[d, "g"] & = & & & B \arrow[d, "g"] \\
{} & A \arrow[r, "f"'] & C & & & A \arrow[r, "f"'] & C
\end{tikzcd}
$$

Bi-pullback: review

and furthermore, given 1-cells $u, v \colon X \rightrightarrows P$ and 2-cells $\alpha : d_0 u \Rightarrow d_0 v$ and $\beta : d_1 u \Rightarrow d_1 v$ in such a way that

$$\begin{tikzcd} fd_0 u \arrow[r, "f.\alpha"] \arrow[d, "\delta.u"'] & fd_0 v \arrow[d, "\delta.v"] \\ gd_1 u \arrow[r, "g.\beta"'] & gd_1 v \end{tikzcd}$$

commutes, there exists a unique 2-cell $\sigma \colon u \Rightarrow v$ such that $d_0 \centerdot \sigma = \alpha$ and $d_1 \centerdot \sigma = \beta$.

back to presentation

Supplemental diagrams

Unpacking them yields the following diagram in $\mathbb{K}$:

$$
\begin{tikzcd}
 & \underline{g}^* E \arrow[rr, "\bar{g}"] \arrow[dl, "\overline{r(\alpha)}"'] \arrow[dd, "\underline{g}^* p"' near start] & & E \arrow[dl, equal] \arrow[dd, "p"] \\
\underline{f}^* E \arrow[rr, "\bar{f}"' near start] \arrow[dd, "\underline{f}^* p"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{r}_{\alpha} \Downarrow"] & & E \arrow[dd, "p" near start] \arrow[ur, phantom, "\bar{\alpha} \Nearrow"] & \\
 & A \arrow[rr, "\underline{g}" near start] \arrow[dl, equal] & & S \arrow[dl, equal] \\
A \arrow[rr, "\underline{f}"'] & & S \arrow[ur, phantom, "\underline{\alpha} \Nearrow"] &
\end{tikzcd}
$$

where obvious diagram of 2-cells commutes.

Supplemental diagrams

Unpacking τ_f yields the following diagram in $\mathbb{K}$:

$$
\begin{tikzcd}
\underline{f}^*E \arrow[r, "\overline{I(id_{\underline{f}})}"] \arrow[r, bend right, "1"'] \arrow[r, phantom, shift right=3, "\Uparrow\overline{\tau_f}"] \arrow[rr, bend left, "\bar{f}"] \arrow[rr, phantom, shift left=5, "\Uparrow\overline{id_{\underline{f}}}"] \arrow[d, "\underline{f}^*p"'] & \underline{f}^*E \arrow[r, "\bar{f}"] \arrow[d, "\underline{f}^*p"] \arrow[dr, phantom, "\cong"] & E \arrow[d, "p"] \\
A \arrow[r, dotted, "1"] \arrow[r, bend right, "1"'] \arrow[r, phantom, shift right=3, "\Uparrow id"] & A \arrow[r, "\underline{f}"] & S
\end{tikzcd}
$$

We also get

$$\overset{\blacktriangledown}{r}_{\underline{f}} \circ (\underline{f}^*p \centerdot \overline{\tau_f}) = id_{\underline{f}^*p}$$

$$\overline{id_{\underline{f}}} \circ (\bar{f}\,\overline{\tau_f}) = id_{\bar{f}}$$

Supplemental diagrams

Unpacking $\tau_{\alpha,\beta}$ yields the following diagram in $\mathbb{K}$:

$$
\begin{tikzcd}
\underline{h}^* E \arrow[r, "\overline{r(\beta)}"] \arrow[rr, bend left, "\overline{r(\beta\circ\alpha)}"] \arrow[d, "\underline{h}^* p"'] \arrow[dr, phantom, "\cong"] & \underline{g}^* E \arrow[r, "\overline{r(\alpha)}"] \arrow[u, phantom, "\cong"] \arrow[d, "\underline{g}^* p"] \arrow[dr, phantom, "\cong"] & \underline{f}^* E \arrow[d, "\underline{f}^* p"] \\
A \arrow[r, "1"] & A \arrow[r, "1"] & A
\end{tikzcd}
$$

Furthermore, we get

$$\overline{r_{\beta\alpha}} \circ (\overline{f} \centerdot \overline{\tau}_{\alpha,\beta}) = \overline{\beta} \circ (\overline{\alpha} \centerdot \overline{r(\beta)})$$

$$\overset{\blacktriangledown}{r}_{\beta\alpha} \circ (\underline{f}^* p \centerdot \overline{\tau}_{\alpha,\beta}) = \overset{\blacktriangledown}{r}_{\beta} \circ (\overset{\blacktriangledown}{r}_{\alpha} \centerdot \overset{\blacktriangledown}{r}_{\beta})$$

pack

$\overline{r(\alpha \centerdot k)}$ is isomorphic to the bi-pullback of $\overline{r(\alpha)}$ along $\overline{k_f}$, which is to say the top left vertical square of the diagram commutes up to an isomorphism.

$$
\begin{tikzcd}
 & (\underline{g}\,\underline{k})^*E \arrow[rrrr, "\overline{gk}"] \arrow[dl, "\overline{r(\alpha\centerdot k)}"'] \arrow[dd, dotted] \arrow[rrr, phantom, "\overline{\alpha\centerdot k}\ \Nearrow"] & & & & E \arrow[dl, equal] \arrow[dd, "1"] \\
(\underline{f}\,\underline{k})^*E \arrow[rrrr, "\overline{fk}"'] \arrow[dd, dotted, "\cong"'] \arrow[dddd, bend right=60, "(\underline{f}\,\underline{k})^*p"', "\cong"] & & & & E \arrow[dd] & \\
 & \underline{k}^*\underline{g}^*E \arrow[rr, "\overline{k_g}"] \arrow[dl, "\underline{k}^*\overline{r(\alpha)}"'] \arrow[dd] & & \underline{g}^*E \arrow[rr, dotted, "\overline{g}"] \arrow[dl, "\overline{r(\alpha)}"'] \arrow[dd] \arrow[rr, phantom, "\overline{\alpha}\ \Nearrow"'] & & E \arrow[dl, equal] \arrow[dd, "p"] \\
\underline{k}^*\underline{f}^*E \arrow[rr, "\overline{k_f}"'] \arrow[dd, "\underline{k}^*\underline{f}^*p"'] & & \underline{f}^*E \arrow[rr, "\overline{f}"'] \arrow[dd, "\underline{f}^*p"'] & & E \arrow[dd, "p"'] & \\
 & B \arrow[rr, "\underline{k}"] \arrow[dl, equal] & & A \arrow[rr, "\underline{g}"] \arrow[dl, equal] \arrow[rr, phantom, "\underline{\alpha}\ \Nearrow"'] & & S \arrow[dl, equal] \\
B \arrow[rr, "\underline{k}"'] & & A \arrow[rr, "\underline{f}"'] & & S &
\end{tikzcd}
$$

Supplemental diagrams

$$
\begin{tikzcd}
D \arrow[rrr, bend left, "\overline{y}"] \arrow[drr, bend right, "\overline{x}"'] \arrow[dd, "\underline{g}^* p \circ \overline{y}"'] \arrow[ddrr, phantom, "\blacktriangledown X \Downarrow"] & & & \underline{g}^* E \arrow[dl, dotted, "\overline{r(\alpha)}"'] \arrow[dr, "\overline{g}"] \arrow[dd] & \\
& & \underline{f}^* E \arrow[rr, bend right, "\overline{f}"'] \arrow[dd, "\underline{f}^* p"'] & & E \arrow[dd, "p"] \\
A \arrow[rrr, dotted, "\cong"] \arrow[drr, bend right, "1"'] & & & A \arrow[dl, "1"] \arrow[dr, "\underline{g}"] \arrow[ur, phantom, "\overline{\alpha} \Nearrow"] & \\
& & A \arrow[rr, bend right, "\underline{f}"'] \arrow[urr, phantom, "\underline{\alpha} \Nearrow"] & & S
\end{tikzcd}
$$

pack