## Street for pedestrians

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November 29, 2017

#### Abstract

The singular aim of these notes is to give a less condensed and more clear explanation of Ross Street's significant paper (Street 1974). We highly encourage reader to also refer to (Kock 1995) and (Lack 2000).

## 1 Pseudo algebras for strict 2-monads

DEFINITION 1.1. Let  $\mathcal{K}$  be a 2-category and  $(T: \mathcal{K} \to \mathcal{K}, i: 1 \Rightarrow T, m: T^2 \Rightarrow T)$  a strict 2-monad on  $\mathcal{K}$ . A **pseudo-algebra** of T consists of

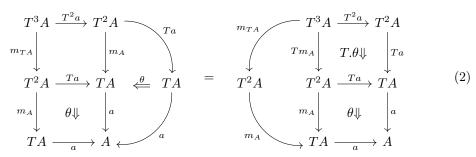
- i. a 0-cell A in K
- ii. a 1-cell  $a:TA\to A$
- iii. invertible 2-cells  $\zeta\colon 1_A\Rightarrow a\circ i_A$  and  $\theta\colon a\circ Ta\Rightarrow a\circ m_A$

subject to the following coherence axioms:

$$(\theta \cdot m_{TA}) \circ (\theta \cdot T^2 a) = (\theta \cdot T m_A) \circ (a \cdot T \theta)$$

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expressed by equality of pasting diagrams:



and

$$(\theta \cdot Ti_A) \circ (a \cdot T\zeta) = id_a = (\theta \cdot i_{TA}) \circ (\zeta \cdot a)$$

expressed by equality of pasting diagrams:

$$TA \xrightarrow{1_{TA}} TA$$

$$TA \xrightarrow{a} A$$

DEFINITION 1.2. Suppose  $(a, \zeta_A, \theta_A) : TA \to A$  and  $(b, \zeta_B, \theta_B) : TB \to B$  are pseudo-algebras of a 2-monad T. A **lax morphism** from a to b consists of a 1-cell  $f: A \to B$  and a 2-cell  $\check{f}$ 

$$TA \xrightarrow{Tf} TB$$

$$\downarrow a \qquad \qquad \downarrow b$$

$$A \xrightarrow{f} B$$

in such a way that

- $f \cdot \zeta_A = (\check{f} \cdot i_A) \circ (\zeta_B \cdot f)$  and
- $(f \cdot \theta_A) \circ (\check{f} \cdot Ta) \circ (b \cdot T\check{f}) = (\check{f} \cdot m_A) \circ (\theta_B \cdot T^2 f)$

DEFINITION 1.3. A 2-monad  $T: \mathcal{K} \to \mathcal{K}$  is said to be **lax idempotent** if given any two (pseudo) T-algebras  $a: TA \to A$ ,  $b: TB \to B$  and a 1-cell  $f: A \to B$ , there exists a unique 2-cell  $\check{f}: b \circ Tf \Rightarrow f \circ a$  rendering  $(f, \check{f})$  a lax morphism of pseudo T-algebras.

$$TA \xrightarrow{Tf} TB$$

$$\downarrow a \qquad \qquad \check{f} \Downarrow \qquad \downarrow b$$

$$A \xrightarrow{f} B$$

Remark 1.4. Dually, reverse the direction of  $\check{f}$  in definition 1.3, then we get the notion of **co-lax idempotent** monad.

### 2 KZ-monads

DEFINITION 2.1. A 2-monad  $T: \mathcal{K} \to \mathcal{K}$  is said to be KZ-monad<sup>1</sup> if  $m \dashv i \cdot T$  in the 2-category  $[\mathcal{K}, \mathcal{K}]$  with identity counit.

REMARK 2.2. Dual to the definition above, we define a monad T to be a **co-KZ-monad** by requiring  $i \cdot T \dashv m$  with identity unit.

Suppose T is a co-KZ-monad and  $i \cdot T \dashv m$ . In particular unit of this adjunction is identity since  $m \circ (i \cdot T) = 1$ . Moreover, the identity 2-cell

$$T \xrightarrow{1} id \Downarrow \int_{m}^{m} T \xrightarrow{T.i} T^{2}$$

has a mate

$$T \xrightarrow{1} T$$

$$\downarrow \qquad \tau \Downarrow \qquad \downarrow i.T$$

$$T \xrightarrow{T.i} T^{2}$$

$$(4)$$

with property that  $m \cdot \tau = id_{1_T}$ . Suppose  $a \colon TA \to A$  is a pseudo algebra. We now would like to calculate the composite 2-cell

$$TA \xrightarrow{\tau_A} T^2A \xrightarrow{a \circ Ta} TA$$

$$T_{i_A} T^2A \xrightarrow{a \circ m_A} TA$$

In the diagram below, since  $m_A \circ \tau = id$ , the left column of 2-cells collapses to identity, and therefore we have

$$TA \xrightarrow{1} TA \xrightarrow{a} A$$

$$\downarrow \qquad \qquad \downarrow i_{TA} \qquad \downarrow i_{A} \qquad \downarrow i_{A}$$

$$TA \xrightarrow{Ti_{A}} T^{2}A \xrightarrow{Ta} TA \underset{\downarrow}{\downarrow} 1_{A} \qquad = \qquad TA \xrightarrow{a} A$$

$$\downarrow \qquad \qquad \downarrow m_{A} \theta \Downarrow \qquad \downarrow a \qquad \qquad \downarrow 1_{A}$$

$$TA \xrightarrow{a} A$$

$$TA \xrightarrow{a} A$$

<sup>&</sup>lt;sup>1</sup>KZ: short for 'Kock-Zöberlein'

$$\theta \cdot \tau_A = \zeta^{-1} \cdot a$$

On the other hand, we can compose row-wise instead, and we get

$$\theta \cdot \tau_A = (\theta \cdot Ti_A) \circ (a \circ Ta \cdot \tau_A) = (a \cdot T\zeta^{-1}) \circ (a \circ Ta \cdot \tau_A)$$

Thus, in the end, we have

$$TA \xrightarrow{r_A} T^2A \xrightarrow{Ta} TA \xrightarrow{a} A = TA \xrightarrow{a} A \xrightarrow{Ta} TA \xrightarrow{a} A$$

$$TA \xrightarrow{r_A} T^2A \xrightarrow{Ta} TA \xrightarrow{a} A \xrightarrow{Ta} TA \xrightarrow{a} A$$

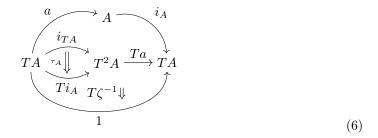
$$TA \xrightarrow{Ta} TA \xrightarrow{a} A \xrightarrow{Ta} TA \xrightarrow{a} TA \xrightarrow{$$

LEMMA 2.3. Let T be a KZ-monad, and A an object of K. Then any pseudo T-algebra is left adjoint to  $i_A$ . Conversely, if  $i_A$  has a left adjoint with invertible counit then this left adjoint is a pseudo T-algebra.

REMARK 2.4. This observation requires a bit of conceptual explanation: for a KZ-monad T, any object admits at most one pseudo T-algebra structure, up to unique isomorphism. So a KZ-monad is a nicely-behaved pseudo monad whose algebras are 'property-like' in the sense that the structure is (a reflective) left adjoint to the unit. Similarly, for a co-KZ-monad T the structure a is right adjoint to the unit  $i_A$  and the invertible unit of this adjunction given by  $\zeta\colon 1\Rightarrow ai_A$  in diagram 1.

$$TA \underbrace{\downarrow}_{a} A$$

What about counit of  $i_A \dashv a$ ? Here is a calculation<sup>2</sup> of counit using mate  $\tau_A$  introduced in diagram 4.



PROPOSITION 2.5. Any KZ-monad (resp. co-KZ-monad) is lax idempotent (resp. co-lax idempotent). (Street 1974) (Kock 1995)

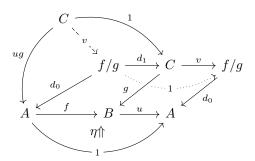
<sup>&</sup>lt;sup>2</sup>The dual of this situation, i.e. unit in the case of KZ-monad, is calculated in page 112 of (Street 1974).

## 3 A general useful lemma in 2-categories

There is an innocent looking yet quite important proposition in (Street 1974) which may be overlooked in first reading of the paper.<sup>3</sup> This is proposition 6 in that paper. We state it here.

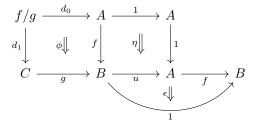
PROPOSITION 3.1. Suppose  $f: A \to B$  is a 1-cell with right adjoint u, unit  $\eta$ , and counit  $\epsilon$  in a 2-category  $\mathcal{K}$  with comma objects. For any 1-cell  $g: C \to B$ , the unique filling arrow  $v: C \to f/g$  obtained by factoring  $\epsilon \cdot g$  through (strict) comma square  $\langle f/g, d_0, d_1, \phi \rangle$  is right adjoint to  $d_1$  with counit identity.

The 1-cell v in the proposition is uniquely determined by equations  $d_1v=1$ ,  $d_0v=ug$ , and  $\phi \cdot v=\epsilon \cdot g$ . Moreover, the proposition states that we can lift the 2-cell  $\eta$  in the lower part of the diagram to a 2-cell  $1 \Rightarrow vd_1$  in the upper part.



*Proof.* We first construct the would-be unit  $\beta$  of adjunction  $d_1 \dashv v$ . Using the fact  $(\epsilon \cdot f) \circ (f \cdot \eta) = 1$  in chasing the diagram below, we obtain:

$$(\phi \cdot vd_1) \circ (fu \cdot \phi) \circ (f \cdot \eta \cdot d_0) = (\epsilon \cdot gd_1) \circ (fu \cdot \phi) \circ (f \cdot \eta \cdot d_0) = \phi$$



We (uniquely) define  $\beta: 1 \Rightarrow vd_1$  to be the unique 2-cell with

$$d_0 \cdot \beta = (u \cdot \phi) \circ (\eta \cdot d_0)$$

$$d_1 \cdot \beta = 1$$
(7)

One readily verifies that with id and  $\beta$ ,  $d_1$  and v satisfy triangle equations of adjunction.  $\square$ 

<sup>&</sup>lt;sup>3</sup>Unfortunately, this occurred in the case of author.

## 4 Fibrations as pseudo-algebras of a co-KZ-monad

Let K be a representable 2-category. Define K/B to be the strict slice 2-category over B, meaning the morphism triangles commute up to equality. (Street 1974) constructs KZ-monads  $L, R: K/B \rightrightarrows K/B$ . The idea is, for a morphism  $p: E \to B$ , an algebra  $R(p) \to p$  (resp.  $L(p) \to p$ ) if it exist, corresponds to the fibration structure on p (resp. opfibration structure). We will only present explicit construction and calculation for the case of fibration<sup>4</sup> and thus, we will mainly concern ourselves with 2-monad R. However, when necessary, we will comment on the dual results for the case of opfibrations. We now define 2-monad R: It takes an object (E,p) to (B/p,R(p)) where

$$\begin{array}{ccc}
B/p & \xrightarrow{\hat{d}_1} & E \\
R(p) & \phi_p \uparrow & \downarrow^p \\
B & \xrightarrow{1} & B
\end{array}$$
(8)

is a comma square.

Remark 4.1. 2-cell  $\phi_p$  can be constructed as follows:

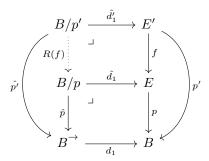
$$B/p \xrightarrow{\hat{d_1}} E$$

$$B/p \xrightarrow{\hat{d_1}} E$$

$$\downarrow^p$$

The action of R on morphisms is given as follows:

If  $f: (E', p') \to (E, p)$  is a 1-cell in  $\mathcal{K}/B$ , then define R(f) to be the unique 1-cell with  $\hat{d}_1 \circ R(f) = f \circ \hat{d}'_1$  and  $\hat{p} \circ R(f) = \hat{p}'$ .

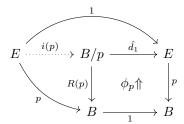


 $<sup>^{4}</sup>$ Unlike Street's paper whereby he works with opfibration structures and thus chooses to work with 2-monad L.

Similarly if  $\sigma: f \Rightarrow g$  is a 2-cell in  $\mathcal{K}/B$ , then we have a unique induced 2-cell  $R(\sigma): R(f) \Rightarrow R(g)$  with  $\hat{d}_1 \cdot R(\sigma) = \sigma \cdot \hat{d}'_1$  and  $\hat{p} \cdot R(\sigma) = id_{\hat{n}'}$ .

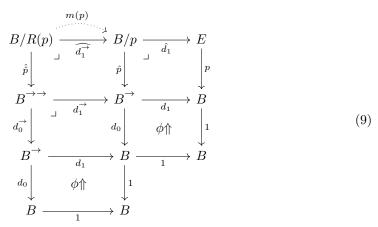
PROPOSITION 4.2. 2-functor  $R: \mathcal{K}/B \to \mathcal{K}/B$  is a 2-monad.

The unit of monad  $i: id \Rightarrow R$  at (E, p) is given by the unique arrow  $i(p): E \to B/p$  with property that  $R(p) \circ i(p) = p$  and  $\hat{d}_1 \circ i(p) = 1_E$ , and moreover  $\phi_p \cdot i(p) = id_p$ , all inferred by universal property of comma object B/p.



It also follows that  $\hat{d}_1 \dashv i(p)$  with identity counit. Indeed, i(p) is v in proposition 3.1, when f = 1 and g = p. From there, we also get the unit  $\beta$  of adjunction with  $R(p) \cdot \beta = \phi_p$ .

The multiplication  $m \colon R^2 \Rightarrow R$  of monad at (E,p) is given by the unique arrow  $m(p) \colon B/R(p) \to B/p$ 



with property that  $R(p) \circ m(p) = R^2(p)$  and  $\hat{d}_1 \circ m(p) = \hat{d}_1 \circ \widehat{d}_1^{\rightarrow}$ , and moreover  $\phi_p \cdot m(p) = (\phi_p \cdot \widehat{d}_1^{\rightarrow}) \circ (\phi \cdot \widehat{d}_0^{\rightarrow} \hat{p}) = (\phi_p \cdot \widehat{d}_1^{\rightarrow}) \circ \phi_{R(p)}$ , all inferred by universal property of comma object B/p.

Proposition 4.3. 2-monad  $R: \mathcal{K}/B \to \mathcal{K}/B$  is a co-KZ-monad.

*Proof.* We have to show that  $i \cdot T \dashv m$ .

Now, we would like to see what an algebra  $a: R(p) \to p$  in  $\mathcal{K}/B$  looks like. The fact that

a is a morphism in  $\mathcal{K}/B$  provides us with a morphism a which makes the diagram

$$\begin{array}{ccc}
B/p & \xrightarrow{a} & E \\
& & \swarrow p \\
& & & B
\end{array} \tag{10}$$

commute. Moreover, by remark 2.4 R being a co-KZ-monad generates an adjunction  $i(p) \dashv a$ whose unit is the invertible 2-cell  $\zeta(p)$ :  $1 \Rightarrow a \circ i(p)$ 

$$E \xrightarrow{i(p)} B/p \xrightarrow{a} E$$

$$p \xrightarrow{R(p)} p$$

$$A$$

$$(11)$$

such that  $p \cdot \zeta(p) = id_p$ .

In the example below we investigate how the construction above look like when we choose 2-category of (locally small) categories as our working 2-category.

Example 4.4. Let's take  $\mathcal{K} = \mathfrak{Cat}$  to be the strict 2-category of categories, functors, and natural transformations. First and foremost, for a functor  $p: E \to B$ , the comma category B/p is given as a category whose objects are pairs  $\langle f: a \to p(e); e \rangle$  where f is morphism in B: $^5$ 

$$b_0 \xrightarrow{f} b_1$$

Morphisms of B/p are of the form

$$b_0 \xrightarrow{f} b_1 \xrightarrow{h_1} p$$

$$c_0 \xrightarrow{q} c_1$$

R(p) as in diagram (8) takes pair  $\langle f;e\rangle$  to  $b_0=\mathrm{dom}(f)$ , and  $\hat{d}_1$  is simply the second projection; it takes  $\langle f; e \rangle$  to e. The unit  $i(p) \colon E \to B/p$  sends an object e of E to the object

$$p(e) = p(e)$$

$$p(e) = p(e)$$

We also note that  $\widehat{d_1^{\rightarrow}}$  (as in diagram 9) is given by the action

$$b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2 \qquad \mapsto \qquad \qquad b_1 \xrightarrow{g} b_2$$

and multiplication m(p) given by

$$b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2 \qquad \mapsto \qquad b_0 \xrightarrow{g \circ f} b_2$$

Now, suppose that  $\mathfrak{a}: R(p) \to p$  is a pseudo algebra for 2-monad R. By commutativity of diagram 10 we know that  $p(\mathfrak{a}\langle f; e \rangle) = \text{dom}(f)$ . So we draw

$$\begin{array}{ccc}
\mathfrak{a}\langle f; e \rangle \\
\downarrow & \\
b_0 & \xrightarrow{f} & b_1
\end{array}$$

As observed in diagram 11 we get an isomorphism lift of identity in the base:

$$e \xrightarrow{\zeta(p)(e)} \mathfrak{a}\langle 1_{p(e)}; e \rangle$$

$$\downarrow^{p} \qquad \qquad \downarrow^{p}$$

$$p(e) = p(e)$$

Observe that functors  $R(i(p))\colon B/p\to B/R(p)$  and  $i(R(p))\colon B/p\to B/R(p)$  are given as follows:

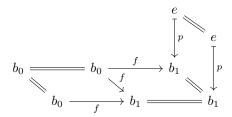
$$R(i(p)): \quad \bigcup_{b_0 \xrightarrow{f} b_1}^{e} \qquad \longmapsto \qquad \bigcup_{b_0 \xrightarrow{f} b_1}^{e} \qquad \bigcup_{b_0 \xrightarrow{f} b_1}^{e}$$

and

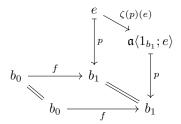
$$i(R(p)):$$

$$b_0 \xrightarrow{f} b_1 \qquad \longmapsto \qquad b_0 \xrightarrow{g} b_1 \xrightarrow{g} b_1$$

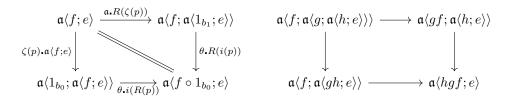
and there is a natural transformations  $\tau: i(R(p)) \Rightarrow R(i(p))$ 



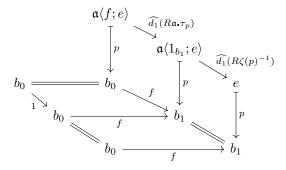
Indeed,  $\tau$  is the mate 2-cell given in diagram 4. We also keep in mind that  $R(\mathfrak{a}) \circ R(i(p))(\langle f;e \rangle) = \langle f;\mathfrak{a}\langle 1_{b_1};e \rangle \rangle$ , and hence  $R(\zeta(p))$  is illustrated as in below:



In addition, invertible 2-cell  $\theta(p)$ :  $\mathfrak{a} \circ R(\mathfrak{a}) \Rightarrow \mathfrak{a} \circ m(p)$  provides us with an isomorphism  $\mathfrak{a}\langle f; \mathfrak{a}\langle g; e \rangle \rangle \to \mathfrak{a}\langle gf; e \rangle$ . Now, we study the coherence equations 1.1 in our case, which state that the following diagrams commute:



Furthermore, the counit of adjunction  $i(p) \dashv \mathfrak{a}$ , as computed in diagram 6, gives us the lift  $\tilde{f} = \hat{d}_1((R\mathfrak{a} \cdot \tau_p) \circ R\zeta(p)^{-1})$  of f:



It remains to prove that  $\tilde{f}$  as defined is cartesian. One can try to prove this directly. However, we prove this in a more general setting in the next section.

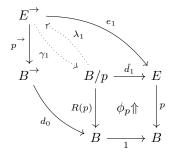
## 5 Chevalley criterion

For an object E of 2-category  $\mathcal{K}$ , comma structure  $\langle E^{\rightarrow}, e_0, e_1, \phi_E : e_0 \Rightarrow e_1 \rangle$  provides us with 1-cells

$$E^{\to} \xrightarrow{\stackrel{e_1}{\longleftarrow}} E$$

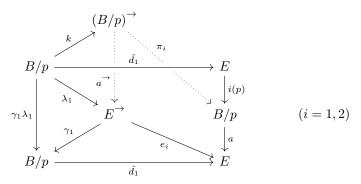
where  $e_1 \dashv \Delta \dashv e_0$ .

DEFINITION 5.1. The 2-cell  $p \cdot \phi_E : d_0 p^{\rightarrow} = pe_0 \Rightarrow pe_1$  factors through universal 2-cell  $\phi_p$  via a unique 1-cell  $\gamma_1$ . We say a 1-cell p in  $\mathcal{K}$  satisfies **Chevalley criterion** if  $\gamma_1$  has a right adjoint  $\lambda_1$  with isomorphism counit.



Given a pseudo algebra  $a \colon R(p) \to p$ , we construct a right adjoint  $\lambda_1$  and show that the counit of adjunction is isomorphism. Hence p satisfies Chevalley criterion. Note that the unit  $\mathfrak{u}$  of adjunction  $\hat{d}_1 \dashv i(p)$  defines a unique 1-cell  $k \colon B/p \to (B/p)^{\to}$  obtained by factoring  $\mathfrak{u}$  through comma square  $\langle B/p, \pi_0, \pi_1, \phi_{B/p} \rangle$  of  $(B/p)^{\to}$ . Define  $\lambda_1 \colon = a^{\to} \circ k$ . A diagram chase shows that the front square in below commutes:

$$\hat{d}_1 \gamma_1 \lambda_1 = e_1 \lambda_1 = e_1 a \stackrel{\rightarrow}{} k = a \pi_1 k = a i(p) \hat{d}_1 \tag{12}$$



We also note that

$$R(p)\gamma_1\lambda_1 = d_0p \stackrel{\rightarrow}{\lambda}_1 = pe_0\lambda_1 = pe_0a \stackrel{\rightarrow}{k} = pa\pi_0k = pa = R(p)$$
  

$$\phi_p \cdot (\gamma_1\lambda_1) = p \cdot \phi_E \cdot \lambda_1 = pa \cdot \phi_{B/p} \cdot k = pa \cdot \mathfrak{u} = R(p) \cdot \mathfrak{u} = \phi_p$$
(13)

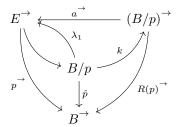
Now, equations (12) and (13), and definition of R(ai(p)) altogether prove that  $\gamma_1 \lambda_1 = R(ai(p))$ . So, we have

$$\gamma_1 \circ \lambda_1 = R(a) \circ R(i(p)) = R(a \circ i(p))$$

and we shall show that counit  $\varepsilon \colon \gamma_1 \circ \lambda_1 \Rightarrow 1$  is given by  $R(\zeta(p)^{-1})$  which is invertible.

Now, we will construct the unit and show that counit and unit satisfy triangle equations of adjunction.

To be completed!



EXAMPLE 5.2. Let  $p: E \to B$  be a cloven Grothendieck fibration. We will show that p satisfies Chevalley criteria. Let  $\mathfrak{s}: B^{\to} \to B$  and  $\mathfrak{t}: B^{\to} \to B$  be the source and target functors, respectively. Note that the objects of  $p \downarrow B = \mathfrak{s}^*(E)$  are pairs  $\langle e', f : p(e') \to b \rangle$  where f is morphism in B.

$$\begin{array}{c}
e' \\
\downarrow \\
a \xrightarrow{f} b
\end{array}$$

 $(e' \mapsto a \text{ indicates that } p(e') = a.)$  Similarly, the objects of  $B \downarrow p = \mathfrak{t}^*(E)$  are pairs  $\langle e, f : a \to p(e) \rangle$  where f is morphism in B.

$$a \xrightarrow{f} b$$

Note that the data of a cloven Grothendieck fibration includes structure of a cleavage, that is a choice of cartesian lifts:

$$\rho_{a,b}: \prod_{\operatorname{Hom}(a,b)} \prod_{e \in E_b} \sum_{e' \in E_a} \mathcal{C}art_E(e',e)$$

For all pairs of objects a, b, satisfying:

$$\operatorname{snd} \rho_{a,c}(g \circ f, e) \cong \operatorname{snd} \rho_{b,c}(g, e) \circ \operatorname{snd} \rho_{a,b}(f, \operatorname{fst} \rho_{b,c}(g, e))$$

$$\operatorname{snd} \rho_{b,b}(1_{Pe}, e) \cong 1_{e}$$

$$\tag{14}$$

We denote by  $\operatorname{Pull}_f(e)$  the domain of cartesian lift i.e.  $\operatorname{fst} \rho_{a,b}(f,e)$  and by  $\widetilde{f}$  the cartesian lift itself i.e.  $\operatorname{snd} \rho_{a,b}(f,e)$ .

$$\operatorname{Pull}_{f}(e) \xrightarrow{\widetilde{f}} e$$

$$\downarrow^{p}$$

$$a \xrightarrow{f} b$$

The functors  $\gamma_0 \colon E^{\to} \to \mathfrak{s}^*(E)$  and  $\gamma_1 \colon E^{\to} \to \mathfrak{t}^*(E)$  are defined as follows: for any object  $u \colon e \to e'$  in  $E^{\to}$ , we define  $\gamma_0(u) = \langle \mathfrak{s}(u), p(u) \rangle$ , and  $\gamma_1(u) = \langle \mathfrak{t}(u), p(u) \rangle$ . Definitions of  $\gamma_i$  (i=0,1) on morphisms is rather straightforward: If  $u_0 \colon d \to d'$  and  $u_1 \colon e \to e'$  are in  $E^{\to}$  and  $\langle h, h' \rangle$  is a morphism from  $u_0$  to  $u_1$  in  $E^{\to}$ , then  $\gamma_0(\langle h, h' \rangle) = \langle h, \langle p(h), p(h') \rangle \rangle$ . Similarly,  $\gamma_1(\langle h, h' \rangle) = \langle h', \langle p(h), p(h') \rangle \rangle$ . Moreover,  $\lambda_1 \colon \mathfrak{t}^*E \to E^{\to}$  is defined on objects as  $\lambda_1 \langle f \colon a \to b, e \rangle = \widetilde{f}$ , and on morphisms by assigning to  $\langle u, \langle h, k \rangle \rangle$ , morphism  $\langle \overline{h}, u \rangle \colon \widetilde{f}_0 \to \widetilde{f}_1$ , where  $\overline{h}$  is the a unique lift of h which make the upper square commute.  $\overline{h}$  is obtained from cartesian property of  $\widetilde{f}_1$ . (Note that although  $\overline{h}$  is a lift of h it may not be in the cleavage.)

$$\begin{array}{ccc} \operatorname{Pull}_{f_0}(e_0) & \stackrel{\widetilde{f_0}}{\longrightarrow} e_0 \\ & & \downarrow^u \\ \operatorname{Pull}_{f_1}(e_1) & \stackrel{\widetilde{f_0}}{\longrightarrow} e_1 \end{array}$$

We now show that  $\lambda_1$  is right adjoint to  $\gamma_1$ . Notice that the counit of adjunction is identity as it is readily observed that  $\gamma_1 \circ \lambda_1 = id_{\mathfrak{t}^*E}$ . For obtaining the unit  $\eta: Id_{E^{\to}} \to \lambda_1 \circ \gamma_1$ , take any object  $u: e_0 \to e_1$  of  $E^{\to}$ . We have  $\lambda_1 \circ \gamma_1(f) = p(f)$ , and we define  $\eta(f)$  as  $\langle id_{p(e_0)}, id_{e_1} \rangle : f \to \lambda_1 \circ \gamma_1(f)$ , where  $id_{p(e_0)}$  is the unique vertical lift of  $id_{p(e_0)}$  which makes the following triangle commute:

$$e_0 \xrightarrow{id_{p(e_0)}} f$$

$$\operatorname{Pull}_{p(f)}(e_1) \xrightarrow{\widetilde{p(f)}} e_1$$

We have to verify that triangle identities of adjunction hold:

$$E \xrightarrow{} = E \xrightarrow{} E \xrightarrow{} E \xrightarrow{} E \xrightarrow{} K^*E \xrightarrow{} K^*E$$

Observe that  $\gamma_1 = \gamma_1 \circ \lambda_1 \gamma_1$  and  $\gamma_1 \cdot \eta = id_{\gamma_1}$ , and this proves the first pasting identity. Similarly,  $\lambda_1 = \lambda_1 \circ \gamma_1 \circ \lambda_1$  and  $\eta \cdot \lambda_1 = id_{\lambda_1}$ , and hence we have the second pasting identity.

DEFINITION 5.3. For a category B, define 2-category  $\mathbf{Fib}(B)$  of fibrations over B whose 0-cells are Grothendieck fibrations, whose 1-cells are fibred functors over B (i.e. those functors over B which preserve cartesian morphisms), and 2-cells are vertical natural transformations (i.e. transformations over B). Compositions are usual composition of functors and natural transformations.

REMARK 5.4. Example 4.4 can be encapsulated as follows: The forgetful 2-functor  $U: \mathbf{Fib}(B) \to \mathfrak{Cat}/B$  is 2-monadic: the *free fibration* of a functor  $p: E \to B$  is fibration  $R(p): B/p \to B$ ; cleavage (aka fibration structure) on p is uniquely (in fact unique up to unique isomorphism) determined by a pseudo-algebra structure for 2-monad R = UF. Strict algebra structures of R correspond to splitting fibration structures on p.

$$F \bigcirc U$$

$$\mathfrak{Cat}/B \bigcirc R$$

We also note that for a category B the domain functor  $\operatorname{cod}: B^{\to} \to B$  is the free Grothendieck fibration on identity functor  $1: B \to B$ ; that is  $\operatorname{dom} = R(1)$ . In more explanatory terms this fact states that

We also note that for a category B with pullbacks the codomain functor cod:  $B \to B$  is the free Grothendieck fibration with existential quantifiers on identity functor 1:  $B \to B$ ;

## References

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