Problem Set 8

STAT39000 Stochastic Calculus

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Problem 1 (5.7)

1. Let $r = 0.05$ and $R_t = e^{rt}$. Then, $\tilde{S}_t = R_t^{-1} S_t$. By stochastic product rule,

$$\begin{aligned} d\tilde{S}_t &= S_t dR_t^{-1} + R_t^{-1} dS_t + d\langle R^{-1}, S\rangle_t \\ &= S_t \left(-rR_t^{-1} dt\right) + R_t^{-1}\left(3S_t dt + S_t dB_t\right) \\ &= R_t^{-1} S_t (3 - r) dt + R_t^{-1} S_t dB_t \\ &= \tilde{S}_t (2.95 dt + dB_t) \\ &= \tilde{S}_t dW_t \end{aligned}$$

where W_t is the standard Brownian motion in measure Q.

2. $V = S_2^2$.
i) $V \geq 0$, by definition.
ii) $\tilde{V} = R_2^{-1} S_2^2 = e^{-2r} S_2^2 = e^{2r} \tilde{S}_2^2$. Note that, $\tilde{S}_t$ is a geometric Brownian motion in Q. Therefore, $\tilde{S}_t = \tilde{S}_0 \exp\left\{-\frac{1}{2}t + W_t\right\}$ and thus $\tilde{S}_2 = \tilde{S}_0 \exp\left\{-1 + W_2\right\}$. Then,

$$\mathbb{E}_Q[\tilde{V}^2] = \mathbb{E}_Q[e^{4r} \tilde{S}_2^4] = \tilde{S}_0^4 e^{4r} e^{-4+4^2 \frac{2}{2}} = e^{4r+12} < \infty$$

since $\tilde{S}_2 \sim \text{Lognormal}\,(-1, 2)$ in Q, as $\tilde{S}_t$ is a geometric Brownian motion w.r.t. Q. Therefore, V is a contingent claim.

3. Define $\tilde{V}_t = E_Q[\tilde{V} \mid \mathcal{F}_t]$, which will make it a square-integrable martingale. Then, Then,

$$\begin{aligned} \tilde{V}_t &= E_Q[R_2^{-1} S_2^2 \mid \mathcal{F}_t] \\ &= e^{2r} E_Q[\tilde{S}_2^2 \mid \mathcal{F}_t] \\ &= e^{2r} E_Q[\tilde{S}_0^2 \exp\{-2 + 2W_2\} \mid \mathcal{F}_t] \\ &= e^{2r-2} \tilde{S}_0^2 E_Q[e^{2(W_2 - W_t + W_t)} \mid \mathcal{F}_t] \\ &= e^{2r-2} \tilde{S}_0^2 e^{2W_t} \mathbb{E}_Q[e^{2(W_2 - W_t)}] \\ &= e^{2r-2} \tilde{S}_0^2 e^{2W_t} e^{2(2-t)} \\ &= \tilde{S}_0^2 \exp\{2r - 2 + 2W_t + 2(2-t)\} \end{aligned}$$

4. By Ito's lemma,

$$d\tilde{V}_t = 2\tilde{S}_0^2 \exp\{2r - 2 + 2W_t + 2(2-t)\} dW_t = 2\tilde{V}_t dW_t$$

which is consistent with the fact that $\tilde{V}_t$ is a martingale in Q.

5. Note that, assuming self-financing, we should have weights (a_t, b_t) such that

$$dV_t = a_t dS_t + b_t dR_t$$

Using the formula in the text

$$\therefore a_t = \frac{2\tilde{V}_t}{\tilde{S}_t},\ b_t = \tilde{V}_t - 2\tilde{V}_t = -\tilde{V}_t$$

6. Note that $V_t = e^{rt}\tilde{V}_t$. Therefore,

$$V_t = e^{rt}\tilde{S}_0^2 \exp\{2r - 2 + 2W_t + 2(2-t)\}$$

Problem 2 (5.9)

1. The SDE should be the same for $\tilde{S}_t$ as in the previous problem.

$$d\tilde{S}_t = \tilde{S}_t dW_t$$

where W_t is the standard Brownian motion in measure Q.

2. $V = \int_0^2 sS_s ds$.
i) $V \geq 0$, by definition, since $S_t \geq 0$.
ii) $\tilde{V} = R_2^{-1} \int_0^2 sS_s ds = e^{-2r} \int_0^2 sS_s ds = \int_0^2 s\tilde{S}_s ds$. Note that

$$\left(\int_0^2 s\tilde{S}_s ds\right)^2 \leq \left(\int_0^2 2\tilde{S}_s ds\right)^2 = 4\left(\int_0^2 \tilde{S}_s ds\right)^2$$

which holds since s is positive and $\tilde{S}_t$ is positive on $[0, 2]$. Then,

$$\mathbb{E}_Q[\tilde{V}^2] \leq \mathbb{E}_Q\left[4\left(\int_0^s \tilde{S}_s ds\right)^2\right] < \infty$$

since $\tilde{S}_t$ is a martingale w.r.t. Q. Therefore, V is a contingent claim.

3. Define $\tilde{V}_t = E_Q[\tilde{V} \mid \mathcal{F}_t]$, which will make it a square-integrable martingale. Then,

$$\tilde{V}_t = E_Q\left[e^{-2r}\int_0^2 sS_s ds \mid \mathcal{F}_t\right],\ 0 \leq t \leq 2$$

$$\begin{aligned}
e^{2r}\tilde{V}_t &= E_Q\left[\int_0^t sS_s ds + \int_t^2 sS_s ds \mid \mathcal{F}_t\right] \\
&= \int_0^t sS_s ds + E_Q\left[\int_t^2 sS_s ds \mid \mathcal{F}_t\right] \\
&= \int_0^t sS_s ds + \int_t^2 sE_Q[S_s \mid \mathcal{F}_t] ds \ (\because \text{Fubini's theorem}) \\
&= \int_0^t sS_s ds + \int_t^2 se^{rs}\tilde{S}_t ds \ (\because \tilde{S}_t \text{ is a martingale w.r.t. } Q) \\
&= \int_0^t sS_s ds + \tilde{S}_t \int_t^2 se^{rs} ds \\
&= \int_0^t sS_s ds + \tilde{S}_t \left(\frac{s}{r}e^{rs}\Big|_t^2 - \int_t^2 \frac{1}{r}e^{rs} ds \right) \\
&= \int_0^t sS_s ds + \tilde{S}_t \left(\left(\frac{2}{r} - \frac{1}{r^2}\right) e^{2r} - \left(\frac{t}{r} - \frac{1}{r^2}\right) e^{rt} \right)
\end{aligned}$$

$$\Rightarrow \tilde{V}_t = e^{-2r}\int_0^t sS_s ds + e^{-2r}\tilde{S}_t \left(\left(\frac{2}{r} - \frac{1}{r^2}\right) e^{2r} - \left(\frac{t}{r} - \frac{1}{r^2}\right) e^{rt} \right)$$

4. Applying the Ito's lemma and using the knowledge that $\tilde{V}_t$ is a martingale in measure Q gives

$$d\tilde{V}_t = e^{-2r}\tilde{S}_t \left(\left(\frac{2}{r} - \frac{1}{r^2}\right) e^{2r} - \left(\frac{t}{r} - \frac{1}{r^2}\right) e^{rt} \right) dW_t$$

5. Let $A_t = e^{-2r}\tilde{S}_t \left(\left(\frac{2}{r} - \frac{1}{r^2}\right) e^{2r} - \left(\frac{t}{r} - \frac{1}{r^2}\right) e^{rt}\right)$. Using the formula given in the textbook which follows the process I have carried out in the previous problem, the portfolio weights are

$$\begin{cases} a_t = \frac{A_t}{\tilde{S}_t} = e^{-2r}\left(\left(\frac{2}{r} - \frac{1}{r^2}\right) e^{2r} - \left(\frac{t}{r} - \frac{1}{r^2}\right) e^{rt}\right) \\ b_t = \tilde{V}_t - A_t = e^{-2r}\int_0^t sS_s ds \end{cases}$$

6. I already found the value V_t in the course of my derivations above. Thus, I simply restate it.

$$V_t = \int_0^t sS_s ds + \tilde{S}_t \left(\left(\frac{2}{r} - \frac{1}{r^2}\right) e^{2r} - \left(\frac{t}{r} - \frac{1}{r^2}\right) e^{rt} \right) dW_t$$

Problem 3 (5.11)

1. $N \sim N(0,1),\ Z = e^{aN+y} \rightarrow g(z) = \frac{1}{az}\phi\left(\frac{\log z - y}{a}\right)$

Proof. Let F_Z be the distribution function of Z. Then,

$$\begin{aligned}
F_Z(z) &= \mathbb{P}\{Z \le z\} \\
&= \mathbb{P}\{e^{aN+y} \le z\} \\
&= \mathbb{P}\{aN + y \le \log z\} \\
&= \mathbb{P}\left\{N \le \frac{\log z - y}{a}\right\} \\
&= \Phi\left(\frac{\log z - y}{a}\right)
\end{aligned}$$

Then,

$$g(z) = \frac{\partial}{\partial z} F_Z(z) = \frac{\partial}{\partial z}\Phi\left(\frac{\log z - y}{a}\right) = \frac{1}{az}\phi\left(\frac{\log z - y}{a}\right)$$

$\square$

2. $\int_x^\infty (z-x)g(z)dz = e^{y+a^2/2}\Phi\left(\frac{y-\log x + a^2}{a}\right) - x\Phi\left(\frac{y-\log x}{a}\right)$

Proof.

$$\begin{aligned}
\int_x^\infty (z-x)g(z)dz &= \int_x^\infty \frac{(z-x)}{az}\phi\left(\frac{\log z - y}{a}\right)dz \\
&= \int_{\frac{-y+\log x}{a}}^\infty (e^{at+y} - x)\phi(t)dt \ \left(\because t = \frac{-y+\log z}{a},\ dt = \frac{1}{az}dz\right) \\
&= e^y \int_{\frac{-y+\log x}{a}}^\infty e^{at}\phi(t)dt - x\int_{\frac{-y+\log x}{a}}^\infty \phi(t)dt \\
&= e^y \cdot \mathbb{E}\left[e^{at}\mathbb{1}\left\{t \ge \frac{-y+\log x}{a}\right\}\right] - x\left(1 - \Phi\left(\frac{-y+\log x}{a}\right)\right) \\
&= e^y \cdot e^{\frac{a^2}{2}}\left(1 - \Phi\left(\frac{-y+\log x}{a} - a\right)\right) - x\Phi\left(\frac{y-\log x}{a}\right) \\
&= e^{y+\frac{a^2}{2}}\Phi\left(\frac{y-\log x + a^2}{a}\right) - x\Phi\left(\frac{y-\log x}{a}\right)
\end{aligned}$$

$\square$

3. From Example 5.5.1, we have $a = \sigma\sqrt{T-t}$, $y = \log\tilde{S}_t - \frac{a^2}{2}$, $s = T-t$, and $x = \tilde{K}$. Then, from the results in parts 1 and 2,

$$\begin{aligned}
\tilde{V}_t &= \tilde{S}_t \cdot \Phi\left(\frac{\log\tilde{S}_t - \frac{a^2}{2} - \log\tilde{K} + a^2}{a}\right) - \tilde{K}\cdot\Phi\left(\frac{\log\tilde{S}_t - \frac{a^2}{2} - \log\tilde{K}}{a}\right) \\
&= \tilde{S}_t \cdot \Phi\left(\frac{\log\frac{\tilde{S}_t}{\tilde{K}} + \frac{a^2}{2}}{a}\right) - \tilde{K}\cdot\Phi\left(\frac{\log\frac{\tilde{S}_t}{\tilde{K}} - \frac{a^2}{2}}{a}\right)
\end{aligned}$$

which is consistent with the details in the text.

Problem 4 (6.2) $Y_t \sim PP(2)$.

1. $Y_3 \sim Poi(2 \cdot 3)$.

$$\mathbb{P}\{Y_3 \geq 2\} = 1 - \mathbb{P}\{Y_3 < 2\} = 1 - \frac{e^{-6}6^0}{0!} - \frac{e^{-6}6^1}{1!} \approx 0.9826$$

2. $Y_4 \sim Poi(2 \cdot 4)$.

$$\mathbb{P}\{Y_4 \geq Y_1 + 2 \mid Y_1 = 4\} = \mathbb{P}\{Y_4 - Y_1 \geq 2 \mid Y_1 = 4\} = \mathbb{P}\{Y_4 - Y_1 \geq 2\} = 1 - \frac{e^{-6}6^0}{0!} - \frac{e^{-6}6^1}{1!} \approx 0.9826$$

3.

$$\begin{aligned}
\mathbb{P}\{Y_1 = 1 \mid Y_3 = 4\} &= \frac{\mathbb{P}\{Y_1 = 1, Y_3 = 4\}}{\mathbb{P}\{Y_3 = 4\}} \\
&= \frac{\mathbb{P}\{Y_3 - Y_1 = 3, Y_1 = 1\}}{\mathbb{P}\{Y_3 = 4\}} \\
&= \frac{\mathbb{P}\{Y_3 - Y_1 = 3\}\mathbb{P}\{Y_1 = 1\}}{\mathbb{P}\{Y_3 = 4\}} \\
&= \frac{e^{-4}4^3}{3!} \cdot \frac{e^{-2}2^1}{1!} \Big/ \frac{e^{-6}6^4}{4!} \\
&= \frac{4^4 \cdot 2^1}{6^4} = 2\left(\frac{2}{3}\right)^4 = \frac{32}{81} \approx 0.3951
\end{aligned}$$

4. Using our favorite trick.

$$\begin{aligned}
E[X_t \mid \mathcal{F}_s] &= E[Y_t - a(t) \mid Y_s] \\
&= E[Y_t - Y_s + Y_s - a(t) \mid Y_s] \\
&= \mathbb{E}[Y_t - Y_s] + Y_s - a(t) \\
&= 2(t-s) + Y_s - a(t) \; (\because Y_t - Y_s \sim Poi(2(t-s)))
\end{aligned}$$

In order for X_t to be a martingale, we need

$$X_s = Y_s - a(s) = 2(t-s) + Y_s - a(t) = Y_s + (2t - a(t)) - 2s$$

$$\therefore a(t) = 2t$$

In general, for a Poisson process with rate λ, $X_t = Y_t - \lambda t$ will produce a martingale.

Problem 5 (6.3)

1. $\mu^{\#}$ is the measure given by the standard normal distribution. Note that $\mu = \lambda\mu^{\#} = 2\mu^{\#}$.

$$d\mu = 2d\mu^{\#} = 2\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}}$$

2. Let $N_t \sim PP(2)$ and $Y_1, Y_2, \ldots \overset{iid}{\sim} N(0,1)$. Then, $X_t = Y_1 + Y_2 + \cdots + Y_{N_t}$.

$$\mathbb{E}[X_t] = \sum_{k=0}^{\infty} \mathbb{P}\{N_t = k\} E[X_t \mid N_t = k] = \sum_{k=0}^{\infty} \frac{e^{-2t}(2t)^k}{k!} \left(\sum_{j=1}^{k} \mathbb{E}[Y_j] \right) = 0$$

3. Similarly,

$$\mathbb{E}[X_t^2] = \sum_{k=0}^{\infty} \mathbb{P}\{N_t = k\} E[X_t^2 \mid N_t = k] = \sum_{k=0}^{\infty} \frac{e^{-2t}(2t)^k}{k!} \left(\sum_{j=1}^{k} \mathbb{E}[Y_j^2] \right) = 2t$$

where the second-to-last equality follows from the fact that Y_j are *iid*.

4. Let L be the generator. Then, for a function g,

$$\begin{aligned} Lg(x) &= \lim_{\Delta t \downarrow 0} \frac{E[g(X_{\Delta t}) \mid X_0 = x] - g(x)}{\Delta t} \\ &= \int_{-\infty}^{\infty} [g(x+y) - g(x)] d\mu(y) \\ &= \int_{-\infty}^{\infty} [g(x+y) - g(x)] \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} dy \end{aligned}$$

5. Let $g(x) = x^2$. Then,

$$\begin{aligned} E[g(X_t)^2 \mid X_0 = x] &= E[X_t^4 \mid X_0 = x] \\ &= x^4 + \Delta t \cdot \int_{-\infty}^{\infty} [(x+y)^4 - x^4] \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} dy + o(\Delta t) \\ &= x^4 + \Delta t \int_{-\infty}^{\infty} (4x^3 y + 6x^2 y^2 + 4xy^3 + y^4) \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} dy + o(\Delta t) < \infty \end{aligned}$$

Note that this is finite, since all the moments of the normal distribution are finite. Thus, we can apply the formula from the text to define M_t as follows.

$$M_t = X_t^2 - \int_0^t Lg(X_s) ds$$

Then, M_t is a square-integrable martingale.