STAT31440 Applied Analysis

Topics Covered up to Midterm

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1 Definitions

- $\limsup / \liminf$ of sets

$$\limsup_{i\to\infty} A_i = \bigcap_{j=1}^{\infty}\left(\bigcup_{i=j}^{\infty} A_i\right), \quad \liminf_{i\to\infty} A_i = \bigcup_{j=1}^{\infty}\left(\bigcap_{i=j}^{\infty} A_i\right)$$

- Metric space
  A metric space (X, d) consists of a non-empty set X, $d : X \times X \to [0, \infty)$ s.t.
  1. $d(x, y) = d(y, x), \forall x, y \in X$ (symmetry)
  2. $d(x, y) = 0 \Rightarrow x = y, \forall x, y \in X$.
  3. $d(x, z) \le d(x, y) + d(y, z), \forall x, y, z \in X$ (triangle inequality).
- Diameter
  (X, d): metric space, $A \subset X$, then

$$\operatorname{diam} A := \begin{cases} \sup_{x,y\in A} d(x, y) & , A \neq \phi \\ 0 & , A = \phi \end{cases}$$

  and we say A is bounded if $\operatorname{diam} A < \infty$.
- Normed linear space
  Let E be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We say that E is a normed linear space if $\exists \|\cdot\| : E \to [0, \infty)$ s.t.
  1. $\|x\| = 0 \iff x = 0 \in E, \forall x \in E$.
  2. $\|\alpha x\| = |\alpha|\|x\|, \forall x \in E, \alpha \in \mathbb{F}$.
  3. $\|x + y\| \le \|x\| + \|y\|, \forall x, y \in E$.
- ℓ^p space

$$\ell^p(\mathbb{N}^*) := \left\{ (x_1, \dots, x_n, \dots) : x_i \in \mathbb{R}, \forall i \text{ and } \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} < \infty \right\}$$

- p-norm

$$\|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$$

- Convergence of sequences
  $(x_i)_{i\ge1} \subset X$, a sequence, converges to $x_* \in X$ if $\forall \varepsilon > 0, \exists n \ge 1$ s.t. $\forall i \ge n, d(x_i, x_*) < \varepsilon$.

- Cauchy sequence
  $(x_i) \subset X$ a sequence in X is a Cauchy sequence if for all $\varepsilon > 0$, $\exists m \geq 1$ for all $i, j \geq m$, $d(x_i, x_j) < \varepsilon$.
- Complete metric space
  A metric space (X, d) is complete if every Cauchy sequence in X converges in X.
- Banach space
  If a normed linear space E is complete w.r.t. the metric $d(x, y) = \|x - y\|$, then $(E, \|\cdot\|)$ is called a Banach space.
- Convergence of series
  If $(S_n)_{n\geq 1}$ defined as $S_n := \sum_{i=1}^{n} x_i$ converges to $s \in \mathbb{R}$, $\sum_{i=1}^{\infty} x_i$ is said to converge to s.
- Absolute convergence of series
  $\sum_{i=1}^{\infty} x_i$ is said to be absolutely convergent if $\sum_{i=1}^{\infty}|x_i|$ converges in $\mathbb{R}$.
- Upper/lower bound
  $A \subset \mathbb{R}$ has an upper bound $M \in \mathbb{R}$, lower bound $L \in \mathbb{R}$ if $x \in A \Rightarrow x \leq M$, $x \in A \Rightarrow x \geq L$ and A is said to be bounded from above (below) if such an M (L) exists.
- Supremum/infimum
  An upper bound M for a set $A \subset \mathbb{R}$ is a least upper bound (supremum) if $M \leq M'$ for all upper bounds M' of A. Similarly, a lower bound L of a set $A \subset \mathbb{R}$ is a greatest lower bound (infimum) if $L \geq L'$ for all lower bounds L' of A.
- $\limsup/\liminf$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \inf\{\sup\{x_k : k \geq n\} : n \geq 1\}$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \sup\{\inf\{x_k : k \geq n\} : n \geq 1\}$$

- Continuity
  $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in \mathbb{R}, |x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.
- Uniform continuity
  $f : X \to Y$ is uniformly continuous on X if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in X, d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$.
- Sequential continuity
  X, Y: metric spaces. $f : X \to Y$ is sequentially continuous at $x \in X$ if $\forall (x_n)_{n\geq 1} \subset X$ s.t. $x_n \to x$ as $n \to \infty$, the sequence $(f(x_n))_{n\geq 1}$ converges to $f(x) \in Y$ as $n \to \infty$.
- Upper/Lower semicontinuity
  A function $f : X \to \mathbb{R}$ is upper semicontinuous on X if $\forall (x_n)_{n\geq 1} \subset X$ such that $x_n \to x$ for some $x \in X$ implies $f(x) \geq \limsup_{n\to\infty} f(x_n)$.
  Similarly, $f : X \to \mathbb{R}$ is lower semicontinuous on X if $\forall (x_n)_{n\geq 1} \subset X$, $x_n \to x, x \in X$ implies $f(x) \leq \liminf_{n\to\infty} f(x_n)$.
- Open/Closed ball
  The open ball $B_r(x) = B(x; r)$ is the set $B_r(x) := \{y \in X : d(x, y) < r\}$ and closed ball $\overline{B_r}(x) := \{y \in X : d(x, y) \leq r\}$.
- Open/Closed sets
  $G \subset X$ is an open set if for every $x \in G$, $\exists r > 0$ s.t. $B_r(x) \subset G$. A set $F \subset X$ is closed in X if $X \setminus F$ is open.
- Topology on a set
  τ is a topology on X if the family τ of open subsets of X satisfies
  1. $\phi, X \in \tau$.
  2. $A, B \in \tau \Rightarrow A \cap B \in \tau$.

3. $\{A_i : i \in I \text{ an arbitrary family of elements of } \tau\} \Rightarrow \bigcup_{i\in I} A_i \in \tau$.

  X equipped with τ is called a topological space.

- Convergence in topological space
  $(x_n)_{n\geq 1} \subset X$ converges to $x \in X$ for a topological space (X, τ) if for all $A \in \tau$ with $x \in A$, $\exists N \geq 1$ s.t. $\forall n \geq N, x_n \in A$.

- Measure zero
  $A \subset \mathbb{R}$ is said to have measure zero if for every $\varepsilon > 0$ there is a countable collection of open intervals (I_n) s.t. $A \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{i=1}^{\infty} length(I_n) < \varepsilon$.

- Closure
  The closure of a set A in a metric (or topological) space X is

$$\overline{A} = \bigcap_{A\subset F\subset X, F:\text{closed}} F$$

  which is the smallest closed set containing A.

- Dense in a metric space
  (X, d): metric space. $A \subset X$ is dense in X if $\overline{A} = X$.

- Separable
  (X, d): metric space is separable if X contains a countable dense subset.

- Isometry/Isomorphism
  X, Y: metric space. $i : X \to Y$ is an isometry if

$$d(i(x_1), i(x_2)) = d(x_1, x_2), \forall x_1, x_2 \in X.$$

  If i is an isometry that is surjective (onto), it is a (metric space) isomorphism.

- Completion of a metric space
  Given metric space (X, d), another metric space $(\tilde{X}, \tilde{d})$ is a completion of X if

  1. $\exists i : X \to \tilde{X}$ an isometry.
  2. $i(X)$ is dense in $\tilde{X}$.
  3. $(\tilde{X}, \tilde{d})$ is complete.

- Equivalence relation
  A relation $\sim$ defines an equivalence relation if it is

  1. reflexive: $a \sim a, \forall a$
  2. symmetric: $a \sim b \iff b \sim a, \forall a, b$
  3. transitive: $a \sim b, b \sim c \Rightarrow a \sim c, \forall a, b, c$

- Sequential compactness
  X: metric space. $K \subset X$ is sequentially compact if every sequence in K has a subsequence which converges to a point in K.

- Open cover
  X: metric space, $A \subset X$. A collection $\{G_\alpha\}_{\alpha\in I}$ of subsets of X is said to cover A if

$$A \subset \bigcup_{\alpha\in I} G_\alpha$$

  If every G_α is open, we say $\{G_\alpha\}$ is an open cover of A.

- ε-net
  For $\varepsilon > 0$ and $A \subset X$, a subset $E = \{x_\alpha : \alpha \in I\}$, with I: arbitrary index set, is a ε-net for A if $\{B_\varepsilon(x_\alpha) : \alpha \in I\}$ is an open cover of A, i.e., $A \subset \bigcup_{\alpha\in I} B_\varepsilon(x_\alpha)$.
  If I is finite and E is an ε-net, then E is a finite ε-net.

- Totally bounded
  X: metric space. $A \subset X$ is totally bounded if for every $\varepsilon > 0$ there exists a finite ε-net for A.
- Compactness
  X: metric space. $K \subset X$ is compact if every open cover of K has a finite subcover.

2 Useful Facts

- Cauchy-Schwarz inequality
$$x \cdot y \leq \|x\|\|y\|$$
- $(X, d_X), (Y, d_Y)$: two metric spaces
  $\Rightarrow X \times Y$ is also a metric space with product metric defined as
$$d(u, v) = d_X(u_X, v_X) + d_Y(u_Y, v_Y), u = (u_X, u_Y) \in X \times Y, v = (v_X, v_Y) \in X \times Y$$
- If E: normed linear space, then E is a metric space with $d(x, y) = \|x - y\|, \forall x, y \in E$, i.e., the norm-induced metric.
- E: normed linear space, $(x_n)_{n\geq 1}$: a sequence in E.
  If $x_n \to x \in E$ for some $x \in E$, then
$$\lim_{n\to\infty} \|x_n\| = \|x\|$$
  i.e., $\|\cdot\|$ is continuous on E.
- Hölder's inequality (in $\mathbb{R}^n$)
  $x, y \in \mathbb{R}^n$, $1 \leq p < \infty, 1 \leq q < \infty, \frac{1}{p} + \frac{1}{q} = 1$ (conjugate exponents). Then,
$$\sum_{j=1}^{n} |x_j y_j| \leq \|x\|_p \|y\|_q.$$
- (x_n): Cauchy $\Rightarrow (x_n)$: bounded.
- (x_n): converges $\Rightarrow (x_n)$: Cauchy.
- A normed linear space may be equipped with multiple different norms.
- $\sum x_n$: absolutely convergent $\Rightarrow \sum x_n$: convergent
- In $\mathbb{R}$ or a Banach space:
$$\sum x_i \to s \iff \forall \varepsilon > 0, \exists N \geq 1 \text{ s.t. } |x_{n+1} + \cdots + x_{n+p}| < \varepsilon, \forall n \geq N, p \geq 1$$
  (a version of Cauchy criterion).
- Existence of inf/sup for bounded sets $A \subset \mathbb{R} \iff$ completeness of $\mathbb{R}$.
- lim inf, lim sup always defined for sequences in $\mathbb{R}$.
$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$
- $$x_n \to x \iff \liminf_{n\to\infty} = \limsup_{n\to\infty} = x$$
- $[a, b]$: closed, bounded interval in $\mathbb{R}$, f: continuous on $[a, b] \Rightarrow f$: uniformly continuous on $[a, b]$.

- $f : \mathbb{R}^n \to \mathbb{R}^n$ is affine if
$$f(tx + (1-t)y) = tf(x) + (1-t)f(y), \ \ \forall x, y \in \mathbb{R}^n, 0 \le t \le 1$$
Every affine function is uniformly continuous on $\mathbb{R}^n$ and can be written as $f : x \mapsto Ax + b$ for $A \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^n), b \in \mathbb{R}^n$.
- X, Y: metric spaces. $f : X \to Y, x \in X, f$: continuous at $x \Rightarrow f$: sequentially continuous at x.
- f: continuous on $X \iff f$: both upper and lower semicontinuous.
- X, Y: metric spaces, $f : X \to Y$, continuous on $X \iff$ for every $G \subset Y$ open, $f^{-1}(G)$ is open in X.
- Open mapping theorem
E, F: Banach spaces, $T : E \to F$, continuous, linear, surjective $\Rightarrow T$: open map, i.e., maps open sets to open sets.
- E, F : Banach spaces, $T : E \to F$, continuous, linear, bijective $\Rightarrow T^{-1}$: continuous.
- f: continuous on $X \iff \forall F \subset Y$ closed $f^{-1}(F) \subset X$ is closed.
- Finite unions of closed sets are closed; arbitrary intersections of closed sets are closed.
- Infinite intersections of open sets may not be open; infinite unions of closed sets may not be closed.
- Every open set $G \subset \mathbb{R}$ can be written as a countable union of disjoint open intervals.
- (X, d): metric space. $F \subset X$: closed $\iff$ for every sequence $(x_n) \subset X$ convergent in X, if $x_n \in F$ for all $n \ge 1$, then $\lim_{n\to\infty} x_n \in F$.
- (X, d): complete metric space, $F \subset X$ is a complete metric space (w.r.t. induced metric space) $\iff F$ is a closed set in X.
- Sequential equivalent of closure
$$\overline{A} = \{x \in X : \exists (a_n)_{n\ge1} \subset A, a_n \to x\}.$$
- Any isometry $i : X \to Y$ is injective.
- Uniqueness of completion
(X, d): metric space. If $(\tilde{X}_1, \tilde{d}_1), (\tilde{X}_2, \tilde{d}_2)$ are two completions of X, then they are isomorphic.
- Equivalence class of Cauchy sequences
$$(x_n) \sim (y_n) \iff d(x_n, y_n) \to 0, n \to \infty$$
- Bolzano-Weierstrass Theorem
Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
- Heine-Borel Theorem
$A \subset \mathbb{R}^n$: sequentially compact $\iff A$: closed, bounded.
- Theorem:
(X, d): metric space. X: sequentially compact $\iff X$: complete and totally bounded.
- Theorem:
X: metric space. $K \subset X$: sequentially compact $\iff K$: compact.