

STAT31430 Applied Linear Algebra

Topics Covered up to Midterm

Seung Chul Lee

1 Definitions

- Linearly independent:

$$\forall \alpha_1, \dots, \alpha_n \in \mathbb{K}, \quad \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0 \Rightarrow \alpha_1 = \dots = \alpha_n = 0$$

- Orthonormal:

$$\langle y_i, y_j \rangle = 0, \quad \forall i \neq j, \quad \|y_i\| = 1, \quad \forall i$$

- Kernel:

For $A \in \mathcal{M}_{n,p}(\mathbb{K})$,

$$\ker(A) = \{x \in \mathbb{K}^p : Ax = 0\} \subset \mathbb{K}^p$$

- Image:

For $A \in \mathcal{M}_{n,p}(\mathbb{K})$,

$$\operatorname{im}(A) = \{Ax : x \in \mathbb{K}^p\} \subset \mathbb{K}^n$$

- Dimension:

The number of elements in a spanning linearly independent set of vectors, i.e., a basis.

- Rank:

$$\operatorname{rank} A = \dim(\operatorname{im} A)$$

- Trace:

$$A = (a_{ij})_{1 \leq i, j \leq n}, \quad \operatorname{tr}(A) = \sum_{i=1}^n a_{ii}$$

- Permutation:

$\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that it is both injective and surjective, i.e., bijective.

- Determinant:

For $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$,

$$\det(A) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^n a_{i, \sigma(i)}$$

where $\varepsilon(\sigma) = (-1)^{p(\sigma)}$, the signature of σ , and $p(\sigma) = \sum_{1 \leq i < j \leq n} \operatorname{Inv}_\sigma(i, j)$, the inversion counter.

- Adjoint/conjugate transpose/Hermitian transpose:

For $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{C})$, $A^* \in \mathcal{M}_n(\mathbb{C})$ given by $A^* = \overline{A}^\top = (\overline{a_{ji}})$

- $A \in \mathcal{M}_n(\mathbb{C})$ is

- self-adjoint (or Hermitian) if $A = A^*$.
- unitary if $A^{-1} = A^*$, i.e., $AA^* = A^*A = I$.
- normal if $AA^* = A^*A$.

- $A \in \mathcal{M}_n(\mathbb{R})$ is
 - symmetric (= self-adjoint) if $A = A^\top$.
 - orthogonal (= unitary) if $A^{-1} = A^\top$, i.e., $AA^\top = A^\top A = I$.
 - normal if $AA^\top = A^\top A$.

- Characteristic polynomial:

For $A \in \mathcal{M}_n(\mathbb{C})$,

$$P_A : \mathbb{C} \rightarrow \mathbb{C}, \quad P_A(\lambda) = \det(A - \lambda I)$$

- Eigenvalues:

The roots of the characteristic polynomial, i.e.,

$$\lambda \in \mathbb{C} \text{ s.t. } \det(A - \lambda I) = 0$$

- Spectrum:

$$\sigma(A) = \{\lambda \in \mathbb{C} : \det(A - \lambda I) = 0\}$$

- Algebraic multiplicity: The largest k such that

$$P_A(z) = (z - \lambda)^k Q(z)$$

- Eigenvector:

A nonzero vector $x \in \mathbb{C}^n$ s.t. $Ax = \lambda x$ for some $\lambda \in \sigma(A)$.

- Spectral radius:

For $A \in \mathcal{M}_n(\mathbb{C})$, the spectral radius of A is

$$\rho(A) := \max_{\lambda \in \sigma(A)} |\lambda|$$

- Eigenspace:

For $\lambda \in \sigma(A)$, $A \in \mathcal{M}_n(\mathbb{C})$, the eigenspace of \mathbb{C}^n associated to λ is

$$E_\lambda := \ker(A - \lambda I) = \{x \in \mathbb{C}^n : Ax = \lambda x\}$$

- Generalized eigenspace:

$$F_\lambda := \bigcup_{k \geq 1} \ker(A - \lambda I)^k$$

- Matrix polynomial:

For polynomial $P \in \mathbb{C}[x] = \{a_0 + a_1x + a_2x^2 + \cdots + a_dx^d : a_1, \dots, a_d \in \mathbb{C}, d \geq 0\}$ and $A \in \mathcal{M}_n(\mathbb{C})$, then $P : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C})$ determined by

$$P(A) = a_0I + a_1A + a_2A^2 + \cdots + a_dA^d$$

is the corresponding matrix polynomial.

- Direct sum:

If $F_1, \dots, F_p \subset \mathbb{C}^n$ are subspaces, we write

$$\mathbb{C}^n = \bigoplus_{i=1}^p F_i$$

if any $x \in \mathbb{C}^n$ can be written uniquely as $x = \sum_{i=1}^p x_i$, $x_i \in F_i$, $1 \leq i \leq p$.

- Reduction to triangular form:

$A \in \mathcal{M}_n(\mathbb{C})$ can be reduced to upper (lower) triangular form if $\exists P \in \mathbb{M}_n(\mathbb{C})$ nonsingular and an upper (lower) triangular matrix T s.t. $A = PTP^{-1}$.

- Similar matrices:
 A and T are similar matrices if $\exists P$ invertible s.t. $A = PTP^{-1}$.
- Diagonalizability:
 A is said to be diagonalizable if $A = PDP^{-1}$ for suitable P and D diagonal.
- Rayleigh quotient:
 $A \in \mathcal{M}_n(\mathbb{C})$ self-adjoint (Hermitian). The Rayleigh quotient is the function $R_A : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{R}$ defined by

$$R_A(x) = \frac{\langle Ax, x \rangle}{\langle x, x \rangle}$$

- Positive definiteness:
 $A \in \mathcal{M}_n(\mathbb{C})$: Hermitian is positive definite if every eigenvalue $\lambda \in \sigma(A)$ satisfies $\lambda > 0$.
- Positive semidefiniteness:
 $A \in \mathcal{M}_n(\mathbb{C})$: Hermitian is positive definite if every eigenvalue $\lambda \in \sigma(A)$ satisfies $\lambda \geq 0$.
- Singular values:
The singular values of $A \in \mathcal{M}_{m,n}(\mathbb{C})$ are the square roots of the eigenvalues of A^*A .
- Moore-Penrose pseudoinverse:
Given a matrix $A \in \mathcal{M}_{m,n}(\mathbb{C})$ with SVD $A = V\tilde{\Sigma}U^*$, the pseudoinverse $A^\dagger \in \mathcal{M}_{n,m}(\mathbb{C})$ is the matrix

$$A^\dagger = U\tilde{\Sigma}^\dagger V^*, \quad \tilde{\Sigma}^\dagger = \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_{n,m}(\mathbb{R})$$

- Fundamental spaces of matrices:

$$A \in \mathcal{M}_{m,n}(\mathbb{R}) = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_m \end{bmatrix}$$

- Column space: $\text{col}(A) = \text{span}\{a_1, \dots, a_n\}$
- Kernel or null space: $\ker(A) = \text{null}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$
- Row space: $\text{row}(A) = \text{span}\{\tilde{a}_1, \dots, \tilde{a}_m\} = \text{col}(A^\top)$
- Left null space: $\ker(A^\top) = \{y \in \mathbb{R}^m : A^\top y = 0\}$

- Norm:

A norm $\|\cdot\| : \mathbb{K}^d \rightarrow [0, \infty)$ is a function satisfying

- positive definiteness: $\|x\| \geq 0$ with $\|x\| = 0$ iff $x = 0$, $\forall x \in \mathbb{K}^d$.
- homogeneity: $\|\lambda x\| = |\lambda|\|x\|$, $\forall x \in \mathbb{K}^d, \lambda \in \mathbb{K}$
- triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathbb{K}^d$

- Inner product:

$\langle \cdot, \cdot \rangle$ an inner product on $V \times V \rightarrow \mathbb{C}$ is a map satisfying

- $\langle v, v \rangle \geq 0$, $\forall v \in V$
- $\langle \alpha_1 w_1 + \alpha_2 w_2, v \rangle = \alpha_1 \langle w_1, v \rangle + \alpha_2 \langle w_2, v \rangle$, $w_1, w_2, v \in V, \alpha_1, \alpha_2 \in \mathbb{C}$
- $\langle v, v \rangle = 0 \iff v = 0 \in V$
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$, $\forall v, w \in V$

- Euclidean norm:

$$\|x\|_2 = \left(\sum_{i=1}^d |x_i|^2 \right)^{\frac{1}{2}}$$

- p -norm:

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty$$

- Weighted p -norm:

$$\|x\|_{p,w} = \left(\sum_{i=1}^d w_i |x_i|^p \right)^{\frac{1}{p}}, \quad w = (w_1, \dots, w_d), \quad w_i > 0, \quad \forall i = 1, \dots, d$$

- Norm using matrix:

For A : real, positive definite, symmetric matrix,

$$\|x\|_A = (x^\top A x)^{\frac{1}{2}} = \left(\sum_{i,j=1}^n a_{ij} x_i x_j \right)^{\frac{1}{2}}$$

defines a norm.

- ∞ -norm:

$$\|x\|_\infty = \max_{1 \leq i \leq d} |x_i| \left(= \lim_{p \rightarrow \infty} \|x\|_p \right)$$

- **Frobenius norm** (Euclidean, Schur norm):

$A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$,

$$\|A\|_F = \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}$$

- (Hölder) q -norm ($q \geq 1$):

$$\|A\|_{\ell^q} = \|A\|_{H,q} = \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^q \right)^{\frac{1}{q}}$$

- Infinity norm (∞ -norm):

$$\|A\|_{\ell^\infty} = \|A\|_{H,\infty} = \max_{1 \leq i,j \leq n} |a_{ij}|$$

- **Matrix norm**:

A norm $\|\cdot\|$ on $\mathcal{M}_n(\mathbb{K})$ is a matrix norm if for all $A, B \in \mathcal{M}_n(\mathbb{K})$, $\|AB\| \leq \|A\| \|B\|$

- **Subordinate (induced) norm**:

Let $\|\cdot\|_*$ be a vector norm on \mathbb{K}^n . Then, the norm

$$\|A\|_* = \sup_{x \in \mathbb{K}^n \setminus \{0\}} \frac{\|Ax\|_*}{\|x\|_*}$$

is a matrix norm on $\mathcal{M}_n(\mathbb{K})$ which is said to be "subordinate" to the vector norm.

- Operator norm:

$$\|A\|_{a,b} = \sup_{x \neq 0} \frac{\|Ax\|_b}{\|x\|_a}$$

for $\|\cdot\|_a$ norm on \mathbb{C}^n , $\|\cdot\|_b$ norm on \mathbb{C}^m , $A \in \text{Lin}(\mathbb{C}^m, \mathbb{C}^n)$. (Not necessarily matrix norms.)

2 Useful Facts

- Gram-Schmidt Orthogonalization:
Let $\{x_1, \dots, x_n\}$ be a linearly independent set of vectors in \mathbb{K}^d . Then, \exists orthonormal family $\{y_1, \dots, y_n\} \subset \mathbb{K}^d$ s.t. $\text{span}\{y_1, \dots, y_p\} = \text{span}\{x_1, \dots, x_p\}$, $\forall 1 \leq p \leq n$.
- Dimensionality result:
Let $A \subset \mathbb{K}^d$ be a subspace. If $\{v_1, \dots, v_k\}$, $\{w_1, \dots, w_l\}$ are two sets of basis vectors for A , then $k = l$.
- For $A \in \mathcal{M}_n(\mathbb{K})$, TFAE
 - A is invertible, i.e., $\exists B \in \mathcal{M}_n(\mathbb{K})$ s.t. $AB = BA = I$.
 - $\ker(A) = \{0\}$
 - $\text{im}(A) = \mathbb{K}^n$
 - $\exists B \in \mathcal{M}_n(\mathbb{K})$ s.t. $AB = I_n$ (left inverse)
 - $\exists B \in \mathcal{M}_n(\mathbb{K})$ s.t. $BA = I_n$ (right inverse)
- Property of trace:
 $A, B \in \mathcal{M}_n(\mathbb{K})$, $\text{tr}(AB) = \text{tr}(BA)$.
- Properties of determinants:
 - $A, B \in \mathcal{M}_n(\mathbb{K})$, $\det(AB) = \det(A)\det(B) = \det(BA)$.
 - $A \in \mathcal{M}_n(\mathbb{K})$, $\det(A) = \det(A^\top)$
 - $A \in \mathcal{M}_n(\mathbb{K})$ is invertible iff $\det(A) \neq 0$.
- Property of triangular matrices:
 - $T \in \mathcal{M}_n(\mathbb{K})$ lower triangular. If T^{-1} exists, it is also a lower triangular matrix with diagonal entries given as reciprocals of diagonal entries of T .
 - If T' is lower triangular, TT' is also lower triangular with diagonal entries being products of diagonal entries of T and T' .
- Inner products and matrices:
 $x, y \in \mathbb{C}^d$,

$$\langle Ax, y \rangle = \langle x, A^*y \rangle$$
- Block matrices:
 $A = (A_{I,J})$, $B = (B_{I,J})$ for some partition (n_I) . Then, $C = AB$ also has block structure $(C_{I,J})$ with

$$C_{I,J} = \sum_{k=1}^P A_{I,K} B_{K,J} \text{ for } 1 \leq I, J \leq P$$
- $\det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \det(A)\det(B)$
- $\lambda \in \sigma(A)$ implies \exists eigenvector associated to λ , i.e., $\ker(A - \lambda I) \neq \{0\}$.
- If $\exists x \neq 0$ with $Ax = \lambda x$, then λ is an eigenvalue of A .
- **Invariance of eigenvalues:**
Both the characteristic polynomial and eigenvalues are invariant under change of basis, i.e., for any $Q \in \mathcal{M}_n(\mathbb{C})$ invertible,

$$P_{Q A Q^{-1}} = P_A, \quad \sigma(Q A Q^{-1}) = \sigma(A).$$
- If A : Hermitian, then all its eigenvalues are real.

- **Lemma:**
If $x \in \mathbb{C}^d$ satisfies $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, then $P(A)x = P(\lambda)x$ for all polynomial $P \in \mathbb{C}[x]$.
In particular, $\lambda \in \sigma(A) \Rightarrow P(\lambda) \in \sigma(P(A))$.
- **Cayley-Hamilton Thm:**
Given $A \in \mathcal{M}_n(\mathbb{C})$. Let $P_A \in \mathbb{C}[x]$ be the characteristic polynomial of A . Then, $P_A(A) = 0$.
- **Spectral Decomposition (Spectral Thm):**
Suppose $A \in \mathcal{M}_n(\mathbb{C})$ has p distinct eigenvalues $\lambda_1, \dots, \lambda_p$ with each λ_i having algebraic multiplicity n_i . Then, the generalized eigenspaces F_{λ_i} satisfy $\dim F_{\lambda_i} = n_i$.
- **Proposition:**
Any matrix $A \in \mathcal{M}_n(\mathbb{C})$ can be reduced to (upper) triangular form.
- **Schur Factorization:**
For all $A \in \mathcal{M}_n(\mathbb{C})$, $\exists U \in \mathcal{M}_n(\mathbb{C})$ unitary (i.e., $UU^* = U^*U = I$) s.t. $T = U^{-1}AU$ is triangular.
- **Proposition:**
If $A \in \mathcal{M}_n(\mathbb{C})$ has p distinct eigenvalues $\lambda_1, \dots, \lambda_p$, then A is diagonalizable.
- **Thm:**
 $A \in \mathcal{M}_n(\mathbb{C})$ is normal $\iff \exists U \in \mathcal{M}_n(\mathbb{C})$ unitary s.t. $A = U \operatorname{diag}\{\lambda_1, \dots, \lambda_n\} U^{-1}$.
- **Thm:**
 $A \in \mathcal{M}_n(\mathbb{C})$ is self-adjoint (Hermitian) $\iff A$: diagonalizable w.r.t. an orthonormal basis and has real eigenvalues.
- **Thm:**
 $A \in \mathcal{M}_n(\mathbb{C})$ self-adjoint. The smallest eigenvalue λ_1 of A satisfies

$$\lambda_1 = \min_{x \in \mathbb{C}^n \setminus \{0\}} R_A(x) = \min_{x \in \mathbb{C}^n, \|x\|=1} \langle Ax, x \rangle$$

and the minimum value is attained for at least one eigenvector $x \neq 0$.

- **Proposition:**
 $A \in \mathcal{M}_n(\mathbb{C})$ self-adjoint with eigenvalues $\lambda_1, \dots, \lambda_n$ in increasing order. Then, for $i = 2, \dots, n$,

$$\lambda_i = \min_{x \perp \operatorname{span}\{x_1, \dots, x_{i-1}\}} R_A(x)$$

where $\{x_1, \dots, x_n\}$ are eigenvectors of A associated to eigenvalues $(\lambda_1, \dots, \lambda_n)$, respectively.

- **Courant-Fisher Thm:** $A \in \mathcal{M}_n(\mathbb{C})$ self-adjoint with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. For all $i = 1, \dots, n$,

$$\lambda_i = \max_{\{a_1, \dots, a_{i-1}\} \subset \mathbb{C}^n} \min_{x \perp \operatorname{span}\{a_1, \dots, a_{i-1}\}} R_A(x)$$

- **SVD Factorization:**

Let $A \in \mathcal{M}_{m,n}(\mathbb{C})$ be a matrix having r positive singular values $\mu_1 \geq \mu_2 \geq \dots \mu_r > 0$.

Set $\Sigma = \operatorname{diag}\{\mu_1, \dots, \mu_r\}$ and $\tilde{\Sigma} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_{m,n}(\mathbb{R})$.

Then, there exist unitary matrices $U \in \mathcal{M}_n(\mathbb{C})$, $V \in \mathcal{M}_m(\mathbb{C})$ s.t.

$$A = V \tilde{\Sigma} U^*$$

- **Properties of the pseudoinverse:**

i) If $\operatorname{rank}(A) = n \leq m$

$$A^\dagger = (A^* A)^{-1} A^*$$

so that if A is square and nonsingular, then $AA^\dagger = A^\dagger A = I$ and $A^\dagger = A^{-1}$.

ii) A^\dagger is the unique matrix X s.t. all of the following hold

1. $AXA = A$
2. $XAX = A$
3. $XA = (XA)^*$
4. $AX = (AX)^*$

iii) Minimum length solution to $Ax = b \Rightarrow x^\dagger = A^\dagger b$.

• Properties of fundamental spaces:

- $\dim(\ker A) = n - \text{rank } A$ (rank-nullity thm)
- $\dim(\text{row } A) = \text{rank } A \leq n$
- $\dim(\ker A^\top) = m - \text{rank } A$
- $\ker A = \text{row}(A)^\perp$
- $\ker A^\top = \text{col}(A)^\perp$

• Polar decomposition:

For all $A \in \mathcal{M}_n(\mathbb{R})$, there exists orthogonal Q and $S \in \mathcal{M}_n(\mathbb{R})$ symmetric and positive semidefinite s.t. $A = QS$. If A is invertible, S is positive definite.

• Comparing norms:

- For $p \geq 1$, $x \in \mathbb{K}^d$,

$$|x_i| \leq \left(\sum_{i=1}^d |x_i|^p \right)^{\frac{1}{p}}, \quad \forall i \Rightarrow \|x\|_\infty \leq \|x\|_p$$

- For $p \geq 1$, $x \in \mathbb{K}^d$,

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^d \|x\|_\infty^p \right)^{\frac{1}{p}} = \|x\|_\infty d^{\frac{1}{p}}$$

- For $x \in \mathbb{K}^d$,

$$\|x\|_2 = \left(\sum_{i=1}^d |x_i|^2 \right)^{\frac{1}{2}} \leq \sum_{i=1}^d (|x_i|^2)^{\frac{1}{2}} = \sum_{i=1}^d |x_i| = \|x\|_1$$

• Properties of vector norms:

- $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$ and $|\|x\| - \|y\|| \leq \|x - y\|$.

In particular, $x \mapsto \|x\|$ is uniformly (Lipschitz) continuous.

- On \mathbb{R}^d , Cauchy-Schwarz: $x \cdot y \leq \|x\|_2 \|y\|_2$

• **Equivalence of vector norms:**

E : finite dimensional vector space. All norms on E are equivalent in the sense that for all norms $\|\cdot\|$, $\|\cdot\|'$, $\exists c, C > 0$ s.t. $c\|x\| \leq \|x\|' \leq C\|x\|$ for all $x \in E$.

- Frobenius norm is a matrix norm. $\|\cdot\|_{\ell^\infty}$ is not a matrix norm.

- Properties of subordinate norms:

- All subordinate matrix norms are matrix norms. Not all matrix norms are subordinate to a vector norm. (e.g., Frobenius norm)
- By homogeneity, for $A \in \mathcal{M}_n(\mathbb{K})$,

$$\|A\|_* = \sup_{\substack{x \in \mathbb{K}^n \\ \|x\|_* = 1}} \|Ax\|_* = \sup_{\substack{x \in \mathbb{K}^n \\ \|x\|_* \leq 1}} \|Ax\|_*$$

- $\|I_n\|_* = 1$ for all vector norms $\|\cdot\|_*$, generating a subordinate norm.

- Proposition:

Let $\|\cdot\|$ be a subordinate matrix norm on $\mathcal{M}_n(\mathbb{K})$. Then, for $A \in \mathcal{M}_n(\mathbb{K})$, $\exists x_A \in \mathbb{K}^n \setminus \{0\}$ s.t.

$$\|A\| = \frac{\|Ax_A\|}{\|x_A\|}$$

- $\tilde{x}_A = \frac{x_A}{\|x_A\|} \Rightarrow \exists x_{\max}$ with $\|x_{\max}\| = 1$ s.t. $\|Ax_{\max}\| = \|A\|$.

- **Property of 1-norm:**

Let $A \mapsto \|A\|_1$ denote the matrix norm subordinate to $\|\cdot\|_1$ on \mathbb{K}^n . Then, for $A \in \mathcal{M}_n(\mathbb{K})$,

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$$

i.e., the largest column sum.

- **Property of ∞ -norm:**

Let $A \mapsto \|A\|_\infty$ denote the matrix norm subordinate to $\|\cdot\|_\infty$ on \mathbb{K}^n . Then, for $A \in \mathcal{M}_n(\mathbb{K})$,

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$$

i.e., the largest row sum.

- **Property of 2-norm:**

Let $\|\cdot\|_2$ be the matrix norm subordinate to $\|\cdot\|_2$ for $A \in \mathcal{M}_n(\mathbb{K})$. This is also called the spectral norm. Then, $\forall A \in \mathcal{M}_n(\mathbb{K})$,

$$\|A\|_2 = \|A^*\|_2 = \mu_1$$

where $\mu_1 \leq \mu_2 \leq \dots \leq \mu_r > 0$ are nonzero singular values of A for $A \neq 0$.

- Lemma:

If $U \in \mathcal{M}_n(\mathbb{C})$ is unitary ($UU^* = U^*U = I$), then for all $A \in \mathcal{M}_n(\mathbb{C})$,

$$\|UA\|_2 = \|AU\|_2 = \|A\|_2$$

- Properties of spectral radius:

- $A \mapsto \rho(A)$ is not a norm on $\mathbb{C}^{n \times n}$.
- If $A \in \mathcal{M}_n(\mathbb{C})$ is a normal matrix, then $\|A\|_2 = \rho(A)$.
- If $A \mapsto \|A\|$ is a matrix norm defined on $\mathcal{M}_n(\mathbb{C})$, then $\rho(A) \leq \|A\|$ for all $A \in \mathcal{M}_n(\mathbb{C})$.
- Given $A \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$, there exists a subordinate matrix norm $B \mapsto \|B\|_{A,\varepsilon}$ s.t. $\|A\|_{A,\varepsilon} \leq \rho(A) + \varepsilon$.

- **Proposition:**

Let $A = V\tilde{\Sigma}U^*$ be an SVD factorization of $A \in \mathcal{M}_{m,n}(\mathbb{C})$ with r nonzero singular values of A arranged in decreasing order.

For each $1 \leq k \leq r$, the matrix $A_k = \sum_{i=1}^k \mu_i v_i u_i^*$ satisfies

$$\|A - A_k\|_2 \leq \|A - X\|_2$$

for all $X \in \mathcal{M}_{m,n}(\mathbb{C})$ with $\text{rank } X = k$. Moreover, $\|A - A_k\|_2 = \mu_{k+1}$.