Street for pedestrians

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Abstract

The singular aim of these notes is to give a less condensed and more clear explanation of Ross Street's significant paper (Street 1974). We highly encourage reader to also refer to (Kock 1995) and (Lack 2000).

1 Pseudo algebras for strict 2-monads

Definition *1.1.* Let $\mathcal{K}$ be a 2-category and $(T\colon \mathcal{K} \to \mathcal{K}, i\colon 1 \Rightarrow T, m\colon T^2 \Rightarrow T)$ a strict 2-monad on $\mathcal{K}$. A ***pseudo-algebra*** of T consists of

i. a 0-cell A in $\mathcal{K}$,

ii. a 1-cell $\mathfrak{a}\colon TA \to A$ which we call structure map,

iii. and invertible 2-cells $\zeta\colon 1_A \Rightarrow \mathfrak{a} \circ i_A$ and $\theta\colon \mathfrak{a} \circ T\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$,

$$\begin{tikzcd} A \arrow[d, "i_A"'] \arrow[dr, "1"] \arrow[d, phantom, shift left=4ex, "\Swarrow\zeta"] & \\ TA \arrow[r, "\mathfrak{a}"'] & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\ TA \arrow[r, "\mathfrak{a}"'] & A \end{tikzcd} \tag{1}$$

subject to the following coherence axioms:

$$(\theta \centerdot m_{TA}) \circ (\theta \centerdot T^2\mathfrak{a}) = (\theta \centerdot Tm_A) \circ (\mathfrak{a} \centerdot T\theta)$$

expressed by equality of pasting diagrams:

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$$
\begin{tikzcd}
T^3A \arrow[r, "T^2\mathfrak{a}"] \arrow[d, "m_{TA}"'] & T^2A \arrow[d, "m_A"] \arrow[dr, bend left, "T\mathfrak{a}"] & \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] & TA \arrow[l, phantom, "\overset{\theta}{\Longleftarrow}"] \arrow[dl, bend left, "a"] \\
TA \arrow[r, "\mathfrak{a}"'] & A &
\end{tikzcd}
=
\begin{tikzcd}
 & T^3A \arrow[dl, bend right, "m_{TA}"'] \arrow[r, "T^2\mathfrak{a}"] \arrow[d, "Tm_A"'] \arrow[dr, phantom, "T.\theta\Downarrow"] & T^2A \arrow[d, "T\mathfrak{a}"] \\
T^2A \arrow[dr, bend right, "m_A"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
 & TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\tag{2}
$$

and

$$(\theta \centerdot Ti_A) \circ (\mathfrak{a} \centerdot T\zeta) = id_{\mathfrak{a}} = (\theta \centerdot i_{TA}) \circ (\zeta \centerdot \mathfrak{a})$$

expressed by equality of pasting diagrams:

$$
\begin{tikzcd}
TA \arrow[r, "1_{TA}"] \arrow[d, "Ti_A"'] \arrow[dd, bend right=60, "1_{TA}"'] \arrow[dr, phantom, "T.\zeta\Downarrow"] & TA \arrow[d, "1_{TA}"] \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "1_{TA}"'] & A \arrow[d, "1_A"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "i_{TA}"'] \arrow[dd, bend right=60, "1_{TA}"'] & A \arrow[d, "i_A"] \arrow[dd, bend left=60, "1_A"] \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[r, phantom, "\overset{}{\underset{\zeta}{\Longleftarrow}}"] \arrow[d, "\mathfrak{a}"] & {} \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\tag{3}
$$

DEFINITION *1.2.* Suppose $(\mathfrak{a}, \zeta_A, \theta_A) : TA \to A$ and $(\mathfrak{b}, \zeta_B, \theta_B) : TB \to B$ are pseudo-algebras of a 2-monad T. A ***lax morphism*** from $\mathfrak{a}$ to $\mathfrak{b}$ consists of a 1-cell $f \colon A \to B$ and a 2-cell $\check{f}$

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\mathfrak{a}"'] \arrow[dr, phantom, "\check{f}\Downarrow"] & TB \arrow[d, "\mathfrak{b}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

in such a way that

- $f \centerdot \zeta_A = (\check{f} \centerdot i_A) \circ (\zeta_B \centerdot f)$ expressing the following pasting equality

$$
\begin{tikzcd}
 & A \arrow[dl, "i_A"'] \arrow[ddl, bend left, "1"] & \\
TA \arrow[d, "\mathfrak{a}"'] \arrow[r, phantom, "\zeta\Uparrow"] & & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
 & A \arrow[rr, "f"] \arrow[dl, "i_A"'] & & B \arrow[dl, "i_B"'] \arrow[ddl, bend left, "1"] \\
TA \arrow[rr, "Tf"'] \arrow[d, "\mathfrak{a}"'] \arrow[drr, phantom, "\check{f}\Downarrow"] & & TB \arrow[d, "\mathfrak{b}"] \arrow[r, phantom, "\zeta\Uparrow"] & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}
$$

and

- $(f \centerdot \theta_A) \circ (\check{f} \centerdot T\mathfrak{a}) \circ (\mathfrak{b} \centerdot T\check{f}) = (\check{f} \centerdot m_A) \circ (\theta_B \centerdot T^2 f)$ expressing the following pasting equality

$$
\begin{tikzcd}
 & T^2A \arrow[rr, "T^2f"] \arrow[dl, "T\mathfrak{a}"'] \arrow[dd] \arrow[drr, phantom, "\Swarrow T\check{f}"] & & T^2B \arrow[dl, "T\mathfrak{b}"] \\
TA \arrow[rr, "Tf"] \arrow[dd, "\mathfrak{a}"'] \arrow[dr, phantom, "\theta\Downarrow"] & & TB \arrow[dd, "\mathfrak{b}"] & \\
 & TA \arrow[dl, "\mathfrak{a}"] \arrow[r, phantom, "\check{f}\Downarrow"] & & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}
=
\begin{tikzcd}
 & T^2A \arrow[rr, "T^2f"] \arrow[dd, "m_A"'] & & T^2B \arrow[dl, "T\mathfrak{b}"'] \arrow[dd, "m_B"] \\
 & & TB \arrow[dd] \arrow[r, phantom, "\theta\Downarrow"] & \\
 & TA \arrow[rr, "Tf"] \arrow[dl, "\mathfrak{a}"'] \arrow[d, phantom, "\Swarrow\check{f}"] & & TB \arrow[dl, "\mathfrak{b}"] \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}
$$

Definition *1.3*. A 2-monad $T : \mathcal{K} \to \mathcal{K}$ is said to be ***lax idempotent*** if given any two (pseudo) T-algebras $\mathfrak{a}\colon TA \to A$, $\mathfrak{b}\colon TB \to B$ and a 1-cell $f\colon A \to B$, there exists a unique 2-cell $\check{f}\colon \mathfrak{b} \circ Tf \Rightarrow f \circ \mathfrak{a}$ rendering $(f, \check{f})$ a lax morphism of pseudo T-algebras.

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\mathfrak{a}"'] \arrow[dr, phantom, "\check{f}\Downarrow"] & TB \arrow[d, "\mathfrak{b}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Remark *1.4*. Dually, reverse the direction of $\check{f}$ in definition 1.3, then we get the notion of ***co-lax idempotent*** monad.

2 KZ-monads

Definition 2.1. A 2-monad $T : \mathcal{K} \to \mathcal{K}$ is said to be ***KZ- monad***[1] if $m \dashv i \centerdot T$ in the 2-category $[\mathcal{K}, \mathcal{K}]$ with identity counit.

Remark 2.2. Dual to the definition above, we define a monad T to be a ***co-KZ-monad*** by requiring $i \centerdot T \dashv m$ with identity unit.

Suppose T is a co-KZ-monad and $i \centerdot T \dashv m$. In particular unit of this adjunction is identity since $m \circ (i \centerdot T) = 1$. Moreover, the identity 2-cell

$$\begin{tikzcd} T \arrow[r, "1"] & T \\ T \arrow[u, "1"] \arrow[r, "T.i"'] \arrow[ur, phantom, "id\Downarrow"] & T^2 \arrow[u, "m"'] \end{tikzcd}$$

has a mate

$$\begin{tikzcd} T \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\lambda\Downarrow"] & T \arrow[d, "i.T"] \\ T \arrow[r, "T.i"'] & T^2 \end{tikzcd} \tag{4}$$

with properties $m.\lambda = id_{1_T}$ and $\lambda.i = id_{(T.i)\circ i}$. These identity follow from triangle identities of adjunction $i \centerdot T \dashv m$ and$(i \centerdot T) \circ i = (T \centerdot i) \circ i$ by naturality of i.

Suppose $\mathfrak{a} \colon TA \to A$ is a pseudo algebra for T. We would like to calculate the composite 2-cell

$$\begin{tikzcd} TA \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\Downarrow\lambda_A"] & T^2A \arrow[r, bend left, "\mathfrak{a} \circ T\mathfrak{a}"] \arrow[r, bend right, "\mathfrak{a} \circ m_A"'] \arrow[r, phantom, "\Downarrow\theta"] & TA \end{tikzcd}$$

In the diagram below, since $m_A \circ \lambda_A = id$, the left column of 2-cells collapses to identity, and therefore we have

$$\begin{tikzcd} TA \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\lambda\Downarrow"] & TA \arrow[r, "\mathfrak{a}"] \arrow[d, "i_{TA}"] & A \arrow[d, "i_A"] \arrow[dd, bend left=60, "1_A"] \\ TA \arrow[r, "Ti_A"] \arrow[d, "1"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \arrow[r, phantom, "\overset{\Leftarrow}{\zeta}"] & {} \\ TA \arrow[r, "1"'] & TA \arrow[r, "\mathfrak{a}"'] & A \end{tikzcd} = \begin{tikzcd} TA \arrow[r, "\mathfrak{a}"] \arrow[d, "1_{TA}"'] & A \arrow[d, "1_A"] \\ TA \arrow[r, "\mathfrak{a}"'] & A \end{tikzcd}$$

$$\theta \centerdot \lambda_A = \zeta^{-1} \centerdot \mathfrak{a}$$

[1] KZ: short for 'Kock-Zöberlein'

On the other hand, we can compose row-wise instead, and we get

$$\theta \centerdot \lambda_A = (\theta \centerdot Ti_A) \circ (\mathfrak{a} \circ T\mathfrak{a} \centerdot \lambda_A) = (\mathfrak{a} \centerdot T\zeta^{-1}) \circ (\mathfrak{a} \circ T\mathfrak{a} \centerdot \lambda_A)$$

Thus, in the end, we have

$$\begin{tikzcd}
TA \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\lambda_A \Downarrow"] \arrow[rr, bend right=70, "1"'] \arrow[rr, phantom, bend right=30, "T\zeta^{-1}\Downarrow"] & T^2A \arrow[r, "T\mathfrak{a}"] & TA \arrow[r, "\mathfrak{a}"] & A
\end{tikzcd}
=
\begin{tikzcd}
 & & TA \arrow[dr, "\mathfrak{a}"] & \\
TA \arrow[r, "\mathfrak{a}"] & A \arrow[ur, "i_A"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\zeta^{-1}\Downarrow"] & & A
\end{tikzcd} \tag{5}$$

Lemma *2.3.* Let T be a KZ-monad, and A an object of $\mathcal{K}$. Then any pseudo T-algebra on A is left adjoint to unit i_A. Conversely, if i_A has a left adjoint with invertible counit then this left adjoint is a pseudo T-algebra.

Remark *2.4.* This observation requires a bit of conceptual explanation: for a KZ-monad T, any object admits at most one pseudo T-algebra structure, up to unique isomorphism. So a KZ-monad is a nicely-behaved 2-monad whose algebras are 'property-like' in the sense that the structure is a (reflective) left adjoint to the unit. Similarly, for a co-KZ-monad T the structure $\mathfrak{a}$ is right adjoint to the unit i_A and the invertible unit of this adjunction is given by $\zeta\colon 1 \Rightarrow \mathfrak{a} i_A$ in diagram (1).

$$\begin{tikzcd}
TA \arrow[r, bend right, "\mathfrak{a}"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "i_A"']
\end{tikzcd}$$

What about counit of $i_A \dashv \mathfrak{a}$? Here is a calculation[2] of counit using mate λ_A introduced in diagram 4.

$$\begin{tikzcd}
 & A \arrow[dr, bend left, "i_A"] & \\
TA \arrow[ur, bend left, "\mathfrak{a}"] \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\lambda_A \Downarrow"] \arrow[rr, bend right=70, "1"'] \arrow[rr, phantom, bend right=30, "T\zeta^{-1}\Downarrow"] & T^2A \arrow[r, "T\mathfrak{a}"] & TA
\end{tikzcd} \tag{6}$$

We prove the triangle identities of adjunction with proposed unit and counit:

$$\begin{aligned}
(\mathfrak{a} \centerdot T\zeta^{-1} \circ (T\mathfrak{a} \centerdot \lambda_A)) \circ (\zeta \centerdot \mathfrak{a}) &= (\zeta^{-1} \centerdot \mathfrak{a}) \circ (\zeta \centerdot \mathfrak{a}) && \{\text{by equality of pasting diagrams (5)}\} \\
&= id_{\mathfrak{a}} && \{\text{factoring out } \mathfrak{a}\}
\end{aligned}$$

[2]The dual of this situation, i.e. unit in the case of KZ-monad, is calculated in page 112 of (Street 1974).

Also,

$$\begin{aligned}((T\zeta^{-1} \circ (T\mathfrak{a} \centerdot \lambda_A)) \centerdot i_A) \circ (i_A \centerdot \zeta) &= (T\zeta^{-1} \centerdot i_A) \circ (i_A \centerdot \zeta) && \{\lambda_A \centerdot i_A = id\} \\ &= (i_A \centerdot \zeta^{-1}) \circ (i_A \centerdot \zeta) && \{\text{2-naturality of } i \colon 1 \Rightarrow T\} \\ &= id_{i_A} && \{\text{factoring out } i_A\}\end{aligned}$$

In (Street 1974), we also see a converse of remark above.

Lemma *2.5*. Suppose $T \colon \mathcal{K} \to \mathcal{K}$ is a co-KZ 2-monad and suppose a 0-cell A, a 1-cell $\mathfrak{a} \colon TA \to A$, and an isomorphism 2-cell $\zeta \colon 1 \Rightarrow \mathfrak{a} \circ i_A$ are given in $\mathcal{K}$, and furthermore, ζ^{-1} satisfies pasting equality (5). We have:

1. ζ is the unit for an adjunction $i_A \dashv a$ whose counit is given by $(T\zeta^{-1}) \circ (T\mathfrak{a} \centerdot \lambda_A)$ (composite 2-cell in (6)).

2. The 2-cell $\theta \colon \mathfrak{a} \circ T\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$, obtained by taking double mate of $\lambda_A \centerdot i_A = id$, is isomorphism.

$$\begin{tikzcd} T^2A & TA \arrow[l, "Ti_A"'] \\ TA \arrow[u, "i_{TA}"] \arrow[ur, phantom, "id\Uparrow"] & A \arrow[l, "i_A"] \arrow[u, "i_A"'] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\ TA \arrow[r, "\mathfrak{a}"'] & A \end{tikzcd}$$

3. 2-cell θ enriches $(A, \mathfrak{a}, \zeta)$ with the structure of a pseudo T-algebra.

Proposition *2.6*. Any KZ-monad (resp. co-KZ-monad) is lax idempotent (resp. co-lax idempotent). (Street 1974) (Kock 1995)

3 A general useful lemma in 2-categories

There is an innocent looking yet quite important proposition in (Street 1974) which may be overlooked in first reading of the paper.[3] This is proposition 6 in that paper. We state it here.

Proposition *3.1*. Suppose $f \colon A \to B$ is a 1-cell with right adjoint u, unit η, and counit ϵ in a 2-category $\mathcal{K}$ with comma objects. For any 1-cell $g \colon C \to B$, the unique filling arrow $v \colon C \to f/g$ obtained by factoring $\epsilon \centerdot g$ through (strict) comma square $\langle f/g, d_0, d_1, \phi \rangle$ is right adjoint to d_1 with counit identity.

[3]Unfortunately, this occurred in the case of author.

The 1-cell v in the proposition is uniquely determined by equations $d_1 v = 1$, $d_0 v = ug$, and $\phi \centerdot v = \epsilon \centerdot g$. Moreover, the proposition states that we can lift the 2-cell η in the lower part of the diagram to a 2-cell $1 \Rightarrow vd_1$ in the upper part.

$$\begin{tikzcd}
& C \arrow[rrd, bend left, "1"] \arrow[rd, dashed, "v"] \arrow[dd, bend right, "ug"'] & & & \\
& & f/g \arrow[r, "d_1"] \arrow[dl, "d_0"'] \arrow[rr, dotted, bend right, "1"] & C \arrow[r, "v"] \arrow[dl, "g"'] & f/g \arrow[dl, "d_0"] \\
& A \arrow[r, "f"] \arrow[rr, bend right=50, "1"'] \arrow[rr, phantom, shift right=3ex, "\eta\Uparrow"] & B \arrow[r, "u"] & A &
\end{tikzcd}$$

Proof. We first construct the would-be unit β of adjunction $d_1 \dashv v$. Using the fact $(\epsilon \centerdot f) \circ (f \centerdot \eta) = 1$ in chasing the diagram below, we obtain:

$$(\phi \centerdot vd_1) \circ (fu \centerdot \phi) \circ (f \centerdot \eta \centerdot d_0) = (\epsilon \centerdot gd_1) \circ (fu \centerdot \phi) \circ (f \centerdot \eta \centerdot d_0) = \phi$$

$$\begin{tikzcd}
f/g \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\phi\Downarrow"] & A \arrow[r, "1"] \arrow[d, "f"'] \arrow[dr, phantom, "\eta\Downarrow"] & A \arrow[d, "1"] & \\
C \arrow[r, "g"'] & B \arrow[r, "u"'] \arrow[rr, bend right=50, "1"'] & A \arrow[r, "f"'] \arrow[d, phantom, "\epsilon\Downarrow"] & B \\
& & {} &
\end{tikzcd}$$

We (uniquely) define $\tau_1 \colon 1 \Rightarrow vd_1$ to be the unique 2-cell with

$$\begin{aligned} d_0 \centerdot \tau_1 &= (u \centerdot \phi) \circ (\eta \centerdot d_0) \\ d_1 \centerdot \tau_1 &= 1 \end{aligned} \tag{7}$$

One readily verifies that with id and τ_1, d_1 and v satisfy triangle equations of adjunction. $\square$

Remark *3.2.* A useful special case of above proposition is when f and g are both identity 1-cells $1 \colon E \to E$. In that case $f/g \simeq E^{\to}$ and $v = i_E$. The unit $\tau_1 \colon 1_{B^{\to}} \Rightarrow i_E \circ e_1$ is the unit of familiar adjunction $e_1 \dashv i_E$. In the case when 2-category $\mathcal{K}$ is 2-category of (small) categories, $\tau_1(u) = (u, 1)$ for any $u \colon b_0 \to b_1$ in $E^{\to}$.

$$\begin{tikzcd}
e_0 \arrow[r, "u"] \arrow[d, "u"'] & e_1 \arrow[d, "1"] \\
e_1 \arrow[r, "1"'] & e_1
\end{tikzcd}$$

similarly, the dual of proposition 3.1 when applied to $f = g = 1$ gives i_E as left adjoint of $e_0 \colon E^{\to} \to E$. The unit of this adjunction is identity, making e_0 a retraction. The counit is given by the unique 2-cell $\tau_0 \colon i_E \circ e_0 \Rightarrow 1_{E^{\to}}$ defined by the equations $e_0 \tau_0 = 1$ and $e_1 \tau_0 = \phi$. In particular, in 2-category of small categories we have $\tau_0(u) = (1, u)$.

4 Fibrations as pseudo-algebras of a co-KZ-monad

Let $\mathcal{K}$ be a representable 2-category. Define $\mathcal{K}/B$ to be the strict slice 2-category over B, meaning the morphism triangles commute up to equality. (Street 1974) constructs KZ-monads $L, R\colon \mathcal{K}/B \rightrightarrows \mathcal{K}/B$. The idea is, for a morphism $p\colon E \to B$, an algebra $R(p) \to p$ (resp. $L(p) \to p$) if it exist, corresponds to the fibration structure on p (resp. opfibration structure). We will only present explicit construction and calculation for the case of fibration[4] and thus, we will mainly concern ourselves with 2-monad R. However, when necessary, we will comment on the dual results for the case of opfibrations. We now define 2-monad R: It takes an object (E, p) to $(B/p, R(p))$ where

$$\begin{tikzcd} B/p \arrow[r, "\hat{d_1}"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\ B \arrow[r, "1"'] & B \end{tikzcd} \tag{8}$$

is a comma square.

Remark *4.1.* 2-cell ϕ_p can be constructed as follows:

$$\begin{tikzcd} B/p \arrow[r, "\hat{d_1}"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\ B \arrow[r, "1"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} B/p \arrow[r, "\hat{d_1}"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ B^{\to} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi\Uparrow"] & B \arrow[d, "1"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$

The action of R on morphisms is given as follows:

If $f\colon (E', p') \to (E, p)$ is a 1-cell in $\mathcal{K}/B$, then define $R(f)$ to be the unique 1-cell with $\hat{d_1} \circ R(f) = f \circ \hat{d_1'}$ and $\hat{p} \circ R(f) = \hat{p'}$.

$$\begin{tikzcd} B/p' \arrow[r, "\hat{d_1'}"] \arrow[d, dotted, "R(f)"'] \arrow[dd, bend right=60, "\hat{p'}"'] \arrow[dr, phantom, "\lrcorner"] & E' \arrow[d, "f"] \arrow[dd, bend left=60, "p'"] \\ B/p \arrow[r, "\hat{d_1}"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ B^{\to} \arrow[r, "d_1"'] & B \end{tikzcd}$$

[4] Unlike Street's paper whereby he works with opfibration structures and as a result, he chooses to work with 2-monad L on $\mathcal{K}/B$ which takes p to $L(p) := p/B$.

Similarly if $\sigma\colon f \Rightarrow g$ is a 2-cell in $\mathcal{K}/B$, then we have a unique induced 2-cell $R(\sigma)\colon R(f) \Rightarrow R(g)$ with $\hat{d}_1 \centerdot R(\sigma) = \sigma \centerdot \hat{d}_1'$ and $\hat{p} \centerdot R(\sigma) = id_{\hat{p}'}$.

Proposition 4.2. 2-functor $R\colon \mathcal{K}/B \to \mathcal{K}/B$ is a 2-monad.

The unit of monad $i : id \Rightarrow R$ at (E, p) is given by the unique arrow $i(p) : E \to B/p$ with property that $R(p) \circ i(p) = p$ and $\hat{d}_1 \circ i(p) = 1_E$, and moreover $\phi_p \centerdot i(p) = id_p$, all inferred by universal property of comma object B/p.

$$\begin{tikzcd}
E \arrow[r, dotted, "i(p)"] \arrow[rr, bend left, "1"] \arrow[dr, bend right, "p"'] & B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\
 & B \arrow[r, "1"'] & B
\end{tikzcd}$$

It also follows that $\hat{d}_1 \dashv i(p)$ with identity counit. Indeed, $i(p)$ is v in proposition 3.1, when $f = 1$ and $g = p$. From there, we also get the unit $\tau_1(p)$ of adjunction with $R(p) \centerdot \tau_1(p) = \phi_p$.

The multiplication $m\colon R^2 \Rightarrow R$ of monad at 0-cell (E, p) is given by the unique arrow $m(p)\colon B/R(p) \to B/p$

$$\begin{tikzcd}
B/R(p) \arrow[r, "\widehat{d_1^{\to}}"'] \arrow[r, dotted, bend left, "m(p)"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner"] & B/p \arrow[r, "\hat{d}_1"'] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\
B^{\to\to} \arrow[r, "d_1^{\to}"'] \arrow[d, "d_0^{\to}"'] \arrow[dr, phantom, "\lrcorner"] & B^{\to} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi\Uparrow"] & B \arrow[d, "1"] \\
B^{\to} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi\Uparrow"] & B \arrow[r, "1"'] \arrow[d, "1"] & B \\
B \arrow[r, "1"'] & B &
\end{tikzcd} \tag{9}$$

with the property that $R(p) \circ m(p) = R^2(p)$ and $\hat{d}_1 \circ m(p) = \hat{d}_1 \circ \widehat{d_1^{\to}}$, and moreover $\phi_p \centerdot m(p) = (\phi_p \centerdot \widehat{d_1^{\to}}) \circ (\phi \centerdot d_0^{\to} \hat{p}) = (\phi_p \centerdot \widehat{d_1^{\to}}) \circ \phi_{R(p)}$, all inferred by universal property of comma object B/p.

Proposition 4.3. 2-monad $R\colon \mathcal{K}/B \to \mathcal{K}/B$ is a co-KZ-monad.

Proof. We have to show that $i \centerdot T \dashv m$. □

Now, we would like to see what a pseudo algebra $\mathfrak{a}\colon R(p) \to p$ in $\mathcal{K}/B$ looks like. The fact that $\mathfrak{a}$ is a morphism in $\mathcal{K}/B$ provides us with a morphism $\mathfrak{a}$ which makes the diagram

$$\begin{tikzcd} B/p \arrow[rr, "\mathfrak{a}"] \arrow[dr, "R(p)"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd} \tag{10}$$

commute. Moreover, by remark 2.4 R being a co-KZ-monad generates an adjunction $i(p) \dashv a$ whose unit is the invertible 2-cell $\zeta \colon 1 \Rightarrow \mathfrak{a} \circ i(p)$

$$\begin{tikzcd} E \arrow[rr, bend left, "1"] \arrow[r, "i(p)"] \arrow[dr, "p"'] & B/p \arrow[r, "\mathfrak{a}"] \arrow[d, "R(p)"] \arrow[u, phantom, "\Downarrow\zeta"] & E \arrow[dl, "p"] \\ & \mathcal{A} & \end{tikzcd} \tag{11}$$

such that $p \centerdot \zeta = id_p$.

In the example below we investigate how the construction above look like when we choose 2-category of (locally small) categories as our working 2-category.

Example 4.4. Let's take $\mathcal{K} = \mathfrak{Cat}$ to be the strict 2-category of categories, functors, and natural transformations. First and foremost, for a functor $p \colon E \to B$, the comma category B/p is given as a category whose objects are pairs $\langle f \colon b \to p(e), e \rangle$ where f is morphism in B :[5]

$$\begin{tikzcd} & e \arrow[d, mapsto, "p"] \\ b_0 \arrow[r, "f"'] & b_1 \end{tikzcd}$$

Morphisms of B/p are of the form

$$\begin{tikzcd} & e \arrow[d, mapsto, "p"] \arrow[dr, "\tilde{h_1}"] & \\ & & e' \arrow[d, mapsto, "p"] \\ b_0 \arrow[r, "f"] \arrow[dr, "h_0"'] & b_1 \arrow[dr, "h_1"] & \\ & c_0 \arrow[r, "g"'] & c_1 \end{tikzcd}$$

Functor $R(p)$ as in diagram (8) takes pair $\langle f, e \rangle$ to $b_0 = \mathrm{dom}(f)$, and $\hat{d_1}$ is simply the second projection; it takes $\langle f, e \rangle$ to e. The unit of monad R at (E, p), i.e. $i(p) \colon E \to B/p$, takes an object e of E to the object

[5] $e \mapsto b_1$ indicates that $p(e) = b_1$.

$$\begin{tikzcd} & e \arrow[d, "p"] \\ p(e) \arrow[r, equal] & p(e) \end{tikzcd}$$

and $\tau_1(p)\colon 1_{B/p} \Rightarrow i(p) \circ \hat{d}_1$ induces a morphisms $B/p \to B/p^{\to}$ which takes an object of B/p in above to

$$\begin{tikzcd} & & e_1 \arrow[dr, equal] \arrow[dd, "p"] & \\ & & & e_1 \arrow[dd, "p"] \\ b_0 \arrow[rr, "f"] \arrow[dr, "f"'] & & b_1 \arrow[dr, equal] & \\ & b_1 \arrow[rr, equal] & & b_1 \end{tikzcd}$$

We also note that $\widehat{d_1^{\to}}$ (as in diagram 9) is given by the action

$$\begin{tikzcd} & & e \arrow[d] \\ b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_2 \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & e \arrow[d] \\ b_1 \arrow[r, "g"'] & b_2 \end{tikzcd}$$

and multiplication $m(p)$ given by

$$\begin{tikzcd} & & e \arrow[d] \\ b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_2 \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & e \arrow[d] \\ b_0 \arrow[r, "g \circ f"'] & b_2 \end{tikzcd}$$

Now, suppose that $\mathfrak{a}\colon R(p) \to p$ is a pseudo algebra for 2-monad R. By commutativity of diagram 10 we know that $p(\mathfrak{a}\langle f, e\rangle) = \mathrm{dom}(f)$. So we draw

$$\begin{tikzcd} \mathfrak{a}\langle f, e\rangle \arrow[d, "p"'] & \\ b_0 \arrow[r, "f"'] & b_1 \end{tikzcd}$$

As observed in diagram 11 we get an isomorphism lift of identity in the base:

$$\begin{tikzcd} e \arrow[r, "\zeta(e)"] \arrow[d, "p"'] & \mathfrak{a}\langle 1_{p(e)}, e\rangle \arrow[d, "p"] \\ p(e) \arrow[r, equal] & p(e) \end{tikzcd}$$

Observe that functors $R(i(p))\colon B/p \to B/R(p)$ and $i(R(p))\colon B/p \to B/R(p)$ are given as follows:

$$
\begin{tikzcd}
R(i(p)) : & & e \arrow[d, |->] & & & & e \arrow[d, |->] \\
& b_0 \arrow[r, "f"'] & b_1 & \longmapsto & b_0 \arrow[r, "f"'] & b_1 \arrow[r, equal] & b_1
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
i(R(p)) : & & e \arrow[d, |->] & & & & e \arrow[d, |->] \\
& b_0 \arrow[r, "f"'] & b_1 & \longmapsto & b_0 \arrow[r, equal] & b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}
$$

and the mate 2-cell λ as in diagram (4) appears as a natural transformations in this case where $\lambda_p \colon i(R(p)) \Rightarrow R(i(p))$ can be illustrated as

$$
\begin{tikzcd}
& & & & e \arrow[dd, |->, "p"] \arrow[dr, equal] & \\
& & & & & e \arrow[dd, |->, "p"] \\
b_0 \arrow[rr, equal] \arrow[dr, equal] & & b_0 \arrow[rr, "f"] \arrow[dr, "f"] & & b_1 \arrow[dr, equal] & \\
& b_0 \arrow[rr, "f"'] & & b_1 \arrow[rr, equal] & & b_1
\end{tikzcd}
$$

We keep in mind that $R(\mathfrak{a}) \circ R \centerdot i(p)\langle f, e\rangle = \langle f, \mathfrak{a}\langle 1_{b_1}, e\rangle\rangle$, and hence $R(\zeta)\langle f, e\rangle$ is illustrated in below:

$$
\begin{tikzcd}
& & e \arrow[dd, |->, "p"] \arrow[dr, "\zeta(e)"] & \\
& & & \mathfrak{a}\langle 1_{b_1}, e\rangle \arrow[dd, |->, "p"] \\
b_0 \arrow[rr, "f"] \arrow[dr, equal] & & b_1 \arrow[dr, equal] & \\
& b_0 \arrow[rr, "f"'] & & b_1
\end{tikzcd}
\tag{12}
$$

In addition, invertible 2-cell $\theta(p) \colon \mathfrak{a} \circ R(\mathfrak{a}) \Rightarrow \mathfrak{a} \circ m(p)$ provides us with an isomorphism $\mathfrak{a}\langle f, \mathfrak{a}\langle g, e\rangle\rangle \to \mathfrak{a}\langle gf, e\rangle$. Now, we study coherence equations (2) and (3) in our case, which state that for any morphism $f \colon b_0 \to b_1$ together with any object e in E over b_1, the following diagram (in the fibre over b_0) commute :

$$
\begin{tikzcd}
\mathfrak{a}\langle f, e\rangle \arrow[r, "\mathfrak{a}\centerdot R(\zeta)"] \arrow[d, "\zeta\centerdot\mathfrak{a}"'] \arrow[dr, equal] & \mathfrak{a}\langle f, \mathfrak{a}\langle 1_{b_1}, e\rangle\rangle \arrow[d, "\theta\centerdot R(i(p))"] \\
\mathfrak{a}\langle 1_{b_0}, \mathfrak{a}\langle f, e\rangle\rangle \arrow[r, "\theta\centerdot i(R(p))"'] & \mathfrak{a}\langle f \circ 1_{b_0}, e\rangle
\end{tikzcd}
$$

and, for every chain of morphisms $b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2 \xrightarrow{h} b_3$ in B and any object e in E over b_3, the diagram (in the fibre over b_0)

$$\begin{tikzcd}
\mathfrak{a}\langle f, \mathfrak{a}\langle g, \mathfrak{a}\langle h, e\rangle\rangle\rangle \arrow[r] \arrow[d] & \mathfrak{a}\langle gf, \mathfrak{a}\langle h, e\rangle\rangle \arrow[d] \\
\mathfrak{a}\langle f, \mathfrak{a}\langle gh, e\rangle\rangle \arrow[r] & \mathfrak{a}\langle hgf, e\rangle
\end{tikzcd}$$

commutes. Finally, the counit of adjunction $i(p) \dashv \mathfrak{a}$, as computed in diagram 6, gives us the lift $\tilde{f} = \widehat{d_1}((R\mathfrak{a} \centerdot \lambda_p) \circ R\zeta^{-1})$ of f:

$$\begin{tikzcd}
 & & & \mathfrak{a}\langle f, e\rangle \arrow[dr, "\widehat{d_1}(R\mathfrak{a}\centerdot\lambda_p)"] \arrow[d, mapsto, "p"] & & \\
 & & & b_0 \arrow[dr, "f"] & \mathfrak{a}\langle 1_{b_1}, e\rangle \arrow[dr, "\widehat{d_1}(R\zeta^{-1})"] \arrow[d, mapsto, "p"] & \\
b_0 \arrow[urrr, equal] \arrow[dr, "1"'] & & & & b_1 \arrow[dr, equal] & e \arrow[d, mapsto, "p"] \\
 & b_0 \arrow[urrr, "f"'] \arrow[dr, equal] & & & & b_1 \\
 & & b_0 \arrow[urrr, "f"'] & & &
\end{tikzcd}$$

It remains to prove that $\tilde{f}$ as defined is cartesian. One can try to prove this directly. However, we prove this in a more general setting in the next section.

5 Chevalley criterion

Suppose p is a 0-cell in $\mathcal{K}/B$. There is a unique derived 1-cell Γ_1 with properties $R(p)\Gamma_1 = d_0 p^{\rightarrow}$, $\hat{d_1}\Gamma_1 = e_1$, and $\phi_p \centerdot \Gamma_1 = p \centerdot \phi_E$.

$$\begin{tikzcd}
E^{\rightarrow} \arrow[d, "p^{\rightarrow}"'] \arrow[dr, dotted, "\Gamma_1"] \arrow[drr, bend left, "e_1"] & & \\
B^{\rightarrow} \arrow[dr, bend right, "d_0"'] & B/p \arrow[r, "\hat{d_1}"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p \Uparrow"] & E \arrow[d, "p"] \\
 & B \arrow[r, "1"'] & B
\end{tikzcd}$$

Lemma 5.1. We have $\hat{d_1}\Gamma_1 \centerdot \tau_0 = \phi_E$ and $R(p)\Gamma_1 \centerdot \tau_0 = id_{R(p)\Gamma_1}$ and from these it follows that $(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0) = i(p) \centerdot \phi_E$, by 2-dimensional universal property of B/p.

Proof. The first identity holds since $e_1 \centerdot \tau_0 = \phi_E$ due to universal property of comma object $E^{\rightarrow}$. For the second identity observe that $R(p)\Gamma_1 \centerdot \tau_0 = pe_0 \centerdot \tau_0 = id_{pe_0} = id_{R(p)\Gamma_1}$, by one of triangle identity of adjunction $i_E \dashv e_0$. Now, notice that

$$\hat{d}_1[(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0)] = \phi_E = \hat{d}_1[i(p) \centerdot \phi_E]$$

$$R(p)[(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0)] = R(p) \centerdot \tau_1(p) \centerdot \Gamma_1 = \phi_p \centerdot \Gamma_1 = p \centerdot \phi_E = R(p)[i(p) \centerdot \phi_E]$$

□

Definition *5.2.* We say a 1-cell $p\colon E \to B$ in $\mathcal{K}$ satisfies ***Chevalley criterion*** if Γ_1 has a right adjoint Λ_1 $\mathcal{K}/B$ with isomorphism counit. Sometimes we call such an adjunction $\Gamma_1 \dashv \Lambda_1$ a Chevalley adjunction.

Proposition *5.3.* There is a bijection between collection of 1-cells $p\colon E \to B$ equipped with an R-pseudo algebra $(\mathfrak{a}, \zeta, \theta)$ and collection of Chevalley adjoints $(\Gamma_1, \Lambda_1, \epsilon, \eta)$. Moreover, if pseudo-algebra is normalized then counit ϵ is identity.[6]

Proof. Given a pseudo algebra $\mathfrak{a}\colon R(p) \to p$, we construct a right adjoint Λ_1 and show that the counit of adjunction is isomorphism. Hence p satisfies Chevalley criterion. Note that the unit $\tau_1(p)$ of adjunction $\hat{d}_1 \dashv i(p)$ defines a unique 1-cell $k\colon B/p \to (B/p)^{\to}$ obtained by factoring $\tau_1(p)$ through comma square $\langle (B/p)^{\to}, \pi_0, \pi_1, \phi_{B/p} \rangle$. Thus, $\pi_0 k = 1_{B/p}$ and $\pi_1 k = i(p)\hat{d}_1$, and $\phi_{B/p} \centerdot k = \tau_1(p)$. Define $\Lambda_1 \colon = \mathfrak{a}^{\to} \circ k$. We note that

$$\begin{aligned}
e_0\Lambda_1 &= e_0\mathfrak{a}^{\to}k && \{\text{definition of } \Lambda_1\} \\
&= \mathfrak{a}\pi_0 k && \{\text{definition of } \mathfrak{a}^{\to}\} \\
&= \mathfrak{a} && \{\text{definition of } k\}
\end{aligned} \tag{13}$$

This establishes that Λ_1 is indeed a 1-cell in $\mathcal{K}/B$, since $d_0 p^{\to}\Lambda_1 = pe_0\Lambda_1 = p\mathfrak{a} = R(p)$. Also, a diagram chase shows that the front square in the diagram below commutes:

$$\begin{aligned}
\hat{d}_1\Gamma_1\Lambda_1 &= e_1\Lambda_1 && \{\text{definition of } \Gamma_1\} \\
&= e_1\mathfrak{a}^{\to}k && \{\text{definition of } \Lambda_1\} \\
&= \mathfrak{a}\pi_1 k && \{\text{definition of } \mathfrak{a}^{\to}\} \\
&= \mathfrak{a}i(p)\hat{d}_1 && \{\text{definition of } k\}
\end{aligned} \tag{14}$$

[6] However, the converse of this statement is not true in general as one can observe in the construction of adjunction from

$$
\begin{tikzcd}
 & (B/p)^{\to} \arrow[d, dotted, "\mathfrak{a}^{\to}"] \arrow[ddr, dotted, "\pi_i"] & \\
B/p \arrow[ur, "k"] \arrow[rr, "\hat{d}_1"] \arrow[dr, "\Lambda_1"'] \arrow[dd, "\Gamma_1\Lambda_1"'] & & E \arrow[d, "i(p)"] \\
 & E^{\to} \arrow[dl, "\Gamma_1"'] \arrow[dr, "e_i"] & B/p \arrow[d, "\mathfrak{a}"] \\
B/p \arrow[rr, "\hat{d}_1"'] & & E
\end{tikzcd}
\qquad (i = 1, 2)
$$

We also note that

$$
\begin{gathered}
R(p)\Gamma_1\Lambda_1 = d_0 p^{\to}\Lambda_1 = R(p) \\
\phi_p \centerdot (\Gamma_1\Lambda_1) = p \centerdot \phi_E \centerdot \Lambda_1 = p\mathfrak{a} \centerdot \phi_{B/p} \centerdot k = p\mathfrak{a} \centerdot \tau_1(p) = R(p) \centerdot \tau_1(p) = \phi_p
\end{gathered} \tag{15}
$$

Equations (14) and (15), and definition of $R(\mathfrak{a}i(p))$ altogether prove that

$$\Gamma_1 \circ \Lambda_1 = R(\mathfrak{a} \circ i(p)) = R(\mathfrak{a}) \circ R(i(p))$$

and we shall show that counit $\epsilon\colon \Gamma_1 \circ \Lambda_1 \Rightarrow 1$ is given by $R(\zeta^{-1})$ which is invertible.[7] Also notice that $p\hat{d}_1 \centerdot \epsilon = p\hat{d}_1 \centerdot R(\zeta^{-1}) = p \centerdot \zeta^{-1} \centerdot \hat{d}_1 = id_{p\hat{d}_1}$, and $R(p) \centerdot \epsilon = R(p) \centerdot R(\zeta^{-1}) = id_{R(p)}$. This guarantees that the counit lives in $\mathcal{K}/B$. Moreover, definition of $R(\zeta)$ implies that $\phi_p \centerdot \epsilon = \phi_p$. Now, we propose the unit; define the 2-cell $\eta\colon 1 \Rightarrow \Lambda_1 \circ \Gamma_1$ to be the unique 2-cell with

$$
\begin{gathered}
e_0 \centerdot \eta = (\mathfrak{a}\Gamma_1 \centerdot \tau_0) \circ (\zeta \centerdot e_0) \\
e_1 \centerdot \eta = \zeta \centerdot e_1
\end{gathered} \tag{16}
$$

Note that the vertical composition of 2-cells in (16) makes sense since $\mathfrak{a}i(p)e_0 = \mathfrak{a}\Gamma_1 i_E e_0$ which holds as one can easily see that $\Gamma_1 i_E = i(p)$. Furthermore, $e_0 \centerdot \eta$ and $e_1 \centerdot \eta$ are compatible in the sense that

$$
\begin{aligned}
(\phi_E \centerdot \Lambda_1\Gamma_1) \circ (e_0\eta) &= (\phi_E \centerdot \mathfrak{a}^{\to} k\Gamma_1) \circ (e_0\eta) && \{\text{definition of } \Lambda_1\} \\
&= (\mathfrak{a}\phi_{B/p} \centerdot k\Gamma_1) \circ (e_0\eta) && \{\text{definition of } \mathfrak{a}^{\to}\} \\
&= (\mathfrak{a}\tau_1(p) \centerdot \Gamma_1) \circ (e_0\eta) && \{\text{definition of } k\} \\
&= (\mathfrak{a}\tau_1(p) \centerdot \Gamma_1) \circ (\mathfrak{a}\Gamma_1 \centerdot \tau_0) \circ (\zeta \centerdot e_0) && \{\text{substituting } e_0 \centerdot \eta\} \\
&= \mathfrak{a}(\tau_1(p) \centerdot \Gamma_1 \circ \Gamma_1 \centerdot \tau_0) \circ (\zeta \centerdot e_0) && \{\text{factoring out } \mathfrak{a}\} \\
&= (\mathfrak{a}i(p) \centerdot \phi_E) \circ (\zeta \centerdot e_0) && \{\text{Lemma 5.1}\} \\
&= (\zeta \centerdot e_1) \circ \phi_E && \{\text{exchange rule}\} \\
&= (e_1\eta) \circ (\phi_E) && \{\text{substituting } e_1 \centerdot \eta\}
\end{aligned}
$$

[7]When $\mathcal{K} = \mathfrak{Cat}$, $R(\zeta)$ is illustrated in diagram 12.

In the next step, we prove that proposed unit[8] η and counit ϵ satisfy triangle equations of adjunction. To prove the first identity, we notice that

$$R(p) \centerdot [(\epsilon \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \eta)] = [R(p) \centerdot (\epsilon \centerdot \Gamma_1)] \circ [R(p) \centerdot (\Gamma_1 \centerdot \eta)] = (id_{R(p)}\Gamma_1) \circ (pe_0 \centerdot \eta) = id_{R(p)\Gamma_1}$$

where the last identity follows from the fact that $pe_0 \centerdot \eta = id_{pe_0} = id_{R(p)\Gamma_1}$. Similarly, we have

$$\hat{d_1} \centerdot [(\epsilon \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \eta)] = (\zeta^{-1} \centerdot \hat{d_1}\Gamma_1) \circ (e_1 \centerdot \eta) = (\zeta^{-1} \centerdot e_1) \circ (\zeta \centerdot e_1) = id_{\hat{d_1}\Gamma_1}$$

Therefore, $(\epsilon \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \eta) = id_{\Gamma_1}$. To prove the second identity, $(\Lambda_1 \centerdot \epsilon) \circ (\eta \centerdot \Lambda_1) = id_{\Lambda_1}$, we first prove the following lemma:

Lemma *5.4.* $\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1 = R(\mathfrak{a}) \centerdot \lambda_p$

Proof. First we verify that the domain and codomain of these 2-cells match.

$$\begin{tikzcd} B/p \arrow[r, "\Lambda_1"] & E^{\rightarrow} \arrow[r, bend left, "e_0"] \arrow[rr, bend right, "1"'] & E \arrow[d, phantom, "\tau_0\Downarrow"] \arrow[r, bend left, "i_E"] & E^{\rightarrow} \arrow[r, "\Gamma_1"] & B/p \end{tikzcd}$$

Indeed,

$$\Gamma_1 i_E e_0 \Lambda_1 = i(p) e_0 \Lambda_1 = i(p)\mathfrak{a} = R(\mathfrak{a}) i(R(p))$$

and as we observed earlier $\Gamma_1 \circ \Lambda_1 = R(\mathfrak{a})R(i(p))$. So, the domain and codomain of $\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1$ and $R(\mathfrak{a}) \centerdot \lambda_p$ agree. The lemma follows from identities in below in conjunction with comma property of B/p for 2-cells.

$$\hat{d_1} \centerdot (\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1) = \phi_E \centerdot \Lambda_1 = \mathfrak{a}\tau_1(p) = \mathfrak{a}\widehat{d_1^{\rightarrow}} \centerdot \lambda_p = \hat{d_1} \centerdot R(\mathfrak{a}) \centerdot \lambda_p$$

$$R(p) \centerdot (\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1) = id_{pe_0} \centerdot \lambda_1 = id = R^2(p) \centerdot \lambda_p = R(p)R(\mathfrak{a}) \centerdot \lambda_p$$

□

[8] Perhaps, it is illuminating to see what this unit look like in the case of $\mathcal{K} = \mathfrak{Cat}$. Indeed, for a morphism $f\colon e_0 \to e_1$ in $E^{\rightarrow}$, $\eta(f)$ is given as follows:

$$\begin{tikzcd} e_0 \arrow[r, "\zeta e_0(f)"] \arrow[d, "f"'] & \mathfrak{a}\langle 1_{p(e_0)}, e_0\rangle \arrow[r, "\mathfrak{a}\Gamma_1\tau_0(f)"] & \mathfrak{a}\langle p(f), e_1\rangle \arrow[d, "\Lambda_1\Gamma_1(f)"] \\ e_1 \arrow[rr, "\zeta e_1(f)"'] & & \mathfrak{a}\langle 1_{p(e_1)}, e_1\rangle \end{tikzcd}$$

Using lemma above we have,

$$e_0 \centerdot [(\Lambda_1 \centerdot \epsilon) \circ (\eta \centerdot \Lambda_1)] = (\mathfrak{a} \centerdot \epsilon) \circ ((\mathfrak{a}\Gamma_1\tau_0) \circ (\zeta e_0)) \centerdot \Lambda_1 = (\mathfrak{a} \centerdot R(\zeta^{-1})) \circ (\mathfrak{a}R(\mathfrak{a}) \centerdot \lambda_p) \circ (\zeta\mathfrak{a}) = (\zeta^{-1}\mathfrak{a}) \circ (\zeta\mathfrak{a}) = id_{e_0\Lambda_1}$$

The penultimate equality comes from equality of pasting diagrams 5. Similarly, using the fact that $e_1\Lambda_1 = ai(p)\hat{d}_1$, we get

$$e_1 \centerdot [(\Lambda_1 \centerdot \epsilon) \circ (\eta \centerdot \Lambda_1)] = (\mathfrak{a}i(p)\hat{d}_1 \centerdot \epsilon) \circ (\zeta \centerdot e_1\Lambda_1) = (\mathfrak{a}i(p)\zeta^{-1}\hat{d}_1) \circ (\zeta \centerdot \mathfrak{a}i(p)\hat{d}_1) = id_{e_1\Lambda_1}$$

The last identity is by exchange law of horizontal-vertical composition of 2-cells. From these two equations we deduce the second adjunction identity.

Conversely, suppose we are given a Chevalley adjunction, that is to say an adjunction $\Gamma_1 \dashv \Lambda_1$ over B:

$$\begin{tikzcd}
E^{\to} \arrow[loop above, "\eta"] \arrow[rr, shift left, "\Gamma_1"] \arrow[rr, phantom, "\perp"] \arrow[dr, "pe_0"'] & & B/p \arrow[loop above, "\epsilon"] \arrow[ll, shift left, "\Lambda_1"] \arrow[dl, "R(p)"] \\
& B &
\end{tikzcd}$$

such that the counit ϵ is an isomorphism, $R(p)\Gamma_1 = pe_0$, $pe_0\Lambda_1 = R(p)$, $R(p) \centerdot \epsilon = id_{R(p)}$, and $pe_0 \centerdot \eta = id_{pe_0}$. We define pseudo-algebra $\mathfrak{a} \colon B/p \to E$ as composite $e_0\Lambda_1$. Note that $p\mathfrak{a} = pe_0\Lambda_1 = R(p)\Gamma_1\Lambda_1 = R(p)$, since the adjunction $\Gamma_1 \dashv \Lambda_1$ takes place in $\mathcal{K}/B$. We propose $e_1\eta i_E$ for ζ. First we prove that $\eta \centerdot i_E$ is invertible and thence ζ is invertible. Using $\tau_1 \centerdot i_E = id$, we have $(i_E\hat{d}_1\epsilon \centerdot i(p)) \circ (\tau_1 \centerdot \Lambda_1\Gamma_1 i_E) \circ (\eta \centerdot i_E) = id_{i_E}$. This is illustrated in the following pasting equality:

$$\begin{tikzcd}
E^{\to} \arrow[r, "1"] \arrow[dr, "\Gamma_1"'] \arrow[r, phantom, shift right=3, "\eta\Downarrow"] & E^{\to} \arrow[r, "e_1"] \arrow[rrr, bend left, "1"] \arrow[rrr, phantom, bend left=20, "\tau_1\Downarrow"] \arrow[dr, "\Gamma_1"] & E \arrow[r, "i_E"] & E^{\to} \\
E \arrow[u, "i_E"] \arrow[r, "i_p"'] & B/p \arrow[u, "\Lambda_1"'] \arrow[r, "1"'] \arrow[r, phantom, shift left=3, "\epsilon\Downarrow"] & B/p \arrow[r, "\hat{d}_1"'] & E \arrow[u, "i_E"']
\end{tikzcd}
=
\begin{tikzcd}
E^{\to} \arrow[r, "1"] & E^{\to} \\
E \arrow[u, "i_E"] \arrow[r, "1"'] \arrow[ur, phantom, "id\Downarrow"] & E \arrow[u, "i_E"']
\end{tikzcd}$$

Same pasting equality shows that $(i_E\hat{d}_1 \centerdot \epsilon \centerdot i(p)) \circ (\tau_1 \centerdot \Lambda_1\Gamma_1 i_E)$ is 2-sided inverse of $\eta \centerdot i_E$. Whiskering with e_1 reveals inverse of ζ:

$$\zeta^{-1} = (e_1 i_E \hat{d}_1 \centerdot \epsilon \centerdot i(p)) \circ (e_1 \centerdot \tau_1 \centerdot \Lambda_1\Gamma_1 i_E) = \hat{d}_1 \centerdot \epsilon \centerdot i(p)$$

since $e_1 \centerdot \tau_1 = id_{e_1}$. Indeed, ζ^{-1} is the counit of composite adjunction in below:

$$
\begin{tikzcd}
E \arrow[r, bend left, "i_E"] \arrow[r, phantom, "\perp"] & E^{\to} \arrow[l, bend left, "e_0"] \arrow[r, bend left, "\Gamma_1"] \arrow[r, phantom, "\perp"] & B/p \arrow[l, bend left, "\Lambda_1"] \arrow[r, bend left, "\hat{d}_1"] \arrow[r, phantom, "\perp"] & E \arrow[l, bend left, "i(p)"]
\end{tikzcd}
$$

Notice that we have proved that ζ is invertible regardless of invertibility of ϵ. Also, obviously if ϵ is identity then so is ζ. To finish the proof, by lemma 2.5 it suffices to prove that $\zeta^{-1} = \hat{d}_1 \centerdot \epsilon \centerdot i(p)$ satisfies pasting equality in (5), and moreover, $i(p) \dashv a$ with ζ and $R(\zeta^{-1}) \circ (R(\mathfrak{a}) \centerdot \lambda_p)$ as unit and counit of this adjunction respectively.

□

Example 5.5. We now return to prove our promise at the end of example 4.4. We would like to show that $\tilde{f}$, obtained as whiskering $\hat{d}_1$ with counit of $i(p) \dashv \mathfrak{a}$, is indeed cartesian. Here, we appeal to the bijection $\mathrm{Hom}_{B/p}(\Gamma_1(g), \langle f, e_1\rangle) \cong \mathrm{Hom}_{E^{\to}}(g, \Lambda_1\langle f, e_1\rangle)$ natural in $g\colon d_0 \to d_1$ in $E^{\to}$ and $\langle f, e_1\rangle$ in B/p. This bijection states that any diagram of the form

$$
\begin{tikzcd}
 & & d_1 \arrow[d, mapsto, "p"] \arrow[dr, "k"] & \\
 & & & e_1 \arrow[dd, mapsto, "p"] \\
p(d_0) \arrow[r, "p(g)"] \arrow[dr, "h_0"] & & p(d_1) \arrow[dr, "h_1"] & \\
 & b_0 \arrow[rr, "f"] & & b_1
\end{tikzcd}
$$

where the square in base commutes and k lies above h_1 can be (uniquely) completed to the diagram below:

$$
\begin{tikzcd}
d_0 \arrow[rr, "g"] \arrow[dr, dashed, "\bar{h}_0"] \arrow[dd, mapsto] & & d_1 \arrow[dr] \arrow[drr, "k"] \arrow[dd, dotted, mapsto] & & \\
 & \mathfrak{a}\langle f, e_1\rangle \arrow[rr] \arrow[dd, mapsto] & & \mathfrak{a}\langle 1_{b_1}, e_1\rangle \arrow[r, "\zeta^{-1}e_1"'] \arrow[dd, mapsto] & e_1 \arrow[dd, mapsto] \\
p(d_0) \arrow[rr, dotted] \arrow[dr, "h_0"] & & p(d_1) \arrow[dr] \arrow[drr, dotted] & & \\
 & b_0 \arrow[rr, "f"] & & b_1 \arrow[r, equal] & b_1
\end{tikzcd}
$$

Taking g to be identity we obtain the usual condition which expresses cartesian property of lift $\tilde{f}$. Also, one can easily show that unique morphism $\bar{h_0}$ over h_0 is calculated by the expression $(e_0\Lambda_1(h_0, h_1, k)) \circ (\mathfrak{a}\Gamma_1\tau_0(g)) \circ (\zeta e_0(g))$.

Example 5.6. Let $p\colon E \to B$ be a cloven Grothendieck fibration. Note that the data of a cloven Grothendieck fibration includes structure of a cleavage, that is a choice of cartesian lifts:

$$\rho_{a,b} : \prod_{\mathrm{Hom}(a,b)} \prod_{e \in E_b} \sum_{e' \in E_a} \mathcal{C}art_E(e', e)$$

where $\mathcal{C}art_E(e', e)$ denotes the set of cartesian morphisms from e' to e.

We say p is *split* if for all pairs of objects a, b:

$$\texttt{snd}\,\rho_{a,c}(g \circ f, e) = \texttt{snd}\,\rho_{b,c}(g, e) \circ \texttt{snd}\,\rho_{a,b}(f, \texttt{fst}\,\rho_{b,c}(g, e))$$

and we say p is *normal* if for all objects e in E:

$$\texttt{snd}\,\rho_{pe,pe}(1_{pe}, e) = 1_e$$

We have the following correspondence:

$$\left\{ \begin{array}{c} \text{cleavages} \\ \text{of } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{pseudo-algebras} \\ (\mathfrak{a}, \zeta, \theta) \text{ of } R \text{ at } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{right adjoints of } \Gamma_1 \\ \text{with isomorphism counit} \end{array} \right\}$$

It follows that any two cleavages of p are isomorphic in a unique way.

Definition *5.7.* For a category B, define 2-category $\mathbf{Fib}(B)$ of fibrations over B whose 0-cells are Grothendieck fibrations, whose 1-cells are fibred functors over B (i.e. those functors over B which preserve cartesian morphisms), and 2-cells are vertical natural transformations (i.e. transformations over B). Compositions are usual composition of functors and natural transformations.

Remark *5.8.* Example 4.4 can be encapsulated as follows: The forgetful 2-functor $U \colon \mathbf{Fib}(B) \to \mathfrak{Cat}/B$ is 2-monadic: the ***free fibration*** of a functor $p \colon E \to B$ is fibration $R(p) \colon B/p \to B$; cleavage (aka fibration structure) on p is uniquely (in fact unique up to unique isomorphism) determined by a pseudo algebra structure for 2-monad $R = UF$. Strict algebra structures of R correspond to splitting fibration structures on p.

$$\begin{tikzcd}
\mathbf{Fib}(B) \arrow[d, bend left, "U"] \\
\mathfrak{Cat}/B \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"] \arrow[loop right, "R"]
\end{tikzcd}$$

We also note that for a category B the domain functor cod: $B^{\to} \to B$ is the free Grothendieck fibration on identity functor $1 \colon B \to B$; that is dom $= R(1)$. In more explanatory terms this fact states that

We also note that for a category B with pullbacks the codomain functor cod: $B^{\to} \to B$ is the free Grothendieck fibration *with existential quantifiers* on identity functor $1 \colon B \to B$;

References

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