First Year Progress Report

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Abstract

Within the first chapters, we will introduce basic concepts and few main examples of presheaves and give the definition of discrete fibration of a category relating it to category of presheafs. Second chapter is an accessible and introductory review of abstract homotopy theory. Using the constructions introduced in chapters 1, and 2, in Chapter 3 we assign to every model category, a homotopy 2-category which will serve as a useful example of 2-categories, besides $\mathfrak{Cat}$, to bear in mind.

In chapter 4, we introduce preliminary notions of theory of 2-categories and bicategories most notably a review of the notions of weighted limits. In the last chapter we give a general definition fibrations within 2-categories based on the idea of representability illustrated in chapter 4. Chapter 5 contains an account of fundamental concept of fibrations, starting from discrete fibrations and generalising to Grothendieck fibrations, Street fibrations and fibrations internal to 2-categories. We also give some speculations as to what a proper n-fibration for higher categories should be defined. Last chapter contains several concluding remarks that weave together the local and global constructions introduced in earlier parts. We also introduce new references for further reading in recent research directions. [1].

Main references are given at the beginning of each chapter and whenever necessary as reference to a proof which is omitted.

[1]There will inevitably be errors in these notes which are all due to the author's shortcoming, and he would highly appreciate your help if you spot them and email to sxh617@cs.bham.ac.uk

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1

Few words on presheaves

1.1 What is a presheaf?

In this chapter, we define the notions presheaves and sheaves and recall some basic facts about them supplanted by various examples. Here, we are only concerned with presheaves and sheaves of sets. Basically, a presheaf over a category $\mathcal{C}$ is a family of sets parametrised by objects of $\mathcal{C}$ functorially. To get from presheaves to sheaves, we introduce gluing condition for presheaves in section 1.2. Finally, in the last section, we give another point of view of sheaves: sheaves as étale bundles over spaces.

Definition *1.1.1*. A presheaf on the category $\mathcal{C}$ is a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

Example *1.1.2*. Suppose X is a topological space. Let $\mathcal{O}(X)$ be the poset of open sets of X. We can look at $\mathcal{O}(X)$ as a category where there is exactly one morphism $U \to V$ precisely when $U \subset V$ and no morphism otherwise. A presheaf on X is a functor $F : \mathcal{O}(X)^{op} \to \mathbf{Set}$. This gives us the maps $\rho_U^V : F(V) \to F(U)$ between sets with $\rho_U^V \circ \rho_V^W = \rho_U^W$ whenever $U \subset V \subset W$, and $\rho_U^U = id_{F(U)}$

Remark. For any open subset U of X, we usually refer to the elements of $F(U)$ as *local sections* and to elements of $F(X)$ as *global sections*.

Example *1.1.3*. Suppose X is a topological space. Let I be the unit interval with standard topology. (subspace topology $I \subset \mathbb{R}$). Define a preasheaf on the interval I as follows:

$$P(U) = \{f : U \to X | f \text{ is continuous and} f(0) = f(1) \text{ if } 0, 1 \in U\}$$

The local sections give us unbounded path in X and global sections are loops in X. So, $P(I)$= set of loops in X

Remark. We will see later(1.2.2) that the presheaf in Example 1.1.3 is not a sheaf.

Example *1.1.4*. Suppose G is a group. A G-set X can be viewed as a presheaf over its delooping groupoid $\mathbb{B}G$, i.e. the groupoid with only one object pt and morphisms, elements of group G:

$$\begin{aligned} \hat{X} : \mathbb{B}G^{op} &\longrightarrow \mathbf{Set} \\ pt &\longmapsto X \\ (pt \xrightarrow{g} pt) &\mapsto (X \xrightarrow{\hat{g}} X) \end{aligned}$$

where $\hat{g}$ is the action of element $g \in G$ on X.

Example *1.1.5*. An important class of presheaves are **representable** presheaves. For any object U of a category $\mathcal{C}$ the functor $\mathrm{Hom}(-, U)$ is a presheaf on $\mathcal{C}$. We denote this functor by h_U and some other times by yU . Any functor naturally isomorphic to h_U is called a *representable* functor.

Remark. The presheaf $\hat{X} : \mathbb{B}G^{op} \to \mathbf{Set}$ in example 1.1.4 is representable if and only if $U(G) \cong \mathbb{B}G(pt, pt) \cong X$ as sets, where $U(G)$ is the underlying set of group G.

Definition *1.1.6*. For any category $\mathcal{C}$, there is a category whose objects are presheaves on $\mathcal{C}$ and whose morphism are natural transformations between presheaves. We denote this category as $\mathbf{Psh}(\mathcal{C})$.

Remark. Example 2 may convince us that we can think of an element X of $\mathbf{Psh}(\mathcal{C})$ as a rule that assigns to each test space $U \in \mathcal{C}$ a set $X(U)$ of allowed maps from U into the generalised space X.

The following theorem makes this observation more precise.

Proposition 1.1.7. Suppose $\mathcal{C}$ is a locally small category. The functor $y : \mathcal{C} \to \mathbf{Psh}(\mathcal{C})$ is full and faithful, and it preserves limits. The functor y is called the ***Yoneda embedding***.

Moreover, if $\mathcal{C}$ is a small and $\mathcal{E}$ is a cocomplete category and $F : \mathcal{C} \to \mathcal{E}$ is a functor between them, then there is a cocontinuous functor (unique up to natural isomorphism) $F_!$ such that $F_! \circ y \cong F$, i.e. the following diagram commutes.

$$\begin{tikzcd} \mathbf{Psh}(\mathcal{C}) \arrow[r, dashed, "F_!"] & \mathcal{E} \\ \mathcal{C} \arrow[u, "y"] \arrow[ur, "F"'] & \end{tikzcd}$$

One can prove that functor $F_!$ is a left Kan extension of F along y.

Proposition 1.1.8. The category $\mathbf{Psh}(\mathcal{C})$ is cocomplete and it is the free cocompletion of category $\mathcal{C}$. Every element of the generalised spaces $\mathbf{Psh}(\mathcal{C})$ is obtained as a canonical colimit of representables.

In fact, for $P \in \mathbf{Psh}(\mathcal{C})$,

$$P \cong Colim(\int_{\mathcal{C}} P \xrightarrow{\pi_1} \mathcal{C} \xrightarrow{y} \mathbf{Psh}(\mathcal{C}))$$

Where $\int_{\mathcal{C}} P$ is the category of elements of $\mathcal{C}$ and the first map is projection and the second map is the Yoneda embedding.

Remark. The functor $F_!$ can be expressed as a coend by virtue of the fact that it's a left Kan extension of F along Yoneda.

$$F_!(P) = Lan_y^F(P) = \int^{c \in \mathcal{C}} \mathbf{Psh}(\mathcal{C})(yc, P).Fc$$

The adjunction gives us following natural bijection:

$$\mathcal{E}(Lan_y^F P, e) \cong \mathbf{Psh}(\mathcal{C})(P, \mathcal{E}(F-, e))$$

Where P is a presheaf on $\mathcal{C}$, e is any object of $\mathcal{E}$.

Definition 1.1.9. A functor $S : \mathcal{E} \to \mathcal{B}$ is a ***discrete fibration*** if for every object e of $\mathcal{E}$ and every morphism $f : b \to Se$ there exists a unique lift $b' \to e$ in $\mathcal{E}$.

Proposition *1.1.10*. The following diagram is a pullback of categories.

$$\begin{tikzcd}
(\int_{\mathcal{C}} P)^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_* \arrow[d] \\
\mathcal{C}^{\mathrm{op}} \arrow[r] & \mathbf{Set}
\end{tikzcd}$$

This establishes following equivalence of categories:

$$DiscFib(\mathcal{C}) \simeq [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$$

And we can think of the projection map $\mathbf{Set}_* \to \mathbf{Set}$ as the universal discrete fibration.

1.2 Why do we need sheaves?

In the last section we observed that presheaves on a category $\mathcal{C}$ are as good as discrete fibrations on $\mathcal{C}$. However, if there is a notion of topology (covering) defined in category $\mathcal{C}$, then presheaves over $\mathcal{C}$ may lose this information. The remedy is to pass over to sheaves; a certain subclass of presheaves which respect the covering relations of domain category.

The process of passing from presheaves to sheaves on categories is called sheafification and can be done in various ways. However, for conceptual reasons here, first, we define sheaves over topological spaces and then via localization, we localize the category of presheaves at the class of local isomorphism.

1.2.1 Idea of sheaf

Suppose X is a topological space. Let U be any open subset of X and $\mathcal{U} = \{U_i\}$ be any covering for U, that is $U = \bigcup U_i$. Suppose also that F is a presheaf on X and we are given a compatible set of local sections $s_i \in F(U_i)$, and $\rho^{U_i}_{U_{ij}}(s_i) = s_i|_{U_{ij}} = s_j|_{U_{ij}} = \rho^{U_j}_{U_{ij}}(s_j)$ where $U_{ij} = U_i \cap U_j$.

Up to here, we are given a compatible local data for any covering by the presheaf F. The sheaf condition says we should be able to find a unique global extension of this local data; that is a global section s on U that matches every local section s_i when restricted to U_i

Let's formulate this idea more precisely.

Definition 1.2.1. Suppose F is a presheaf on topological space X. F is said to be a *sheaf* precisely when for any open subset U of X and any covering $\mathcal{U} = \{U_i\}$ of U, the following diagram is an equalizer:

$$F(U) \xrightarrow{e} \prod_i F(U_i) \underset{g}{\overset{f}{\rightrightarrows}} \prod_{i,j} F(U_{ij})$$

where $f(\{s_i\}) = \{s_i|_{U_{ij}}\}$ and $f(\{s_i\}) = \{s_j|_{U_{ij}}\}$

1.2.2 Some examples of sheaves

The examples below will exhibit the important informal principle that whenever a presheaf on space X is defined only by some local properties of maps out of space X, then the presheaf is actually a sheaf.

Example 1.2.2. The presheaf in 1.1.3 is **not** a sheaf. It's quite easy to see this. Take $U_1 = \mathbb{I} - \{1\}$ and $U_2 = \mathbb{I} - \{0\}$. Both U_1 and U_2 are open and they form a covering for the interval $\mathbb{I}$. However, since both of the subsets does not possess at least one of two points 0 or 1. Take any two compatible local sections $s_1 \in U_1$ and $s_2 \in U_2$ with $s_1(0) \neq s_2(1)$. The only extension of $\{s_1, s_2\}$ could be

$$s(t) = \begin{cases} s_1(t) & \text{, if } t \in U_1 \\ s_2(t) & \text{, if } t \in U_2 \smallsetminus U_1 \end{cases}$$

Notice that s is continuous since s_1 and s_2 are continuous and they agree on $U_1 \cap U_2$. But $s \notin P(\mathbb{I})$ because the endpoints of the path are distinct. So, in this case, there is no global section which extends the local sections s_1 and s_2.

Example 1.2.3. Suppose M is a fixed topological space, e.g. a smooth manifold.The following presheaf is a sheaf and it is referred to as the sheaf of local continuous functions on X with values in M.

$$\mathcal{O}_X^M(U) = \{f : U \to M | f \text{ continuous}\} \quad \text{where} \quad U \subset X$$

Remark. Suppose S is a discrete topological space. We have a natural bijection for every open subset U of X:

$$\mathrm{Hom}(\pi_0 U, S) \cong \mathcal{O}_X^S(U)$$

Which basically says a map to a discrete space from U is determined by a family of maps on connected components of U.

Example 1.2.4 (*Sheaf of holomorphic functions*). We can go beyond the sheaf of local continuous functions and put more structure on outgoing maps from open subsets of X. Suppose U is an open subset of $\mathbb{C}^n$. The following defines a the sheaf of holomorphic functions on $\mathbb{C}^n$:

$$H(U) = \{h : U \to \mathbb{C} |\ h : \text{holomorphic}\}$$

Definition 1.2.5. A continuous map $p : Y \to X$ is a ***bundle*** over X. Also if p is surjective then we call p a *fibration* of spaces. The set $p^{-1}x$ is called the fibre over x.

Example 1.2.6. Another important class of sheaves is sheaf of local sections of a bundle. Suppose $p : Y \to X$ is a bundle. For every open subset U of X, define

$$\Gamma_p(U) = \{s : U \to Y | s \text{ continuous and } p \circ s = i_U\}$$

Where $i_U : U \to X$ is the inclusion map. In other words, $\Gamma(U)$ consists of all of maps s which render following digram commutative:

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ U \arrow[ur, "s"] \arrow[r, "i"'] & X \end{tikzcd}$$

Warning! It's worth emphasising that the sheaf condition does not say that every local section extends to a global section.

Proposition 1.2.7. The sheaf $\mathcal{O}_X^Y$ and the sheaf of local sections Γ_{π_2}, where $\pi_2 : Y \times X \to X$ is the projection map, are isomorphic.

Proof. Let $U \xrightarrow{s} Y$ be an element of $\mathcal{O}_X^Y(U)$. Then the following diagram is commutative.

$$\begin{tikzcd} & Y \times X \arrow[d, "\pi_2"] \\ U \arrow[ur, "<s,i>"] \arrow[r, "i"'] & X \end{tikzcd}$$

And $< s, i >$ is the corresponding element in $\Gamma_{\pi_2}(U)$. Also, if t is section in $\Gamma_{\pi_2}(U)$, then the corresponding section in $\mathcal{O}_X^Y$ is $\pi_1 \circ t$ and these two maps form a bijection of sets: $\mathcal{O}_X^Y(U) \cong \Gamma_{\pi_2}(U)$.

EXAMPLE *1.2.8.* *(Skyscraper sheaf)* Suppose A is a set and (X, x) is in $\mathbf{Top}_*$. Define

$$S^x(U) = \begin{cases} A & \text{, if } x \in U \\ \{*\} & \text{, otherwise} \end{cases}$$

1.3 Étale maps and étale space of a presheaf

First, let's start with the definition of étale bundle.

DEFINITION *1.3.1.* A bundle $p : Y \to X$ is called an ***étale bundle*** whenever for every $y \in Y$ there exists an open neighbourhood U_y containing the point y such that $p(U_y)$ is open in X and $p\big|_{U_y} : U_y \cong p(U_y)$ is a homeomorphism of spaces.

The idea here is basically to construct, for every presheaf $\mathcal{F}$ on X, an étale bundle over X in a canonical way. Often times, the space $\mathrm{Et}(\mathcal{F})$ over X constructed in this way in non-Hausdorff. Then we use the sheaf of sections for this étale bundle as sheafification of the presheaf $\mathcal{F}$.

$$\begin{tikzcd} & \mathrm{Et}(\mathcal{F}) \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, "i"'] & X \end{tikzcd}$$

Let $\Gamma_\pi^{\mathrm{Et}(\mathcal{F})}$ be the sheaf of sections for the bundle π. Then consider the functor which assigns to a presheaf $\mathcal{F}$ the sheaf $\Gamma_\pi^{\mathrm{Et}(\mathcal{F})}$. In fact, it can be proved that this functor is left adjoint to inclusion functor $i : \mathbf{Sh}(X) \to \mathbf{Psh}(X)$ there is the unit of an adjunction, and the unit of adjunction is called *sheafification*.

Details of this construction can be found in [9][Chapter 2]. Also, sections 5.1 and 5.2 in [15] deals explicitly with the connection between sheaves and étale bundles in the context of geometric logic.

DEFINITION *1.3.2.* A sheaf P on a topological space X is ***locally constant*** if there exists an open cover of X such that the restriction of P to each open set in the cover is a constant sheaf.

If the space X is locally connected, locally constant sheaves on X are, up to isomorphism, precisely the sheaves of sections of covering spaces $\pi : \mathrm{Et}(P) \to X$.

Such a locally constant sheaf is a constant sheaf if and only the covering π is trivial. So any non trivial covering will give you a non-constant but locally constant sheaf. A simple example is the sheaf of sections of the two sheeted non trivial covering of the unit circle $S^1 \to S^1 : e^{i\theta} \mapsto e^{2i\theta}$

2

Basics of simplicial homotopy theory

Introduction:

This chapter is a minimalistic account of abstract homotopy theory in which we introduce notions of simplicial category, geometric realisation, nerve construction, homotopies of simplicial sets, Kan complexes, and their fibrations. We will come back to an abstract notion of fibrations between categories in chapter 5 which has strong affinity to Kan fibration introduced here. The last section is a brief introduction Quillen model categories. The major references are [5], [6].

2.1 Simplicial category

In this section, we briefly sketch the basic ingredients of simplicial homotopy theory.

Definition 2.1.1. The *augmented simplicial category* Δ_+ whose objects are finite ordinals (possibly empty) $\mathbf{n} = \{0, 1, \ldots, n-1\}$ and whose morphims are order-preserving maps.

Remark. The object $\mathbf{0}$ is initial and the object $\mathbf{1}$ is terminal in the category Δ_+. Also, Δ_+ is a strict monoidal category. The tensor product is given by addition of ordinals:

$$+ : \Delta_+ \times \Delta_+ \to \ \Delta_+$$

$\mathbf{n} + \mathbf{m} = \{0, \ldots, n-1\} + \{0, \ldots, m-1\} := \{0, 1, \ldots, n+m-1\}$
Also, if $f : \mathbf{n} \to \mathbf{n}'$ and $g : \mathbf{m} \to \mathbf{m}'$, then $f + g : \mathbf{n} + \mathbf{m} \to \mathbf{n}' + \mathbf{m}'$.
where

$$(f+g)(i) = \begin{cases} f(i) & \text{, if } i = 0, 1, \ldots, n-1 \\ (n') + g(i-n) & \text{, otherwise} \end{cases}$$

With this structure $(\Delta_+, +, \mathbf{0})$ is a strict monoidal category. Let $\sigma_0 : \mathbf{1} + \mathbf{1} \to \mathbf{1}$ and $\delta_0 : \mathbf{0} \to \mathbf{1}$ be the unique morphisms to terminal object $\mathbf{1}$. $< \mathbf{1}, \sigma_0, \delta_0 >$ forms a monoid in the monoidal category Δ_+.

Moreover, it's universal in the following sense: Given a monoid $< M, \mu, \eta >$ in a strict monoidal category $(\mathcal{B}, \otimes, I)$ there is a unique strict monoidal functor $F : (\Delta_+, +, \mathbf{0}) \to (\mathcal{B}, \otimes, I)$ such that $F(\mathbf{1}) = M$, $F\sigma_0 = \mu$, and $F\delta_0 = \eta$.

$$\begin{tikzcd}
\mathbf{0} \arrow[r, "\delta_0"] \arrow[d, dashed, mapsto] & \mathbf{1} \arrow[d, dashed, mapsto] & \mathbf{1}+\mathbf{1} \arrow[l, "\sigma_0"'] \arrow[d, dashed, mapsto] \\
I \arrow[r, "\eta"'] & M & M \otimes M \arrow[l, "\mu"]
\end{tikzcd}$$

Definition 2.1.2. There are exactly $n+1$ injective monotone $f : \mathbf{n} \to \mathbf{n+1}$. We denote them by δ_i where $i = 0, \ldots, n$, and we call them *coface* maps. $\delta_i : \mathbf{n} \to \mathbf{n+1}$ skips $i \in \mathbf{n+1}$. They form the following δ-complex:

$$\mathbf{0} \longrightarrow \mathbf{1} \rightrightarrows \mathbf{2} \Rrightarrow \mathbf{3} \ \overrightarrow{\Rrightarrow}\ \mathbf{4} \qquad \cdots$$

There are exactly n surjective monotone $g : \mathbf{n+1} \to \mathbf{n}$. We denote them by σ_j where $j = 0, \ldots, n-1$ and we call them *codegeneracy* maps. $\sigma_j : \mathbf{n+1} \to \mathbf{n}$ sends both $j, j+1 \in \mathbf{n+1}$ to $j \in \mathbf{n}$. They form the following σ-complex:

$$\mathbf{1} \longleftarrow \mathbf{2} \leftleftarrows \mathbf{3} \Lleftarrow \mathbf{4} \cdots$$

Remark. Putting together coface and codegeneracy morphisms, we get the following complex:

$$\begin{tikzcd}
\mathbf{0} \arrow[r, "\delta_0"] & \mathbf{1} \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] & \mathbf{2} \arrow[l, "\sigma_0" description] \arrow[r, shift left=4, "\delta_0"] \arrow[r, "\delta_1" description] \arrow[r, shift right=4, "\delta_2"'] & \mathbf{3} \arrow[l, shift right=2, "\sigma_0"'] \arrow[l, shift left=2, "\sigma_1"] \cdots \in \Delta_+ .
\end{tikzcd} \tag{2.1}$$

For any n, viewing ordinal $\mathbf{n}$ as a category $\{0 \leq 1 \leq \ldots \leq n-1\}$, we observe that morphisms of Δ_+ become functors, and furthermore, we have the following chain of adjoint functors $\mathbf{n} \to \mathbf{n}+\mathbf{1}$ in 2.1:

$$\delta_{n+1} \dashv \sigma_n \dashv \delta_n \dashv \ldots \sigma_0 \dashv \delta_0$$

where the unit of $\delta_{i+1} \dashv \sigma_i$ and the counit of $\sigma_i \dashv \delta_i$ are identities.

Proposition 2.1.3. Any morphism in Δ_+ can be written as a composition of δ_i's and σ_j's. More precisely, an arrow $f : \mathbf{n} \to \mathbf{n}'$ has a unique decomposition as follows:

$$f = \delta_{i_1} \circ \ldots \circ \delta_{i_k} \circ \sigma_{j_1} \circ \ldots \circ \sigma_{j_l}$$

where

$$n' > i_1 > \ldots > i_k \geq 0$$

$$0 \leq j_1 < \ldots < j_l < n-1$$

and

$$n' = n - l + k$$

Definition 2.1.4. The *simplicial category* Δ is the full subcategory of Δ_+ whose objects are all the positive ordinals. So, we have the embedding $i : \Delta \hookrightarrow \Delta_+$. Category Δ does not possess an initial object, but it still has the terminal object $\mathbf{1}$.

In order to have a compatible notation with that of topologists, we introduce a shifting functor $[-] : \Delta_+ \to \Delta$, where $[\mathbf{n}] := \mathbf{1} + \mathbf{n}$, and $[f] := 1 + f$ for any morphism f in Δ_+. For instance, $[\mathbf{0}] = \{0\}$, $[\mathbf{1}] = \{0, 1\}$, etc. which are meant to capture the topological concepts of 0-cells (points), 1-cells (paths), etc. universally.

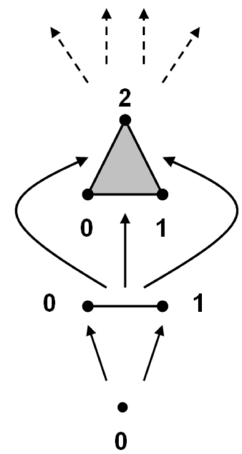


Figure 2.1.1: *A geometric illustration of coface maps of simplicial category* Δ

Remark. From now on, our standard notation for objects of category Δ is $[\mathbf{n}] = \{0, 1, \ldots, n\}$ $(n \geq 0)$.

As we have seen already, coface and codegeneracy maps generate all morphisms in the simplicial category. Furthermore, they satisfy certain relations which is expresses as commutativity of following diagrams:

$$\begin{tikzcd} & {[n]} \arrow[dr, "\delta_j^{n+1}"] & \\ {[n-1]} \arrow[ur, "\delta_i^n"] \arrow[dr, "\delta_{j-1}^n"'] & & {[n+1]} \\ & {[n]} \arrow[ur, "\delta_i^{n+1}"'] & \end{tikzcd}$$

with

$$0 \leq i < j \leq n+1$$

$$\begin{tikzcd}[column sep=small]
& {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\
{[n+1]} \arrow[ur, "\sigma_i^n"] \arrow[dr, "\sigma_{j+1}^n"'] & & {[n-1]} \\
& {[n]} \arrow[ur, "\sigma_i^{n-1}"'] &
\end{tikzcd}$$

with

$$0 \leq i \leq j \leq n-1$$

$$\begin{tikzcd}[column sep=small]
& {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\
{[n-1]} \arrow[ur, "\delta_i^n"] \arrow[dr, "\sigma_j^{n-2}"'] & & {[n-1]} \\
& {[n-2]} \arrow[ur, "\delta_{i-1}^{n-1}"'] &
\end{tikzcd}$$

with

$$0 < j+1 < i \leq n$$

$$\begin{tikzcd}[column sep=small]
& {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\
{[n-1]} \arrow[ur, "\delta_i^n"] \arrow[dr, "\sigma_{j-1}^{n-2}"'] & & {[n-1]} \\
& {[n-2]} \arrow[ur, "\delta_i^{n-1}"'] &
\end{tikzcd}$$

with

$$0 \leq i < j \leq n-1$$

And finally,

$$\begin{tikzcd}
 & {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\
{[n-1]} \arrow[ur, "\delta_j^n"] \arrow[rr, "\mathrm{id}_{[n-1]}"] \arrow[dr, "\delta_{j+1}^n"'] & & {[n-1]} \\
 & {[n]} \arrow[ur, "\sigma_j^{n-1}"'] &
\end{tikzcd}$$

with

$$0 \leq j \leq n-1$$

Proposition 2.1.5. The opposite category of augmented simplicial category Δ_+ is isomorphic to the category of non-empty ordinals and first-and-last-element preserving order-preserving functions. In other words, Δ_+^{op} is generated by the complex obtained by leaving off the top and bottom coface morphisms δ_0 and δ_n at each level in 2.1:

$$\mathbf{1} \xleftarrow{\sigma_0} \mathbf{2} \underset{\xleftarrow{\sigma_1}}{\overset{\xleftarrow{\sigma_0}}{\xrightarrow{\delta_1}}} \mathbf{3} \ \cdots \ \in \Delta_+.$$

Or equivalently,

$$[\mathbf{0}] \xleftarrow{\sigma_0} [\mathbf{1}] \underset{\xleftarrow{\sigma_1}}{\overset{\xleftarrow{\sigma_0}}{\xrightarrow{\delta_1}}} [\mathbf{2}] \ \cdots \ \in \Delta.$$

And, Δ^{op} is generated by complex of the form below:

$$\bullet \ \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \ \bullet \ \begin{array}{c}\longleftarrow\\[-6pt]\longrightarrow\\[-6pt]\longleftarrow\\[-6pt]\longrightarrow\\[-6pt]\longleftarrow\end{array} \ \bullet \ \cdots$$

We will soon see in the section of nerve construction that the left top and bottom arrows correspond to domain and codomain maps of 1-cells, the left middle arrow to identity map of 0-cells, and the right middle arrow to the composition of composable 1-cells.

Definition 2.1.6. The presheaf category $\mathbf{Psh}(\Delta) = \mathbf{Set}^{\Delta^{op}}$ is called ***the category of simplicial sets*** and its objects are called *simplicial sets*. Our notation for the category of simplicial sets will be *s***Set**.

- Given a simplicial object X in $\mathcal{C}$ we obtain a sequence of objects $X_n = X[n]$ endowed with the morphisms $d_j = X(\delta_j) : X_n \to X_{n-1}$ and $s_j = X(\sigma_j)$:

$X_n \to X_{n+1}$. These morphisms satisfy the dual of the relations between cofaces and codegeneracies. So, we call d_i and s_j face maps and degeneracy maps respectively.

- A morphism between simplicial objects X and X' is basically a natural transformation between them.
- Elements of sets X_n are called n-cells of X.

We can define ***simplicial objects*** in a similar way by changing the codomain category **Set** with other structured categories.

Remark. Generally, $s\mathcal{A}$ denotes the category of simplicial objects in a category $\mathcal{A}$.

- $s^2\mathbf{Set} = s(s\mathbf{Set})$ is the category of bisimplicial sets.
- $s\mathbf{Grp}$ is the category of simplicial groups.
- $s\mathbf{Ring}$ is the category of simplicial rings.
- etc.

Proposition 2.1.7 (*Density theorem*). Every simplicial set is canonically a colimit of standard simplices. That is:

$$\varinjlim \Delta^n \cong X$$

where $\Delta^n = y[n]$ is a representable simplicial set.

Proof. Apply Grothendieck construction to the simplicial category Δ and the category $\mathbf{Set}$ and you have:

$$\varinjlim_{x \in \mathbf{el}(X)} \Delta^n \cong X$$

$\square$

2.2 Nerve and geometric realisation

We define the nerve of small categories and geometric realisation of simplicial sets. The nerve construction gives us a functor from $\mathfrak{Cat}$ to $s\mathbf{Set}$. Simplicial sets which are constructed as a nerve of small category are precisely the ones with a unique inner horn filling property.

Definition 2.2.1. Suppose $\mathcal{C}$ is any small category. Define the ***nerve*** of $\mathcal{C}$ to be the simplicial set $N\mathcal{C}$ which has:

- vertices of $N\mathcal{C}$ as objects of $\mathcal{C}$
- edges of $N\mathcal{C}$ as arrows of $\mathcal{C}$
- faces of $N\mathcal{C}$ as pairs of composable arrows in $\mathcal{C}$, ...

More precisely,

$$N\mathcal{C}_n = \{\text{strings of composable arrows } c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n\}$$

The face map $d_i \colon N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ takes

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$$

to

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_{i-1}} c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} c_n$$

for $0 \leq i \leq n$ and leaves out first (resp. last arrow) for $i = 0$ (resp. $i = n$).

And the degeneracy map $s_i : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ sends

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$$

to

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_i} c_i \xrightarrow{id} c_i \xrightarrow{f_{i+1}} c_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} c_n$$

The following proposition is a direct corollary of Yoneda lemma.

Proposition 2.2.2. $N\mathcal{C}_n \cong \mathrm{Map}_{s\mathbf{Set}}(\Delta^n, N\mathcal{C})$

Example 2.2.3. Let G be a group, and let BG be the simplicial group defined as follows. Let $BG_n = G^{\times n}$, the product of G with itself n times. $G^{\times 0}$ is just the trivial group $\{e\}$. For an element $(g_1, \dots, g_n) \in BG_n$, let

$$\begin{aligned}
d_0(g_1, \dots, g_n) &= (g_2, \dots, g_n) \\
d_i(g_1, \dots, g_n) &= (g_1, \dots, g_i g_{i+1}, \dots g_n) \quad \text{if } 0 < i < n \\
d_n(g_1, \dots, g_n) &= (g_1, \dots, g_{n-1}) \\
s_i(g_1, \dots, g_n) &= (g_1, \dots, g_i, e, g_{i+1}, \dots, g_n).
\end{aligned}$$

This defines a simplicial group. It's obvious that the simplicial set BG (forget group structure) is the nerve of category $\mathcal{C}_G$ with exactly one object (let's call it G) and morphisms to be elements of G with composition as multiplication (in the opposite order). So, notationally, $BG = N(\mathcal{C}_G)$.

Definition 2.2.4 (*Left Kan extension*). Let $\mathcal{E}$ be a locally small, cocomplete category and $F : \Delta \to \mathcal{E}$ any functor. We define the functor $R : \mathcal{E} \to$ **sSet** by $(Re)_n = \mathcal{E}(F[n], e)$. Then, Re is a simplicial set. R admits a left adjoint L which is obtained by the Grothendieck construction. The functor $L :$ **sSet** $\to \mathcal{E}$ is known as ***left Kan extension*** of F.

$$\begin{tikzcd} & \textbf{sSet} \arrow[dr, dotted, "L"] & \\ \Delta \arrow[ur, "y"] \arrow[rr, "F"'] & & \mathcal{E} \end{tikzcd}$$

L can be computed as a colimit:

$$LX = \int^n X_n.F[n]$$

Definition 2.2.5 (*Standard topological simplex*). We define the functor $\Delta : \Delta \to \mathbf{Top}$ as follows:

- On objects: it sends $[n]$ to the standard (n-dim) topological simplex

$$\Delta_n = \{x_0e_0 + x_1e_1 + \ldots + x_ne_n | \sum_{0 \le i \le n} x_i = 1, x_i \geq 0\} \subseteq \mathbb{R}^{n+1}$$

- On morphisms: it sends $\alpha : [n] \to [m]$ to

$$\Delta_\alpha(p) = \sum_{0 \le i \le n} t_i e_{\alpha(i)}$$

where $p = t_0e_0 + t_1e_1 + \ldots + t_ne_n$ is a point of Δ^n

We look at the action of Δ on coface and codegeneracy maps: Let's set $\Delta(\delta_i) = d^i$ and $\Delta(\sigma_i) = s^i$.

Then

$$d^i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

Also,

$$s^i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, \ldots, t_{n+1})$$

And of course, this data is enough to know the action of the functor on every arrow of category Δ.

Definition 2.2.6 (*(Total) singular complex of a space*). Let Y be any topological space. We define a simplicial set SY by setting $SY_n = \mathbf{Top}(\Delta_n, Y)$ to be the set of continuous maps from the standard topological n-simplex to the space Y. In algebraic topology, we call elements of SY_n the n-simplices of Y.

SY together with the following face and degeneracy maps form a simplicial set which we call the total singular complex of Y.

$$d_i : \mathbf{Top}(\Delta_{n+1}, Y) \to \mathbf{Top}(\Delta_n, Y)$$

$$s_i : \mathbf{Top}(\Delta_{n-1}, Y) \to \mathbf{Top}(\Delta_n, Y)$$

- $d_i(p)$ is called i-th face of singular complex p.
- $s_i(q)$ is called degenerate singular n-simplex.

Now, we are ready to define the geometric realisation of a simplicial set. In Definition 10, let $\mathcal{E}$ be $\mathbf{Top}$ and F be $\Delta : \Delta \to \mathbf{Top}$, the standard simplex functor.

Definition 2.2.7 (*Geometric realisation*). The functor L which is the left adjoint to the total singular complex functor $S : \mathbf{Top} \to s\mathbf{Set}$ is called the geometric realisation of a (simplicial complex) and is denoted by $| - | : s\mathbf{Set} \to \mathbf{Top}$. It's computed on objects by

$$|X| = \int^n X_n \times \Delta_n$$

Example 2.2.8. Let G be a group and BG the nerve of category $\mathcal{C}_G$ with exactly one object (let's call it G) and morphisms to be elements of G with composition as multiplication. Then $|BG|$ is called the classifying space of the group G, and it can be shown that $|BG|$ is the classifying space for G-bundles meaning that there is a universal principal G-bundle

$p : EG \to |BG|$. That is every principal G-bundle $p : Y \to X$ is obtained by pulling back p along a morphism $f : X \to |BG|$.

More concisely, this says the following diagram is a pullback in $\mathbf{Top}_c$, where $\mathbf{Top}_c$ is the cartesian closed category of compactly generated spaces.

$$\begin{tikzcd} Y \arrow[r, "f^*"] \arrow[d, "p^*"'] & EG \arrow[d, "p"] \\ X \arrow[r, "f"'] & {|BG|} \end{tikzcd}$$

2.3 Homotopies of simplicial sets

Consider the simplicial sets $\Delta[0]$ and $\Delta[1]$. Recall that there are two morphisms

$$e_0, e_1 : \Delta[0] \longrightarrow \Delta[1],$$

coming from the morphisms $[0] \to [1]$ mapping 0 to an element of $[1] = \{0, 1\}$. Recall also that each set $\Delta[1]_k$ is finite. We can form the product

$$U \times \Delta[1]$$

for any simplicial set U. Note that $\Delta[0]$ has the property that $\Delta[0]_k = \{*\}$ is a singleton for all $k \geq 0$. Hence $U \times \Delta[0] = U$. Thus e_0, e_1 above gives rise to morphisms

$$e_0, e_1 : U \to U \times \Delta[1].$$

Suppose that U and V are two simplicial sets. Let $a, b : U \to V$ be two morphisms in $s\mathbf{Set}$.

1. We say a morphism
$$h : U \times \Delta[1] \longrightarrow V$$
is a *homotopy connecting* a *to* b if $a = h \circ e_0$ and $b = h \circ e_1$.

2. We say morphisms a and b are *homotopic* if there exists a homotopy connecting a to b or a homotopy connecting b to a.

For every $n \geq 0$ let us write

$$\Delta[1]_n = \{\alpha_0^n, \dots, \alpha_{n+1}^n\}$$

where $\alpha_i^n : [n] \to [1]$ is the map such that

$$\alpha_i^n(j) = \begin{cases} 0 & \text{if} \quad j < i \\ 1 & \text{if} \quad j \geq i \end{cases}$$

Thus

$$h_n : (U \times \Delta[1])_n = \coprod U_n \cdot \alpha_i^n \longrightarrow V_n$$

has a component $h_{n,i} : U_n \to V_n$ which is the restriction to the summand corresponding to α_i^n for all $i = 0, \dots, n+1$.

In the situation above, we have the following relations:

1. We have $h_{n,0} = b_n$ and $h_{n,n+1} = a_n$.
2. We have $d_j^n \circ h_{n,i} = h_{n-1,i-1} \circ d_j^n$ for $i > j$.
3. We have $d_j^n \circ h_{n,i} = h_{n-1,i} \circ d_j^n$ for $i \leq j$.
4. We have $s_j^n \circ h_{n,i} = h_{n+1,i+1} \circ s_j^n$ for $i > j$.
5. We have $s_j^n \circ h_{n,i} = h_{n+1,i} \circ s_j^n$ for $i \leq j$.

Conversely, given a system of maps $h_{n,i}$ satisfying the properties listed above, then these define a morphism h which is a homotopy between a and b.

2.4 Kan complexes

There is a very important full subcategory of $s\mathbf{Set}$ which plays a crucial role in abstract homotopy theory of simplicial sets. The idea of a Kan complex is a homotopical abstraction of the structures in the nerve of n-groupoids for all $n \leq \infty$. In this way, we can regard Kan complexes as a homotopical model for ∞-groupoids/homotopy types. In this section, we give a precise definition of Kan complexes and in the next section we will see that they are fibrant objects with respect to Quillen model structure on simplicial sets.

Definition 2.4.1. As a simplicial complex, the k-***th horn*** $|\Lambda^n_k|$ on the n-simplex $|\Delta^n|$ is the subcomplex of $|\Delta^n|$ obtained by removing the interior of $|\Delta^n|$ and the interior of the face $d_k\Delta^n$. We let Λ^n_k refer to the associated simplicial set.

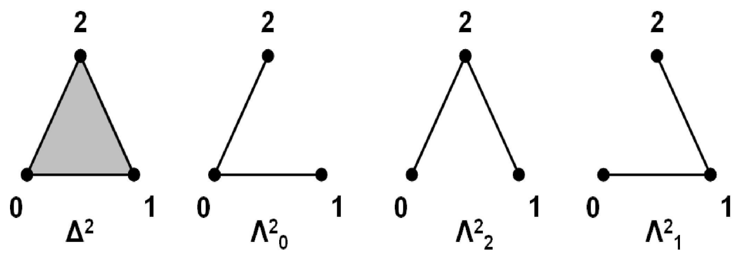


Figure 2.4.1: *The three horns of* $|\Delta|^2$

Definition 2.4.2. The simplicial object X satisfies the *extension condition* or *Kan condition* if any morphism of simplicial sets $\Lambda^n_k \to X$ can be extended to a simplicial morphism $\Delta^n \to X$. A simplicial set satisfying Kan extension condition is often called a ***Kan complex***.

- The standard simplices Δ^n, $n > 0$ are not Kan complexes.
- However, Δ^0 is a Kan complex.
- $N(\mathcal{C})$ is a Kan complex if and only if the category $\mathcal{C}$ is a groupoid.
- Any simplicial group is a Kan complex. [Thm 2.2. J.C.Moore seminars on algebraic homotopy theory]

Given a topological space Y, the simplicial set $S(Y)$ *does* satisfy the Kan extension condition. Consider any morphism of simplicial sets $f\colon \Lambda^n_k \to S(Y)$. This is the same as specifying for each $(n-1)$-face, $d_i\Delta^n$, $i \neq k$, of Δ^n a singular simplex $\sigma_i\colon |\Delta^{n-1}| \to Y$. Every other simplex of Λ^n_k is a face or a degeneracy of a face of one of these $(n-1)$-simplices, and so the rest of the map f is determined by this data. Furthermore, the compatibility conditions coming from the simplicial set axioms ensure that the topological maps σ_i piece together to yield, collectively, a continuous function $f\colon |\Lambda^n_k| \to Y$. This map extends to all of $|\Delta^n|$.

To get some intuition about the simplicial setting and simplicial (combinatorial) homotopy theory, we can compare it to classical (topological) homotopy theory:

Top. htpy theory	**Simp. htpy theory**
Interval $\mathbb{I}$	Δ^1
Topological space X	Simplicial set X
Continuous maps $X \xrightarrow{f} Y$	Simplicial maps $X \xrightarrow{\theta} Y$
Path: $\mathbb{I} \xrightarrow{p} X$	Path: $\Delta^1 \xrightarrow{p} X$
Initial point: $p(0)$	Initial point: $p \circ d_1$
Final point: $p(1)$	Final point: $p \circ d_0$
...	...

The following definition should seem natural now: Two 0-simplices a and b of the simplicial set X are said to be *in the same path component* of X if there is a path p with initial point a and final point b.

If X is a Kan complex, then "being in the same path component" is an equivalence relation. In this case, without any ambiguity, we can denote the set of path component of X by:

$$\pi_0 X = \text{the set of path components of } X.$$

2.5 Kan complexes as fibrant objects

As promised before, we will introduce the notion of Quillen model structure for categories and in particular, a Quillen model structure for the category of simplicial sets. The idea here is that categories endowed with Quillen model structures are nice enough to admit an abstract homotopy theory and inside these categories, talking about path object, cylinder object and homotopy equivalence of maps make sense. For our purposes, there are two examples of Quillen model structures which we will be interested in: Quillen model structure on category $\mathbf{Top}_c$ (category of compactly generated spaces) and $s\mathbf{Set}$ (category of simplicial sets).

Definition 2.5.1. The category $\mathcal{K} = (\mathcal{C}, \mathcal{F}, \mathcal{W})$ with finite limits and colimits endowed with three classes of morphism (named cofibrations, fibrations, and weak equivalences resp.) satisfying the following axioms is called a ***Quillen model category***.

- (2-of-3) For any two composable maps $f : X \to Y$ and $g : Y \to Z$, if any two of $\{f, g, g \circ f\} \in \mathcal{W}$ then so is the third.

- (Stable under retract) If the diagram below is commutative and in addition $u \circ i = id_X$ and $v \circ j = id_Y$ (under this condition, we say f is a retract of g),

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & X' \arrow[r, "u"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "j"] & Y' \arrow[r, "v"] & Y \end{tikzcd}$$

  Then if $g \in \{\mathcal{C}/\mathcal{F}/\mathcal{W}\}$ then so is f. So, cofibrations, fibrations, and weak equivalences are stable under retract.

- (Lifting) Suppose i is a cofibration and p is a fibration. Then the following commutative diagram has a diagonal filler (or lifting) whenever either of i or p is a weak equivalence.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

- (Factorisation) Any map f in $\mathcal{K}$ can be factored in each of two ways:
  (1) j a trivial cofibration and p a fibration
  (2) j a cofibration and p a trivial fibration

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

  where a map is called a *trivial fibration* when it is a fibration and a weak equivalence and it is called a *trivial cofibration* when it is a cofibration and weak equivalence.

Definition 2.5.2. An object X in $\mathcal{K}$ is called a ***fibrant object*** if the unique map $X \to 1$ is a fibration and a *cofibrant object* if the unique map $0 \to X$ is a cofibration.

Definition 2.5.3 (*Path object*). Let $\mathcal{K}$ be a Quillen model category and X an object of $\mathcal{K}$ Let the following be a factorisation of the diagonal map into a trivial cofibration j followed by a fibration p.

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "\Delta"'] & & P \arrow[dl, "p"] \\ & X \times X & \end{tikzcd}$$

We call (P, j, p) a *path object* for X.

DEFINITION 2.5.4. Suppose X and Y are cofibrant and fibrant objects in a Quillen model category $\mathcal{K}$ resp. Also, suppose we have two maps $f, g : X \to Y$ between them. We say f and g are ***homotopic*** if there is a path object (P, j, p) for Y and a commutative diagram as in the below:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "<f,g>"'] & & P \arrow[dl, "p"] \\ & Y \times Y & \end{tikzcd}$$

The map h exhibits a *right homotopy* between f and g.

In what follows we give two important examples of model structures; a model structure for category of compactly generated topological spaces and a model structure for the category of simplicial sets.

EXAMPLE 2.5.5 (*Quillen model structure for* $\mathbf{Top}_c$). The category of weakly Hausdorff compactly generated spaces can be equipped with a Quillen model structure:

- Fibrations are *Serre fibrations*.
- Cofibrations are the maps with *left lifting property* with respect to trivial fibrations.
- Weak equivalences are weak homotopy equivalences.
- cofibrant objects are retract of relative cell complexes (e.g. CW-complexes.)
- The canonical path object of a (compactly generated) topological space X is $[I, X]$.
- Every object is fibrant.
- Homotopy of maps is ordinary notion of homotopy for topological spaces.

Note Recall that Serre fibrations are maps $p : E \to X$ with covering homotopy property, i.e. for any commutative square with i being inclusion there exist a diagonal filler which lifts n-dimensional homotopies (paths, homotopies, etc) in X to the ones in E.

$$\begin{tikzcd} \Delta^n \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ \Delta^n \times \mathbb{I} \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

where $i(x) = (x, 0)$ for every $x \in \Delta^n$. A lot more details on Quillen model structure on $\mathbf{Top}_c$ can be found in [9].

EXAMPLE 2.5.6. **Quillen model structure for $s\mathbf{Set}$** The category of simplicial sets can be equipped with a Quillen model structure:

- Fibrations are *Kan fibrations*.
- Cofibrations are monomorphism. (level-wise injections)
- Weak equivalences are homotopy equivalences of their corresponding geometric realisations.
- fibrant objects are Kan complexes (section **??**).
- Path object of X is obtained by copowering of X with the following diagram:

$$\Delta[0] \coprod \Delta[0] \overset{d_0, d_1}{\hookrightarrow} \Delta[1] \overset{\simeq}{\to} \Delta[0]$$

- Every object is cofibrant.
- Homotopy of maps is homotopy of maps of simplicial sets introduced in section 2.3

Note Recall that Kan fibrations are maps $p : E \to X$ with right lifting property i.e. for any commutative square with i being inclusion there exist a diagonal filler which lifts n-dimensional simplices in X to the ones in E to fill the horns in E.

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

3
Localization and homotopy category

Introduction:

In this chapter, we will introduce the notion of localization of categories and will provide a setting in which both homotopy 2-category of spaces (e.g. topological spaces, simplicial sets, quasi-categories, etc) and sheafification arise as localization at a proper class of morphism, called localizing class, in suitable categories.

3.1 Localizing classes

Definition 3.1.1. Let $\mathcal{A}$ be a category. Also, suppose $\mathcal{W}$ is a wide subcategory of $\mathcal{A}$, i.e. every object of $\mathcal{A}$ is in $\mathcal{W}$.

An span

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

where $s \in mor(\mathcal{W})$ is called a *left roof*, and a co-span

$$\begin{tikzcd} & K & \\ M \arrow[ur, "g"] & & N \arrow[ul, "t"'] \end{tikzcd}$$

is called a *right roof*.

DEFINITION 3.1.2. Suppose $\mathcal{W}$ is a wide subcategory of $\mathcal{A}$. We call $(\mathcal{A}, \mathcal{W})$ a calculus of fractions whenever the following are satisfied:

Right Ore condition Every right roof can be completed by a left roof to a commutative square. More explicitly,for every morphism of g of $\mathcal{A}$ and every morphism t in $\mathcal{W}$ there exist a morphism f of $\mathcal{A}$ and s of $\mathcal{W}$ which make the diagram below commutative:

$$\begin{tikzcd} & L \arrow[dl, dotted, "s"'] \arrow[dr, dotted, "f"] & \\ M \arrow[dr, "g"'] & & N \arrow[dl, "t"] \\ & K & \end{tikzcd}$$

Left Ore condition Every left roof can be completed by a right roof to a commutative square. that is for every morphism of f of $\mathcal{A}$ and every morphism s in $\mathcal{W}$ there exist morphisms g of $\mathcal{A}$ and t of $\mathcal{W}$ which make the diagram below commutative:

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M \arrow[dr, dotted, "g"'] & & N \arrow[dl, dotted, "t"] \\ & K & \end{tikzcd}$$

- For any pair of morphisms $f, g : M \to N$ there is a morphism s in $\mathcal{W}$ such that $s \circ f = s \circ g$ iff there is a morphism t in $\mathcal{W}$ such that $f \circ t = g \circ t$.

Remark. Sometimes $\mathcal{W}$ is referred to as a localizing class for $\mathcal{A}$.

Remark. Going from $(\mathcal{A}, \mathcal{W})$ to $(\mathcal{A}^{\mathrm{op}}, \mathcal{W}^{\mathrm{op}})$, we can switch between left roofs of $\mathcal{A}$ and right roofs of $\mathcal{A}^{\mathrm{op}}$. There are rather trivial duality theorems corresponding to this switch which basically permit us to only work with left roofs and deduce dual results for the right ones. Based on this observation, we only work with left roofs afterwards.

Remark. Right Ore condition gives us a way of generating a left roof from a right roof and left Ore condition gives us a way of generating a right roof from a left one. However, the choices are not unique. We will soon fix this problem by introducing a notion of equivalence between left roofs.

The goal now is to make a category out of left roofs which should formalise fractions in categories. For this purpose, view a ring R as an additive category ΣR with exactly one object and morphisms the elements of ring R, whereby composition of morphisms given by multiplication in the ring. Moreover, the enrichment structure over $\mathcal{Ab}$ comes from addition operation in R. In this scenario, if we take $\mathcal{W}$ to be the set of non-zero elements of the ring, we arrive at the fields of fractions of the ring by formally inverting non-zero elements and taking their equivalence classes $\frac{f}{s}$. We can represent the fraction by a left roof.

$$\begin{tikzcd} & * \arrow[dl, "s"'] \arrow[dr, "f"] & \\ * & & * \end{tikzcd}$$

At this stage, we introduce composition of left roofs:
For two left roofs

$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "f"] & & & B \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & Y & & Z \end{tikzcd}$$

with $s, t \in \mathcal{W}$ and $f, g \in \mathcal{A}$, by right Ore condition, we can complete the right roof $\langle f, Y, t \rangle$ to a commutative square by a left roof $\langle u, C, h \rangle$ and we define the composition as the left roof $\langle su, C, gh \rangle$

$$
\begin{tikzcd}
& & C \arrow[dl, dotted, "u"'] \arrow[dr, dotted, "h"] & & \\
& A \arrow[dl, "s"'] \arrow[dr, "f"] & & B \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}
$$

We wish to get a composition mapping $Roof(X, Y) \times Roof(Y, Z) \to Roof(X, Z)$ out of this procedure; however, the filling square depends on the choice of C, u and h. To fix this problem, we have to identify all such left roofs. For this purpose we need a notion of equivalence between left roofs.

3.2 Equivalence of roofs

In this section we introduce a notion of equivalence relation between left roofs which will solve two of the problems we encountered in previous section.

Definition *3.2.1.* Suppose $\langle s, A, f\rangle$ and $\langle t, B, g\rangle$ are two left roofs ,in the category $\mathcal{A}$, represented by

$$
\begin{tikzcd}
& A \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y
\end{tikzcd}
$$

A *relation* H between $\langle s, A, f\rangle$ and $\langle t, B, g\rangle$ is an object H with two morphisms $p : H \to A$ and $q : H \to B$ such that $sp = tq$ is in $\mathcal{W}$ and moreover the following diagram commutes, i.e. $fp = gq$.

$$
\begin{tikzcd}
& H \arrow[ddl, "l"'] \arrow[d, dotted, "p"] \arrow[ddd, "q"] \arrow[ddr, "r"] & \\
& A \arrow[dl, dotted, "s"'] \arrow[dr, dotted, "f"] & \\
X & & Y \\
& B \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}
\tag{3.1}
$$

Note that the diagram 5.15 commutes and we can think of $\langle l, H, r\rangle$ as a left roof which is a common refinement of left roofs $\langle s, A, f\rangle$ and $\langle t, B, g\rangle$.

Now, suppose $F : \mathcal{A} \to \mathcal{B}$ is a functor which transports diagram 5.15 into a commutative diagram in $\mathcal{B}$. Moreover, assume F sends all of morphisms in $\mathcal{W}$ to invertible morphisms in $\mathcal{B}$, i.e. $F(s)$ is an isomorphism in $\mathcal{B}$. Then, we have

$$F(t) \circ F(q) = F(tq) = F(sp) = F(s) \circ F(p)$$

Since $F(s)$, $F(t)$,$F(sp)$ and $F(tq)$ are all invertible morphisms in $\mathcal{B}$, we conclude that $F(p)$ and $F(q)$ must also be invertible. In this case,

$$\begin{aligned} F(f) \circ F(s)^{-1} = F(f) \circ F(p) \circ F(p)^{-1} \circ F(s)^{-1} = \\ F(g) \circ F(q) \circ F(q)^{-1} \circ F(t)^{-1} = F(g) \circ F(t)^{-1} \end{aligned}$$

This shows the following diagram commutes in $\mathcal{B}$:

$$\begin{tikzcd}
 & F(A) \arrow[dl, "F(s)"'] \arrow[dr, "F(f)"] & \\
F(X) \arrow[ur, dotted, "F(s)^{-1}"'] \arrow[dr, dotted, "F(t)^{-1}"] & & F(Y) \\
 & F(B) \arrow[ul, "F(t)"] \arrow[ur, "F(g)"'] &
\end{tikzcd}$$

So, indeed F sends related left roofs to equal fractions in $\mathcal{B}$.

Proposition 3.2.2. The relation on the roofs is an equivalence relation.

Proof. It is a straightforward proof. □

Proposition 3.2.3. There is a category $\mathcal{A}[\mathcal{W}^{-1}]$ which has the objects of $\mathcal{A}$ as its objects and morphisms are the equivalence classes of left roofs. Composition is defined the same way as in section 3.1. except it is on equivalence classes of roofs instead of roofs themselves.

Let's see what the identity of composition is:
Suppose an equivalence class in $\mathrm{Map}_{\mathcal{A}[\mathcal{W}^{-1}]}(X, Y)$ is represented by the roof

$$\begin{tikzcd}
& A \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

Let's compose it with a an equivalence class represented by the trivial left roof on Y

$$\begin{tikzcd}
& Y \arrow[dl, "id"'] \arrow[dr, "id"] & \\
Y & & Y
\end{tikzcd}$$

the result is the left roof $\langle s, A, f\rangle$:

$$\begin{tikzcd}
& & A \arrow[dl, dotted, "id"'] \arrow[dr, dotted, "f"] & & \\
& A \arrow[dl, "s"'] \arrow[dr, "f"] & & Y \arrow[dl, "id"'] \arrow[dr, "id"] & \\
X & & Y & & Y
\end{tikzcd}$$

This proves $\langle id, Y, id\rangle$ is indeed an identity morphism on the right on Y. Similarly $\langle id, X, id\rangle$ is identity at the left on X.

Now, we have to check associativity of composition:

3.3 Localization functor

Proposition 3.3.1. Suppose $(\mathcal{A}, \mathcal{W})$ is a calculus of fractions. Then, there is a canonical functor $Q : \mathcal{A} \to \mathcal{A}[\mathcal{W}^{-1}]$ which is identity on objects and assign to each morphism $f : M \to N$ in $\mathcal{A}$ a morphism in $\mathcal{A}[\mathcal{W}^{-1}]$ represented by the following roof:

$$\begin{tikzcd}
& M \arrow[dl, "id"'] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd}$$

Proof. Note that $Q(id) = id$. Also, if $f : M \to N$ and $g : N \to P$ are two composable morphisms in $\mathcal{A}$ then their composition in $\mathcal{A}[\mathcal{W}^{-1}]$ is represented by the outer edges of following diagram:

$$
\begin{tikzcd}
& & M \arrow[dl, dotted, "id"'] \arrow[dr, dotted, "f"] & & \\
& M \arrow[dl, "id"'] \arrow[dr, "f"] & & N \arrow[dl, "id"'] \arrow[dr, "g"] & \\
M & & N & & P
\end{tikzcd}
$$

which represents $Q(g \circ f)$. Hence, $Q(g \circ f) = Q(g) \circ Q(f)$. $\square$

Proposition 3.3.2. The canonical quotient functor $Q : \mathcal{A} \to \mathcal{A}[\mathcal{W}^{-1}]$ makes the morphism in $\mathcal{W}$ invertible, and it is initial among all functors $F : \mathcal{A} \to \mathcal{B}$ which sends $mor(\mathcal{W})$ into invertible morphisms in $\mathcal{B}$. That is for any such functor F there is a unique functor G which render the following diagram commutative:

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathcal{A}[\mathcal{W}^{-1}] \arrow[d, dotted, "\exists ! G"] \\
& \mathcal{B}
\end{tikzcd}
\tag{3.2}
$$

Moreover,

$$(-) \circ Q : Fun(\mathcal{A}[\mathcal{W}^{-1}], \mathcal{B}) \to Fun(\mathcal{A}, \mathcal{B})$$

is fully faithful for every category $\mathcal{B}$.

Example 3.3.3. Suppose R is a commutative ring. Then ΣR is an additive category with exactly one object R. Take a prime ideal P of R. Then $(\Sigma R, \Sigma R - P)$ is a calculus of fractions and $Q : \Sigma R \to \Sigma R[\Sigma(R - P)^{-1}]$ is the well-known localization of ring R at a prime ideal P that we know from commutative algebra. The elements of $\Sigma R[\Sigma(R - P)^{-1}]$ are fraction of the form $\dfrac{a}{s}$ such that $s \notin P$, and two fractions $\dfrac{a}{s}$ and $\dfrac{b}{t}$ are equal whenever there are u and v in R such that $su = tv$ and $au = bv$, and this holds if and only if there exists some w in $R - P$ such that $w(at - bs) = 0$.

Example 3.3.4. Suppose M is a commutative monoid. We can consider $\mathcal{M}$ as a category with one object and composition is multiplication of the monoid M. Then $(\mathcal{M}, \mathcal{M})$ form a calculus of fractions and $\mathcal{M}[\mathcal{M}^{-1}]$ is isomorphic to the Grothendieck group of M that we know from K-theory. The most simple example is $\mathbb{N}[\mathbb{N}^{-1}] \cong \mathbb{Z}$.

Proposition 3.3.5. Let $(\mathcal{K}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ be a Quillen model category (e.g. $\mathbf{Top}_c$, $s\mathbf{Set}$, Quasi-categories, etc.) Let $\mathcal{K}_{fc}$ be the subcategory of fibrant-cofibrant objects in $\mathcal{K}$. Then,

$(\mathcal{K}_{fc}, \mathcal{W})$ admits a calculus of fractions and furthermore, the homotopy category $Ho(\mathcal{K})$ is equivalent to $\mathcal{K}_{fc}[\mathcal{W}^{-1}]$.

PROPOSITION 3.3.6. Let $(\mathcal{C}, \mathbb{J})$ be a site, and $\mathcal{W}$ the collection of $\mathbb{J}$-local isomorphism in $\mathbf{Psh}(\mathcal{C})$. Then $(\mathbf{Psh}(\mathcal{C}), \mathcal{W})$ admits a calculus of fractions and furthermore, $\mathbf{Psh}(\mathcal{C})[\mathcal{W}^{-1}] \cong \mathbf{Sh}(\mathcal{C}, \mathbb{J})$.

Proof. Insert the proof here!

$\square$

4

Basics of 2-Categories

Introduction:

In this chapter we give an introduction to the theory of 2-categories which will be an essential background for the next chapters. By no means, our account of 2-categories will be comprehensive. However, we will define and prove everything which will be used subsequently. A word on notations: throughout the rest of this thesis and particularly in this chapter, we will use $\mathcal{C}at$ for $\mathbf{1}$-category of (small) categories and functors, $\mathfrak{Cat}$ for the 2-category of categories, functors and natural transformations, and $\mathbf{2}\mathfrak{Cat}$ for the 3-category of 2-categories, strict 2-functors, 2-natural transformations, and modifications. In general, $\mathbf{n}\,\mathfrak{Cat}$ stands for (n+1)-category of n-categories.

4.1 What is a 2-category?

Definition *4.1.1.* A **2-category** is a $\mathcal{C}at$-enriched category, where $\mathfrak{Cat}$ is the category of small categories and functors.

Let's try and expand the above definition in more details and see what enrichments structure grants us.

Suppose $\mathcal{K}$ is a 2-category. Since $\mathcal{C}at$ is a cartesian closed category, and since $\mathcal{K}$ is enriched over $\mathfrak{Cat}$, we have that the following diagram commutes (associativity)

$$\begin{tikzcd}
\mathcal{K}(z,w) \times \mathcal{K}(y,z) \times \mathcal{K}(x,y) \arrow[r] \arrow[d] & \mathcal{K}(z,w) \times \mathcal{K}(x,z) \arrow[d] \\
\mathcal{K}(y,w) \times \mathcal{K}(x,y) \arrow[r] & \mathcal{K}(x,w)
\end{tikzcd}$$

This implies for any 1-cells (or morphisms) $f : x \to y$, $g : y \to z$, and $h : z \to w$ that $h \circ (g \circ f) = (h \circ g) \circ f$.

And for 2-cells α, β, and γ:

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
z \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\gamma"] & w
\end{tikzcd}$$

$\gamma(\beta\alpha) = (\gamma\beta)\alpha$.

Note that this composition of 2-cells comes from composition structure of enrichment given by tensor product. We call this horizontal composition of 2-cells, and occasionally may use $\beta \centerdot \alpha$ to be emphatic about composition of α and β being horizontal.

Also, from commutativity of the following diagrams, we conclude that

- $f \circ 1_x = f = 1_y \circ f$
- $\alpha \centerdot \tau_x = \alpha = \tau_y \centerdot \alpha$

for any 1-cell $f : x \to y$ and any 2-cell $\alpha : f \to g : x \to y$. Note that τ_x and τ_y are identity 2-cells between 1_x and 1_y, respectively.

If we use weaker conditions to express associativity of 1-cells and compositions with identity, we get the notion of a **bicategory**. We summarize the difference between 2-categories and bicategories in the following table:

2-category	Bicategory
associativity of composition of 1-cells $h(gf) = (hg)f$	an invertible 2-cell $\eta : h(gf) \to (hg)f$ natural in f, g, h
$f \circ 1_x = f = 1_y \circ f$	an invertible 2-cells $\theta : f \circ 1_x \to f$, $\gamma : 1_y \circ f \to f$

Definition 4.1.2. A 2-*functor* F between 2-categories $\mathcal{K}$ and $\mathcal{L}$ is a $\mathcal{C}at$-enriched functor.

4.1.1 A different definition of 2-Categories

We can organize the data of a 2-category in somewhat different way. This reorganization has few advantages:

- It makes definitions of functors and natural transformations naturally better understood.
- Coherence axioms become diagram chase and diagram commutativity.
- It's in the style of definition of higher categories (i.e. simplicial categories)
- It enables us to define an internal 2-category to any category with finite limits.

First we define an internal category inside a finitely complete category.

Definition 4.1.3. Suppose $\mathcal{E}$ is finitely complete category. An ***internal category*** $\mathbf{C}$ in $\mathcal{E}$ consists of following **data**:

- An object of *objects* $\mathbf{C}_0$ in $\mathcal{E}$
- An object of *morphisms* $\mathbf{C}_1$ in $\mathcal{E}$
- A source/domain and target/codomain morphisms $s, t : \mathbf{C}_1 \to \mathbf{C}_0$ in $\mathcal{E}$.
- A morphism (of identities) $i : \mathbf{C}_0 \to \mathbf{C}_1$ in $\mathcal{E}$.
- A composition morphism $\mu : \mathbf{C}_1 \,{}_t\!\times_s \mathbf{C}_1 \to \mathbf{C}_1$ in $\mathcal{E}$, where $\mathbf{C}_1 \,{}_t\!\times_s \mathbf{C}_1$ is the object of composable morphisms given by the following pullback

$$\begin{tikzcd} \mathbf{C}_1 \,{}_t\!\times_s \mathbf{C}_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{C}_1 \arrow[d, "s"] \\ \mathbf{C}_1 \arrow[r, "t"'] & \mathbf{C}_0 \end{tikzcd}$$

subject to the following **axioms:**

1. $s \circ i = id_{\mathbf{C}_0} = t \circ i$; that is diagram below commutes

$$
\begin{tikzcd}
\mathbf{C}_0 \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, "1"'] & \mathbf{C}_1 \arrow[d, "s"] \\
\mathbf{C}_1 \arrow[r, "t"'] & \mathbf{C}_0
\end{tikzcd}
$$

2. $t \circ \mu = t \circ \pi_1$ and $s \circ \mu = s \circ \pi_0$; that is diagram below commutes

$$
\begin{tikzcd}
\mathbf{C}_1 \arrow[d, "t"'] & \mathbf{C}_1 \,{}_t\!\times_s \mathbf{C}_1 \arrow[l, "\pi_1"'] \arrow[r, "\pi_0"] \arrow[d, "\mu"'] & \mathbf{C}_1 \arrow[d, "s"] \\
\mathbf{C}_0 & \mathbf{C}_1 \arrow[l, "t"] \arrow[r, "s"'] & \mathbf{C}_0
\end{tikzcd}
$$

3. The associativity law for composition of morphisms expressed by commutativity of diagram below

$$
\begin{tikzcd}
\mathbf{C}_1 \,{}_t\!\times_{s\pi_0} \mathbf{C}_1 \,{}_t\!\times_s \mathbf{C}_1 \arrow[r, "\mu \,{}_t\!\times_s 1"] \arrow[d, "1 \,{}_t\!\times_s \mu"'] & \mathbf{C}_1 \,{}_t\!\times_s \mathbf{C}_1 \arrow[d, "\mu"] \\
\mathbf{C}_1 \,{}_t\!\times_s \mathbf{C}_1 \arrow[r, "\mu"'] & \mathbf{C}_1
\end{tikzcd}
$$

4. The left and right unit laws for composition of morphisms expressed by commutativity of diagram below

$$
\begin{tikzcd}
\mathbf{C}_0 \,{}_1\!\times_s \mathbf{C}_1 \arrow[r, "i \,{}_1\!\times_s 1"] \arrow[dr, "p_0"'] & \mathbf{C}_1 \,{}_t\!\times_s \mathbf{C}_1 \arrow[d, "\mu"'] & \mathbf{C}_1 \,{}_t\!\times_s \mathbf{C}_0 \arrow[l, "1 \,{}_t\!\times_1 i"'] \arrow[dl, "p_1"] \\
& \mathbf{C}_1 &
\end{tikzcd}
$$

We use abbreviation $\mathbf{C}_2 := \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1$ in $\mathcal{E}$. $\mathbf{C}_2$ is the object of internal composable morphims. So, we can demonstrate an internal category by three objects and six morphisms between them

$$
\begin{tikzcd}
\mathbf{C}_0 \arrow[r, "i" description] & \mathbf{C}_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & \mathbf{C}_2 \arrow[l, "\mu" description] \arrow[l, shift right=2, "\pi_0"'] \arrow[l, shift left=2, "\pi_1"]
\end{tikzcd}
$$

Definition 4.1.4. Suppose $\mathcal{C}$ is a category with finite limits. We can define an ***internal 2-category*** $\mathcal{K}$ in $\mathcal{C}$ in the following way: The data for $\mathcal{K}$ consists of

- An object of *objects* $\mathcal{K}_0$ in $\mathcal{C}$
- An object of *morphisms* $\mathcal{K}_1$ in $\mathcal{C}$
- An object of *2-cells* $\mathcal{K}_2$ in $\mathcal{C}$
- The domain and codomain maps: $s_0, t_0 : \mathcal{K}_1 \to \mathcal{K}_0$, and also $s_1, t_1 : \mathcal{K}_2 \to \mathcal{K}_1$.
- Identity map $i : \mathcal{K}_0 \to \mathcal{K}_1$ on 0-cells and $\tau : \mathcal{K}_1 \to \mathcal{K}_2$ on 1-cells.
- Composition of 1-cells given by $m : \mathcal{K}_1 \times_{\mathcal{K}_0} \mathcal{K}_1 \to \mathcal{K}_1$ in $\mathcal{C}$, where the pullback is a pullback s_0, and t_0.
- Vertical composition of 2-cells by $\mu : \mathcal{K}_2 \times_{\mathcal{K}_1} \mathcal{K}_2 \to \mathcal{K}_2$, where the pullback is the pullback of s_1 and t_1.
- Right and left whiskering given by $\mu_r : \mathcal{K}_2 \times_{\mathcal{K}_0} \mathcal{K}_1 \to \mathcal{K}_2$ and $\mu_l : \mathcal{K}_1 \times_{\mathcal{K}_0} \mathcal{K}_2 \to \mathcal{K}_2$ where the pullbacks are got by pulling back s_0, $t_0 s_1$ and t_0, $s_0 s_1$.

So, a structure for an internal 2-category can be summarized in

$$
\begin{tikzcd}
\mathcal{K}_0 \arrow[r] & \mathcal{K}_1 \arrow[l, shift right=2] \arrow[l, shift left=2] & \mathcal{K}_2 \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[l, leftarrow]
\end{tikzcd}
$$

and morphisms m, μ, μ_l, μ_r.

Besides, we need to express the appropriate axioms for this data:

- $(\mathcal{K}_0, \mathcal{K}_1, s_0, t_0, i, m)$ form a category internal in $\mathcal{C}$.
- $(\mathcal{K}_1, \mathcal{K}_2, s_1, t_1, \tau, \mu)$ form a category internal in $\mathcal{C}$.
- For right and left whiskering we get following commutative diagrams:

$$
\begin{tikzcd}
\mathcal{K}_2 \times_{\mathcal{K}_0} \mathcal{K}_1 \arrow[r, shift left, "s_1 \times_{\mathcal{K}_0} id"] \arrow[r, shift right, "t_1 \times_{\mathcal{K}_0} id"'] \arrow[d, "\mu_r"] & \mathcal{K}_1 \times_{\mathcal{K}_0} \mathcal{K}_1 \arrow[d, "m"] \\
\mathcal{K}_2 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & \mathcal{K}_1
\end{tikzcd}
$$

$$\begin{tikzcd}
\mathcal{K}_1 \times_{\mathcal{K}_0} \mathcal{K}_2 \arrow[r, shift left, "id\times_{\mathcal{K}_1} s_1"] \arrow[r, shift right, "id\times_{\mathcal{K}_1} t_1"'] \arrow[d, "\mu_l"] & \mathcal{K}_1 \times_{\mathcal{K}_0} \mathcal{K}_1 \arrow[d, "m"] \\
\mathcal{K}_2 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & \mathcal{K}_1
\end{tikzcd}$$

- There is a right and left action of 1-cells on appropriate 2-cells by right and left whiskering and it is expressed as commutativity of diagrams below:

$$\begin{tikzcd}
\mathcal{K}_2 \times_{\mathcal{K}_0} \mathcal{K}_1 \times_{\mathcal{K}_0} \mathcal{K}_1 \arrow[r, "\mu_r \times_{\mathcal{K}_0} 1"] \arrow[d, "1\times_{\mathbf{C}_0}\mu"'] & \mathcal{K}_2 \times_{\mathcal{K}_0} \mathcal{K}_1 \arrow[d, "\mu_r"] \\
\mathcal{K}_2 \times_{\mathcal{K}_0} \mathcal{K}_1 \arrow[r, "\mu_r"] & \mathcal{K}_2
\end{tikzcd}$$

  Note that commutativity here implies: $(hg)\alpha = h(g\alpha)$ for the diagram

$$\begin{tikzcd}
x \arrow[r, bend left, "f_2"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Uparrow\alpha"] & y \arrow[r, "g"] & z \arrow[r, "h"] & w
\end{tikzcd}$$

  Similarly we need to other commutative diagrams similar to the one above to say: $(h\alpha)g = h(\alpha g)$ and $(\alpha)hg = (\alpha h)g$

Remark. Notice that by composition of whiskerings we can arrive at horizontal composition of 2-cells.

Definition 4.1.5. A *2-functor* between the 2-categories $\mathcal{K}$ and $\mathcal{L}$ internal in the category $\mathcal{C}$ consists of morphisms $F_0 : \mathcal{K}_0 \to \mathcal{L}_0$, $F_1 : \mathcal{K}_1 \to \mathcal{L}_1$, and $F_2 : \mathcal{K}_2 \to \mathcal{L}_2$ in $\mathcal{C}$ which map objects to objects , 1-cells to 1-cells and 2-cells to 2-cells such that they preserve composition and identity up to canonical invertible 2-cells.

Suppose $\mathcal{C}$ is a $\mathcal{V}$-enriched category and $h : \mathcal{V} \to \mathcal{V}'$ is a lax-monoidal functor. We can construct a $\mathcal{V}'$-enriched category $\mathcal{C}_h$ with:

- $\mathbf{Ob}(\mathcal{C}) := \mathbf{Ob}(\mathcal{C}_h)$
- $\mathcal{C}_h(x, y) := h(\mathcal{C}(x, y))$ for any pair of objects x, y in $\mathcal{C}$.

- Composition $I_{\mathcal{V}'} \to h(I_{\mathcal{V}}) \to h(\mathcal{C}(x,x)) = \mathcal{C}_h(x,x)$ in $\mathcal{V}'$ defines the unit map of $\mathcal{C}_h$.
- Composition $h(\mathcal{C}(x,y)) \times h(\mathcal{C}(y,z)) \to h(\mathcal{C}(x,y) \times \mathcal{C}(y,z)) \to h(\mathcal{C}(x,z))$ defines the composition map of $\mathcal{C}_h$.

Definition *4.1.6*. For a 2-category $\mathcal{K}$, representable functor $\mathrm{Hom}(1,-) : \mathcal{C}at \to \mathbf{Set}$, which sends a small category C to a set isomorphic to set of objects of C , induces an enriched functor from the $\mathfrak{Cat}$-enriched category $\mathcal{K}$ to a $\mathbf{Set}$-enriched category $\mathcal{K}_0$ which is called ***the underlying category*** of $\mathcal{K}$. We have:

- $Ob(\mathcal{K}_0) = Ob(\mathcal{K})$
- $\mathcal{K}_0(A,B) := \mathrm{Hom}(1, \mathcal{K}(A,B)) \cong Ob(\mathcal{K}(A,B))$

4.2 Examples of 2-categories and bicategories

Example *4.2.1*. Any topological space X can be made into a bicategory. The 0-cells are points of X, 1-cells are paths in X and 2-cells are homotopies between paths. Notice that in this way X is not a 2-category since associativity is up to isomorphism and not strict equality: for paths α, β, γ we have $\gamma \circ (\beta \circ \alpha) \cong (\gamma \circ \beta) \circ \alpha$. In fact, since homotopies and homotopies between homotopies are all invertible this is indeed a bigroupoid and is denoted by $\pi_{\leq 2}(X)$.

Example *4.2.2*. There is a bicategory of topological spaces. Here the 0-cells are spaces X, 1-cells are continuous maps $f : X \to Y$, and 2-cells are homotopies $H : f \to g$ between two maps f and g; more explicitly, homotopies are given by continuous functions $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for every $x \in X$. In a similar way, one constructs the bicategory of pointed-topological spaces.

Example *4.2.3*. Suppose $\mathcal{E}$ is finitely complete category. There is bicategory $\mathfrak{Cat}(\mathcal{E})$ of internal categories in $\mathcal{E}$, internal functors and natural transformations.

Example *4.2.4*. For any category $\mathcal{C}$ there is an associated span bicategory $\mathbf{Span}(\mathcal{C})$. Set of 0-cells is $\mathbf{Ob}(\mathcal{C})$, hom-set of 1-cells $\mathbf{Span}(\mathcal{C})(A,B)$ consists of spans between A and B, that is:

$$\begin{tikzcd} & S \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\ A & & B \end{tikzcd}$$

and a 2-cell between spans $\langle d_0, S, d_1\rangle$ and $\langle e_0, T, e_1\rangle$ is morphism $h : S \to T$ in $\mathcal{C}$ such that $e_i \circ h = d_i$ for $i = 1, 2$.

$$\begin{tikzcd}
 & S \arrow[dl, "d_0"'] \arrow[dr, "d_1"] \arrow[dd, "h"] & \\
A & & B \\
 & T \arrow[ul, "e_0"] \arrow[ur, "e_1"'] &
\end{tikzcd}$$

4.3 Comma categories and comma objects

We start from 1-category $\mathfrak{Cat}$:

Definition 4.3.1. Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories and $f : \mathcal{C} \to \mathcal{E}$ and $g : \mathcal{D} \to \mathcal{E}$ are functors between them. The **comma category** of f and g, denoted as $f \downarrow g$, has as its objects all triples (c, d, α) where $c \in Ob(\mathcal{C})$, $d \in Ob(\mathcal{D})$, and $\alpha : f(c) \to g(d)$ is an arrow in $\mathcal{E}$, and the set of morphisms between any two of these objects consists of pairs $(\gamma, \lambda) : (c, d, \alpha) \to (x, y, \beta)$ where $\gamma : c \to x$ in $\mathcal{C}$, $\lambda : d \to y$ in $\mathcal{D}$ such that the following square commutes in $\mathcal{E}$:

$$\begin{tikzcd}
f(c) \arrow[r, "\alpha"] \arrow[d, "f(\gamma)"'] & g(d) \arrow[d, "g(\lambda)"] \\
f(x) \arrow[r, "\beta"'] & g(y)
\end{tikzcd}$$

Remark. Note that we obtain forgetful functors $d_0 : f \downarrow g \to \mathcal{C}$ and $d_1 : f \downarrow g \to \mathcal{D}$ and a natural transformation $\theta : f \circ d_0 \Rightarrow g \circ d_1$, as shown in the diagram below:

$$\begin{tikzcd}
f \downarrow g \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\swArrow\theta"] & \mathcal{C} \arrow[d, "f"] \\
\mathcal{D} \arrow[r, "g"'] & \mathcal{E}
\end{tikzcd} \tag{4.1}$$

where $\theta_{\langle c,d,\alpha\rangle} = \alpha$. Moreover, $f \downarrow g$ is universal in the following sense: given a category $\mathcal{X}$ and and functors $u_0 : \mathcal{X} \to \mathcal{C}$ and $u_1 : \mathcal{X} \to \mathcal{D}$ together with a 2-cell $\delta : f \circ u_0 \Rightarrow g \circ u_1$, there is a unique functor $v : \mathcal{X} \to f \downarrow g$ such that $d_0 \circ v = u_0$, $d_1 \circ v = u_1$, and $\delta = \theta \circ \tau_v$:

$$\begin{tikzcd}
f \circ u_0 \arrow[rr, bend left, "\delta"] \arrow[dr, equal] & \arrow[d, phantom, "\|"] & g \circ u_1 \\
& f \circ d_0 \circ v \arrow[r, "\theta \centerdot v"'] & g \circ d_1 \circ v \arrow[u, equal]
\end{tikzcd}$$

Remark. The comma category above can also be realized as following pullback in the category $\mathfrak{Cat}$:

$$\begin{tikzcd}
f \downarrow g \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}^I \arrow[d, "d_0 \times d_1"] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "f \times g"'] & \mathcal{E} \times \mathcal{E}
\end{tikzcd}$$

.

Next, we construct slice categories as comma categories:

Example 4.3.2. For any two categories $\mathcal{E}$ and $\mathcal{B}$ and any functor $P : \mathcal{E} \to \mathcal{B}$, we denote by $\mathcal{B}/P$ the comma category of P, and $id_{\mathcal{B}}$.

$$\begin{tikzcd}
\mathcal{B}/P \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\overset{\theta}{\swarrow}"] & \mathcal{B} \arrow[d, "id"] \\
\mathcal{E} \arrow[r, "P"'] & \mathcal{B}
\end{tikzcd}$$

Its objects are the morphism $b \to P(e)$ and its arrows are commutative squares of the form

$$\begin{tikzcd}
b \arrow[r] \arrow[d] & P(e) \arrow[d] \\
b' \arrow[r] & P(e')
\end{tikzcd}$$

Setting $\mathcal{E} = 1$ and $P = B$ an object of $\mathcal{B}$, we obtain the slice category $\mathcal{B}/B$.

Example 4.3.3. One can regard the comma category $f \downarrow g$ as an object of category $\mathbf{Span}(\mathfrak{Cat})(\mathcal{C}, \mathcal{D})$ equipped with bijection

$$\mathbf{Span}(\mathfrak{Cat})(\mathcal{C},\mathcal{D})(\langle u_0,\mathcal{X},u_1\rangle,\langle d_0, f\downarrow g, d_1\rangle)\cong \mathbf{Fun}(\mathcal{C},\mathcal{D})(fu_0, gu_1)$$

and moreover, for any two 1-cells v and v', and $\gamma : d_0 \circ v \Rightarrow d_0 \circ v'$ and $\lambda : d_1 \circ v \Rightarrow d_1 \circ v'$ such that composites

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "v"] \arrow[d, "v'"'] \arrow[dr, phantom, "\Swarrow\lambda"] & f\downarrow g \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\Swarrow\theta"] & \mathcal{C} \arrow[d, "f"] \\
f\downarrow g \arrow[r, "d_1"'] & \mathcal{D} \arrow[r, "g"'] & \mathcal{E}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
f\downarrow g \arrow[r, "d_0"] \arrow[dr, phantom, "\Searrow\gamma"] & \mathcal{C} \arrow[r, "f"] \arrow[dr, phantom, "\Searrow\theta"] & \mathcal{E} \\
\mathcal{X} \arrow[u, "v"] \arrow[r, "v'"'] & f\downarrow g \arrow[u, "d_0"'] \arrow[r, "d_1"'] & \mathcal{D} \arrow[u, "g"']
\end{tikzcd}$$

are equal there exists a unique 2-cell $\alpha : v \Rightarrow v'$ such that $\gamma = d_0 \circ \alpha$ and $\lambda = d_1 \circ \alpha$.

4.4 Representability and 2-categorical (co)limits

In this section, we will talk about the importance of the notion of representability in 1-categorical and 2-categorical settings. We start with the following definition.

Definition *4.4.1*. A functor $F : \mathcal{C} \to \mathbf{Set}$ is **representable** whenever there is an object A in the category $\mathcal{C}$ with a natural isomorphism $\phi : F \cong \mathrm{Hom}(A, -)$. In this situation, we say F is represented by object A. A presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is **representable** when there is an object B in the category $\mathcal{C}$ with a natural isomorphism $\psi : P \cong \mathrm{Hom}(-, B)$. In this case, We say P is represented by object B.

Note. We usually use notations $k_A = \mathrm{Hom}(A, -)$ and $h_B = \mathrm{Hom}(-, B)$. The functors h and k are, respectively, Yoneda and dual-of-Yoneda embeddings.

Remark. By Yoneda lemma, the representing object is determined uniquely up to canonical isomorphism for a given representable functor (resp. presheaf).

There are many reasons why representable functors and representable presheaves are so important in category theory and higher category theory. Suppose we want to define a certain object such as a limit, colimit, exponential, etc in a given category $\mathcal{C}$. One elegant approach is to use representable functor (resp. presheaves) which has this desired object as its representing object. Yoneda lemma ensures us that this object will be unique up to canonical isomorphism.

EXAMPLE 4.4.2. As an example, fix a category $\mathcal{C}$ and two objects A and B. Take the functor $h_A \times h_B : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. If this functor is represented by object C, then $\mathrm{Hom}(X, C) \cong \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$, naturally in X. Now, if the binary product of A and B exists in $\mathcal{C}$, then this exactly gives the definition of product of A and B in $\mathcal{C}$, and moreover, $X \cong A \times B$. So, representabilty of the above-mentioned presheaf is equivalent to the existence of a product of A and B in $\mathcal{C}$. We can even start from this point and define products of two objects this way; the representing object, if it exists, for the functor $\mathrm{Hom}(-, A) \times \mathrm{Hom}(-, B) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

NOTE. Observe from above that $k_X = \mathrm{Hom}(X, -)$ preserves binary products, and in general all (small) limits, if they exist in category $\mathcal{C}$. In fact, if $\mathcal{J}$ is a small category and $D : \mathcal{J} \to \mathcal{C}$ is a diagram in $\mathcal{C}$, then

$$k_X(\lim_{\mathcal{J}} D) \cong \lim_{\mathcal{J}}(k_X D)$$

where the right-hand limit is computed in the category $\mathbf{Set}$.

Example 4.4.2 is an instance of a more general phenomenon. We can extend this to the general case of limits and colimits.

EXAMPLE 4.4.3. Suppose $D : \mathcal{J} \to \mathcal{C}$ is a diagram in the category $\mathcal{C}$. Note that the set of cones in $\mathcal{C}$ with the vertex A is exactly the set of natural transformations between the constant functor at one-point set $* : \mathcal{J} \to \mathbf{Set}$ and D, more formally, the set $\mathrm{Hom}(*, \mathcal{C}(A, D(-)))$. For a given cone $L \in \mathrm{Hom}(*, \mathcal{C}(A, D(-)))$ with vertex A and any map $f : j \to j'$ in $\mathcal{J}$, the commutativity of naturality square of L ensures the commutativity of the following triangle:

$$\begin{tikzcd} & A \arrow[dl, "L(j)"'] \arrow[dr, "L(j')"] & \\ D(j) \arrow[rr, "D(f)"] & & D(j') \end{tikzcd}$$

A **limit** for the functor D is a representing object for the functor

$$\mathrm{Hom}(*, \mathcal{C}(-, D)) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$
$$A \mapsto \mathrm{Hom}(*, \mathcal{C}(A, D(-)))$$

Now, we wish to generalize the above definition of representable functor to include categories enriched over monoidal closed categories. First, note that the enrichment structure gives us $\mathcal{M}ap(A, X) \in Ob(\mathcal{V})$ for any two objects A, and X in $\mathcal{C}$. As a result, we can construct the enriched functor $\mathcal{M}ap(A, -) : \mathcal{C} \to \mathcal{V}$ which sends object X of $\mathcal{C}$ to $\mathcal{M}ap(A, X)$ in $\mathcal{V}$. The action of this functor on morphisms is determined by the following map in $\mathcal{V}$,

$$\mathcal{M}ap(X, Y) \to [\mathrm{Map}(A, X), \mathrm{Map}(A, Y)]$$

which is a right adjunct to the composition map

$$\mathcal{M}ap(X, Y) \otimes \mathrm{Map}(A, X) \to \mathcal{M}ap(A, Y)$$

Definition 4.4.4. Let $\mathcal{V}$ be a closed monoidal category and $\mathcal{C}$ a category enriched over $\mathcal{V}$. $F\colon \mathcal{C} \to \mathcal{V}$ is a co-representable functor if it is enriched-naturally isomorphic to $\mathcal{M}ap(A, -)$ for some object A of $\mathcal{C}$.

Note. If $\mathcal{V}$ is symmetric monoidal closed, then we can form the contravariant functor version of the above mapping functor, i.e. $\mathrm{Map}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ and define that an enriched functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ is representable whenever there is an object A in $\mathcal{C}$ such that $\mathrm{Map}(-, A)$ is enriched-naturally isomorphic to F.

Next application of representability is also very important particularly in defining new objects in mathematics with higher structures. Let us give a basic example of this phenomenon. Suppose we want to define a group internal to any category with binary product and terminal object. One way is to write in the style of the data + coherence axioms, that is to pick out one object from our category $\mathcal{C}$; one object G, meant to signify elements of the group, and three maps $G \times G \to G$, $G \to G$, and $1 \to G$, the multiplication morphism, the inverse morphism, and the constant morphism (which gives identity element of the group) respectively. We also have to write down right coherence conditions between these morphism. For more sophisticated structures such as topological groups, Lie groups, spectra, etc. internal to categories with enough structures, this approach soon gets ineffective and tiresome.

A more elegant approach which was pioneered by Grothendieck was the use of representable functors and liftings. For instance, suppose we want to define a group internal to a category $\mathcal{C}$ with products and terminal object. For an object A to be a group in $\mathcal{C}$ it will be necessary and sufficient that we can find a unique lifting $\tilde{A} : \mathcal{C}^{\mathrm{op}} \to$ **Grp** of the representable functor $\mathrm{Hom}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$:

$$\begin{tikzcd} & \mathbf{Grp} \arrow[d, "U"] \\ \mathcal{C}^{\mathrm{op}} \arrow[r, "y_A"'] \arrow[ur, dashed, "\tilde{A}"] & \mathbf{Set} \end{tikzcd}$$

where U is the forgetful functor.

One example of such lifting is the fundamental group from algebraic topology.

Example 4.4.5. Let $\mathbf{hTop}_*$ be the category with objects as pointed topological spaces and morphisms as homotopy classes of base-point preserving maps. The co-representable functor $\mathrm{Hom}((S^1, *), -)$ computes, for every pointed spaces (X, x_0), the set of loops in X starting at x_0. The lifting computes the fundamental group of the pointed space.

$$\begin{tikzcd} & \mathbf{Grp} \arrow[d, "U"] \\ \mathbf{hTop} \arrow[r, "y_{S^1}"'] \arrow[ur, dashed, "\pi_1"] & \mathbf{Set} \end{tikzcd}$$

Having in mind our definition of group in the above, this suggests that S^1 must be a co-group in the category $\mathbf{hTop}_*$. Indeed this is true, and the co-multiplication map is $S^1 \to S^1 \vee S^1$.

Note. For S^n, where $n \geq 2$, we have the following lifting:

$$\begin{tikzcd} & \mathcal{Ab} \arrow[d, "U"] \\ \mathbf{hTop} \arrow[r, "y_{S^n}"'] \arrow[ur, dashed, "\pi_n"] & \mathbf{Set} \end{tikzcd}$$

Example 4.4.6. Suppose a pair $(\mathbf{C}_1, \mathbf{C}_0)$ is an internal category in some category $\mathcal{E}$ as in 4.1.3. The representable functor $\mathrm{Hom}(-, \mathbf{C}_0)$ can be lifted via the forgetful functor from categories to sets:

$$\begin{tikzcd} & \mathcal{C}at \arrow[d, "U"] \\ \mathcal{C}^{\mathrm{op}} \arrow[r, "y_{\mathbf{C}_0}"'] \arrow[ur, dashed, "\mathbf{C}"] & \mathbf{Set} \end{tikzcd}$$

where $\mathbf{C}$ is a functor whose value at any object C of $\mathcal{C}$ is a category whose set of objects is $\mathrm{Hom}(C, A_0)$ and whose set of morphisms is $\mathrm{Hom}(C, A_1)$.

Now that we have seen some application of representability in category theory, let's jump one level up and see how we can employ this beautiful notion in the world of 2-categories. The main difference is that in thw world of 2-categories there will be two ways to say whether a 2-functor is representable, either using isomorphism of hom-categories or equivalence of hom-categories and precisely this different choices account for strict and weak structures of representing objects. A limit of digram in a category, viewed as a representing object for an appropriate $\mathbf{Set}$-functor, generalises to the notion of weighted limit of a weighted diagram in a 2-category, defined as representing object of a $\mathfrak{Cat}$-valued 2-functor.

Definition 4.4.7. Suppose $\mathcal{J}$ is a small 2-category and $\mathcal{K}$ is a 2-category. Moreover, let $D\colon \mathcal{J} \to \mathcal{K}$ and $W\colon \mathcal{J} \to \mathfrak{Cat}$ be 2-functors. A DIAGRAM OF SHAPE J WITH WEIGHT W in $\mathcal{K}$ consists of

$$\begin{tikzcd} \mathcal{J} \arrow[r, "D"] \arrow[dr, "W"'] & \mathcal{K} \\ & \mathfrak{Cat} \end{tikzcd}$$

where 2-functor D represents the diagram, and W specifies a weight $W(j)$ for each object $j \in \mathcal{J}_0$ and weight transformer $W(f)$ to each morphism $j \xrightarrow{f} j'$ in $\mathcal{J}$. A (LAX) WEIGHTED CONE over weighted diagram (D, W) with vertex $A \in \mathcal{K}_0$ is given by the following data:

- a functor $L(j) : W(j) \to \mathcal{K}(A, D(j))$ for every $j \in \mathcal{J}_0$
- a natural transformation $L(f)\colon D(f)_* \circ L(j) \Rightarrow L(j') \circ W(f)$, for every morphism $f : j \to j'$ in $\mathcal{J}$.

$$\begin{tikzcd} W(j) \arrow[r, "L(j)"] \arrow[d, "W(f)"'] \arrow[dr, phantom, "\Downarrow L(f)"] & \mathcal{K}(A, D(j)) \arrow[d, "D(f)_*"] \\ W(j') \arrow[r, "L(j')"'] & \mathcal{K}(A, D(j')) \end{tikzcd} \tag{4.2}$$

We form the category $\mathfrak{Cone}(A, (D, W))$ of lax weighted cones over (D, W) with vertex A. Objects of this category are 2-natural transformations $L\colon W \Rightarrow \mathcal{K}(A, D(-))$ as given in above, and a morphism between two such 2-natural transformations L and L' is a modification, that is for each $j \in \mathcal{J}_0$, a natural transformation $m(j)\colon L(j) \to L'(j)$ such that

$$L'(f) \circ (D(f)_* \centerdot m(j)) = (m(j') \centerdot W(f)) \circ L(f) \tag{4.3}$$

This identity is exhibited in diagram 4.4 by stating first traversing along the front face and then bottom face yields the same result as traversing the top face followed by back face.

$$\begin{tikzcd}
 & W(j) \arrow[rr, "L'(j)"] \arrow[dd] \arrow[dr, phantom, "\overset{m(j)}{\nearrow}"] & & \mathcal{K}(A, D(j)) \arrow[dd, "D(f)_*"] \\
W(j) \arrow[ur, equal] \arrow[rr, "L(j)"] \arrow[dd, "W(f)"'] & & \mathcal{K}(A, D(j)) \arrow[ur, equal] \arrow[dd] & \\
 & W(j') \arrow[rr, "L'(j')"] \arrow[dr, phantom, "\overset{m(j')}{\nearrow}"] & & \mathcal{K}(A, D(j')) \\
W(j') \arrow[ur, equal] \arrow[rr, "L(j')"'] & & \mathcal{K}(A, D(j')) \arrow[ur, equal] &
\end{tikzcd} \tag{4.4}$$

Indeed, category $\mathfrak{Cone}(A, (D, W))$ just so constructed is indeed a functor category; that is:

$$\mathfrak{Cone}(A, (D, W)) \cong \mathbf{Fun}(\mathcal{J}, \mathfrak{Cat})(W, \mathcal{K}(A, D)) \tag{4.5}$$

Definition *4.4.8.* A **(lax) weighted limit** for the weighted diagram (D, W) is a representing object $\lim_W D \in \mathcal{K}_0$ for the 2-functor

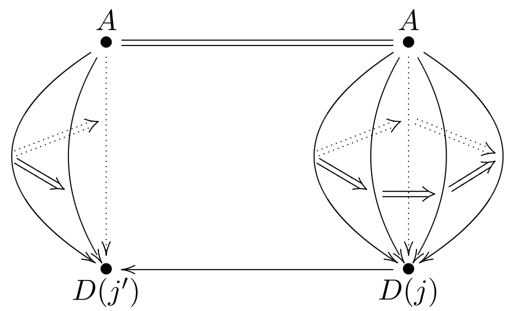
$$\begin{aligned} \mathfrak{Cone}(D, W) : \mathcal{K}^{\text{op}} &\to \mathfrak{Cat} \\ X &\mapsto \mathfrak{Cone}(X, (D, W)) \end{aligned}$$

This is equivalent to say that there is an equivalence of categories

$$\Phi_X : \mathcal{K}(X, \lim_W D) \simeq \mathfrak{Cone}(X, (D, W)) : \Psi_X \tag{4.6}$$

natural in X.

Remark. It is enlightening to contrast weighted limits with 1-categorical limits. In the former case, a cone over a diagram $D\colon \mathcal{J} \to \mathcal{C}$ is given by a vertex $A \in \mathbf{Ob}(\mathcal{C})$, and for each $j \in \mathbf{Ob}(\mathcal{J})$ a *single* morphism $A \to D(j)$ preserved by the natural action of morphisms $f\colon j \to j'$ in $\mathcal{J}$. And a limit is the universal such cone over D. Whereas in the case of weighted limits, we instead ask for a category of morphisms $A \to D(j)$ for each j in $\mathcal{J}$, and moreover the action of 1-cells and 2-cells of $\mathcal{J}$ induces functors and natural transformations between such categories. The picture below illustrates this situation for a 1-cell $f\colon j \to j'$ in $\mathcal{J}$.



Note. There are several important variations of this definition which provides us with stricter structures. More precisely, the level of strictness of our weighted limits supervenes upon the strictness structure of $\mathbf{Fun}(\mathcal{J}, \mathfrak{Cat})$. We enumerate some important variations from the most strict to the least.

Strictness	D	W	$L(f)$	Φ
Conical limit	*strict*	1 (*constant*)	$=$	$\cong$
(Strict) weighted limits	*strict*	*strict*	$=$	$\cong$
Pseudo weighted limit	*strict*	*pseudo*	$\cong$	$\cong$
Lax weighted limit	*strict*	*lax*	$\Rightarrow$	$\cong$
Weighted bilimit	*strict*	*pseudo*	$\cong$	$\simeq$
Lax weighted bilimit	*strict*	*lax*	$\Rightarrow$	$\simeq$

By example 4.4.3, it is easy to see that conical limits in a 2-category $\mathcal{K}$ are exactly the ordinary limits. Furthermore, we remark that the paper [11] deals only with strict weighted limits but [12] is mostly concerned with psuedo weighted limits.

Example 4.4.9. We construct pseudo-pullbacks as strict weighted limits. Let $\mathcal{J}$ be the category generated by 0-cells and 1-cells in below:

$$\begin{tikzcd}
& 2 & \\
0 \arrow[ur] & & 1 \arrow[ul]
\end{tikzcd}$$

And, let the diagram D be the functor that sends 0-cells and 1-cells of $\mathcal{J}$ to the following opspan in $\mathcal{K}$:

$$\begin{tikzcd}
& C & \\
A \arrow[ur, "f"] & & B \arrow[ul, "g"']
\end{tikzcd}$$

Also, let the weight functor W send 0-cells and 1-cells of $\mathcal{J}$ to the following opspan in $\mathfrak{Cat}$:

$$\begin{tikzcd}
& \mathbb{I} & \\
pt \arrow[ur, "i_0"] & & pt \arrow[ul, "i_1"']
\end{tikzcd}$$

where *pt* is the category with only one objects and no non-trivial arrow and $\mathbb{I}$ is the interval groupoid that is the groupoid with two objects and arrow between them. We claim a (strict) weighted limit of (D, W) is a pseudo-pullback of f and g in $\mathcal{K}$. For a 0-cell X in $\mathcal{K}$, a cone with apex X over opspan $\langle f, C, g\rangle$ is specified by functors $L(j)\colon W(j) \to \mathcal{K}(X, D(j))$ satisfying the naturality condition of weighted cone in the diagram 4.2. $L(0)$ and $L(1)$ give us 1-cells $X \xrightarrow{l_0} A$ and $X \xrightarrow{l_1} B$, respectively, and $L(2) : \mathbb{I} \to \mathcal{K}(X, C)$ specifies two 1-cells and an *iso* 2-*cell* λ between them. The domain and codomain 1-cells of $L(2)$ must be equal to fl_0 and gl_1, resp. according to naturality of L.

$$\begin{tikzcd}
X \arrow[r, "l_1"] \arrow[d, "l_0"'] \arrow[dr, dotted, bend left] \arrow[dr, dotted, bend right] \arrow[dr, phantom, "\cong_\lambda"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

Now, universal property of $\lim_W D$ says that for any 1-cell $h : X \to Y$ the following diagram commutes:

$$\begin{tikzcd}
\mathcal{K}(Y, \lim_W D) \arrow[r, "\cong"] \arrow[d, "h^*"'] & \mathfrak{Cone}(Y, (D, W)) \arrow[d, "\mathfrak{Cone}(h)"] \\
\mathcal{K}(X, \lim_W D) \arrow[r, "\cong"'] & \mathfrak{Cone}(X, (D, W))
\end{tikzcd}$$

Observe that $\Phi_{\lim_W D}(1_{\lim_W D})$ is the limiting cone $\langle \lim_W D, d_0, d_1, \delta \rangle$, where δ is an isomorphism 2-cell, and commutativity of the above diagram for object $Y := \lim_W D$ implies that $\Phi_X(m) = \langle X, d_0 u, d_1 u, \delta \centerdot u \rangle$ for any 1-cell $u : X \to \lim_W D$. So, we know how to explicitly compute Φ after all. On the other hand, for any cone $L = \langle X, l_0, l_1, \lambda \rangle$, $\Psi_X(L) \colon X \to \lim_W D$ is the unique morphism with $\Phi_X \circ \Psi_X = id$, in other words $d_0 \circ \Psi_X(L) = l_0$, $d_1 \circ \Psi_X(L) = l_1$, and $\delta \centerdot \Psi(L) = \lambda$.

$$\begin{tikzcd}
X \arrow[dr, dotted, "\Psi(L)"] \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] & & \\
& \lim_W D \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\cong_\delta"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

There is another part to the universal property of colimit cone which involves morphisms of cones. Suppose L and L' are both objects of $\mathfrak{Cone}(X, (D, W))$ and modification $m \colon L \Rrightarrow L'$ is a morphism of cones. The data of modification m provides us with 2-cells $m(0) : l_0 \Rightarrow l'_0 : X \to A$ and $m(1) : l_1 \Rightarrow l'_1 : X \to B$. Equations 4.3 in our strict case are tantamount to commutativity of diagram below:

$$\begin{tikzcd}
f l_0 \arrow[r, "f \centerdot m(0)"] \arrow[d, "\lambda"'] & f l'_0 \arrow[d, "\lambda'"] \\
g l_1 \arrow[r, "g \centerdot m(1)"'] & g l'_1
\end{tikzcd}$$

That is all about a morphism m of cones L and L' in $\mathbf{Fun}(\mathcal{J}, \mathfrak{Cat})(W, \mathcal{K}(X, D))$. We get a unique 2-cell $\Psi(m) : \Psi(L) \Rightarrow \Psi(L')$ which generates $m(0)$ and $m(1)$ by post-horizontal-composition with d_0 and d_1 respectively. Put slightly differently, given 1-cells $u, v : X \rightrightarrows \lim_W D$ and 2-cells $\alpha : d_0 u \Rightarrow d_0 v$ and $\beta : d_1 u \Rightarrow d_1 v$ in such a way that

$$\begin{tikzcd} fd_0u \arrow[r, "f.\alpha"] \arrow[d, "\delta.u"'] & fd_0v \arrow[d, "\delta.v"] \\ gd_1u \arrow[r, "g.\beta"'] & gd_1v \end{tikzcd}$$

commutes, there exists a unique 2-cell $\sigma\colon\, u \Rightarrow v$ such that $d_0 \centerdot \sigma = \alpha$ and $d_1 \centerdot \sigma = \beta$.

EXAMPLE *4.4.10.* We construct comma objects in 2-categories as strict weighted limits. Let $\mathcal{K}$ be a 2-category and $\mathcal{J}$ be the category illustrated below:

$$\begin{tikzcd} & 2 & \\ 0 \arrow[ur, "f"] & & 1 \arrow[ul, "g"'] \end{tikzcd}$$

And, let the diagram D be the functor which maps $\mathcal{J}$ to the following opspan in $\mathcal{K}$:

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] & & B \arrow[ul, "g"'] \end{tikzcd}$$

Also, define the weight W as the functor which maps $\mathcal{J}$ to the following opspan in $\mathfrak{Cat}$:

$$\begin{tikzcd} & {[1]} & \\ {[0]} \arrow[ur, "\delta_0"] & & {[0]} \arrow[ul, "\delta_1"'] \end{tikzcd}$$

where $[n] = \{0 \to 1 \to 2 \to \cdots \to n\}$ is a (poset) category. We claim a (strict) weighted limit of (D, W) is a comma object f/g in $\mathcal{K}$. A (strict) cone over the opspan $\langle f, C, g\rangle$ is given by 1-cells $x_a : X \to A$ and $x_b : X \to B$ and a 2-cell $\theta : fx_a \Rightarrow gx_b$:

$$\begin{tikzcd} X \arrow[r, "x_a"] \arrow[d, "x_b"'] \arrow[dr, phantom, "\overset{\theta}{\swArrow}"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

By universal property of limit cone, we get a unique morphism $u\colon\, X \to \lim_W D$ with

$\theta_0 \centerdot u = \theta$.

$$
\begin{tikzcd}
X \arrow[drr, bend left, "x_a"] \arrow[dr, dotted, "u"] \arrow[ddr, bend right, "x_b"'] & & \\
& \lim_W D \arrow[r, "d_a"] \arrow[d, "d_b"] \arrow[dr, phantom, "\theta_0 \Swarrow"] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}
$$

Suppose $L = \langle X, x_a, x_b, \theta \rangle$ and $L' = \langle X, x'_a, x'_b, \theta' \rangle$ are both weighted cones with apex X and a morphism from L to L' is given by modification $m\colon L \Rrightarrow L'$. Equation 4.3 becomes

$$f \centerdot m_0 = m_2 \centerdot \delta_0$$

and

$$g \centerdot m_1 = m_2 \centerdot \delta_1$$

Together, they yield the commutativity of diagram below:

$$
\begin{tikzcd}
fx_a \arrow[r, "f \centerdot m_0"] \arrow[d, "\theta"'] & fx'_a \arrow[d, "\theta'"] \\
gx_b \arrow[r, "g \centerdot m_1"'] & gx'_b
\end{tikzcd}
$$

In such a situation, the unique 2-cell $\Psi(m) : \Psi(L) \Rightarrow \Psi(L')$ generates m_0 and m_1 by post-horizontal-composition with d_a and d_b respectively.

Example 4.4.11. If $\mathcal{K}$ is chosen to be the 2-category $\mathfrak{Cat}$ of categories, then the comma object obtained this way agrees with what we described in 4.3.1

Remark. Notice that we can construct comma objects as pseudo-weighted limits. Isomorphisms $L(f)$ in 4.2 specialized to this situation give us two extra 1-cells z and z', a 2-cell τ between them and isomorphisms $\eta : Fx_a \cong z$ and $\zeta : Gx_b \cong z'$. The fact that the second isomorphisms could be inverted gives us a strict cone $\langle x_a, X, x_b; \zeta^{-1}\tau\eta \rangle$. Furthermore, the universal property of the limit cone for both cases of strict and pseudo are essentially the same.

$$
\begin{tikzcd}
X \arrow[r, "x_a"] \arrow[d, "x_b"'] \arrow[dr, dotted, bend left] \arrow[dr, dotted, bend right] \arrow[dr, phantom, "\Swarrow\tau"] & A \arrow[d, "F"] \\
B \arrow[r, "G"'] & C
\end{tikzcd}
$$

Dually, a weighted colimit can be defined by a pair of functors: a diagram functor $D : \mathcal{J} \to \mathcal{K}$ and a weight functor $W : \mathcal{J}^{\mathrm{op}} \to \mathfrak{Cat}$. Thus weighted colimits are the same thing as weighted limits in $\mathcal{K}^{\mathrm{op}}$. As an example we construct a cocomma object.

EXAMPLE 4.4.12. Suppose $\mathcal{K}$ is the 2-category of (small) 2-categories, (possibly lax) 2-functors, and lax natural transformations. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be 2-functors. Then the comma object F/G in $\mathcal{K}$ is a 2-category with

- 0-cells given by triples $\langle A, FA \xrightarrow{f} GB, B\rangle$ where $A \in \mathbf{Ob}(\mathcal{A})$ and $B \in \mathbf{Ob}(\mathcal{B})$ and f is a 1-cell in $\mathcal{C}$.
- 1-cells given by pairs of 1-cells $a\colon A \to A'$ in $\mathcal{A}$ and $b\colon B \to B'$ in $\mathcal{B}$ together with a 2-cell

$$
\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "f"'] \arrow[dr, phantom, "\phi\Uparrow"] & FA' \arrow[d, "f'"] \\
GB \arrow[r, "Gb"'] & GB'
\end{tikzcd}
$$

- 2-cells given by a pair of 2-cells $\alpha : a \Rightarrow a'$ and $\beta : b \Rightarrow b'$ such that the obvious diagram of 2-cells in below commutes:

$$
\begin{tikzcd}
FA \arrow[r, bend left, "Fa'"] \arrow[r, bend right, "Fa"'] \arrow[r, phantom, "\Uparrow F(\alpha)"] \arrow[d, "f"'] \arrow[dr, phantom, "\phi\Uparrow \quad \phi'\Uparrow"] & FA' \arrow[d, "f'"] \\
GB \arrow[r, dotted, bend left, "G(b')"] \arrow[r, bend right, "G(b)"'] \arrow[r, phantom, "\Uparrow G(\beta)"] & GB'
\end{tikzcd}
$$

That is to say

$$
(f' \centerdot F(\alpha)) \circ \phi = \phi' \circ (G(\beta) \centerdot f)
$$

An special case is when F is identity 2-functor on $\mathcal{C}$ and G is a constant 2-functor at some 0-cell C of $\mathcal{C}$. The comma object of F and G is known as slice 2-category $\mathcal{C} \,/\!\!/\, C$. In fact this is the 2-categorical generalization of slice categories of example 4.3.2.

5

Categorical fibrations: Grothendieck and Street

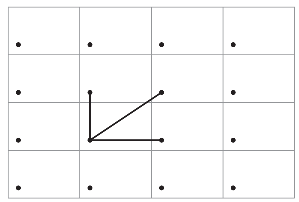
Introduction:

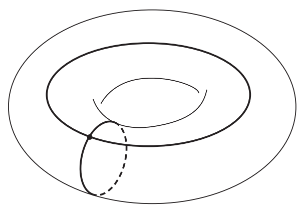
What I shall say in this chapter is neither original nor advanced; the only merit of its presentation is a cohesive hand-pick of results from the interesting and vast theory of categorical fibrations as the necessary background for the upcoming chapters.

In this chapter we define 1-categorical fibrations which occur within the 2-category $\mathfrak{Cat}$. We first cover the case of discrete fibrations which are conceptually easier to grasp, and then we consider the general notion of fibrations between categories. Regarding $\mathfrak{Cat}$ as the mother of 2-categories, we extend the definition of fibration to any arbitrary 2-category. Main references for this chapter are [14] and [2]

5.1 Discrete Fibrations

We recall from algebraic topology that a continuous map $p : E \to B$ is said to be a COVERING MAP (or space E is a COVERING SPACE over B) whenever for every $x \in B$ there is an open neighbourhood U containing x such that $p^{-1}(U) = \amalg_{i \in I} V_i$, a disjoint union of open sets V_i in E such that $p|_{V_i} V_i \cong U$. One example of a covering map is the quotient map $\mathbb{R}^2 \to \mathbb{T}$ where the torus $\mathbb{T}$ is obtained as the quotient space of $\mathbb{R}^2$ by identifying $(x, y) \sim (x+m, y+n)$ for every $m, n \in \mathbb{Z}$. Another well-known examples is the helix-shaped real line over 1-sphere.





Another class of covering spaces are built out of locally constant sheaves. We recall from definition 1.3.2 a sheaf P on a topological space X is locally constant if there exists an open cover of X such that the restriction of P to each open set in the cover is a constant sheaf. If the space X is locally connected, locally constant sheaves on X are, up to isomorphism, precisely the sheaves of sections of covering spaces $\pi : Et(P) \to X$.

Concerning covering maps there is the *unique path lifting* theorem:

THEOREM *5.1.1*. Suppose B is a connected and locally path connected space and $\lambda : I \to B$ a path in B starting at $\lambda(0) = b_0$. Then for each $e \in p^{-1}(b_0)$ there is a unique path $\tilde{\lambda} : I \to E$ with $p(\tilde{\lambda}) = \lambda$. Moreover, if there is a homotopy H between two paths λ and γ (with the same starting and ending points) in the base space B, then there is a unique lift $\tilde{H}$ of homotopy H between the lifts $\tilde{\lambda}$ and $\tilde{\gamma}$ (with the same starting and ending points).

$$\begin{tikzcd} & E \arrow[d, "p"] \\ I \arrow[r, "\lambda"'] \arrow[ur, dotted, "\tilde{\lambda}"] & B \end{tikzcd}$$

We want to generalize this fundamental result covering maps to functors.

Definition 5.1.2. A functor $S : \mathcal{E} \to \mathcal{B}$ is a *discrete fibration* if for every object e of $\mathcal{E}$ and every morphism $f : b \to Se$ in $\mathcal{B}$ there exists a unique lift $\tilde{f} : \tilde{b} \to e$ in $\mathcal{E}$. The category $\mathcal{B}$ is sometimes referred to as the context category of fibration.

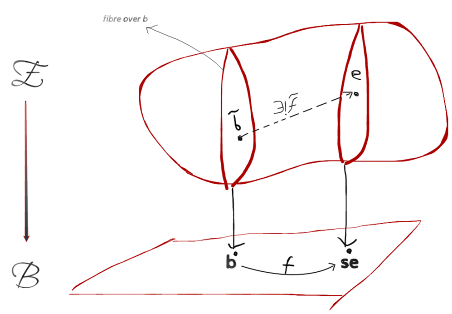


Figure 5.1.1: A a unique lift $\tilde{f}$ of f

Remark. Discrete fibrations over a category $\mathcal{B}$ form a new category denoted by $\mathbf{DFib}(\mathcal{B})$ called the category of discrete fibrations which is a full subcategory of slice category $\mathfrak{Cat}/\mathcal{B}$.

Remark. To every discrete fibration with context category $\mathcal{B}$, we can assign a presheaf functorially. For every object $b \in \mathcal{B}$, consider the (discrete) category $\mathcal{E}_b$ which as its objects has all $e \in \mathcal{E}$ with $Se = b$ and as its morphisms has all $p : e \to e'$ in $\mathcal{E}$ with $S(p) = 1_b$. Sometimes in the literature, they are referred to as *vertical morphisms* over b. Now, assign to S the presheaf $\hat{S}$:

$$\begin{aligned} \hat{S} \colon \mathcal{B}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ b &\longmapsto \mathcal{E}_b \\ (b' \xrightarrow{f} b) &\longmapsto (\mathcal{E}_b \xrightarrow{f^*} \mathcal{E}_{b'}) \end{aligned} \tag{5.1}$$

where f^* maps an object in the fibre of b to $dom(\tilde{f})$, where $\tilde{f}$ is the unique lift $dom(\tilde{f})$ of f.

Let's check functoriality of $\hat{S}$. It is obvious that $1_b^* = 1_{\mathcal{E}_b}$. To prove $\hat{S}$ respects compositions in $\mathcal{B}$, take two arbitrary composable arrows $b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2$, and $x_2 \in \mathcal{E}_{b_2}$. Then, there are unique lifts $\tilde{f} : x_1 \to x_2$ and $\tilde{g} : x_0 \to x_1$ of f, and g, respectively. The uniqueness of lifts implies $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$. So, we have

$$\hat{S}(g \circ f) = dom(\widetilde{g \circ f}) = dom(\tilde{g} \circ \tilde{f}) = \hat{S}(g) \circ \hat{S}(f)$$

. This proves that $\hat{S}$ is indeed a presheaf over base category $\mathcal{B}$.

Remark. The fibre category $\mathcal{E}_b$ fits into the following pullback diagram:

$$\begin{tikzcd} \mathcal{E}_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "S"] \\ 1 \arrow[r, "b"'] & \mathcal{B} \end{tikzcd} \tag{5.2}$$

Note that however, even if the base category is discrete, and each fibre is discrete, it may not be the case that $\mathcal{E}$ is discrete as we can take $\mathcal{E} = [n]$ consisting of $n+1$ objects and n non-trivial arrows between them, and identity functor $id : [n] \to [n]$ as our fibration. Then $\mathcal{E}_k = k \xrightarrow{id} k$ and $\amalg \mathcal{E}_k = \mathbf{n}$ where $\mathbf{n}$ is the discrete category with n objects. The fibred category $[n]$, however, is not discrete. This shows that in general the following morphism of discrete fibrations is not an isomorphism or even equivalence of discrete fibrations. So, in general the following map of fibrations is not necessarily an equivalence:

$$\begin{tikzcd} \amalg \mathcal{E}_b \arrow[rr, hook] \arrow[dr, "\amalg b^* S"'] & & \mathcal{E} \arrow[dl, "S"] \\ & B & \end{tikzcd}$$

Example 5.1.3. By the discussion before the definition of 5.1.2, it is now obvious that for a covering map $p : E \to B$ of topological spaces the fundamental groupoid functor $\Pi_0(p) : \Pi(E) \to \Pi(B)$ is a discrete fibration.

Dually, we can define the notion of opfibraions for categories:

Definition 5.1.4. A functor $S : \mathcal{E} \to \mathcal{B}$ is called a *discrete opfibration* whenever the functor $S^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a discrete fibration. For a category $\mathcal{B}$, discrete opfibrations over $\mathcal{B}$ form a full subcategory of $\mathfrak{Cat}/\mathcal{B}$ which we will denote by $\mathbf{DoFib}(\mathcal{B})$

Remark. Unpacking the above definition, S is a discrete opfibration precisely when for every object e of $\mathcal{E}$ and every morphism $f : Se \to b$ in $\mathcal{B}$ there exists a unique lift $\tilde{f} : e \to \tilde{b}$ in $\mathcal{E}$.

Proposition 5.1.5. The forgetful functor $U : \mathbf{Set}_* \to \mathbf{Set}$ is a discrete opfibration, and the fibre over a set X is the set X itself (view it as a discrete category where objects of X are its elements). In fact, this is a universal discrete opfibration in the category $\mathfrak{Cat}$ in the sense that:

- For every functor $F : \mathcal{B} \to \mathbf{Set}$ the pullback of U along F gives us a discrete opfibration $\pi_F : \int_{\mathcal{B}} F \to \mathcal{B}$ with the fibre over $b \in \mathcal{B}$ being discrete category Fb:

$$\begin{tikzcd} \int_{\mathcal{B}} F \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_* \arrow[d, "U"] \\ \mathcal{B} \arrow[r, "F"'] & \mathbf{Set} \end{tikzcd} \tag{5.3}$$

- Moreover, for any discrete opfibration $S : \mathcal{E} \to \mathcal{B}$, there is a unique functor (up to isomorphism) $\hat{S} : \mathcal{B} \to \mathbf{Set}$ that makes the following diagram a pullback in $\mathfrak{Cat}$:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "V"] \arrow[d, "S"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_* \arrow[d, "U"] \\ \mathcal{B} \arrow[r, dotted, "\hat{S}"'] & \mathbf{Set} \end{tikzcd} \tag{5.4}$$

  where $V(e) = (\mathcal{E}_{Se}, e)$ and $V(e \xrightarrow{f} e') = \mathcal{E}_{Se} \xrightarrow{f_*} \mathcal{E}_{Se'}$ given by taking the codomain of the unique lifting of f.

Proof. First, we introduce the equivalence adjunction $\widehat{(-)} \dashv \int(-)$ or:

$$\begin{tikzcd} \mathbf{DoFib}(\mathcal{B}) \arrow[r, bend left, "\widehat{(-)}"] \arrow[r, phantom, "\perp"] & {[\mathcal{B}, \mathbf{Set}]} \arrow[l, bend left, "\int(-)"] \end{tikzcd} \tag{5.5}$$

where for any discrete opfibration S, $\widehat{S}$ is defined similar to 5.1, except $\widehat{S}$ is defined covariantly. $\int(-)$ is well-known Grothendieck construction. To prove adjoint equivalence, we have to show there is a natural bijection of Hom-sets:

$$[\mathcal{B}, \mathbf{Set}](\widehat{S}, F) \cong_{\phi} \mathbf{DoFib}(S, \int F) \tag{5.6}$$

where $F : \mathcal{B} \to \mathbf{Set}$ is a functor and $S : \mathcal{E} \to \mathcal{B}$ is a discrete opfibration. Note that if $\gamma : \widehat{S} \Rightarrow F$ is a natural transformation, then for any morphism $b \xrightarrow{f} b' \in \mathcal{B}$, we have the commutative square below:

$$\begin{tikzcd}
\mathcal{E}_b \arrow[r, "\gamma_b"] \arrow[d, "f_*"'] & F(b) \arrow[d, "F(f)"] \\
\mathcal{E}_{b'} \arrow[r, "\gamma_{b'}"'] & F(b')
\end{tikzcd}$$

This helps us to define functor

$$\Gamma : \mathcal{E} \to \int_{\mathcal{B}} F$$

by $\Gamma(e \xrightarrow{p} e') := (S(e), \gamma_{S(e)}(e)) \xrightarrow{Sp} (S(e'), \gamma_{S(e')}(e'))$. Using Sp as f in the diagram above, we get the equation

$$F(Sp)(\gamma_{S(e)}(e)) = \gamma_{Se'}((Sp)_*(e))$$

which guarantees that Γ is a well-defined functor. Moreover, $\pi_1 \circ \Gamma = S$. That is, the following square commutes.

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "\Gamma"] \arrow[dr, "S"'] & & \int F \arrow[dl, "\pi_1"] \\
& \mathcal{B} &
\end{tikzcd}$$

Of course, since all of our constructions are natural in S and F, defining $\Gamma = \phi(\gamma)$ (and easily checking it is a bijection), we arrive at the natural bijection 5.6. Now, we observe that the pullback diagram 5.3 is obtained by applying the functor $\int(-)$ and pullback diagram 5.4 is obtained by using functor $\widehat{(-)}$. $\square$

Proposition 5.1.6. Pullback of a discrete opfibration along any functor is a discrete opfibration. So, discrete opfibrations are preserved under taking pullbacks.

Proof. Suppose $S : \mathcal{E} \to \mathcal{B}$ is a discrete opfibration, and $F : \mathcal{C} \to \mathcal{B}$ is a functor. By proposition 5.1.5, we get a pullback square from structure of discrete opfibration $\mathcal{E}$ over $\mathcal{B}$.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "V"] \arrow[d, "S"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_* \arrow[d, "U"] \\
\mathcal{B} \arrow[r, dotted, "\hat{S}"'] & \mathbf{Set}
\end{tikzcd}$$

By pasting two pullback diagrams we get the outer diagram as a pullback in $\mathfrak{Cat}$:

$$\begin{tikzcd}
F^*(\mathcal{E}) \arrow[r] \arrow[d, "F^*(S)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[r] \arrow[d, "S"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_* \arrow[d, "U"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{B} \arrow[r, "\hat{S}"'] & \mathbf{Set}
\end{tikzcd} \tag{5.7}$$

and proposition 5.1.5 implies that there is a unique isomorphism $\Gamma : F^*(\mathcal{E}) \to \int \widehat{S}F$ such that the following diagram commutes

$$\begin{tikzcd}
F^*(\mathcal{E}) \arrow[rr, "\Gamma"] \arrow[dr, "F^*(S)"'] & & \int \widehat{S}F \arrow[dl, "\pi_1"] \\
& \mathcal{C} &
\end{tikzcd} \tag{5.8}$$

which establishes an isomorphism Γ between discrete opfibrations $F^*(S)$ and π_1 in category $\mathbf{DoFib}(\mathcal{B})$. Since π_1 is a discrete opfibration, then so is $F^*(S)$ $\square$

Proposition 5.1.7. The following diagram is a pullback of categories.

$$\begin{tikzcd}
(\int_{\mathcal{B}} P)^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_* \arrow[d] \\
\mathcal{B}^{\mathrm{op}} \arrow[r] & \mathbf{Set}
\end{tikzcd}$$

Proof. By 5.1.6, the left leg of above diagram must be a discrete opfibration and by definition 5.1.4, $\int_{\mathcal{B}} P \to \mathcal{B}$ is a discrete fibration. This establishes following equivalence of categories, similar to the result 5.5 obtained for opfibrations.

$$\mathbf{DFib}(\mathcal{B}) \simeq \mathbf{Psh}(\mathcal{B})$$

□

We remind the reader that a presheaf P on a site $(\mathcal{C}, \mathbb{J})$ is a sheaf if and only if for any object U of $\mathcal{C}$ and any covering sieve $S \in \mathbb{J}(U)$ any matching family $\chi : S \Rightarrow P$ can be uniquely extended to $\overline{\chi} : yU \Rightarrow P$, so that the diagram

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[dr, "\chi"'] & yU \arrow[d, dashed, "\overline{\chi}"] \\
& P
\end{tikzcd}$$

commutes. We can express the condition above in terms of fibrations: $\int \chi$ has a unique extension as a morphism of (discrete) fibred categories over $\mathcal{C}$.

$$\begin{tikzcd}
& \int S \arrow[dl, hook'] \arrow[dr, "\int \chi"] \arrow[dd] & \\
\mathcal{C}/U \cong \int yU \arrow[rr, dotted] \arrow[dr] & & \int P \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

An internal description of a discrete fibration (5.1.2) can also be given. For this purpose, we need to define internal diagrams.

Suppose E and B are categories internal to $\mathcal{A}$ in the spirit of 4.1.4, except that we only have non-trivial 0-cells and 1-cells and all 2-cells are equalities. A functor $s : E \to B$ is a discrete fibration if the following diagram is a pullback square in $\mathcal{A}$.

$$\begin{tikzcd}
E_1 \arrow[r] \arrow[d] & E_0 \arrow[d] \\
B_1 \arrow[r] & B_0
\end{tikzcd}$$

5.2 Grothendieck fibrations

5.2.1 Precartesian and cartesian morphism

Definition 5.2.1. Suppose $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a functor. A morphism $u : X \to Y$ in $\mathcal{X}$ is called $\mathfrak{p}$-*precartesian* whenever for any $\mathcal{X}$-morphism $v : Z \to Y$ such that $\mathfrak{p}(u) = \mathfrak{p}(v)$, there exists a unique morphism w such that $u \circ w = v$. Morphism $u : X \to Y$ is said to be $\mathfrak{p}$-*cartesian* whenever for any $\mathcal{X}$-morphism $v : Z \to Y$ and any $h : \mathfrak{p}(Z) \to \mathfrak{p}(X)$ with $\mathfrak{p}(u) \circ h = \mathfrak{p}(v)$, there exists a unique lift w of h such that $u \circ w = v$.

Nomenclature. In the diagrams we write $X \mapsto A$, for $X \in \mathcal{X}$ and $A \in \mathcal{C}$ to indicate that "X is sitting above A", that is $\mathfrak{p}(X) = A$. Besides, morphisms in the fibre category $\mathcal{X}_C$, that is all morphisms $v : X \to Y$ with $\mathfrak{p}(v) = id_X$, are called *vertical*. Furthermore, when functor $\mathfrak{p}$ is obvious from the context, then we simply use the term cartesian instead of $\mathfrak{p}$-cartesian.

Remark. The definition 5.2.7 essentially says being cartesian means that any completing of $\mathfrak{p}(u)$ and $\mathfrak{p}(v)$ to a commutative triangle in the base category $(\mathcal{C})$ is induced from a completing of u and v to a commutative triangle in the total space category $(\mathcal{X})$ in a unique way:

$$\begin{tikzcd}
W \arrow[d, mapsto] \arrow[dr, dotted, "w"'] \arrow[drr, "v"] & & \\
\mathfrak{p}W \arrow[dr, "h"'] \arrow[drr, "\mathfrak{p}(v)"] & X \arrow[r, "u"'] \arrow[d, mapsto] & Y \arrow[d, mapsto] \\
& \mathfrak{p}X \arrow[r, "\mathfrak{p}(u)"'] & \mathfrak{p}Y
\end{tikzcd} .$$

In the next proposition we list some elementary observations about precartesian and cartesian morphisms:

Proposition 5.2.2. Suppose $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a functor. The following hold:

i. Any cartesian morphism is precartesian.

ii. Any lift w of a morphism $f : B \to \mathfrak{p}X$ factors uniquely through any precartesian lift u of f via a unique vertical morphism.

$$
\begin{tikzcd}
W \arrow[d, mapsto] \arrow[dr, dotted, "v"'] \arrow[drr, "w"] & & \\
B \arrow[dr, equal, "id"'] \arrow[drr] & Y \arrow[r, "u"'] \arrow[d, mapsto] & X \arrow[d, mapsto] \\
& B \arrow[r, "f"'] & SX
\end{tikzcd}
.
$$

iii. Precartesian lifts, if they exists, are unique up to unique isomorphism. suppose $u : Y \to X$ and $w : Z \to X$ are both cartesian lifts of $f : B \to \mathfrak{p}X$ in $\mathcal{E}$. Then $\mathfrak{p}(u) = \mathfrak{p}(w)$ and since u is precartesian it follows that there is a unique vertical morphism $v : Z \to Y$, with $u \circ v = w$. By a similar argument, we can find another vertical morphism $v' : Y \to Z$ with $w \circ v' = u$.

$$
\begin{tikzcd}
Y \arrow[d, dotted, "v'"'] \arrow[dd, dotted, bend right, "id_Y"'] \arrow[ddr, "u"] & \\
Z \arrow[d, dotted, "v"'] \arrow[dr, "w"] & \\
Y \arrow[r, "u"'] & X
\end{tikzcd}
$$

iv. An immediate consequence of the remark above is that any precartesian vertical arrow in $\mathcal{X}$ is an isomorphism.

v. Any isomorphism is cartesian.

Lemma 5.2.3. An $\mathcal{X}$-morphism $u : X \to Y$ is $\mathfrak{p}$-cartesian if and only if the following commuting square is a pullback diagram in $\mathbf{Set}$ for each W in $\mathcal{X}$:

$$
\begin{tikzcd}
\mathcal{X}(W, X) \arrow[r, "u\circ -"] \arrow[d, "\mathfrak{p}_{W,X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(W, Y) \arrow[d, "\mathfrak{p}_{W,Y}"] \\
\mathcal{C}(\mathfrak{p}W, \mathfrak{p}X) \arrow[r, "\mathfrak{p}(u)\circ -"'] & \mathcal{C}(\mathfrak{p}W, \mathfrak{p}Y)
\end{tikzcd}
$$

Lemma 5.2.4. Suppose $v : Y \to Z$ is an $\mathfrak{p}$-cartesian morphism in $\mathcal{X}$. Any morphism $u : X \to Y$ is $\mathfrak{p}$-cartesian if and only if $v \circ u$ is $\mathfrak{p}$-cartesian.

Proof. Since v is cartesian, by lemma 5.2.3 the right square is cartesian (i.e. a pullback).

So, u is cartesian if and only if the left square is cartesian if and only if the outer rectangle is cartesian if and only if $v \circ u$ is cartesian.

$$\begin{tikzcd}
\mathcal{X}(W,X) \arrow[r, "v\circ-"] \arrow[d, "\mathfrak{p}_{W,X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(W,Y) \arrow[r, "u\circ-"] \arrow[d, "\mathfrak{p}_{W,Y}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(W,Z) \arrow[d, "\mathfrak{p}_{W,Z}"] \\
\mathcal{C}(\mathfrak{p}W,\mathfrak{p}X) \arrow[r, "\mathfrak{p}(v)\circ-"'] & \mathcal{C}(\mathfrak{p}W,\mathfrak{p}Y) \arrow[r, "\mathfrak{p}(u)\circ-"'] & \mathcal{C}(\mathfrak{p}W,\mathfrak{p}Z)
\end{tikzcd}$$

□

Example 5.2.5. For any category $\mathcal{X}$, there is a unique functor $\mathcal{X} \to 1$. All morphisms of $\mathcal{X}$ are vertical, a morphisms is cartesian if and only if it is precartesian is and only if it is an isomorphisms.

Example 5.2.6. For any category $\mathcal{C}$, there is a codomain functor $\mathrm{cod} : \mathcal{C}^{[1]} \to \mathcal{C}$ which sends an object $f : D \to C$ of $\mathcal{C}^{[1]}$ to its codomain C and sends a morphism $\langle v, u\rangle : g \to f$ of $\mathcal{C}^{[1]}$, i.e. a commuting square, to f. The claim is cartesian morphisms in $\mathcal{C}^{[1]}$ are precisely pullback squares in $\mathcal{C}$. First, take pullback square:

$$\begin{tikzcd}
\mathcal{C}^{[1]} \arrow[dd, "\mathrm{cod}"'] & Y \arrow[r, dotted, "g"] \arrow[d, dotted, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "u"] \\
 & D \arrow[r, "f"'] & C \\
\mathcal{C} & D \arrow[r, "f"] & C
\end{tikzcd} \tag{5.9}$$

Now, we need to prove that the morphism $\langle g, f\rangle : v \to u$ sitting over f is cartesian. Suppose $\langle g', f'\rangle : w \to u$ with $f \circ h = f'$ for some $h : E \to D$. These equations render the following diagram commutative:

$$\begin{tikzcd}
Z \arrow[r, "g'"] \arrow[d, "w"'] & X \arrow[d, "u"] \\
E \arrow[r, "f'"'] & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
E \arrow[d, "h"'] \arrow[dr, "f'"] & \\
D \arrow[r, "f"'] & C
\end{tikzcd}$$

Using the universal property of pullback diagram 5.9, we find a unique morphism $k : Z \to Y$ which renders both the top triangle and the left square commuting:

$$
\begin{tikzcd}
Z \arrow[d, "w"'] \arrow[drr, dotted, "k"] \arrow[drrr, bend left, "g'"] & & & \\
E \arrow[drr, "h"'] & & Y \arrow[r, "g"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "u"] \\
& & D \arrow[r, "f"'] & C
\end{tikzcd}
$$

To finish the argument, observe that morphism $\langle k, h\rangle : w \to v$ also satisfies $\langle g, f\rangle \circ \langle k, h\rangle = \langle g', f'\rangle$. Conversely, it's straightforward to see that a cartesian morphisms is a pullback. One can even shown that precartesian morphisms are same as pullbacks. Note that $\mathcal{C}_C^{[1]} = \mathcal{C}/C$, stating the fibred category over $C \in \mathcal{C}$ is the same thing as the slice over C. The cartesian vertical morphisms are exactly the subcategory of $\mathcal{C}/C$, consisting only of isomorphisms.

5.2.2 Prefibrations and fibrations

Definition 5.2.7. A functor $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a *Grothendieck prefibration* (resp. *Grothendieck fibration*) whenever for each $X \in \mathcal{X}$, every morphism $A \xrightarrow{f} \mathfrak{p}X$ in $\mathcal{C}$ has a precartesian (resp. cartesian) lift in $\mathcal{X}$.

To not rely on the axiom of choice, we often require a choice of cartesian lifts to be added to the structure of fibrations we consider:

Definition 5.2.8. A *cleavage* for a (pre)fibration $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a choice for each $X \in \mathcal{X}$ and $f : B \to \mathfrak{p}X$ in $\mathcal{C}$, a (pre)cartesian lift $\rho(f, X) : \rho_f X \to X$ of f in $\mathcal{X}$. More formally, the data of a cleavage is a term ρ of the following dependent type:

$$\rho : \prod_{B,A:\,\mathbf{Ob}(\mathcal{C})} \prod_{f:\,\mathcal{C}(B,A)} \prod_{X:\,\mathcal{X}_A} \sum_{Y:\,\mathcal{X}_B} \mathcal{C}art_{\mathcal{X}}(Y, X)$$

where the type $\mathcal{C}art_{\mathcal{X}}(Y, X)$ is type of all cartesian morphisms from Y to X. If the fibration $\mathfrak{p}$ is equipped with a cleavage ρ, then $(\mathfrak{p}, \rho)$ is called a *cloven fibration*. The cleavage ρ is said to be *splitting* if for any composable pair of morphisms f, g:

$$\rho(g \circ f, X) = \rho(g, X) \circ \rho(f, \rho_g X)$$

And *normal* whenever for every object X in $\mathcal{X}$:

$$\rho(id_{\mathfrak{p}X}, X) = id_X$$

Remark. If we assume the axiom of choice (AC), then every Grothendieck fibration is cloven.

Remark. Sometimes when there is no risk of confusion about the cleavage of a (pre)fibration , we usually use the suppressed notation $\widetilde{f} : \rho_f X \to X$ instead of cartesian lift $\rho(f, X)$ of $f : B \to \mathfrak{p}X$.

Proposition 5.2.9. A cloven prefibration is a cloven fibration if and only if precartesian morphisms are closed under composition.

Proof. *Necessity:* Suppose $\mathfrak{p}$ is a prefibration and morphism $f\colon A \to \mathfrak{p}X$ is given in $\mathcal{C}$. Let $\widetilde{f}$ be a lift of f in the cleavage. Let $u : Z \to X$ be any morphism and let $h : \mathfrak{p}Z \to A$ with $f \circ h = \mathfrak{p}u$. Take $\tilde{h}$ to be a precartesian lift of h in the cleavage. Since precartesian morphisms are closed under composition, we conclude that $\widetilde{f} \circ \tilde{h}$ is again precartesian. Now, since $\mathfrak{p}(\widetilde{f} \circ \tilde{h}) = f \circ h = \mathfrak{p}u$, then u factors through $\widetilde{f} \circ \tilde{h}$ via a unique morphism w. Define $v\colon = \tilde{h} \circ w$. Then $\widetilde{f} \circ v = u$ and $\mathfrak{p}v = h$. This proves existence of factorization of u through $\widetilde{f}$.

$$\begin{tikzcd}
Z \arrow[d, dotted, "w"'] \arrow[rrd, "u"] & & \\
\rho_h \rho_f X \arrow[r, "\tilde{h}"'] & \rho_f X \arrow[r, "\tilde{f}"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{p}Z \arrow[dr, "h"'] \arrow[drr, "\mathfrak{p}(u)"] & & \\
& A \arrow[r, "f"'] & \mathfrak{p}X
\end{tikzcd}$$

For uniqueness, if v' is another such morphism then $\tilde{h} \circ w' = v'$ for a unique w' with $\mathfrak{p}w' = id_{\mathfrak{p}Z}$, because we have $\mathfrak{p}v' = \mathfrak{p}\tilde{h} = h$ and $\tilde{h}$ is precartesian. Now, $\widetilde{f} \circ \tilde{h} \circ w' = u$ which implies $w' = w$ and thence $v' = v$.

Sufficiency: Suppose $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a fibration and $u\colon Y \to X$ and $v\colon Z \to Y$ are both precartesian morphisms in $\mathcal{X}$. We want to prove their composition in again precartesian. To this end, take a morphism $w : W \to X$ with $\mathfrak{p}w = fg$ where $f = \mathfrak{p}u$ and $g = \mathfrak{p}v$. We select $\widetilde{f}$ and $\widetilde{g}$ as cartesian lift of f and g in the cleavage respectively.

By (i) and (iii) of proposition 5.2.2, there are unique vertical morphisms k and l such that $\widetilde{f} = uk$ and $\widetilde{g} = vl$. Moreover k and l are invertible. By (v) of 5.2.2, k is also a cartesian morphism and and by 5.2.4, $\widetilde{f}k^{-1}\widetilde{g}$ is cartesian. In addition, $\mathfrak{p}(\widetilde{f}k^{-1}\widetilde{g}) = fg = \mathfrak{p}w$. Thus, there is unique vertical morphisms m such that $\widetilde{f}k^{-1}\widetilde{g}m = w$. Setting $n := lm$, we observe that $uvlm = u\widetilde{g}m = \widetilde{f}k^{-1}\widetilde{g}m = w$. Moreover, since l is invertible, uniqueness of m guarantees uniqueness of any vertical n with $uvn = w$. Therefore, uv is indeed precartesian.

$$
\begin{tikzcd}
W \arrow[drr, "w"] \arrow[dd, dotted, bend right, "m"'] & & \\
Z \arrow[r, "v"'] & Y \arrow[r, "u"'] & X \\
\rho_g Y \arrow[u, dotted, "l"] \arrow[ur, "\widetilde{g}"'] & \rho_f X \arrow[u, dotted, "k"] \arrow[ur, "\widetilde{f}"'] &
\end{tikzcd}
$$

$$
C \xrightarrow[g]{} B \xrightarrow[f]{} A
$$

□

Example 5.2.10. The unique functor $\mathcal{X} \to 1$ is a Grothendieck fibration. A choice of cartesian lift for each $X \in \mathcal{X}$ is id_X, and with this choice the fibration is indeed a normal split cloven cloven fibration.

Example 5.2.11. For any category $\mathcal{C}$, the codomain functor $\mathrm{cod} : \mathcal{C}^{[1]} \to \mathcal{C}$ is a Grothendieck opfibration and it is a Grothendieck fibration if and only if $\mathcal{C}$ has all pullbacks. The proof of opfibration is trivial. Similarly dom is always a Grothendieck fibration and it is a Grothendieck opfibration if and only if $\mathcal{C}$ has all pushouts.

Corollary 5.2.12. Suppose the following cubic diagram is commutative, and moreover, the side faces corresponding to $u_0 \to u_1$ and $u_2 \to u_3$, and the front face corresponding to $u_1 \to u_3$ in $\mathcal{C}^{[1]}$ are cartesian squares. By 5.2.4, the diagonal face $u_0 \to u_3$ is cartesian square which in turns implies that the rear square $u_0 \to u_2$ is also cartesian.

$$\begin{tikzcd}
 & E \arrow[rr] \arrow[dl] \arrow[dd, "u_0"] \arrow[dr, dotted] & & F \arrow[dl] \arrow[dd, "u_2"] \\
G \arrow[rr] \arrow[dd, "u_1"'] & & H \arrow[dd, "u_3"] & \\
 & A \arrow[rr, "f_0"] \arrow[dl, "g_0"'] \arrow[dr, dotted] & & B \arrow[dl, "g_1"] \\
C \arrow[rr, "f_1"'] & & D &
\end{tikzcd}$$

Example 5.2.13. Every discrete (op)fibration is a Grothendieck (op)fibration. Suppose $\mathfrak{p}$ is a discrete fibration, then take arbitrary $X \in \mathcal{X}$ and $f : A \to \mathfrak{p}X$ in $\mathcal{C}$. We claim the unique lift $\tilde{f}$ is cartesian. To see this assume $r : Z \to X$ is a morphism with $\mathfrak{p}(r) = f \circ h$ for some $h : \mathfrak{p}Z \to A$. Now, $A = \mathfrak{p}\tilde{A}$ and hence, there is a unique lift $\tilde{h} : Z' \to \tilde{A}$ of h.

$$\begin{tikzcd}
Z \arrow[rrd, "r"] \arrow[d, equal] & & \\
Z' \arrow[r, dotted, "\tilde{h}"'] & \tilde{A} \arrow[r, dotted, "\tilde{f}"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{p}Z \arrow[rrd, "\mathfrak{p}(r)"] \arrow[rd, "h"'] & & \\
 & A \arrow[r, "f"'] & \mathfrak{p}X
\end{tikzcd}$$

Notice that $\mathfrak{p}(\tilde{f} \circ \tilde{h}) = f \circ h = \mathfrak{p}(r)$. So, both $\tilde{f} \circ \tilde{h}$ and r are lifts of $\mathfrak{p}(r)$ and have X as their codomain. So, by discreteness of fibration, they must be equal, i.e. $\tilde{f} \circ \tilde{h} = r$, and in particular $Z' = \mathrm{dom}(\tilde{f} \circ \tilde{h}) = \mathrm{dom}(r) = Z$, and $\tilde{h} : Z \to A$ is the unique morphism with $\tilde{f} \circ \tilde{h} = r$.

We saw in proposition 5.1.7 that any presheaf P on a category $\mathcal{C}$ corresponds to a discrete fibration $\int_{\mathcal{C}} P \to \mathcal{C}$, where $\int_{\mathcal{C}}$ is the category of elements of P. A curious case is when the presheaf is representable yX for some X in $\mathcal{C}$. Then $\int_{\mathcal{C}} \cong \mathcal{C}/A$, and we get the familiar fibration $\mathcal{C}/X \to \mathcal{C}$. Note that for $f : B \to A$ and $a : A \to X$, we have $\mathrm{Pull}_f\, a = a \circ f$ which coincides with the notion of pulling back morphism a along f.

Proposition 5.2.14. Grothendieck fibrations are closed under composition and pullback.

Proof. (closed under composition) Suppose $\mathfrak{q} : \mathcal{Y} \to \mathcal{X}$ and $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ are Grothendieck fibrations, and an object $Y \in \mathcal{Y}$ and a morphism $f : A \to \mathfrak{pq}(Y) \in \mathcal{B}$ are given.

$$\begin{tikzcd}
W \arrow[r, dotted, "l"] & Y & & \mathcal{Y} \arrow[d, "\mathfrak{q}"] \\
X \arrow[r, dotted, "u"] & \mathfrak{q}Y & & \mathcal{X} \arrow[d, "\mathfrak{p}"] \\
A \arrow[r, "f"] & \mathfrak{pq}Y & & \mathcal{C}
\end{tikzcd}$$

In the diagram above, u is a $\mathfrak{p}$-cartesian lift of f with codomain $\mathfrak{q}Y$ and l is a $\mathfrak{q}$-cartesian lift of u with codomain x. Because l and u are cartesian morphisms, part (vi) of proposition 5.2.2 tells us that for every $Z \in \mathcal{Y}$, the left and right commuting squares in 5.10 are pullbacks. By pasting them, we have the outer commuting rectangle as a pullback for each $Z \in \mathcal{Y}$, which implies that $\mathfrak{p} \circ \mathfrak{q}$ is a Grothendieck fibration.

$$\begin{tikzcd}
\mathcal{Y}(Z, W) \arrow[r, "\mathfrak{q}_{Z,W}"] \arrow[d, "l\circ -"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(\mathfrak{q}Z, X) \arrow[r, "\mathfrak{p}_{\mathfrak{q}Z,X}"] \arrow[d, "u\circ -"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(\mathfrak{pq}Z, A) \arrow[d, "f\circ -"] \\
\mathcal{Y}(Z, Y) \arrow[r, "\mathfrak{q}_{Z,Y}"'] & \mathcal{X}(\mathfrak{q}Z, \mathfrak{q}Y) \arrow[r, "\mathfrak{p}_{\mathfrak{q}Z,\mathfrak{q}Y}"'] & \mathcal{C}(\mathfrak{pq}Z, \mathfrak{pq}Y)
\end{tikzcd} \tag{5.10}$$

(closed under pullback): Consider a (strict) pullback diagram in $\mathfrak{Cat}$:

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "L"] \arrow[d, "\mathfrak{q}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d, "\mathfrak{p}"] \\
\mathcal{D} \arrow[r, "F"'] & \mathcal{C}
\end{tikzcd} \tag{5.11}$$

where $\mathfrak{p}$ is a Grothendieck fibration. we want to show that $\mathfrak{q}$ is a Grothendieck fibration as well. Let $g : D \to \mathfrak{q}Y$ be a morphism in $\mathcal{D}$. So, $F(g) : F(D) \to \mathfrak{p} \circ L(Y)$, and it has a cartesian lift $\widetilde{F(g)} : X \to L(Y)$ in $\mathcal{X}$. Now, we have $\mathfrak{p}(\widetilde{F(g)}) = F(g)$. Since $\mathcal{Y}$ is the pullback category, we obtain a unique morphism $\tilde{g} : W \to Y$ in $\mathcal{Y}$ with $\mathfrak{q}(\tilde{g}) = g$ and $L(\tilde{g}) = \widetilde{F(g)}$. In particular, $L(W) = X$ and $\mathfrak{q}(W) = D$. It remains to show that $\tilde{g}$ is cartesian. For every Z in $\mathcal{Y}$, we can form the commutative cube below.

$$\begin{tikzcd}
& \mathcal{Y}(Z,W) \arrow[rr, "\tilde{g}\circ -"] \arrow[dl, "L_{Z,W}"'] \arrow[dd] & & \mathcal{Y}(Z,Y) \arrow[dl, "\mathfrak{q}_{Z,Y}"] \arrow[dd, "\mathfrak{q}_{Z,Y}"] \\
\mathcal{X}(LZ,LW) \arrow[rr] \arrow[dd, "\mathfrak{p}_{LZ,LW}"'] & & \mathcal{X}(LZ,LY) \arrow[dd] & \\
& \mathcal{D}(\mathfrak{q}Z,\mathfrak{q}W) \arrow[rr, "g\circ -"] \arrow[dl] & & \mathcal{D}(\mathfrak{q}Z,\mathfrak{q}Y) \arrow[dl, "F_{\mathfrak{q}Z,\mathfrak{q}Y}"] \\
\mathcal{C}(F\mathfrak{q}Z,F\mathfrak{q}W) \arrow[rr, "F(g)\circ -"'] & & \mathcal{C}(\mathfrak{p}LZ,\mathfrak{p}LY) &
\end{tikzcd}$$

The left and right faces are cartesian squares of sets since 5.11 is a cartesian square. The front face is also a cartesian square since $\mathfrak{p}$ is a fibration. Hence, the back face is also cartesian and this implies that $\mathfrak{q}$ is a Grothendieck fibration.

□

We are at a stage to define the category of all Grothendieck fibrations:

Definition 5.2.15. A *(pre)fibration map* between two (pre)fibrations $\mathfrak{q} : \mathcal{Y} \to \mathcal{D}$ and $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ consists of two functors $F : \mathcal{D} \to \mathcal{C}$ and $L : \mathcal{Y} \to \mathcal{X}$ such that

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "L"] \arrow[d, "\mathfrak{q}"'] & \mathcal{X} \arrow[d, "\mathfrak{p}"] \\
\mathcal{D} \arrow[r, "F"'] & \mathcal{C}
\end{tikzcd}$$

commutes, and moreover, L carries (resp.weakly) $\mathfrak{q}$-cartesian morphisms to (resp.weakly) $\mathfrak{p}$- cartesian morphisms. Grothendieck (pre)fibrations and (pre)fibration maps between them form a category which will be denoted by $\mathbf{Fib}$ ($\mathbf{PreFib}$). We also use $\mathbf{Fib}_{\mathcal{C}}$ to denote the subcategory of $\mathbf{Fib}$ which has only fibrations with codomain $\mathcal{C}$ as objects and morphisms are those morphisms of $\mathbf{Fib}$ which have $id_{\mathcal{C}}$ as bottom row functor.

Remark. Note that since F preserves identity morphisms, then L respects vertical morphisms. Thus, a fibration map produces a family of functors on fibre categories $(\mathcal{Y}_D \to \mathcal{X}_{F(D)} | D \in \mathbf{Ob}(\mathcal{D}))$.

Proposition 5.2.16. A functor $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a Grothendieck fibration if and only if $\mathbf{Fun}(\mathcal{E},\mathfrak{p}) : \mathbf{Fun}(\mathcal{E},\mathcal{X}) \to \mathbf{Fun}(\mathcal{E},\mathcal{C})$ is a Grothendieck fibration for any category $\mathcal{E}$

and for any functor $F : \mathcal{F} \to \mathcal{E}$,

$$\begin{tikzcd} \mathbf{Fun}(\mathcal{E}, \mathcal{X}) \arrow[r] \arrow[d] & \mathbf{Fun}(\mathcal{F}, \mathcal{X}) \arrow[d] \\ \mathbf{Fun}(\mathcal{E}, \mathcal{C}) \arrow[r] & \mathbf{Fun}(\mathcal{F}, \mathcal{C}) \end{tikzcd} \tag{5.12}$$

is fibration map.

Proof. We advise the reader to see [**?**][3.6] for a complete proof. □

LEMMA 5.2.17. Consider a (strict) pullback diagram in $\mathfrak{Cat}$:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "L"] \arrow[d, "\mathfrak{q}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d, "\mathfrak{p}"] \\ \mathcal{D} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd} \tag{5.13}$$

If $\langle \mathfrak{p} : \mathcal{X} \to \mathcal{C}, \rho \rangle$ is a cloven Grothendieck fibration and $F : \mathcal{D} \to \mathcal{C}$ has a right adjoint, then $L : \mathcal{Y} \to \mathcal{X}$ also has a right adjoint.

Proof. Suppose G is a right adjoint to F with counit ε. For any object X of $\mathcal{X}$, we can find a cartesian lift $\widetilde{\varepsilon(\mathfrak{p}X)}$ in cleavage ρ:

$$\begin{tikzcd} \rho_{\varepsilon_{\mathfrak{p}X}} X \arrow[r, dotted, "\widetilde{\varepsilon_{\mathfrak{p}X}}"] \arrow[d, mapsto, "\mathfrak{p}"'] & X \arrow[d, mapsto, "\mathfrak{p}"] \\ FG(\mathfrak{p}X) \arrow[r, "\varepsilon_{\mathfrak{p}X}"'] & \mathfrak{p}X \end{tikzcd}$$

Since $\mathfrak{p}(\rho_{\varepsilon_{\mathfrak{p}X}} X) = F(G\mathfrak{p}(X))$ and the diagram in 5.13 is a pullback, there must be a unique object Y in $\mathcal{Y}$ such that $\mathfrak{q}(Y) = G\mathfrak{p}(X)$ and $L(Y) = \rho_{\varepsilon(\mathfrak{p}X)} X$. Set $R(X) := Y$. So,

$$LR(X) = \rho_{\varepsilon(\mathfrak{p}X)} X \quad \text{and} \quad \mathfrak{q}R(X) = G\mathfrak{p}(X)$$

We want to make R into a functor which is right adjoint to L. Take an arbitrary morphism $u : X \to X'$ in $\mathcal{X}$. Because of naturality of counit ε, the bottom square of diagram below commutes, that is: $\varepsilon_{\mathfrak{p}X'} \circ FG(\mathfrak{p}(u)) = \mathfrak{p}(u) \circ \varepsilon_{\mathfrak{p}X}$. Since $\widetilde{\varepsilon_{\mathfrak{p}X'}}$ is a cartesian lift of $\varepsilon_{\mathfrak{p}X'}$ there must be a unique morphism $\rho(u) : \rho_{\varepsilon_{\mathfrak{p}X}} X \to \rho_{\varepsilon_{\mathfrak{p}X'}} X'$ such that $\mathfrak{p}(\rho(u)) = FG(\mathfrak{p}(u))$. Again by the fact that 5.13 is a pullback diagram, we find that the pair $\langle \rho(u), G\mathfrak{p}(u)\rangle$ induces $R(f) : R(X) \to R(X')$ in $\mathcal{Y}$. It is easy to see that R defined in this way is indeed a functor.

$$\begin{tikzcd}
& \rho_{\varepsilon_{\mathfrak{p}X}} X \arrow[rr, "\widetilde{\varepsilon_{\mathfrak{p}X}}"] \arrow[dl, dotted, "\rho(\mathfrak{p}(u))"'] \arrow[dd] & & X \arrow[dl, "u"] \arrow[dd, mapsto, "\mathfrak{p}"] \\
\rho_{\varepsilon(\mathfrak{p}X')} X' \arrow[rr] \arrow[dd, mapsto, "\mathfrak{p}"'] & & X' \arrow[dd] & \\
& FG(\mathfrak{p}X) \arrow[rr] \arrow[dl] & & \mathfrak{p}X \arrow[dl, "\mathfrak{p}(u)"] \\
FG(\mathfrak{p}X') \arrow[rr, "\varepsilon_{\mathfrak{p}X'}"'] & & \mathfrak{p}X' &
\end{tikzcd}$$

Finally, we will show that R is a right adjoint to L. For any $Z \in \mathcal{Y}$, we have the commutative diagram below:

$$\begin{tikzcd}
\mathcal{Y}(Z, RX) \arrow[r, "L_{Z,RX}"] \arrow[d, "\mathfrak{q}_{Z,RX}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(LZ, LRX) \arrow[r, "\widetilde{\varepsilon_{\mathfrak{p}X}} \circ -"] \arrow[d, "\mathfrak{p}_{LZ,LRX}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(LZ, X) \arrow[d, "\mathfrak{p}_{LZ,X}"] \\
\mathcal{D}(\mathfrak{q}Z, G\mathfrak{p}X) \arrow[r, "F_{\mathfrak{q}Z,\mathfrak{q}RX}"'] & \mathcal{C}(F\mathfrak{q}Z, FG\mathfrak{p}X) \arrow[r, "\varepsilon_{\mathfrak{p}X} \circ -"'] & \mathcal{C}(F\mathfrak{q}Z, \mathfrak{p}X)
\end{tikzcd}$$

According to part (vi) of proposition 5.2.2 the right square is a pullback.Also the left square is a pullback due to our first premise in 5.13. Whence the outer rectangle is a pullback diagram; however the composite of the bottom row is a bijection, so the composite on the top row must also be a bijection which proves that R is a right adjoint to L with counit $\widetilde{\varepsilon_{\mathfrak{p}X}}$. ◻

Proposition 5.2.18. suppose $\langle \mathfrak{p}, \rho\rangle : \mathcal{X} \to \mathcal{C}$ is a cloven Grothendieck fibration if and only if for very object $X \in \mathcal{X}$, induced functor $\mathfrak{p}_X : \mathcal{X}/X \to \mathcal{C}/\mathfrak{p}X$ has a right adjoint right inverse.

Proof. We define the right adjoint R_X of $\mathfrak{p}_X$ on objects of $\mathcal{C}/\mathfrak{p}X$ by $\mathfrak{r}_X(A \xrightarrow{f} \mathfrak{p}X) := \rho_f X \xrightarrow{\tilde{f}} X$. Thanks to the universal property of cartesian morphisms, this extends to a functor: Suppose g is a morphism between f_0 and f_1 in $\mathcal{C}/\mathfrak{p}X$.

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f_0"'] & & B \arrow[dl, "f_1"] \\ & \mathfrak{p}X & \end{tikzcd}$$

Since $\mathfrak{r}_X(f_1)$ is cartesian, there is a unique lift $\mathfrak{r}_X(g)\colon \mathfrak{r}_X(f_0) \to \mathfrak{r}_X(f_1)$ of g which renders the following diagram, in $\mathcal{X}$, commutative:

$$\begin{tikzcd} \mathfrak{r}_X(A) \arrow[rr, dotted, "\mathfrak{r}_X(g)"] \arrow[dr, "\mathfrak{r}_X(f_0)"'] & & \mathfrak{r}_X(B) \arrow[dl, "\mathfrak{r}_X(f_1)"] \\ & X & \end{tikzcd}$$

So, indeed $\mathfrak{r}_X(g)$ is a morphisms in $\mathcal{X}/X$. The unit of adjunction $\mathfrak{p}_X \dashv \mathfrak{r}_X$ is the natural transformation $\eta^X : 1_{\mathcal{X}/X} \Rightarrow \mathfrak{r}_X \circ \mathfrak{p}_X$ is defined component-wise as $\eta^X(Y \xrightarrow{u} X)$ to be unique vertical morphism in from u to $\mathfrak{r}_X \circ \mathfrak{p}_X(u)$ in $\mathcal{X}/X$:

$$\begin{tikzcd} Y \arrow[d, dotted] \arrow[dr, "u"] & \\ \mathfrak{r}_{\mathfrak{p}u}(X) \arrow[r, "\tilde{\mathfrak{p}u}"'] & X \end{tikzcd}$$

Also, it is readily observed that the counit is identity, and triangle identities hold. $\square$

PROPOSITION *5.2.19*. The functor s is a fibration if and only if the canonical functor $E^{[1]} \to B/s$ has right adjoint right inverse.

NOTE. The category $E^{[1]}$ is cotensor of E with simplex category $[1]$. Also, you may recognize $E^{[1]}$ as the arrow category of E.

5.2.3 Fibrations and indexed categories

We now begin to describe a process which associates to a normal split cloven Grothendieck fibration a 2-functor, to a cloven Grothendieck fibration a pseudo-functor, and to a cloven Grothendieck prefibration a lax functor.

Suppose $\langle \mathfrak{p} : \mathcal{X} \to \mathcal{C}, \rho \rangle$ is a cloven prefibration. We define $\mathfrak{X}\colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{Cat}$ as follows: For an object A of $\mathcal{C}$, we define $\mathfrak{X}(A)$ to be the fibre category whose objects and morphisms are objects and morphisms of $\mathcal{X}$ which are mapped to A and id_A by $\mathfrak{p}$, respectively. Note that for any morphism $f : A \to B$, we get a functor $\mathfrak{X}(f) : \mathfrak{X}(B) \to \mathfrak{X}(A)$ sending Y to $\rho_f Y$ and $u : Y \to Y'$ a morphism in $\mathfrak{X}(B)$ to $\rho_f(u)$, the unique vertical morphism which makes the following diagram commute:

$$
\begin{tikzcd}
& \rho_f Y \arrow[rr, "\rho(f,Y)"] \arrow[dl, dotted, "\rho_f(u)"'] \arrow[ddl, mapsto] & & Y \arrow[dl, "u"'] \arrow[ddl, mapsto] \\
\rho_f Y' \arrow[rr, "\rho(f,Y')"] \arrow[d, mapsto] & & Y' \arrow[d, mapsto] & \\
A \arrow[rr, "f"] & & B &
\end{tikzcd}
$$

Now suppose $f \colon A \to B$ and $g \colon B \to C$ are morphisms in $\mathcal{C}$. We have $\mathfrak{X}(gf)(Z) = \rho_{gf} Z$ and $\mathfrak{X}(f) \circ \mathfrak{X}(g)(Z) = \rho_f \rho_g Z$. Notice that since $\mathfrak{p}(\rho(g,Z) \circ \rho(f, \rho_g Z)) = \mathfrak{p}(\rho(gf, Z)) = gf$, precartesian property of morphisms $\rho(gf, Z)$ yields a unique vertical morphism $v : \rho_f \rho_g Z \to \rho_{gf} Z$ such that $\rho(gf, Z) \circ v = \rho(g, Z) \circ \rho(f, \rho_g Z)$. (The fact that composition of precartesian morphisms may not be precartesian precludes v from being an isomorphism.) All squares in the diagram below commute and this shows the choice of v is natural.

$$
\begin{tikzcd}
& \rho_f \rho_g Z \arrow[rr, "\rho(f,\rho_g Z)"] \arrow[dl, dotted, "\rho_f(\rho_g u)"'] \arrow[dd, "v"] & & \rho_g Z \arrow[rr, "\rho(g,Z)"] \arrow[dl, dotted, "\rho_g(u)"'] & & Z \arrow[dl, "u"'] \arrow[dd, equal] \\
\rho_f \rho_g Z' \arrow[rr, "\rho(f,\rho_g Z')"] \arrow[dd, "v'"'] & & \rho_g Z' \arrow[rr, "\rho(f,Z')"] & & Z' \arrow[dd, equal] & \\
& \rho_{gf} Z \arrow[rrrr, "\rho(gf,Z)"] \arrow[dl, "\rho_{gf}(u)"'] & & & & Z \arrow[dl, "u"'] \\
\rho_{gf} Z' \arrow[rrrr, "\rho(gf,Z')"] & & & & Z' &
\end{tikzcd}
$$

This turns $\mathfrak{X}$ into a lax functor. If $\mathfrak{p}$ was indeed a fibration then similar procedure gives us a pseudo-functor instead. So, we get 2-functors

$$
\begin{aligned}
\mathbf{PreFib}(\mathcal{B}) &\to \mathbf{LaxFun}(\mathcal{B}^{\mathrm{op}}, \mathfrak{Cat}) \\
\mathbf{Fib}(\mathcal{B}) &\to \mathbf{PsFun}(\mathcal{B}^{\mathrm{op}}, \mathfrak{Cat})
\end{aligned}
$$

Indeed, they are biequivalence of 2-categories. Suppose $\mathscr{X} : \mathcal{S}^{\mathrm{op}} \to \mathbf{2}\mathfrak{Cat}$ is a pseudo-functor. We would like to associate a Grothendieck fibration to $\mathscr{X}$ such that fibres are categories isomorphic to $\mathscr{X}(U)$ for objects U in $\mathcal{S}$. This is known as Grothendieck construction and the fired category is denoted by $\mathfrak{Gr}(\mathscr{X})$. The objects of $\mathfrak{Gr}(\mathscr{X})$ are pairs (U, A) where U is an object of $\mathcal{S}$ and A is in an object of category $\mathscr{X}(U)$. A morphism between such two pairs, say (V, B) and (U, A) consists of a morphism $i : V \to U$ in the context category $\mathcal{S}$, and a morphism $f : B \to i^*(A)$ in $\mathscr{X}(U)$. We express the data of a morphism as

$$(V, B) \xrightarrow{(i,f)} (U, A)$$

The composition of two composable morphisms

$$(W, C) \xrightarrow{(j,g)} (V, B) \xrightarrow{(i,f)} (U, A)$$

in $\mathfrak{Gr}(\mathscr{X})$ is given by

$$(W, C) \xrightarrow{(i \circ j,h)} (U, A)$$

where $h := \theta_{i,j}(A) \circ j^*(f) \circ g$.

$$\begin{tikzcd}
\mathscr{X}(W) & \mathscr{X}(V) & \mathscr{X}(U) \\
\\
C \arrow[d, "g"'] & & \\
j^*(B) \arrow[d, "j^*(f)"'] & & \\
j^*i^*(A) \arrow[d, "\theta_{i,j}(A)"'] & B \arrow[d, "f"] & \\
(ij)^*(A) & i^*(A) & A \\
\\
W \arrow[r, "j"'] & V \arrow[r, "i"'] & U
\end{tikzcd} \tag{5.14}$$

It's plain clear that $\Pi_{\mathscr{X}} : \mathfrak{Gr}(\mathscr{X}) \to \mathcal{S}$ sending objects (U, A) of $\mathfrak{Gr}(\mathscr{X})$ to U is a Grothendieck fibration. Moreover, every morphism in $\mathfrak{Gr}(\mathscr{X})$ factors as vertical morphism followed by a horizontal one:

$$
\begin{tikzcd}
(V,B) \arrow[d, "{(id,f)}"'] \arrow[dr, "{(i,f)}"] & \\
(V,i^*(A)) \arrow[r, "{(i,id)}"'] & (U,A)
\end{tikzcd}
$$

Corollary 5.2.20. Since monads in a 2-category $\mathfrak{Cat}$ are nothing but lax functors $1 \to \mathfrak{Cat}$, we conclude from the above equivalence that monads are indeed the same as prefibred categories over the terminal category.

5.2.4 Yoneda's lemma for fibred categories

Suppose $\mathcal{S}$ is a category. The following diagram of 2-categories expresses the relation between some of categories introduced in this chapter so far:

$$
\begin{tikzcd}
& \mathbf{Fun}(\mathcal{S}^{\mathrm{op}}, \mathfrak{Cat}) \arrow[rr, "\simeq"] & & \mathbf{Fib}_{\mathcal{S}} \\
\mathbf{Psh}(\mathcal{S}) \arrow[ur, hook] \arrow[rr, "\simeq"] & & \mathbf{DFib}_{\mathcal{S}} \arrow[ur, hook] & \\
\mathcal{S} \arrow[u, hook, "y_{\mathcal{S}}"] \arrow[uur, dotted] \arrow[rr, equal] & & \mathcal{S} \arrow[u, hook] \arrow[uur] &
\end{tikzcd}
$$

We have an embedding of $\mathcal{S}$ into $\mathbf{Fib}_{\mathcal{S}}$ by sending an object U of $\mathcal{S}$ to the slice fibration $\mathcal{S}/U \to \mathcal{S}$. The following result shows that slice fibrations are representable fibrations:

Proposition 5.2.21. For any object U in $\mathcal{S}$, and any fibred category $\mathfrak{p} : \mathcal{X} \to \mathcal{S}$ over $\mathcal{S}$, we have a family of equivalences of categories

$$\Phi_U : \mathbf{Fib}_{\mathcal{S}}(\mathcal{S}/U, \mathcal{X}) \simeq \mathcal{X}(U) : \Psi_U$$

natural in U.

Proof. Let's give notation $\mathfrak{u} : \mathcal{S}/U \to \mathcal{S}$ to the slice fibration. Now, for a fibration map $G : \mathfrak{u} \to \mathfrak{p}$, define $\Phi(G) := G(U \xrightarrow{1} U)$. Also for a natural transformation $\alpha : G \Rightarrow H$ over id_F, define $\Phi(\alpha) := \alpha(1_U)$. Φ is a functor. To define Ψ on objects, take X in $\mathcal{X}$ over U. We define $\Psi(X) : \mathcal{S}/U \to \mathcal{X}$ as the following functor: $\Psi(X)(V \xrightarrow{f} U) = f^*X$, and for $h : f' \to f$ in $\mathcal{S}/U$, $\Psi(X)(f' \xrightarrow{h} f) = \tilde{h}$. One can check that $\Psi(X)$ is indeed a functor. Moreover, $\mathfrak{p} \circ \Psi(X) = \mathfrak{u}$ and $\Psi(X)$ preserves cartesian morphisms of $\mathcal{S}/U$ (that is every morphism of $\mathcal{S}/U$ since slice fibration is discrete.) by lemma (5.2.4). Ψ can be promoted to a functor. Note that $\Psi \circ \Phi(G) \cong G$ for any fibration map G; sine G sends each morphism of $\mathcal{S}/U$ to a cartesian one in $\mathcal{X}$, $G(f\colon f \to 1)$ is cartesian. Hence $\Psi \circ \Phi(G)(f) = f^*(G_{1_U}) \cong G(f)$.

$\square$

5.2.5 Categories fibred in groupoids

We start by the following observation:

Lemma 5.2.22. Suppose $\mathscr{X} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Grpd}$ is a pseudo-functor. Every morphism in $\mathfrak{Gr}(\mathscr{X})$ is $\Pi_{\mathscr{X}}$-cartesian.

Proof. To prove this, take any morphism $(i, f) : (V, B) \to (U, A)$ in $\mathfrak{Gr}(\mathscr{X})$. Suppose also that $(k, g) : (W, C) \to (U, A)$ in $\mathfrak{Gr}(\mathscr{X})$ such that $i \circ j = k$. Now since $\mathscr{X}$ is evaluated in $\mathbf{Grpd}$, f, and g are isomorphisms and we can define $p : C \to j^*B$ as $p := j^*(f)^{-1} \circ \theta_{i,j}(A)^{-1} \circ g$. It is now straightforward to see that (j, p) is the unique map in $\mathfrak{Gr}(\mathscr{X})$ which makes the upper triangle commute in the diagram below:

$$\begin{tikzcd}
(W,C) \arrow[d, mapsto] \arrow[dr, dotted, "(j,p)"'] \arrow[drr, "(k,g)"] & & \\
W \arrow[dr, "j"'] \arrow[drr, "k"] & (V,B) \arrow[r, "(i,f)"'] \arrow[d, mapsto] & (U,A) \arrow[d, mapsto] \\
& V \arrow[r, "i"'] & U
\end{tikzcd} .$$

$\square$

Definition 5.2.23. For a Grothendieck fibration $\mathscr{X} \to \mathcal{S}$ isomorphic to $\Pi_{\mathscr{X}}$ for a pseudo-functor $\mathscr{X} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Grpd}$ as above, the category $\mathscr{Y}$ is called ***a category fibred in groupoids over $\mathcal{S}$***.

Categories fibred in groupoids have an easier description than categories fibred in categories. We do not need to worry about existence of cartesian lifts since every lift is cartesian because of 5.2.22.

Theorem 5.2.24. $\mathscr{X}$ is category fibred in groupoids over $\mathcal{S}$ with the functor $\pi : \mathscr{X} \to \mathcal{S}$ if and only if

- (Lifting arrows condition) For every arrow $f : V \to U$ in $\mathcal{S}$ and every object X in $\mathscr{X}$ sitting above U, there is an arrow $F : Y \to X$ with $\pi(F) = f$.
- (Lifting triangles condition) Given a commutative triangle in $\mathcal{T}$, and a lift of horn of f and g, there is a unique arrow $H : Y \to Z$ such that $F \circ H = G$ and $\pi(H) = h$.

$$\begin{tikzcd} Z \arrow[dd, dotted, "\exists! H"'] \arrow[dr, "G"] & & & W \arrow[dd, "h"'] \arrow[dr, "g"] & \\ & X & \mapsto & & U \\ Y \arrow[ur, "F"'] & & & V \arrow[ur, "f"'] & \end{tikzcd}$$

Remark. By taking nerves we get quasi-categories $N(\mathscr{X})$ and $N(\mathcal{S})$, and we can express the two lifting conditions as two horn-filling conditions below:

$$\begin{tikzcd} \Lambda^1[1] \arrow[r] \arrow[d, "i"'] & N(\mathscr{X}) \arrow[d, "N(\pi)"] \\ \Delta[2] \arrow[r] \arrow[ur, dashed, "\exists"] & N(\mathcal{S}) \end{tikzcd} \qquad \begin{tikzcd} \Lambda^2[2] \arrow[r] \arrow[d, "i"'] & N(\mathscr{X}) \arrow[d, "N(\pi)"] \\ \Delta[2] \arrow[r] \arrow[ur, dashed, "\exists!"] & N(\mathcal{S}) \end{tikzcd}$$

Theorem 5.2.25. A pseudo-functor $\mathscr{X} : \mathcal{S}^{\mathrm{op}} \to \mathfrak{Cat}$ gives rise to a category fibred in groupoids if and only if it factors through the embedding $\mathbf{Grpd} \hookrightarrow \mathfrak{Cat}$.

$$\begin{tikzcd} & \mathbf{Grpd} \arrow[dr, hook] & \\ \mathcal{S}^{\mathrm{op}} \arrow[ur, dotted] \arrow[rr, "\mathscr{X}"'] & & \mathfrak{Cat} \end{tikzcd}$$

5.2.6 Stacks

The idea of stacks is a categorification of sheaves: given an indexed functor $\mathcal{X} : \mathcal{S}^{\mathrm{op}} \to \mathfrak{Cat}$ and a covering family $\{U_i \to U | i \in I\}$ in $\mathcal{S}$, we would like to see under what conditions we can glue fibre categories $\mathcal{X}(U_i)$ together to get $\mathcal{X}(U)$ up to an equivalence. This condition is known as descent condition and is generalization of matching families for presheaves.

Definition 5.2.26. Suppose $\mathcal{X}$ is a fibred category over site $(\mathcal{S}, \mathbb{J})$ and $R = \{U_i \to U | i \in I\}$ is a covering family for object U in base $\mathcal{S}$. The category $\mathbf{Desc}(\mathcal{S}, R)$ of ***descent data*** for R is constructed as follows:

1. Objects are pairs of families $((X_i)_{i \in I}, (\phi_{ij})_{i,j \in I})$ where X_i is an object of $\mathcal{X}(U_i)$ and $\phi_{ij} : p_i^*(X_i) \to p_j^*(X_j)$ is a morphism in $\mathcal{X}$ where the base diagram is a pullback diagram

$$\begin{tikzcd}
 & & p_i^*(X_i) \arrow[d, "\phi_{ij}"] \arrow[lldd, dotted, "\tilde{p_i}"'] & & \\
 & & p_j^*(X_j) \arrow[dd, mapsto] \arrow[rrd, dotted, "\tilde{p_j}"] & & \\
X_i \arrow[dd, mapsto] & & & & X_j \arrow[dd, mapsto] \\
 & & U_{ij} \arrow[lld, dotted, "p_i"'] \arrow[rrd, dotted, "p_j"] & & \\
U_i \arrow[rrrd, "u_i"] & & & & U_j \arrow[ld, "u_j"'] \\
 & & & U &
\end{tikzcd} \tag{5.15}$$

and ϕ_{ij} satisfy compatibility conditions:

5.3 Fibration internal to 2-categories

Now, we are ready to define fibrations within a 2-category $\mathcal{K}$. Using the techniques we demonstrated in 4.4, we give the following definition from [**?**]:

fibrations are closed under composition and pullback along arbitrary functors,

Definition 5.3.1. A 1-cell $s : E \to B$ in a 2-category $\mathcal{K}$ is a fibration if and only if $\mathcal{K}(X, s)$ is fibration for all $X \in \mathcal{K}_0$, and moreover, for every map $f : Y \to X$ in $\mathcal{K}_1$:

$$\begin{tikzcd}
\mathcal{K}(X,E) \arrow[r] \arrow[d] & \mathcal{K}(Y,E) \arrow[d] \\
\mathcal{K}(X,B) \arrow[r] & \mathcal{K}(Y,B)
\end{tikzcd}$$

is a map of categorical fibrations.

However, to be able to express similar result to 1-categorical case, we need our chosen 2-category to be finitely complete, a constraint which ensures the existence of comma objects. The following proposition is from [**?**]

PROPOSITION 5.3.2. In any finitely complete 2-category $\mathcal{K}$, $s : E \to B$ is fibration if and only if for any generic subobject of B, i.e. $b : X \to B$, the map $i : X \times_B E \to b/s$ has a right adjoint in the 2-category $\mathcal{K}/X$.

$$\begin{tikzcd}
X \times_B E \arrow[dr, dotted, "i"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& b/s \arrow[r, "d_1"] \arrow[d, "d_0"] & B \arrow[d, "s"] \\
& A \arrow[r, "b"'] \arrow[ur, Rightarrow, "\lambda"] & D
\end{tikzcd}$$

PROPOSITION 5.3.3. In any finitely complete 2-category, composition of fibrations is a fibration and the pullback of a fibration along any morphism is a fibration.

5.4 Examples of Grothendieck (op)fibrations in Logic, Algebra, and Geometry

EXAMPLE 5.4.1. The category $\mathbf{Sch}$ of schemes is fibred over the category $\mathbf{Top}$ of topological spaces.

$$\begin{tikzcd}
\mathbf{Sch} \arrow[d] \\
\mathbf{Top}
\end{tikzcd}$$

EXAMPLE 5.4.2. Suppose $\mathcal{C}$ is a locally small left exact category (i.e. finitely complete). Let G be an internal group in $\mathcal{C}$, in particular an object G in $\mathcal{C}$ equipped with three arrows corresponding to multiplication, inversion, and unit of multiplication which are compressed in the morphism $1 + G + G^2 \xrightarrow{\{e,i,m\}} G$, satisfying group axioms. We now introduce category G-$\mathbf{Bun}(\mathcal{C})$ of G-bundles in $\mathcal{C}$, sitting over $\mathcal{C}$. The objects of G-$\mathbf{Bun}(\mathcal{C})$ are pair of morphisms $p : E \to B$ and $\mu : G \times E \to E$ such that the following diagram commutes:

$$\begin{tikzcd} G \times E \arrow[r, "\mu"] \arrow[d, "\pi_2"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

where π_2 is the second projection. This piece of information reads as p is a G-bundle with μ acting on each fibre.

A morphism of bundles should preserve the actions of G on fibres. So, if $p : E \to B$ and $p' : E' \to B'$ are two G-bundles, then we define G-$\mathbf{Bun}(\mathcal{C})(p, p')$ to be pair of morphisms $\psi : B \to B'$ and $\phi : E \to E'$ such that the diagrams below commute:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\psi"'] & B' \end{tikzcd}$$

$$\begin{tikzcd} G \times E \arrow[r, "id\times\phi"] \arrow[d, "\mu"'] & G \times E' \arrow[d, "\mu'"] \\ E \arrow[r, "\phi"'] & E' \end{tikzcd}$$

The first of these diagrams says that ϕ transfers, for every "$b \in B$", the fibre over b into the fibre over $\psi(b)$. The second one says that within these transfers, action of G on fibres is respected, i.e. "$\phi(g.e) = g.\phi(e)$".

we can put the all this data for a morphism $< \phi, \psi >: (p, \mu) \to (p', \mu')$ into a single commutative diagram in below:

$$
\begin{tikzcd}
 & G \times E \arrow[rr, "\mu"] \arrow[dl, "\pi_2"'] \arrow[dd] & & E \arrow[dl, "p"] \arrow[dd, "\phi"] \\
E \arrow[rr, "p"] \arrow[dd, "\phi"'] & & B \arrow[dd, "\psi"] & \\
 & G \times E' \arrow[rr, "\mu'"] \arrow[dl, "\pi_2'"'] & & E' \arrow[dl, "p'"] \\
E' \arrow[rr, "p'"'] & & B' &
\end{tikzcd}
$$

Now, consider the forgetful functor $U : G\text{-}\mathbf{Bun}(\mathcal{C}) \to \mathcal{C}$ sending a G-bundle (p, μ) to B and a bundle map $< \phi, \psi >$ to ψ. We claim U is a Grothendieck fibration. Take any bundle $(p : E \to B, \mu : G \times E \to E)$ in $G\text{-}\mathbf{Bun}(\mathcal{C})$ and any morphism $\psi : X \to B$ in $\mathcal{C}$. We claim that $< p^*\psi, \psi >$ obtained by the following pullback is a U-cartesian lift for ψ.

$$
\begin{tikzcd}
X \times_B E \arrow[r, "p^*\psi"] \arrow[d, "\psi^* p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
X \arrow[r, "\psi"'] & B
\end{tikzcd}
$$

First, we need to find an action of G on object $X \times_B E$. Note that

$$p \circ \mu \circ (id \times p^*\psi) = p \circ \pi_2 \circ (id \times p^*\psi) = p \circ p^*\psi \circ \pi_2 = \psi \circ \psi^* p \circ \pi_2$$

By the universal property of the pullback diagram, there is a unique morphism $\tilde{\mu}$ such that

$$\psi^* p \circ \tilde{\mu} = \psi^* p \circ \pi_2$$

$$\mu \circ (id \times p^*\psi) = p^*\psi \circ \tilde{\mu}$$

$\tilde{\mu}$ is the desired action, the first equation tells us that $\psi^* p$ is indeed a bundle over X and the second equation establishes the fact that $\psi^* p$ is equivariant. Altogether they prove $< p^*\psi, \psi >$ is a lift of ψ.

$$
\begin{tikzcd}
G \times (X \times_B E) \arrow[r, "id\times p^*\psi"] \arrow[dd, "\pi_2"'] \arrow[dr, dotted, "\tilde{\mu}"] & G \times E \arrow[dr, "\mu"] & \\
& X \times_B E \arrow[r, "p^*\psi"'] \arrow[d, "\psi^* p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
X \times_B E \arrow[r, "\psi^* p"'] & X \arrow[r, "\psi"'] & B
\end{tikzcd}
$$

The fact that our chosen lift $< p^*\psi, \psi >$ is gotten by taking pullback (along the morphism ψ) can easily be shown sufficient for proving it is indeed a U-cartesian lift.

5.5 2-FIBRATION OF 2-CATEGORIES

Our main reference is [4]. Suppose $P\colon \mathcal{E} \to \mathcal{B}$ is a 2-functor. We call a 1-cell 1-cartesian if it satisfies the condition of definition 5.2.1. Inspired by lemma 5.2.3 we define 2-cartesian 1-cells in $\mathcal{E}$ as follows.

Definition 5.5.1. A 1-cell $u : X \to Y$ is *2-cartesian* with respect to P whenever for each 0-cell W in $\mathcal{E}$ the following commuting square is a (strict) pullback diagram in $\mathfrak{Cat}$.

$$
\begin{tikzcd}
\mathcal{E}(W, X) \arrow[r, "u_*"] \arrow[d, "P_{W,X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}(W, Y) \arrow[d, "P_{W,Y}"] \\
\mathcal{B}(PW, PX) \arrow[r, "P(u)_*"'] & \mathcal{B}(PW, PY)
\end{tikzcd}
$$

In basic terms this means that u is 1-cartesian and for every 2-cell $\alpha\colon v \Rightarrow w\colon W \to Y$ and every 2-cell $\sigma\colon g \Rightarrow h\colon PW \to PX$ with $P(\alpha) = P(u) \centerdot \sigma$ there is a unique lift ϕ

of σ such that $u \centerdot \phi = \alpha$.

$$\begin{tikzcd}
W \arrow[r, dotted, bend left, "\overline{g}"] \arrow[r, dotted, bend right, "\overline{h}"'] \arrow[r, phantom, "\phi\Downarrow"] \arrow[rr, bend left=60, "v"] \arrow[rr, bend right=60, "w"'] \arrow[rr, phantom, "\Downarrow\alpha"] & X \arrow[r, "u"] & Y
\end{tikzcd}$$

$$P \Big\downarrow$$

$$\begin{tikzcd}
PW \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\sigma\Downarrow"] \arrow[rr, bend left=60, "Pv"] \arrow[rr, bend right=60, "Pw"'] & PX \arrow[r, "Pu"] & PY
\end{tikzcd}$$

Definition 5.5.2. A 2-cell $\alpha\colon u \Rightarrow v\colon X \to Y$ in $\mathcal{E}$ is *cartesian* if it is cartesian as a 1-cell for the functor $P_{x,y}\colon \mathcal{E}(X, Y) \to \mathcal{B}(PX, PY)$.

Definition 5.5.3. A 2-functor $P\colon \mathcal{E} \to \mathcal{B}$ is a *2-fibration* if

1. any 1-cell in $\mathcal{B}$ of the form $f\colon B \to PE$ has a 2-cartesian lift;
2. P is a local fibration, that is for any pair of objects X, Y in $\mathcal{E}$, the functor $P_{X,Y}\colon \mathcal{E}(X, Y) \to \mathcal{B}(PX, PY)$ is a Grothendieck fibration.
3. cartesian 2-cells are closed under pre-composition and post-composition with arbitrary 1-cells.

Remark. The second condition is equivalent to say that for any $g \in E$ and $\alpha\colon f \Rightarrow Pg$, there is a cartesian 2-cell $\sigma\colon f \Rightarrow g$ with $P\sigma = \alpha$;

Now, we extend our definition to fibration between 2-categories. Our guiding principle is the following:

- For any set B we have an isomorphism of categories:

$$\mathbf{Set}/B \cong \mathbf{Fun}(B, \mathbf{Set}) \tag{5.16}$$

which for a fibration $p : E \to B$ is a fibration gives us the functor that sends $b \in B$ to $p^{-1}B \in \mathbf{Set}$

- We observed that there is an equivalence of 2-categories between categorical fibrations (i.e. cloven Grothendieck fibrations) over category $\mathcal{B}$ and pseudo-functors from $\mathcal{B}^{\mathrm{op}}$ to the strict 2-category $\mathfrak{Cat}$.

$$\mathcal{F}ib_{\mathscr{B}} \simeq \mathbf{PsFun}(\mathcal{B}^{\mathrm{op}}, \mathfrak{Cat}) \tag{5.17}$$

We would like to think that whatever our definition of 2-categorical fibrations is there should be a 3-equivalence of the form below:

$$\mathcal{F}ib^2_{\mathscr{B}} \simeq \mathbf{PsFun}(\mathscr{B}^{\mathrm{op}}, \mathbf{2}\mathfrak{Cat}) \tag{5.18}$$

where $\mathbf{2}\mathfrak{Cat}$ is the strict 3-category of 2-categories.

More generally, continuing this way, we would like to define n-fibrations in a way so that we have an $n+1$-equivalence of $n+1$ categories where $\mathbf{n}\,\mathfrak{Cat}$ is the strict $n+1$-category of n-categories.

$$\mathcal{F}ib^n_{\mathscr{B}} \simeq \mathbf{PsFun}(\mathscr{B}^{\mathrm{op}}, \mathbf{n}\,\mathfrak{Cat}) \tag{5.19}$$

Now, we would like to remind the reader that in the case of 1-fibration we characterized cartesian lifts by certain pullbacks in **??**. We imitate that for 2-fibrations:

Definition 5.5.4.

$$\begin{tikzcd}
Hom(x, e') \arrow[r, "S_{x,e'}"] \arrow[d, "p\circ-"'] \arrow[dr, phantom, "\lrcorner", very near start] & Hom(Sx, Se') \arrow[d, "Sp\circ-"] \\
Hom(x, e) \arrow[r, "S_{x,e}"'] & Hom(Sx, Se)
\end{tikzcd}$$

6

Concluding Remarks

Introduction:

In the first section of this chapter we will combine the techniques and results of previous chapters to compare homotopy category of spaces (extracting global data) and sheafification process (passing to local data) by means of localization introduced in chapter 3.

6.1 Local and Global

We think of spaces as having local and non-local properties. Sheaf theory is roughly the study of local properties of spaces and homotopy theory measures non-local aspects of spaces. The following theorems give a unified account of these aspects via localization in two different categories with calculus of fractions.

Theorem 6.1.1. Let $(\mathcal{K}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ be a Quillen model category (e.g. $\mathbf{Top}_c$, $s\mathbf{Set}$, Quasi-categories, etc.) Let $\mathcal{K}_{fc}$ be the subcategory of fibrant-cofibrant objects in $\mathcal{K}$. Then, $(\mathcal{K}_{fc}, \mathcal{W})$ admits a calculus of fractions and furthermore, $Ho(\mathcal{K})$ is equivalent to $\mathcal{K}_{fc}[\mathcal{W}^{-1}]$.

Proof. See [8][page 806, Appendix] $\square$

Theorem *6.1.2.* Let $\mathcal{C}$ be a site, and $\mathcal{W}$ the collection of local isomorphism in $\mathbf{Psh}(\mathcal{C})$. Then $(\mathbf{Psh}(\mathcal{C}), \mathcal{W})$ admits a calculus of fractions and furthermore, $\mathbf{Psh}(\mathcal{C})[\mathcal{W}^{-1}] \cong \mathbf{Sh}(\mathcal{C}, J)$ where J is the unique site corresponding to $\mathcal{W}$.

Proof. See [7][page 399, chapter 16.] $\square$

There are two main roads to take for improving the results in these notes. Both of them revolve around the unified treatment of local and global information about generalised spaces which for us are higher topoi (n-topoi and ∞-topi) . So, first, let's give a definition of $(\infty, 1)$-topos. This definition is due to Charles Rezk and Jacob Lurie. See [8][Chapter6].

Definition *6.1.3.* Let $\mathcal{X}$ be a simplicial set, and $\mathcal{P}(\mathcal{X}) := Fun(\mathcal{X}^{op}, \mathcal{S}p)$. $\mathcal{P}(\mathcal{X})$ is called the ∞-category of presheaves on $\mathcal{X}$.

Definition *6.1.4.* A category $\mathcal{E}$ is a (Grothendieck) topos if it is a left exact localization of the category of presheaves of sets on a small category $\mathcal{C}_0$.

Definition *6.1.5.* Let $\mathcal{X}$ be a quasi-category. We say that $\mathcal{X}$ is an ∞-topos if there exist a small quasi-category $\mathcal{C}$ and an accessible left exact localization $\mathcal{P}(\mathcal{C}) \to \mathcal{X}$.

6.1.1 Higher Topoi and Non-commutative Geometry

One road is the study of fibrations, and classifying spaces for higher topoi, extending the result which are already established in the case of 1-topoi. In [10], the author develops a generalized geometric realization for topoi, and establishes a weak homotopy equivalence between the classifying space and the classifying topos of any small (topological) category. Topos theory is used to determine which structures are classified by "classifying" spaces. There has not yet been an analogue of this construction for higher topoi.

First let's start with the following generalisation example (3.3.4). The Grothendieck group completion of the decategorification of the category of vector bundles on some topological space X produces the group known as the topological K-theory of X. It would like to further study this theorem for principal G-bundles instead of vector bundles and use the construction of [10] to relate the isomorphism classes of G-bundles, on one hand to classifying topoi, and on the other hand to Grothendieck group completion. This is one of the starting point to higher non-commutative geometry.

To make this more clear, we quote G. Segal from [3]:

"Passing from commutative to noncommutative geometry we lose not only the idea of points — for the "points" of a noncommutative algebra are its irreducible representations, which do not usually form a reasonable space — but also, in general, the idea that different parts of the space can be studied independently. It might at first seem that we have lost geometry altogether. That is not the case, but it is true that the category of noncommutative spaces has some features resembling the homotopy category more than the category of spaces. Indeed much of noncommutative geometry is about the homotopy-types of the objects — it is focussed on algebraic-topological ideas like cyclic homology and K-theory".

For the interested reader, more on interplay between abstract homotopy theory, topos theory and non-commutative geometry can be found in [1]. Of course, one of the main connections of non-commutative geometry is to theory of C^*-algebras and quantum theory.

6.1.2 Higher Topoi and Cohesive Homotopy Type Theory

One of the most recent advancement in the field of homotopy type theory is cohesive homotopy type theory, which in a way is to relate homotopy type theory to the internal logic of cohesive higher topoi. Formalizations of some structures in cohesive ∞-topoi in terms of homotopy type theory is the main idea of cohesive homotopy type theory.

In [13], Michael Shulman calls these type theories "spatial" and "cohesive", in which the types have independent topological and homotopical structure. In fact, cohesive homotopy type theory is in a way reconciliation of local and global structures of spaces. The following examples should give us some intuition about the idea here.

The *delooping* of an object X is pointed object $\mathbf{B}X$ such that X is the loop space object of $\mathbf{B}X$:

$$X \simeq \Omega(\mathbf{B}X)$$

If $X = G$ is a groupoid with one object (i.e. a group), then its delooping

- in the category $\mathbf{Top}_c$ is the classifying space $\mathcal{B}G$. (Constructed in Chapter 2 using nerve construction.)
- in the ∞-category of ∞-groupoids is the one-object groupoid $\mathbf{B}G$.

Homotopy hypothesis tells us the geometric realization of the groupoid $\mathbf{B}G$ is equivalent to the classifying space $\mathcal{B}G$:

$$|\mathbf{B}G| \simeq \mathcal{B}G$$

EXAMPLE *6.1.6.* The smooth circle S^1 (as a 1-dimensional closed manifold) in cohesive homotopy type theory is 0-truncated and has 0-type with $\pi_{\mathbf{1}}(S^1)$ is the 1-truncated and 1-type $\mathbf{B}\mathbb{Z}$.

EXAMPLE *6.1.7.* In cohesive homotopy type theory, continuum real line object can be realised as $\mathbb{Z} \hookrightarrow \mathbb{A}^1$ such that $\pi_{\mathbf{1}}(\mathbb{A}^1) \simeq *$.

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