Report 3.9

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On p. 74, Agresti informally justifies the needed convergence to normality of the right-hand side of (3.1.23) when the MLE is substituted for σ. Provide a formal version of his "proof," justifying each of his steps. (Hint: Use Slutsky's theorem.)

3.1.23 states that

$$\frac{g(\widehat{\boldsymbol{\pi}}) - g(\boldsymbol{\pi})}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1).$$

The delta method states that

$$\sqrt{n}[g(Y_n) - g(\theta)] \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2)$$

if $g'(\theta)$ exists and is nonzero, when $\sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0, \sigma^2)$.

Suppose we have counts $\{n_i, i = 1, \dots, c\}$ that follow a multinomial $(n, \{\pi_i\})$ distribution. The sample proportion $\hat{\pi}_i$ has mean and variance

$$E(\hat{\pi}_i) = \pi_i \qquad \text{and} \qquad \text{var}(\hat{\pi}_i) = \pi_i(1 - \pi_i)/n.$$

By consistency of MLE's, we know that $\widehat{\boldsymbol{\pi}} \xrightarrow{p} \boldsymbol{\pi}$, and because of regularity conditions it follows that $\sqrt{n}(\widehat{\boldsymbol{\pi}} - \boldsymbol{\pi}) \xrightarrow{d} N(0, \sigma^2)$.

By Slutsky's theorem, we know that since $\widehat{\sigma} \xrightarrow{p} \sigma$, $\sigma/\widehat{\sigma} \to 1$ (by consistency of MLE's), which gives

$$\sqrt{n}\frac{g(\widehat{\boldsymbol{\pi}}) - g(\boldsymbol{\pi})}{\sigma} = \sqrt{n}\frac{g(\widehat{\boldsymbol{\pi}}) - g(\boldsymbol{\pi})}{\sigma}\frac{\sigma}{\widehat{\sigma}}$$

and (by Slutsky's), since the left part converges to a $N(0,1)$ by the delta method, we know that

$$\sqrt{n}\frac{g(\widehat{\boldsymbol{\pi}}) - g(\boldsymbol{\pi})}{\hat{\sigma}} \xrightarrow{d} N(0,1).$$