Street for pedestrians

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Abstract

The singular aim of these notes is to give a less condensed and more clear explanation of Ross Street's significant paper (Street 1974). We highly encourage reader to also refer to (Kock 1995) and (Lack 2000).

1 Pseudo algebras for strict 2-monads

Definition *1.1.* Let $\mathcal{K}$ be a 2-category and $(T\colon \mathcal{K} \to \mathcal{K}, i\colon 1 \Rightarrow T, m\colon T^2 \Rightarrow T)$ a strict 2-monad on $\mathcal{K}$. A ***pseudo-algebra*** of T consists of

i. a 0-cell A in $\mathcal{K}$

ii. a 1-cell $a : TA \to A$

iii. invertible 2-cells $\zeta\colon 1_A \Rightarrow a \circ i_A$ and $\theta\colon a \circ Ta \Rightarrow a \circ m_A$

$$\begin{tikzcd} A \arrow[d, "i_A"'] \arrow[dr, "1"] \arrow[dr, phantom, "\Leftarrow\zeta", near start, shift right=3] & \\ TA \arrow[r, "a"'] & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd} \tag{1}$$

subject to the following coherence axioms:

$$(\theta \centerdot m_{TA}) \circ (\theta \centerdot T^2a) = (\theta \centerdot Tm_A) \circ (a \centerdot T\theta)$$

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expressed by equality of pasting diagrams:

$$
\begin{tikzcd}
T^3A \arrow[r, "T^2a"] \arrow[d, "m_{TA}"'] & T^2A \arrow[d, "m_A"] \arrow[dr, bend left, "Ta"] & \\
T^2A \arrow[r, "Ta"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "a"] & TA \arrow[l, Rightarrow, "\theta"'] \arrow[dl, bend left, "a"] \\
TA \arrow[r, "a"'] & A &
\end{tikzcd}
=
\begin{tikzcd}
 & T^3A \arrow[r, "T^2a"] \arrow[d, "Tm_A"'] \arrow[dl, bend right, "m_{TA}"'] \arrow[dr, phantom, "T.\theta\Downarrow"] & T^2A \arrow[d, "Ta"] \\
T^2A \arrow[dr, bend right, "m_A"'] & T^2A \arrow[r, "Ta"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "a"] \\
 & TA \arrow[r, "a"'] & A
\end{tikzcd}
\tag{2}
$$

and

$$(\theta \centerdot Ti_A) \circ (a \centerdot T\zeta) = id_a = (\theta \centerdot i_{TA}) \circ (\zeta \centerdot a)$$

expressed by equality of pasting diagrams:

$$
\begin{tikzcd}
TA \arrow[r, "1_{TA}"] \arrow[d, "Ti_A"'] \arrow[dd, bend right=60, "1_{TA}"'] \arrow[dr, phantom, "T.\zeta\Downarrow"] & TA \arrow[d, "1_{TA}"] \\
T^2A \arrow[r, "Ta"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "a"] \arrow[d, "1_{TA}"'] & A \arrow[d, "1_A"] \\
TA \arrow[r, "a"] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "a"] \arrow[d, "i_{TA}"'] \arrow[dd, bend right=60, "1_{TA}"'] & A \arrow[d, "i_A"] \arrow[dd, bend left=60, "1_A"] \\
T^2A \arrow[r, "Ta"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "a"] \arrow[r, phantom, "\overset{\Leftarrow}{\zeta}"] & {} \\
TA \arrow[r, "a"'] & A
\end{tikzcd}
\tag{3}
$$

DEFINITION *1.2.* Suppose $(a, \zeta_A, \theta_A) : TA \to A$ and $(b, \zeta_B, \theta_B) : TB \to B$ are pseudo-algebras of a 2-monad T. A ***lax morphism*** from a to b consists of a 1-cell $f \colon A \to B$ and a 2-cell $\check{f}$

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\check{f}\Downarrow"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

in such a way that

- $f \centerdot \zeta_A = (\check{f} \centerdot i_A) \circ (\zeta_B \centerdot f)$ and
- $(f \centerdot \theta_A) \circ (\check{f} \centerdot Ta) \circ (b \centerdot T\check{f}) = (\check{f} \centerdot m_A) \circ (\theta_B \centerdot T^2 f)$

DEFINITION *1.3.* A 2-monad $T : \mathcal{K} \to \mathcal{K}$ is said to be ***lax idempotent*** if given any two (pseudo) T-algebras $a \colon TA \to A$, $b \colon TB \to B$ and a 1-cell $f \colon A \to B$, there exists a unique 2-cell $\check{f} \colon b \circ Tf \Rightarrow f \circ a$ rendering $(f, \check{f})$ a lax morphism of pseudo T-algebras.

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\check{f}\Downarrow"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Remark *1.4*. Dually, reverse the direction of $\check{f}$ in definition 1.3, then we get the notion of ***co-lax idempotent*** monad.

2 KZ-monads

Definition *2.1*. A 2-monad $T : \mathcal{K} \to \mathcal{K}$ is said to be ***KZ-monad***[1] if $m \dashv i \centerdot T$ in the 2-category $[\mathcal{K}, \mathcal{K}]$ with identity counit.

Remark *2.2*. Dual to the definition above, we define a monad T to be a ***co-KZ-monad*** by requiring $i \centerdot T \dashv m$ with identity unit.

Suppose T is a co-KZ-monad and $i \centerdot T \dashv m$. In particular unit of this adjunction is identity since $m \circ (i \centerdot T) = 1$. Moreover, the identity 2-cell

$$\begin{tikzcd} T \arrow[r, "1"] \arrow[dr, phantom, "id\Downarrow"] & T \\ T \arrow[u, "1"] \arrow[r, "T\centerdot i"'] & T^2 \arrow[u, "m"'] \end{tikzcd}$$

has a mate

$$\begin{tikzcd} T \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\tau\Downarrow"] & T \arrow[d, "i\centerdot T"] \\ T \arrow[r, "T\centerdot i"'] & T^2 \end{tikzcd} \tag{4}$$

with property that $m \centerdot \tau = id_{1_T}$. Suppose $a \colon TA \to A$ is a pseudo algebra. We now would like to calculate the composite 2-cell

$$\begin{tikzcd} TA \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\tau_A\Downarrow"] & T^2A \arrow[r, bend left, "a\circ Ta"] \arrow[r, bend right, "a\circ m_A"'] \arrow[r, phantom, "\Downarrow\theta"] & TA \end{tikzcd}$$

In the diagram below, since $m_A \circ \tau = id$, the left column of 2-cells collapses to identity, and therefore we have

$$\begin{tikzcd} TA \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\tau\Downarrow"] & TA \arrow[r, "a"] \arrow[d, "i_{TA}"] & A \arrow[d, "i_A"] \arrow[dd, bend left=60, "1_A"] \\ TA \arrow[r, "Ti_A"] \arrow[d, "1"'] & T^2A \arrow[r, "Ta"] \arrow[d, "m_A"] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "a"] \arrow[r, phantom, "\zeta"] & {} \\ TA \arrow[r, "1"'] & TA \arrow[r, "a"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "1_{TA}"'] & A \arrow[d, "1_A"] \\ TA \arrow[r, "a"] & A \end{tikzcd}$$

[1] KZ: short for 'Kock-Zöberlein'

$$\theta \centerdot \tau_A = \zeta^{-1} \centerdot a$$

On the other hand, we can compose row-wise instead, and we get

$$\theta \centerdot \tau_A = (\theta \centerdot Ti_A) \circ (a \circ Ta \centerdot \tau_A) = (a \centerdot T\zeta^{-}1) \circ (a \circ Ta \centerdot \tau_A)$$

Thus, in the end, we have

$$\begin{tikzcd}
TA \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\tau_A \Downarrow"] \arrow[rr, bend right=60, "1"'] & T^2 A \arrow[r, "Ta"] \arrow[d, phantom, "T\zeta^{-1}\Downarrow"] & TA \arrow[r, "a"] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "a"] & A \arrow[r, "i_A"] \arrow[rr, bend right, "1"'] & TA \arrow[r, "a"] \arrow[d, phantom, "\zeta^{-1}\Downarrow"] & A
\end{tikzcd}
\tag{5}$$

Lemma *2.3*. Let T be a KZ-monad, and A an object of $\mathcal{K}$. Then any pseudo T-algebra is left adjoint to i_A. Conversely, if i_A has a left adjoint with invertible counit then this left adjoint is a pseudo T-algebra.

Remark *2.4*. This observation requires a bit of conceptual explanation: for a KZ-monad T, any object admits at most one pseudo T-algebra structure, up to unique isomorphism. So a KZ-monad is a nicely-behaved pseudo monad whose algebras are 'property-like' in the sense that the structure is (a reflective) left adjoint to the unit. Similarly, for a co-KZ-monad T the structure a is right adjoint to the unit i_A and the invertible unit of this adjunction given by $\zeta \colon 1 \Rightarrow a i_A$ in diagram 1.

$$\begin{tikzcd}
TA \arrow[r, bend right, "a"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "i_A"']
\end{tikzcd}$$

What about counit of $i_A \dashv a$? Here is a calculation[2] of counit using mate τ_A introduced in diagram 4.

$$\begin{tikzcd}
 & A \arrow[dr, bend left, "i_A"] & \\
TA \arrow[ur, bend left, "a"] \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\tau_A \Downarrow"] \arrow[rr, bend right=60, "1"'] & T^2 A \arrow[r, "Ta"] \arrow[d, phantom, "T\zeta^{-1}\Downarrow"] & TA
\end{tikzcd}
\tag{6}$$

Proposition *2.5*. Any KZ-monad (resp. co-KZ-monad) is lax idempotent (resp. co-lax idempotent). (Street 1974) (Kock 1995)

[2]The dual of this situation, i.e. unit in the case of KZ-monad, is calculated in page 112 of (Street 1974).

3 A general useful lemma in 2-categories

There is an innocent looking yet quite important proposition in (Street 1974) which may be overlooked in first reading of the paper.[3] This is proposition 6 in that paper. We state it here.

Proposition *3.1.* Suppose $f\colon A \to B$ is a 1-cell with right adjoint u, unit η, and counit ϵ in a 2-category $\mathcal{K}$ with comma objects. For any 1-cell $g\colon C \to B$, the unique filling arrow $v\colon C \to f/g$ obtained by factoring $\epsilon_{\bullet} g$ through (strict) comma square $\langle f/g, d_0, d_1, \phi\rangle$ is right adjoint to d_1 with counit identity.

The 1-cell v in the proposition is uniquely determined by equations $d_1 v = 1$, $d_0 v = ug$, and $\phi \mathbin{\scriptstyle\blacksquare} v = \epsilon \mathbin{\scriptstyle\blacksquare} g$. Moreover, the proposition states that we can lift the 2-cell η in the lower part of the diagram to a 2-cell $1 \Rightarrow vd_1$ in the upper part.

$$\begin{tikzcd}
& C \arrow[dr, dashed, "v"] \arrow[ddl, bend right, "ug"'] \arrow[drr, bend left, "1"] & & & \\
& & f/g \arrow[r, "d_1"] \arrow[dll, "d_0"'] \arrow[rr, dotted, bend right, "1"'] & C \arrow[r, "v"] \arrow[dl, "g"'] & f/g \arrow[dl, "d_0"] \\
A \arrow[rr, "f"] \arrow[rrr, bend right=50, "1"'] & & B \arrow[r, "u"] \arrow[d, phantom, "\eta\Uparrow"] & A & \\
& & {} & &
\end{tikzcd}$$

Proof. We first construct the would-be unit β of adjunction $d_1 \dashv v$. Using the fact $(\epsilon \mathbin{\scriptstyle\blacksquare} f) \circ (f \mathbin{\scriptstyle\blacksquare} \eta) = 1$ in chasing the diagram below, we obtain:

$$(\phi \mathbin{\scriptstyle\blacksquare} vd_1) \circ (fu \mathbin{\scriptstyle\blacksquare} \phi) \circ (f \mathbin{\scriptstyle\blacksquare} \eta \mathbin{\scriptstyle\blacksquare} d_0) = (\epsilon \mathbin{\scriptstyle\blacksquare} gd_1) \circ (fu \mathbin{\scriptstyle\blacksquare} \phi) \circ (f \mathbin{\scriptstyle\blacksquare} \eta \mathbin{\scriptstyle\blacksquare} d_0) = \phi$$

$$\begin{tikzcd}
f/g \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\phi\Downarrow"] & A \arrow[r, "1"] \arrow[d, "f"'] \arrow[dr, phantom, "\eta\Downarrow"] & A \arrow[d, "1"] & \\
C \arrow[r, "g"'] & B \arrow[r, "u"'] \arrow[rr, bend right=50, "1"'] & A \arrow[r, "f"'] \arrow[d, phantom, "\epsilon\Downarrow"] & B \\
& & {} &
\end{tikzcd}$$

We (uniquely) define $\beta\colon 1 \Rightarrow vd_1$ to be the unique 2-cell with

$$\begin{aligned} d_0 \mathbin{\scriptstyle\blacksquare} \beta &= (u \mathbin{\scriptstyle\blacksquare} \phi) \circ (\eta \mathbin{\scriptstyle\blacksquare} d_0) \\ d_1 \mathbin{\scriptstyle\blacksquare} \beta &= 1 \end{aligned} \tag{7}$$

One readily verifies that with *id* and β, d_1 and v satisfy triangle equations of adjunction. □

[3] Unfortunately, this occurred in the case of author.

4 Fibrations as pseudo-algebras of a co-KZ-monad

Let $\mathcal{K}$ be a representable 2-category. Define $\mathcal{K}/B$ to be the strict slice 2-category over B, meaning the morphism triangles commute up to equality. (Street 1974) constructs KZ-monads $L, R\colon \mathcal{K}/B \rightrightarrows \mathcal{K}/B$. The idea is, for a morphism $p\colon E \to B$, an algebra $R(p) \to p$ (resp. $L(p) \to p$) if it exist, corresponds to the fibration structure on p (resp. opfibration structure). We will only present explicit construction and calculation for the case of fibration[4] and thus, we will mainly concern ourselves with 2-monad R. However, when necessary, we will comment on the dual results for the case of opfibrations. We now define 2-monad R: It takes an object (E, p) to $(B/p, R(p))$ where

$$\begin{tikzcd} B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\ B \arrow[r, "1"'] & B \end{tikzcd} \tag{8}$$

is a comma square.

REMARK *4.1.* 2-cell ϕ_p can be constructed as follows:

$$\begin{tikzcd} B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\ B \arrow[r, "1"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} B/p \arrow[r, "\hat{d}_1"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, "p"] \\ B^{\to} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi\Uparrow"] & B \arrow[d, "1"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$

The action of R on morphisms is given as follows:

If $f\colon (E', p') \to (E, p)$ is a 1-cell in $\mathcal{K}/B$, then define $R(f)$ to be the unique 1-cell with $\hat{d}_1 \circ R(f) = f \circ \hat{d}'_1$ and $\hat{p} \circ R(f) = \hat{p}'$.

$$\begin{tikzcd} B/p' \arrow[r, "\hat{d}'_1"] \arrow[d, dotted, "R(f)"'] \arrow[dd, bend right=60, "\hat{p}'"'] \arrow[dr, phantom, "\lrcorner" very near start] & E' \arrow[d, "f"] \arrow[dd, bend left=60, "p'"] \\ B/p \arrow[r, "\hat{d}_1"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, "p"] \\ B^{\to} \arrow[r, "d_1"'] & B \end{tikzcd}$$

[4]Unlike Street's paper whereby he works with opfibration structures and thus chooses to work with 2-monad L.

Similarly if $\sigma\colon f \Rightarrow g$ is a 2-cell in $\mathcal{K}/B$, then we have a unique induced 2-cell $R(\sigma)\colon R(f) \Rightarrow R(g)$ with $\hat{d_1} \centerdot R(\sigma) = \sigma \centerdot \hat{d}'_1$ and $\hat{p} \centerdot R(\sigma) = id_{\hat{p}'}$.

Proposition *4.2*. 2-functor $R\colon \mathcal{K}/B \to \mathcal{K}/B$ is a 2-monad.

The unit of monad $i : id \Rightarrow R$ at (E, p) is given by the unique arrow $i(p) : E \to B/p$ with property that $R(p) \circ i(p) = p$ and $\hat{d_1} \circ i(p) = 1_E$, and moreover $\phi_p \centerdot i(p) = id_p$, all inferred by universal property of comma object B/p.

$$
\begin{tikzcd}
E \arrow[r, dotted, "i(p)"] \arrow[rr, bend left, "1"] \arrow[dr, bend right, "p"'] & B/p \arrow[r, "\hat{d_1}"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\
 & B \arrow[r, "1"'] & B
\end{tikzcd}
$$

It also follows that $\hat{d_1} \dashv i(p)$ with identity counit. Indeed, $i(p)$ is v in proposition 3.1, when $f = 1$ and $g = p$. From there, we also get the unit β of adjunction with $R(p) \centerdot \beta = \phi_p$.

The multiplication $m\colon R^2 \Rightarrow R$ of monad at (E, p) is given by the unique arrow $m(p)\colon B/R(p) \to B/p$

$$
\begin{tikzcd}
B/R(p) \arrow[r, "\widehat{d_1^{\to}}"'] \arrow[r, dotted, bend left, "m(p)"] \arrow[d, "\hat{\hat{p}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B/p \arrow[r, "\hat{d_1}"'] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
B^{\to\to} \arrow[r, "d_1^{\to}"'] \arrow[d, "d_0^{\to}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\to} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi\Uparrow"] & B \arrow[d, "1"] \\
B^{\to} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi\Uparrow"] & B \arrow[r, "1"'] \arrow[d, "1"] & B \\
B \arrow[r, "1"'] & B &
\end{tikzcd}
\tag{9}
$$

with property that $R(p) \circ m(p) = R^2(p)$ and $\hat{d_1} \circ m(p) = \hat{d_1} \circ \widehat{d_1^{\to}}$, and moreover $\phi_p \centerdot m(p) = (\phi_p \centerdot \widehat{d_1^{\to}}) \circ (\phi \centerdot d_0^{\to} \hat{p}) = (\phi_p \centerdot \widehat{d_1^{\to}}) \circ \phi_{R(p)}$, all inferred by universal property of comma object B/p.

Proposition *4.3*. 2-monad $R\colon \mathcal{K}/B \to \mathcal{K}/B$ is a co-KZ-monad.

Proof. We have to show that $i \centerdot T \dashv m$. □

Now, we would like to see what an algebra $a\colon R(p) \to p$ in $\mathcal{K}/B$ looks like. The fact that

a is a morphism in $\mathcal{K}/B$ provides us with a morphism a which makes the diagram

$$\begin{tikzcd} B/p \arrow[rr, "a"] \arrow[dr, "R(p)"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd} \tag{10}$$

commute. Moreover, by remark 2.4 R being a co-KZ-monad generates an adjunction $i(p) \dashv a$ whose unit is the invertible 2-cell $\zeta(p)\colon 1 \Rightarrow a \circ i(p)$

$$\begin{tikzcd} E \arrow[r, "i(p)"] \arrow[rr, bend left=60, "1"] \arrow[dr, "p"'] & B/p \arrow[u, phantom, "\Downarrow\zeta(p)"] \arrow[r, "a"] \arrow[d, "R(p)"'] & E \arrow[dl, "p"] \\ & \mathcal{A} & \end{tikzcd} \tag{11}$$

such that $p \centerdot \zeta(p) = id_p$.

In the example below we investigate how the construction above look like when we choose 2-category of (locally small) categories as our working 2-category.

Example 4.4. Let's take $\mathcal{K} = \mathfrak{Cat}$ to be the strict 2-category of categories, functors, and natural transformations. First and foremost, for a functor $p\colon E \to B$, the comma category B/p is given as a category whose objects are pairs $\langle f\colon a \to p(e); e\rangle$ where f is morphism in B:[5]

$$\begin{tikzcd} & e \arrow[d, mapsto, "p"] \\ b_0 \arrow[r, "f"'] & b_1 \end{tikzcd}$$

Morphisms of B/p are of the form

$$\begin{tikzcd} & e \arrow[d, mapsto, "p"] \arrow[dr, "\tilde{h}_1"] & \\ & & e' \arrow[dd, mapsto, "p"] \\ b_0 \arrow[r, "f"] \arrow[dr, "h_0"'] & b_1 \arrow[dr, "h_1"] & \\ & c_0 \arrow[r, "g"'] & c_1 \end{tikzcd}$$

$R(p)$ as in diagram (8) takes pair $\langle f; e\rangle$ to $b_0 = \mathrm{dom}(f)$, and $\hat{d_1}$ is simply the second projection; it takes $\langle f; e\rangle$ to e. The unit $i(p)\colon E \to B/p$ sends an object e of E to the object

$$\begin{tikzcd} & e \arrow[d, mapsto, "p"] \\ p(e) \arrow[r, equal] & p(e) \end{tikzcd}$$

[5] $e \mapsto b_1$ indicates that $p(e) = b_1$.

We also note that $\widehat{\overrightarrow{d_1}}$ (as in diagram 9) is given by the action

$$
\begin{tikzcd}
 & & e \arrow[d, mapsto] & & & e \arrow[d, mapsto] \\
b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_2 & \mapsto & b_1 \arrow[r, "g"'] & b_2
\end{tikzcd}
$$

and multiplication $m(p)$ given by

$$
\begin{tikzcd}
 & & e \arrow[d, mapsto] & & & e \arrow[d, mapsto] \\
b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_2 & \mapsto & b_0 \arrow[r, "g \circ f"'] & b_2
\end{tikzcd}
$$

Now, suppose that $\mathfrak{a}\colon R(p) \to p$ is a pseudo algebra for 2-monad R. By commutativity of diagram 10 we know that $p(\mathfrak{a}\langle f; e\rangle) = \mathrm{dom}(f)$. So we draw

$$
\begin{tikzcd}
\mathfrak{a}\langle f; e\rangle \arrow[d, mapsto, "p"'] & \\
b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}
$$

As observed in diagram 11 we get an isomorphism lift of identity in the base:

$$
\begin{tikzcd}
e \arrow[r, "\zeta(p)(e)"] \arrow[d, mapsto, "p"'] & \mathfrak{a}\langle 1_{p(e)}; e\rangle \arrow[d, mapsto, "p"] \\
p(e) \arrow[r, equal] & p(e)
\end{tikzcd}
$$

Observe that functors $R(i(p))\colon B/p \to B/R(p)$ and $i(R(p))\colon B/p \to B/R(p)$ are given as follows:

$$
\begin{tikzcd}
 & & e \arrow[d, mapsto] & & & & e \arrow[d, mapsto] \\
R(i(p)) : & b_0 \arrow[r, "f"'] & b_1 & \longmapsto & b_0 \arrow[r, "f"'] & b_1 \arrow[r, equal] & b_1
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & & e \arrow[d, mapsto] & & & & e \arrow[d, mapsto] \\
i(R(p)) : & b_0 \arrow[r, "f"'] & b_1 & \longmapsto & b_0 \arrow[r, equal] & b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}
$$

and there is a natural transformations $\tau\colon i(R(p)) \Rightarrow R(i(p))$

$$
\begin{tikzcd}
 & & e \arrow[dd, "p"] \arrow[dr, equal] & \\
 & & & e \arrow[dd, "p"] \\
b_0 \arrow[r, equal] \arrow[dr, equal] & b_0 \arrow[r, "f"] \arrow[d, "f"] & b_1 \arrow[dr, equal] & \\
 & b_0 \arrow[r, "f"'] & b_1 \arrow[r, equal] & b_1
\end{tikzcd}
$$

Indeed, τ is the mate 2-cell given in diagram 4. We also keep in mind that $R(\mathfrak{a}) \circ R(i(p))(\langle f; e\rangle) = \langle f; \mathfrak{a}\langle 1_{b_1}; e\rangle\rangle$, and hence $R(\zeta(p))$ is illustrated as in below:

$$
\begin{tikzcd}
 & & e \arrow[dd, "p"] \arrow[dr, "\zeta(p)(e)"] & \\
 & & & \mathfrak{a}\langle 1_{b_1}; e\rangle \arrow[dd, "p"] \\
b_0 \arrow[r, "f"] \arrow[dr, equal] & & b_1 \arrow[dr, equal] & \\
 & b_0 \arrow[rr, "f"'] & & b_1
\end{tikzcd}
$$

In addition, invertible 2-cell $\theta(p) \colon \mathfrak{a} \circ R(\mathfrak{a}) \Rightarrow \mathfrak{a} \circ m(p)$ provides us with an isomorphism $\mathfrak{a}\langle f; \mathfrak{a}\langle g; e\rangle\rangle \to \mathfrak{a}\langle gf; e\rangle$. Now, we study the coherence equations 1.1 in our case, which state that the following diagrams commute:

$$
\begin{tikzcd}
\mathfrak{a}\langle f; e\rangle \arrow[r, "\mathfrak{a}\centerdot R(\zeta(p))"] \arrow[d, "\zeta(p)\centerdot\mathfrak{a}\langle f;e\rangle"'] \arrow[dr, equal] & \mathfrak{a}\langle f; \mathfrak{a}\langle 1_{b_1}; e\rangle\rangle \arrow[d, "\theta\centerdot R(i(p))"] \\
\mathfrak{a}\langle 1_{b_0}; \mathfrak{a}\langle f; e\rangle\rangle \arrow[r, "\theta\centerdot i(R(p))"'] & \mathfrak{a}\langle f \circ 1_{b_0}; e\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{a}\langle f; \mathfrak{a}\langle g; \mathfrak{a}\langle h; e\rangle\rangle\rangle \arrow[r] \arrow[d] & \mathfrak{a}\langle gf; \mathfrak{a}\langle h; e\rangle\rangle \arrow[d] \\
\mathfrak{a}\langle f; \mathfrak{a}\langle gh; e\rangle\rangle \arrow[r] & \mathfrak{a}\langle hgf; e\rangle
\end{tikzcd}
$$

Furthermore, the counit of adjunction $i(p) \dashv \mathfrak{a}$, as computed in diagram 6, gives us the lift $\tilde{f} = \widehat{d_1}((R\mathfrak{a} \centerdot \tau_p) \circ R\zeta(p)^{-1})$ of f:

$$
\begin{tikzcd}
 & & \mathfrak{a}\langle f; e\rangle \arrow[dd, "p"] \arrow[dr, "\widehat{d_1}(R\mathfrak{a}\centerdot\tau_p)"] & & \\
 & & & \mathfrak{a}\langle 1_{b_1}; e\rangle \arrow[dd, "p"] \arrow[dr, "\widehat{d_1}(R\zeta(p)^{-1})"] & \\
b_0 \arrow[rr, equal] \arrow[dr, "1"'] & & b_0 \arrow[dr, "f"] & & e \arrow[dd, "p"] \\
 & b_0 \arrow[rr, "f"'] \arrow[dr, equal] & & b_1 \arrow[dr, equal] & \\
 & & b_0 \arrow[rr, "f"'] & & b_1
\end{tikzcd}
$$

It remains to prove that $\tilde{f}$ as defined is cartesian. One can try to prove this directly. However, we prove this in a more general setting in the next section.

5 Chevalley criterion

For an object E of 2-category $\mathcal{K}$, comma structure $\langle E^{\rightarrow}, e_0, e_1, \phi_E : e_0 \Rightarrow e_1 \rangle$ provides us with 1-cells

$$\begin{tikzcd} E^{\rightarrow} \arrow[r, shift left=2, "e_1"] \arrow[r, shift right=2, "e_0"'] & E \arrow[l, "\Delta"'] \end{tikzcd}$$

where $e_1 \dashv \Delta \dashv e_0$.

Definition 5.1. The 2-cell $p \centerdot \phi_E \colon d_0 p^{\rightarrow} = pe_0 \Rightarrow pe_1$ factors through universal 2-cell ϕ_p via a unique 1-cell γ_1. We say a 1-cell p in $\mathcal{K}$ satisfies ***Chevalley criterion*** if γ_1 has a right adjoint λ_1 with isomorphism counit.

$$\begin{tikzcd} E^{\rightarrow} \arrow[d, "p^{\rightarrow}"'] \arrow[dr, dotted, shift right, "\gamma_1"'] \arrow[drr, bend left, "e_1"] & & \\ B^{\rightarrow} \arrow[dr, bend right, "d_0"'] & B/p \arrow[ul, dotted, shift right, "\lambda_1"'] \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p \Uparrow"] & E \arrow[d, "p"] \\ & B \arrow[r, "1"'] & B \end{tikzcd}$$

Given a pseudo algebra $a \colon R(p) \to p$, we construct a right adjoint λ_1 and show that the counit of adjunction is isomorphism. Hence p satisfies Chevalley criterion. Note that the unit $\mathfrak{u}$ of adjunction $\hat{d}_1 \dashv i(p)$ defines a unique 1-cell $k \colon B/p \to (B/p)^{\rightarrow}$ obtained by factoring $\mathfrak{u}$ through comma square $\langle B/p, \pi_0, \pi_1, \phi_{B/p} \rangle$ of $(B/p)^{\rightarrow}$. Define $\lambda_1 \colon = a^{\rightarrow} \circ k$. A diagram chase shows that the front square in below commutes:

$$\hat{d}_1 \gamma_1 \lambda_1 = e_1 \lambda_1 = e_1 a^{\rightarrow} k = a \pi_1 k = a i(p) \hat{d}_1 \tag{12}$$

$$\begin{tikzcd} & (B/p)^{\rightarrow} \arrow[d, dotted, "a^{\rightarrow}"'] \arrow[ddrr, dotted, "\pi_i"] & & \\ B/p \arrow[ur, "k"] \arrow[rrr, "\hat{d}_1"] \arrow[dr, "\lambda_1"'] \arrow[dd, "\gamma_1 \lambda_1"'] & & & E \arrow[d, "i(p)"] \\ & E^{\rightarrow} \arrow[dl, "\gamma_1"'] \arrow[drr, "e_i"'] & & B/p \arrow[d, "a"] \\ B/p \arrow[rrr, "\hat{d}_1"'] & & & E \end{tikzcd} \qquad (i = 1, 2)$$

We also note that

$$\begin{aligned} R(p) \gamma_1 \lambda_1 = d_0 p^{\rightarrow} \lambda_1 = pe_0 \lambda_1 = pe_0 a^{\rightarrow} k = pa \pi_0 k = pa = R(p) \\ \phi_p \centerdot (\gamma_1 \lambda_1) = p \centerdot \phi_E \centerdot \lambda_1 = pa \centerdot \phi_{B/p} \centerdot k = pa \centerdot \mathfrak{u} = R(p) \centerdot \mathfrak{u} = \phi_p \end{aligned} \tag{13}$$

Now, equations (12) and (13), and definition of $R(ai(p))$ altogether prove that $\gamma_1\lambda_1 = R(ai(p))$. So, we have

$$\gamma_1 \circ \lambda_1 = R(a) \circ R(i(p)) = R(a \circ i(p))$$

and we shall show that counit $\varepsilon\colon \gamma_1 \circ \lambda_1 \Rightarrow 1$ is given by $R(\zeta(p)^{-1})$ which is invertible.

Now, we will construct the unit and show that counit and unit satisfy triangle equations of adjunction.

To be completed!

$$\begin{tikzcd}
E^{\to} \arrow[dr, bend right] \arrow[ddr, bend right, "p^{\to}"'] & & (B/p)^{\to} \arrow[ll, "a^{\to}"'] \arrow[ddl, bend left, "R(p)^{\to}"] \\
& B/p \arrow[ul, bend right, "\lambda_1"'] \arrow[ur, bend right, "k"'] \arrow[d, "\hat{p}"] & \\
& B^{\to} &
\end{tikzcd}$$

EXAMPLE *5.2*. Let $p\colon E \to B$ be a cloven Grothendieck fibration. We will show that p satisfies Chevalley criteria. Let $\mathfrak{s} : B^{\to} \to B$ and $\mathfrak{t} : B^{\to} \to B$ be the source and target functors, respectively. Note that the objects of $p \downarrow B = \mathfrak{s}^*(E)$ are pairs $\langle e', f : p(e') \to b\rangle$ where f is morphism in B.

$$\begin{tikzcd}
e' \arrow[d, mapsto, "p"'] & \\
a \arrow[r, "f"'] & b
\end{tikzcd}$$

($e' \mapsto a$ indicates that $p(e') = a$.) Similarly, the objects of $B \downarrow p = \mathfrak{t}^*(E)$ are pairs $\langle e, f : a \to p(e)\rangle$ where f is morphism in B.

$$\begin{tikzcd}
& e \arrow[d, mapsto, "p"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}$$

Note that the data of a cloven Grothendieck fibration includes structure of a cleavage, that is a choice of cartesian lifts:

$$\rho_{a,b} : \prod_{\mathrm{Hom}(a,b)} \prod_{e \in E_b} \sum_{e' \in E_a} \mathcal{C}art_E(e', e)$$

For all pairs of objects a, b, satisfying:

$$\begin{gathered}
\texttt{snd}\,\rho_{a,c}(g \circ f, e) \cong \texttt{snd}\,\rho_{b,c}(g, e) \circ \texttt{snd}\,\rho_{a,b}(f, \texttt{fst}\,\rho_{b,c}(g, e)) \\
\texttt{snd}\,\rho_{b,b}(1_{Pe}, e) \cong 1_e
\end{gathered} \tag{14}$$

We denote by $\mathrm{Pull}_f(e)$ the domain of cartesian lift i.e. $\texttt{fst}\,\rho_{a,b}(f,e)$ and by $\widetilde{f}$ the cartesian lift itself i.e. $\texttt{snd}\,\rho_{a,b}(f,e)$.

$$\begin{tikzcd}
\mathrm{Pull}_f(e) \arrow[r, dotted, "\widetilde{f}"] \arrow[d, mapsto, "p"'] & e \arrow[d, mapsto, "p"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}$$

The functors $\gamma_0 \colon E^{\to} \to \mathfrak{s}^*(E)$ and $\gamma_1 \colon E^{\to} \to \mathtt{t}^*(E)$ are defined as follows: for any object $u \colon e \to e'$ in $E^{\to}$, we define $\gamma_0(u) = \langle \mathfrak{s}(u), p(u)\rangle$, and $\gamma_1(u) = \langle \mathtt{t}(u), p(u)\rangle$. Definitions of γ_i (i=0,1) on morphisms is rather straightforward: If $u_0 \colon d \to d'$ and $u_1 \colon e \to e'$ are in $E^{\to}$ and $\langle h, h'\rangle$ is a morphism from u_0 to u_1 in $E^{\to}$, then $\gamma_0(\langle h, h'\rangle) = \langle h, \langle p(h), p(h')\rangle\rangle$. Similarly, $\gamma_1(\langle h, h'\rangle) = \langle h', \langle p(h), p(h')\rangle\rangle$. Moreover, $\lambda_1 \colon \mathtt{t}^*E \to E^{\to}$ is defined on objects as $\lambda_1\langle f \colon a \to b, e\rangle = \widetilde{f}$, and on morphisms by assigning to $\langle u, \langle h, k\rangle\rangle$, morphism $\langle \overline{h}, u\rangle \colon \tilde{f}_0 \to \tilde{f}_1$, where $\overline{h}$ is the a unique lift of h which make the upper square commute. $\overline{h}$ is obtained from cartesian property of $\widetilde{f_1}$. (Note that although $\overline{h}$ is a lift of h it may not be in the cleavage.)

$$\begin{tikzcd}
\mathrm{Pull}_{f_0}(e_0) \arrow[r, "\widetilde{f_0}"] \arrow[d, dotted, "\overline{h}"'] & e_0 \arrow[d, "u"] \\
\mathrm{Pull}_{f_1}(e_1) \arrow[r, "\widetilde{f_1}"'] & e_1
\end{tikzcd}$$

We now show that λ_1 is right adjoint to γ_1. Notice that the counit of adjunction is identity as it is readily observed that $\gamma_1 \circ \lambda_1 = id_{\mathtt{t}^*E}$. For obtaining the unit $\eta : Id_{E^{\to}} \to \lambda_1 \circ \gamma_1$, take any object $u \colon e_0 \to e_1$ of $E^{\to}$. We have $\lambda_1 \circ \gamma_1(f) = \widetilde{p(f)}$, and we define $\eta(f)$ as $\langle \overline{id_{p(e_0)}}, id_{e_1}\rangle : f \to \lambda_1 \circ \gamma_1(f)$, where $\overline{id_{p(e_0)}}$ is the unique vertical lift of $id_{p(e_0)}$ which makes the following triangle commute:

$$\begin{tikzcd}
e_0 \arrow[drr, "f"] \arrow[dr, dotted, "\overline{id_{p(e_0)}}"'] & & \\
& \mathrm{Pull}_{p(f)}(e_1) \arrow[r, "\widetilde{p(f)}"'] & e_1
\end{tikzcd}$$

We have to verify that triangle identities of adjunction hold:

$$\begin{tikzcd}
E^{\to} \arrow[r, equal] \arrow[d, "\gamma_1"'] \arrow[dr, phantom, "\Downarrow\eta"] & E^{\to} \arrow[d, "\gamma_1"] \\
\mathtt{t}^*E \arrow[r, equal] \arrow[ur, "\lambda_1"'] & \mathtt{t}^*E
\end{tikzcd}
=
\begin{tikzcd}
E^{\to} \arrow[d, bend right, "\gamma_1"'] \arrow[d, bend left, "\gamma_1"] \arrow[d, phantom, "="] \\
\mathtt{t}^*E
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathtt{t}^*E \arrow[r, equal] \arrow[d, "\lambda_1"'] & \mathtt{t}^*E \arrow[d, "\lambda_1"] \arrow[dl, phantom, "\eta\Uparrow"] \\
E^{\to} \arrow[r, equal] \arrow[ur, "\gamma_1"] & E^{\to}
\end{tikzcd}
=
\begin{tikzcd}
\mathtt{t}^*E \arrow[d, bend right, "\lambda_1"'] \arrow[d, bend left, "\lambda_1"] \arrow[d, phantom, "="] \\
E^{\to}
\end{tikzcd}
\tag{15}$$

Observe that $\gamma_1 = \gamma_1 \circ \lambda_1 \gamma_1$ and $\gamma_1 \centerdot \eta = id_{\gamma_1}$, and this proves the first pasting identity. Similarly, $\lambda_1 = \lambda_1 \circ \gamma_1 \circ \lambda_1$ and $\eta \centerdot \lambda_1 = id_{\lambda_1}$, and hence we have the second pasting identity.

Definition *5.3*. For a category B, define 2-category $\mathbf{Fib}(B)$ of fibrations over B whose 0-cells are Grothendieck fibrations, whose 1-cells are fibred functors over B (i.e. those functors over B which preserve cartesian morphisms), and 2-cells are vertical natural transformations (i.e. transformations over B). Compositions are usual composition of functors and natural transformations.

Remark *5.4*. Example 4.4 can be encapsulated as follows: The forgetful 2-functor $U\colon \mathbf{Fib}(B) \to \mathfrak{Cat}/B$ is 2-monadic: the ***free fibration*** of a functor $p\colon E \to B$ is fibration $R(p)\colon B/p \to B$; cleavage (aka fibration structure) on p is uniquely (in fact unique up to unique isomorphism) determined by a pseudo-algebra structure for 2-monad $R = UF$. Strict algebra structures of R correspond to splitting fibration structures on p.

$$\begin{tikzcd}
\mathbf{Fib}(B) \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\
\mathfrak{Cat}/B \arrow[u, bend left, "F"] \arrow[loop right, "R"]
\end{tikzcd}$$

We also note that for a category B the domain functor $\mathrm{cod}\colon B^{\to} \to B$ is the free Grothendieck fibration on identity functor $1\colon B \to B$; that is $\mathrm{dom} = R(1)$. In more explanatory terms this fact states that

We also note that for a category B with pullbacks the codomain functor $\mathrm{cod}\colon B^{\to} \to B$ is the free Grothendieck fibration *with existential quantifiers* on identity functor $1\colon B \to B$;

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