Assignment 4

STAT39000 Stochastic Calculus

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Exercise 1

1. $f(t,x) = xe^{-x^2}$.

$$\begin{aligned}
df(t,B_t) &= \partial_x f(t,B_t)dB_t + (\partial_t f(t,B_t) + \frac{1}{2}\partial_{xx} f(t,B_t))dt \text{ (by Ito's lemma version II)} \\
&= (e^{B_t^2} - 2B_t^2 e^{-B_t^2})dB_t + (0 + \frac{1}{2}(-2B_t e^{-B_t^2} - 4B_t e^{-B_t^2} + 4B_t^3 e^{-B_t^2})) \\
&= (e^{-B_t^2} - 2B_t^2 e^{-B_t^2})dB_t + (2B_t^3 e^{-B_t^2} - 3B_t e^{-B_t^2})dt \\
&= e^{-B_t^2}\{(1 - 2B_t^2)dB_t + (2B_t^3 - 3B_t)dt\}
\end{aligned}$$

2. $f(t,x) = xe^{-2tx}\sin t$.

$$\begin{aligned}
df(t,B_t) &= \partial_x f(t,B_t)dB_t + (\partial_t f(t,B_t) + \frac{1}{2}\partial_{xx} f(t,B_t))dt \text{ (by Ito's lemma version II)} \\
&= (e^{-2tB_t}\sin t - 2tB_t e^{-2tB_t}\sin t)dB_t + (-2B_t^2 e^{-2tB_t}\sin t + B_t e^{-2tB_t}\cos t)dt \\
&\quad + \frac{1}{2}(-2te^{-2tB_t}\sin t - 2te^{-2tB_t}\sin t + 4t^2 B_t e^{-2tB_t}\sin t)dt \\
&= e^{-2tB_t}\sin t\{(1 - 2tB_t)dB_t + (B_t\cot t - 2B_t^2 + 2t^2 B_t - 2t)dt\}
\end{aligned}$$

3. $f(t,x) = \cos x + x^2$

$$\begin{aligned}
df(B_t) &= f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt \text{ (by Ito's lemma version I)} \\
&= (-\sin B_t + 2B_t)dB_t + \frac{1}{2}(-\cos B_t + 2)dt \\
&= (2B_t - \sin B_t)dB_t + (1 - \frac{1}{2}\cos B_t)dt
\end{aligned}$$

4. $dX_t = X_t(4dt - 2dB_t) = 4X_t dt - 2X_t dB_t$. Note that

$$df(t,X_t) = (\partial_t f(t,X_t) + 4X_t \partial_x f(t,X_t) + \frac{4X_t^2}{2}\partial_{xx} f(t,X_t))dt - 2X_t \partial_x f(t,X_t)dB_t$$

by Ito's lemma version III.

1) $f(t,x) = xe^{-x^2}$.

$$\begin{aligned}
df(t,X_t) &= (4X_t(e^{-X_t^2} - 2X_t^2 e^{-X_t^2}) + \frac{4X_t^2}{2}(-2X_t e^{-X_t^2} - 4X_t e^{-X_t^2} + 4X_t^3 e^{-X_t^2}))dt - 2X_t(e^{-X_t^2} - 2X_t^2 e^{-2X_t^2})dB_t \\
&= (4X_t e^{-X_t^2} - 8X_t^3 e^{-X_t^2} - 4X_t^3 e^{-X_t^2} - 8X_t^3 e^{-X_t^2} + 8X_t^5 e^{-X_t^2})dt - 2X_t(e^{-X_t^2} - 2X_t^2 e^{-X_t^2})dB_t \\
&= X_t e^{-X_t^2}\{4(1 - 5X_t^2 + 2X_t^4)dt - 2(1 - 2X_t^2)dB_t\}
\end{aligned}$$

2) $f(t,x) = xe^{-2tx}\sin t$.

$$\begin{aligned} df(t,X_t) &= (-2X_t^2 e^{-2tX_t}\sin t + X_t e^{-2tX_t}\cos t)dt + 4X_t(e^{-2tX_t}\sin t - 2tX_t e^{-2tX_t}\sin t)dt \\ &\quad + \frac{4X_t^2}{2}(-2te^{-2tX_t}\sin t - 2te^{-2tX_t}\sin t + 4t^2 X_t e^{-2tX_t}\sin t)dt \\ &\quad - 2X_t(-2X_t^2 e^{-2tX_t}\sin t + X_t e^{-2tX_t}\cos t)dB_t \\ &= X_t e^{-2tX_t}\sin t\{(4 + \cot t - 2X_t - 16tX_t + 8t^2X_t^2)dt - (2 - 4tX_t)dB_t\} \end{aligned}$$

3) $f(t,x) = \cos x + x^2$.

$$\begin{aligned} df(t,X_t) &= (4X_t(-\sin X_t + 2X_t) + 2X_t^2(-\cos X_t + 2))dt - 2X_t(-\sin X_t + 2X_t)dB_t \\ &= 2X_t\{(6X_t - 2\sin X_t - X_t\cos X_t)dt - (2X_t - \sin X_t)dB_t\} \end{aligned}$$

Exercise 2

1. $dX_t = X_t dt + 4X_t dB_t = X_t(1dt + 4dB_t)$. Note that this a geometric Brownian motion with drift 1 and volatility 4. Therefore, the solution has an explicit form as follows:

$$X_t = X_0\exp\{(m - \frac{\sigma^2}{2})t + \sigma B_t\} = X_0\exp\{-7t + 4B_t\}$$

where X_0 is the initial value/condition of the process.

2. $X_0 = 1 \Rightarrow X_t = \exp\{-7t + 4B_t\}$.

$$\mathbb{P}\{X_1 > 6\} = \mathbb{P}\{e^{4B_1 - 7} > 6\} = \mathbb{P}\{4B_1 - 7 > \log 6\} = \mathbb{P}\{B_1 > \frac{\log 6 + 7}{4}\} = 1 - \Phi\left(\frac{\log 6 + 7}{4}\right) \approx 0.0140$$

3. $X_0 = \frac{1}{2} \Rightarrow X_t = \frac{1}{2}\exp\{-7t + 4B_t\}$

$$\mathbb{P}\{X_2 < 7\} = \mathbb{P}\{\frac{1}{2}e^{4B_2 - 14} < 7\} = \mathbb{P}\{4B_2 - 14 < \log 14\} = \mathbb{P}\left\{\frac{B_2}{\sqrt{2}} < \frac{\log 14 + 14}{4\sqrt{2}}\right\} = \Phi\left(\frac{\log 14 + 14}{4\sqrt{2}}\right) \approx 0.9984$$

4. $Y_t = f(t,X_t) = \log X_t$. Then, by Ito's lemma version III,

$$dY_t = df(t,X_t) = (\partial_t f(t,X_t) + X_t\partial_x f(t,X_t) + \frac{16X_t^2}{2}\partial_{xx}f(t,X_t))dt + 4X_t\partial_x f(t,X_t)dB_t$$

$$\begin{aligned} \Rightarrow dY_t &= \left(0 + X_t\cdot\frac{1}{X_t} + 8X_t^2\left(-\frac{1}{X_t^2}\right)\right)dt + 4X_t\cdot\frac{1}{X_t}dB_t \\ &= -7dt + 4dB_t \end{aligned}$$

Note that this is a Brownian motion with $m = -7$ and $\sigma = 4$. It is an intuitive and interesting result that the logarithm of a geometric Brownian motion is a Brownian motion.

Exercise 3

1. $Z_t = X_tY_t$. Using the stochastic product rule,

$$\begin{aligned} dZ_t = d(X_tY_t) &= X_t dY_t + Y_t dX_t + d\langle X, Y\rangle_t \\ &= X_tY_t(3dt - dB_t) + Y_tX_t(dt + 2dB_t) + 2X_t(-Y_t)dt \\ &= Z_t(3dt - dB_t) + Z_t(dt + 2dB_t) - 2Z_t dt \\ &= Z_t(2dt + dB_t) \end{aligned}$$

Note that Z_t is also a geometric Brownian motion.

2. $Z_t = X_t/Y_t$. Let $\tilde{Y}_t = f(t, Y_t) = 1/Y_t$. Then, by Ito's lemma version III,

$$\begin{aligned}
d\tilde{Y}_t = df(t, Y_t) &= (\partial_t f(t, Y_t) + 3Y_t \partial_x f(t, Y_t) + \frac{Y_t^2}{2}\partial_{xx} f(t, Y_t))dt - Y_t \partial_x f(t, Y_t) dB_t \\
&= (0 + 3Y_t \frac{-1}{Y_t^2} + \frac{Y_t^2}{2}\frac{2}{Y_t^3})dt - Y_t \cdot \left(\frac{1}{Y_t^2}\right) dB_t \\
&= -2\tilde{Y}_t dt + \tilde{Y}_t dB_t = \tilde{Y}_t(-2dt + dB_t)
\end{aligned}$$

Note that $\tilde{Y}_t$ is a geometric Brownian motion. Since, $Z_t = X_t \tilde{Y}_t$, I can apply the stochastic product rule:

$$\begin{aligned}
dZ_t = d(X_t \tilde{Y}_t) &= X_t d\tilde{Y}_t + \tilde{Y}_t dX_t + d\langle X, \tilde{Y}\rangle_t \\
&= X_t \tilde{Y}_t(-2dt + dB_t) + \tilde{Y}_t X_t(dt + 2dB_t) + 2X_t \tilde{Y}_t dt \\
&= Z_t(-2dt + dB_t) + Z_t(dt + 2dB_t) + 2Z_t dt \\
&= Z_t(dt + 3dB_t)
\end{aligned}$$

Again, Z_t is also a geometric Brownian motion.

3. Suppose there exists such a function f. Then, $df(X_t) = dB_t$. By Ito's lemma version III,

$$df(X_t) = (X_t f'(X_t) + 2X_t^2 f''(X_t))dt + 2X_t f'(X_t) dB_t$$

By our assumption, we should have both $X_t f'(X_t) + 2X_t^2 f''(X_t) = 0$ and $2X_t f'(X_t) = 1$. Generally, this function should satisfy $xf'(x) + 2x^2 f''(x) = 0$ and $2xf'(x) = 1$. Then,

$$f'(x) = \frac{1}{2x} \Rightarrow f''(x) = -\frac{1}{2x^2}, \quad x\frac{1}{2x} + 2x^2\left(-\frac{1}{2x^2}\right) = -\frac{1}{2} \neq 0$$

where the second derivative should be defined due to $f \in C^2$, but this is a contradiction to our starting assumption. Hence, there does not exist a function $f \in C^2, f : (0, \infty) \to \mathbb{R}$ s.t. $f(X_t) = B_t$.

Exercise 4

1. $Y_t = B_t^2 + 2t = f(t, B_t)$. By Ito's lemma version I,

$$dY_t = df(t, B_t) = (\partial_t f(t, B_t) + \frac{1}{2}\partial_{xx} f(t, B_t))dt + \partial_x f(t, B_t) dB_t = 3dt + 2B_t dB_t$$

$$\Rightarrow d\langle Y\rangle_t = 4B_t^2 dt, \quad d\langle Y, X\rangle_t = 2B_t(-2X_t)dt = -4B_t X_t dt$$

$$\therefore A_t = 4B_t^2, \quad C_t = -4B_t X_t$$

2. $Y_t = e^{X_t} + \int_0^t X_s^3 ds$. Let $f(t, X_t) = e^{X_t}$ and $g(t, X_t) = \int_0^t X_s^3 ds$. Note that g is a drift term and should not contribute to the quadratic variation nor the covariation. Focusing on f first, we have by Ito's lemma version III,

$$\begin{aligned}
df(t, X_t) &= (\partial_t f(t, X_t) + 3X_t^2 \partial_x f(t, X_t) + \frac{4X_t^2}{2}\partial_{xx} f(t, X_t))dt - 2X_t \partial_x f(t, X_t) dB_t \\
&= Q_t dt - 2X_t e^{X_t} dB_t
\end{aligned}$$

where Q_t is some process pertaining to the drift term, which also should not matter. Then, as it was done in the text,

$$\begin{aligned}
(dY_t)^2 &= (d(e^{X_t}) + X_t^3 dt)^2 \\
&= (Q_t dt - 2X_t e^{X_t} dB_t + X_t^3 dt)^2 \\
&= (Q_t dt - 2X_t e^{X_t} dB_t)^2 + 2(Q_t dt - 2X_t e^{X_t} dB_t)X_t^3 dt + (X_t^3 dt)^2 \\
&= Q_t^2 (dt)^2 - 4Q_t X_t e^{X_t}(dt \cdot dB_t) + 4X_t^2 e^{2X_t}(dB_t)^2 + 2Q_t X_t^3 (dt)^2 - 4X_t^4 e^{X_t}(dt \cdot dB_t) + X_t^6 (dt)^2 \\
&= 4X_t^2 e^{2X_t} dt
\end{aligned}$$

$$\Rightarrow d\langle Y\rangle_t = 4X_t^2 e^{2X_t}dt,\;\; d\langle Y,X\rangle_t = 4X_t^2 e^{X_t}dt$$
$$\therefore A_t = 4X_t^2 e^{2X_t},\;\; C_t = 4X_t^2 e^{X_t}$$

3. $Y_t = X_t + \exp\{\int_0^t X_s^3 ds\}$. Let $f(t,X_t) = X_t$ and $g(t,X_t) = \exp\{\int_0^t X_s^3 ds\}$. Note that $df(t,X_t) = dX_t$ and let $Z_t = \int_0^t X_s^3 ds \Rightarrow dZ_t = X_t^3 dt$. Then, by Ito's lemma version III,

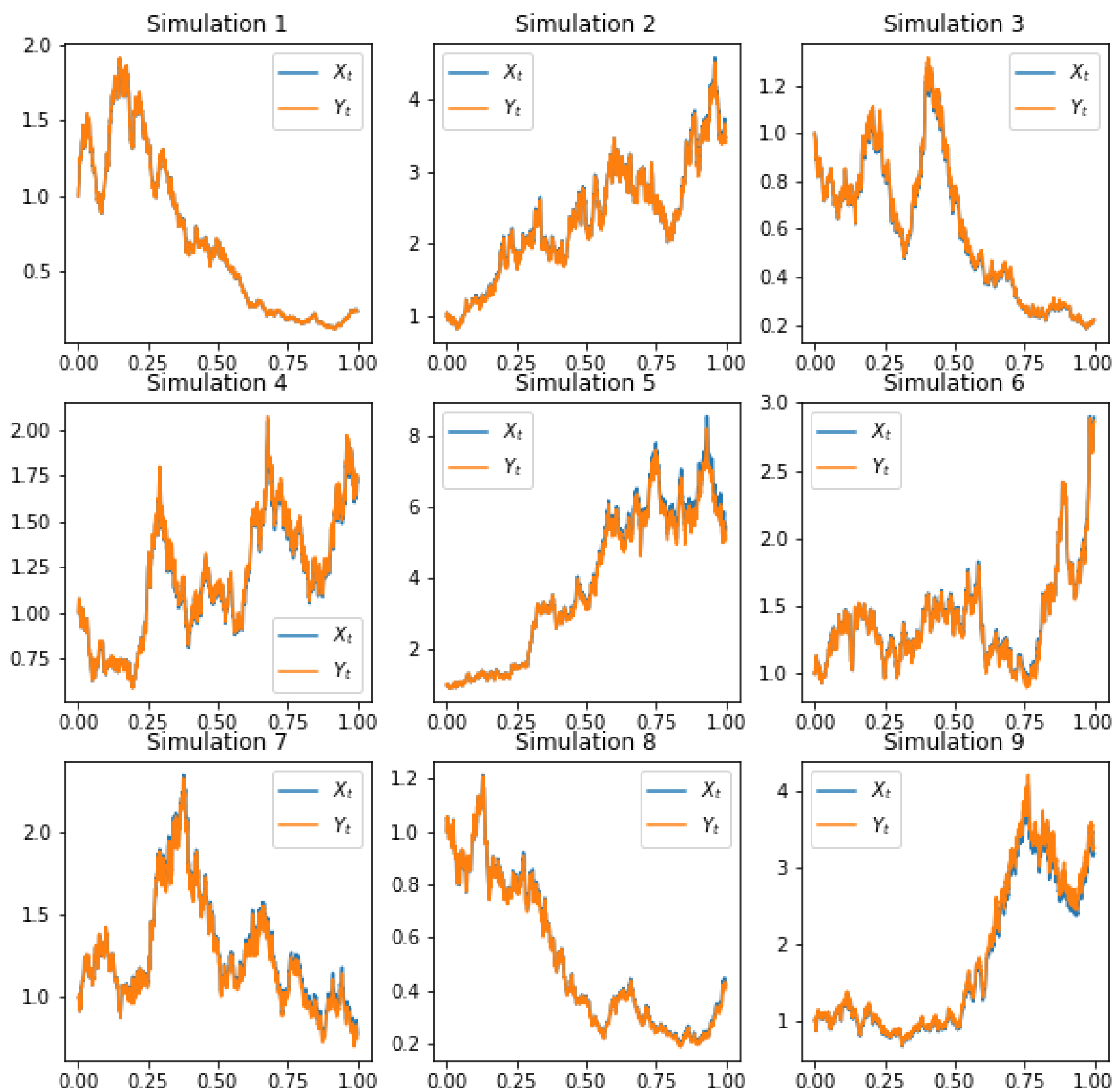
$$\begin{aligned} dg(t,X_t) = d(e^{Z_t}) &= (\partial_t f(t,X_t) + X_t^3 \partial_x f(t,X_t) + \frac{0}{2}\partial_{xx} f(t,X_t))dt + 0\cdot\partial_x f(t,X_t)dB_t \\ &= (0 + X_t^3 e^{Z_t} + 0)dt + 0\cdot dB_t \\ &= X_t^3 \exp\{\int_0^t X_s^3 ds\}dt \end{aligned}$$

Therefore,

$$\begin{aligned} dY_t = df(t,X_t) + dg(t,X_t) &= dX_t + X_t^3\exp\{\int_0^t X_s^3 ds\}dt \\ &= X_t(3X_t dt - 2dB_t) + X_t^3 \exp\{\int_0^t X_s^3 ds\}dt \\ &= (3X_t^2 + X_t^3\exp\{\int_0^t X_s^3 ds\})dt - 2X_t dB_t \end{aligned}$$

$$\Rightarrow d\langle Y\rangle_t = 4X_t^2 dt,\;\; d\langle Y,X\rangle_t = 4X_t^2 dt$$
$$\therefore A_t = C_t = 4X_t^2$$

Exercise 5 I used Python to run the simulation and I provide the code at the end of this exercise. Below is the plot of nine separate simulations I have done.



I find that the blue line (X_t) and the orange line (Y_t) are precisely identical. This is an obvious result, since $Y_t = e^{B_t} = f(B_t)$ and by Ito's lemma version I,

$$dY_t = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt = \frac{1}{2}Y_t dt + Y_t dB_t$$

which is identical to dX_t. Since $Y_0 = e^0 = 1 = X_0$, the two processes must be the same. The intuition behind this exercise seems to be reiterating the fact that the geometric Brownian motion is the exponential of a Brownian motion.

Python code used for this exercise:

[1]:
```
# Necessary packages
import numpy as np
import matplotlib.pyplot as plt
```

[2]:
```
# Define simulation function
def simul_sde(start, end, dt, seed = 123):
    """
    returns a series of floats b_t that follows a standard Brownian motion
    and a series of floats x_t that satisfies the following SDE:
    dX_t = 0.5*X_tdt + X_tdB_t, X_0 = 1

    Arguments
    ---
    start: nonnegative integer. The starting point of the Brownian motion to be simulated.
    end: nonnegative integer. The end point of the Brownian motion to be simulated.
    dt: nonnegative float. The step size.
    seed: integer. Seed to be used for sampling to ensure reproducibility. Default 123.
    """
    length = int((end-start)/dt)
    t = np.linspace(start = start, stop = end, num = length+1)

    np.random.seed(seed)
    z = np.random.normal(0, 1, length)
    b_t = np.zeros(length+1)
    x_t = np.zeros(length+1)
    x_t[0] = 1
    for i in range(len(b_t)-1):
        b_t[i+1] = b_t[i] + np.sqrt(dt)*z[i]
        x_t[i+1] = x_t[i]*(1 + dt/2 + np.sqrt(dt)*z[i])

    return t, b_t, x_t
```

[3]:
```
t = np.zeros((1001, 9))
b = np.zeros((1001, 9))
x = np.zeros((1001, 9))

# Run 9 simulations
for i in range(9):
    t[:, i], b[:, i], x[:, i] = simul_sde(0, 1, 0.001, seed = i)
```

[4]:
```
m, n = 3, 3 # rows, columns of subplots
fig, ax = plt.subplots(3, 3, figsize=(10,10))

for i in range(m):
    for j in range(n):
        ax[i, j].plot(t[:, 3*i+j], x[:, 3*i+j], label = "$X_t$")
        ax[i, j].plot(t[:, 3*i+j], np.exp(b[:, 3*i+j]), label = "$Y_t$")
        ax[i, j].set_title('Simulation {}'.format(3*i+j+1))
        ax[i, j].legend()

plt.savefig("SDE.png")
```