AQFT: Problem sheet 2

Action for two scalars

$$S = \int d^D x \left[-\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}(m^2 - i\varepsilon)\phi^2 \right.$$

$$\left. - \tfrac{1}{2}(\partial\chi)^2 - \tfrac{1}{2}(\mu^2 - i\varepsilon)\chi^2 + \lambda\phi\chi^2 \right]$$

$$= \underbrace{\int d^D x \left[\tfrac{1}{2}\phi(\Box - m^2 + i\varepsilon)\phi + \tfrac{1}{2}\chi(\Box - \mu^2 + i\varepsilon)\chi \right.}_{S_0}$$

$$\underbrace{\left. + \lambda\phi\chi^2 \right]}_{S_I}$$

$$= S_0 + S_I$$

1. Want to find generating functional

$$Z[J,K] = \int \mathcal{D}\phi\,\mathcal{D}\chi\, \exp\left[iS + i\int d^D x \left(J(x)\phi(x) + K(x)\chi(x) \right) \right]$$

$$= \int \mathcal{D}\phi\,\mathcal{D}\chi\; e^{iS_I[\phi,\chi]}$$

$$\exp\left(iS_0[\phi,\chi] + i\int d^D x \left(J(x)\phi(x) + K(x)\chi(x) \right) \right)$$

Can replace $\phi \to \frac{1}{i}\frac{\delta}{\delta J}$ and $\chi \to \frac{1}{i}\frac{\delta}{\delta K}$ in S_I

$$= \exp\left(iS_I\left[\frac{1}{i}\frac{\delta}{\delta J}, \frac{1}{i}\frac{\delta}{\delta K} \right] \right) Z_0[J,K]$$

where $Z_0[J,K]$ is the free generating functional.

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Consider just the ϕ-action. To solve, just consider the classical equations of motion in the presence of a source

$$\frac{\delta \bar{S}_0[\phi;J]}{\delta\phi(x)} = (\Box - m^2 + i\varepsilon)\phi(x) + J(x) = 0$$

The classical solution is

$$\phi_J(x) = -\frac{J(x)}{\Box - m^2 + i\varepsilon}$$

$$= \int d^D y \frac{-J(y)}{\Box - m^2 + i\varepsilon} \delta^{(D)}(x-y)$$

$$= \int d^D y \int d^D k \frac{-J(y)}{\Box - m^2 + i\varepsilon} \cdot e^{ik(x-y)}$$

$$= i \cdot \int d^D y \; J(y) \int d^D k \frac{-i}{k^2 + m^2 - i\varepsilon} e^{ik\cdot(x-y)}$$

$$= i \int d^D y \; J(y) \; G_F(x,y)$$

Then, for fluctuations $\phi = \phi_J + \varphi$

$$\tilde{S}_0[\phi; J] = \tilde{S}_0[\phi_J; J] + \int d^D x \left.\frac{\delta \tilde{S}_0}{\delta\phi(x)}\right|_{\phi=\phi_J} \varphi(x)$$

$$+ \frac{1}{2}\int d^D x \int d^D x' \left.\frac{\delta^2 \tilde{S}_0}{\delta\phi(x)\,\delta\phi(x')}\right|_{\phi=\phi_J} \varphi(x)\,\varphi(x')$$

$$+ \ldots$$

$$= \int d^D x \left[\frac{1}{2}\phi_J(x) \overbrace{(\Box - m^2 + i\varepsilon)\phi_J(x)}^{-J(x)} + J(x)\phi_J(x)\right]$$

$$+ \frac{1}{2}\int d^D x \int d^D y \, (\Box - m^2 + i\varepsilon)\,\delta^{(D)}(x-y)\,\varphi(x)\,\varphi(y)$$

$$= \frac{1}{2}\int d^D x \; \varphi(x)\,(\Box - m^2 + i\varepsilon)\,\varphi(x)$$

$$+ \frac{i}{2}\int d^D x \int d^D y \; J(x)\, G_F(x,y)\, J(y)$$

$\Rightarrow$

$$\tilde{Z}_0[J] = \int D\varphi \; e^{iS_J[\phi = \phi_J + \varphi]}$$

$$= \exp\left[-\frac{1}{2}\int d^D x \int d^D x' \; J(x)\, G_F(x,x')\, J(x')\right]$$

$$\cdot \int D\varphi \; \exp\left[\frac{i}{2}\int d^D x \; \varphi(x)\,(\Box - m^2 + i\varepsilon)\,\varphi(x)\right]$$

$$= \tilde{Z}_0[0] \exp\left[-\frac{1}{2}\int d^D x \int d^D x' \; J(x)\, G_F(x,x')\, J(x')\right]$$

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Together,

drop this usually!

$$Z_0[J,K] = Z_0[0,0]$$

$$\exp\left[-\frac{i}{2}\int d^Dx \int d^Dx'\left(J(x)\, G_F(x,x')\, J(x') + K(x)\, D_F(x,x')\, K(x')\right)\right]$$

where

$$G_F(x,y) = \int d^Dk \;\frac{-i}{k^2+m^2+i\varepsilon}\; e^{ik\cdot(x-y)}$$

$$D_F(x,y) = \int d^Dk \;\frac{-i}{k^2+M^2+i\varepsilon}\; e^{ik\cdot(x-y)}$$

we may therefore write

$$Z[J,K] = \exp\left(i\, S_I\left[\frac{1}{i}\frac{\delta}{\delta J}, \frac{1}{i}\frac{\delta}{\delta K}\right]\right) Z_0[J,K]$$

$$= Z_0[J,K]\, Z_0[J,K]^{-1} \exp\left(i\, S_I\left[\frac{1}{i}\frac{\delta}{\delta J}, \frac{1}{i}\frac{\delta}{\delta K}\right]\right) Z_0[J,K]$$

From the definition of the exponential

$$= Z_0[J,K]\, \exp\left(i\, Z_0[J,K]^{-1} S_I\left[\frac{1}{i}\frac{\delta}{\delta J}, \frac{1}{i}\frac{\delta}{\delta K}\right] Z_0[J,K]\right) I$$

$$= Z_0[J,K]\, \exp\left(i\, S_I\left[Z_0^{-1}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)Z_0,\; Z_0^{-1}\left(\frac{1}{i}\frac{\delta}{\delta K}\right)Z_0\right]\right) I$$

In particular, in DeWitt notation $\left(\frac{\delta}{\delta J(x)} \to \frac{\partial}{\partial J_x} \text{ etc.}\right)$

$$Z_0^{-1}\frac{\delta}{\delta J(x)} Z_0$$

$$= e^{\frac{i}{2} J_y G_{yz} J_z}\; \frac{\partial}{\partial J_x}\; e^{-\frac{i}{2} J_y G_{yz} J_z}\;\bullet$$

$$= e^{\frac{i}{2} J_y G_{yz} J_z}\cdot\left(\frac{\partial}{\partial J_x}\bullet - J_y G_{yx}\right) e^{-\frac{i}{2} J_y G_{yz} J_z}$$

$$= \frac{\partial}{\partial J_x} - G_{xy} J_y \,.$$

and similarly

$$Z_0^{-1} \frac{\delta}{\delta K(x)} Z_0 = \frac{\partial}{\partial K_x} - D_{xy} K_y$$

Then

leave dummy integral!

$$Z[J,K] = Z_0[J,K] \exp\left[i\lambda \int d^Dx \left(\frac{1}{i}\frac{\partial}{\partial J_x} - \frac{1}{i} G_{xy} J_y \right) \left(\frac{1}{i}\frac{\partial}{\partial K_x} - \frac{1}{i} D_{xy} K_y \right)^2 \right] I$$

$$= Z_0[J,K] \exp\left[-\lambda \int d^Dx \left(\frac{\partial}{\partial J_x} - G_{xy} J_y \right) \left(\frac{\partial}{\partial K_x} - D_{xy} K_y \right)^2 \right] I$$

2. Expand in λ,

$$Z_{int}[J,K] = \exp\left[-\lambda \int d^Dx \left(\frac{\partial}{\partial J_x} - G_{xy} J_y \right) \left(\frac{\partial}{\partial K_x} - D_{xy} K_y \right)^2 \right] I$$

$$= 1 - \lambda \int d^Dx \left(\frac{\partial}{\partial J_x} - G_{xy} J_y \right) \left(\frac{\partial}{\partial K_x} - D_{xy} K_y \right)^2 I$$

$$+ \frac{\lambda^2}{2} \left[\int d^Dx \left(\frac{\partial}{\partial J_x} - G_{xy} J_y \right) \left(\frac{\partial}{\partial K_x} - D_{xy} K_y \right)^2 \right]^2 I$$

$$+ O(\lambda^3)$$

At $O(\lambda)$:

$$\left(\frac{\partial}{\partial J_x} - G_{xy} J_y \right) \left(\frac{\partial}{\partial K_x} - D_{xy} K_y \right)^2 I$$

$$= -(G_{xy} J_y) \left(\frac{\partial}{\partial K_x} - D_{xy} K_y \right) (-D_{xz} K_z)$$

$$= (G_{xy} J_y) \left(D_{xx} - (D_{xy} K_y)^2 \right)$$

At $O(\lambda^2)$:

$$\left(\frac{\partial}{\partial J_{x'}} - G_{x'y} J_y \right) \left(\frac{\partial}{\partial K_{x'}} - D_{x'y} K_y \right)^2$$

$$\times \left[\left(\frac{\partial}{\partial J_x} - G_{xy} J_y \right) \left(\frac{\partial}{\partial K_x} - D_{xy} K_y \right)^2 I \right]$$

using the $O(\lambda)$ result

$$= \left(\frac{\partial}{\partial J_{x'}} - G_{x'y} J_y \right) \left(\frac{\partial}{\partial K_{x'}} - D_{x'y} K_y \right)^2$$

$$\times \left[(G_{xy} J_y) \left(D_{xx} - (D_{xy} K_y)^2 \right) \right]$$

consider the J/G part

$$\left(\frac{\partial}{\partial J_{x'}} - G_{x'z} J_z \right) (G_{xy} J_y) = G_{xx'} - (G_{xy} J_y)(G_{x'z} J_z)$$

and the D/K part

$$\left(\frac{\partial}{\partial K_{x'}} - D_{x'z} K_z \right)^2 \left(D_{xx} - (D_{xy} K_y)^2 \right)$$

$$= \left(\frac{\partial}{\partial K_{x'}} - D_{x'z} K_z \right) \left[-2 D_{xx'} (D_{xy} K_y) - (D_{x'y} K_y) \right.$$

careful with dummies!

$$\left. \times \left(D_{xx} - (D_{xy} K_y)^2 \right) \right]$$

$$= \Big[-2 D_{xx'} D_{xx'} - D_{x'x'} \left(D_{xx} - (D_{xa} K_y)^2 \right)$$

$$- (D_{x'z} K_z) \cdot -2 D_{xx'} (D_{xy} K_y)$$

$$- (D_{x'y} K_y) \left(-2 D_{xx'} (D_{xy} K_y) \right)$$

since $x \leftrightarrow x'$ is symmetric.

$$\left. - (D_{x'z} K_z) \left(D_{xx} - (D_{xy} K_y)^2 \right) \right) \Big]$$

$$= \Big[-2 (D_{xx'})^2 - D_{xx} D_{x'x'} + 4 D_{xx'} (D_{xy} K_y)(D_{x'z} K_z)$$

$$+ 2 D_{x'x'} (D_{xy} K_y)^2 - (D_{xy} K_y)^2 (D_{x'z} K_z)^2 \Big]$$

Together,

$$Z_{int}[J, K] = 1 - \lambda \int d^D x \, (G_{xy} J_y) \left(D_{xx} - (D_{xy} K_y)^2 \right)$$

$$+ \lambda^2 \int d^D x \int d^D x' \left[G_{xx'} - (G_{xy} J_y)(G_{x'z} J_z) \right]$$

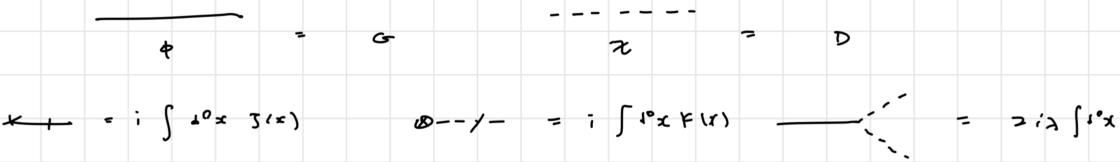
need to multiply out!

$$\Big[-(D_{xx'})^2 - \tfrac{1}{2} D_{xx} D_{x'x'} + 2 D_{xx'} (D_{xu} K_u)(D_{x'v} K_v)$$

$$+ D_{x'x'} (D_{xu} K_u)^2 - \tfrac{1}{2} (D_{xu} K_u)^2 (D_{x'v} K_v)^2 \Big]$$

$$+ O(\lambda^3)$$

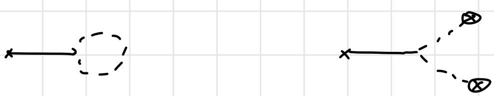
In terms of Feynman diagrams



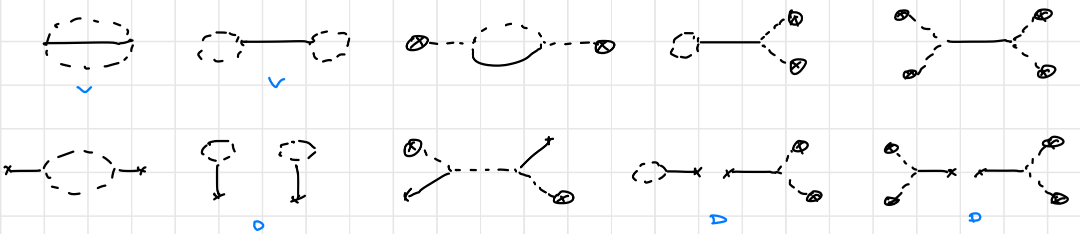
since

$$\mathcal{L} \supset \lambda \phi \chi^2 = 2 \cdot \frac{\lambda}{1! \, 2!} \phi \chi^2$$

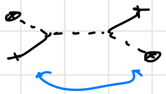
At $O(\lambda)$:



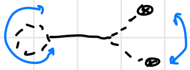
At $O(\lambda^2)$:



can check that Feynman rules reproduce these! For example



$$\sim \quad (2i\lambda)^2 \cdot \frac{1}{2} \cdot i^4 \quad \sim \quad -2\lambda^2$$



$$\sim \quad (2i\lambda)^2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot i^2 \quad \sim \quad 2\lambda^2$$

etc.

3. Now find the effective action:

$$e^{iW[J,K] - iW[0,0]} = \frac{Z[J,K]}{Z[0,0]}$$

In Andrew's convention, set $W[0,0] = 0$!

Can read off the free part, so concentrate on interactions!

$$e^{iW_{int}[J,K]} = \frac{Z_{int}[J,K]}{Z_{int}[0,0]}$$

Start by noting that

$$Z_{int}[0,0] = 1 + \lambda^2 \int d^D x \int d^D x' \, G_{xx'} \left[-(D_{xx'})^2 - \frac{1}{2} D_{xx} D_{x'x'} \right] + O(\lambda^3)$$

$$Z_{int}[J,K] = 1 + O(\lambda)$$

so that

$$\frac{Z_{int}[J,K]}{Z_{int}[0,0]} = Z_{int}[J,K] \cdot \left(1 + \lambda^2 \dots \right)^{-1} + O(\lambda^3)$$

$$= Z_{int}[J,K] \left(1 - \lambda^2 \dots \right) + O(\lambda^3)$$

$$= Z_{int}[J,K] + \lambda^2 \int d^D x \int d^D x' \, G_{xx'} \left[(D_{xx'})^2 + \frac{1}{2} D_{xx} D_{x'x'} \right] + O(\lambda^3)$$

This precisely cancels the vacuum diagrams! Write this as

$$= A_{(0)} + \lambda A_{(1)} + \lambda^2 A_{(2)} + O(\lambda^3)$$

Also expand the LHS

$$e^{iW_{int}[J,K]} = e^{i(W_{(0)} + \lambda W_{(1)} + \lambda^2 W_{(2)} + O(\lambda^3))}$$

where $W_{(0)} = 0$ is fixed by the exponential. Then

$$= 1 + i(\lambda W_{(1)} + \lambda^2 W_{(2)} + \ldots)$$

$$- \frac{1}{2}(\lambda W_{(1)} + \lambda^2 W_{(2)} + \ldots)^2 + \ldots$$

$$= 1 + i\lambda W_{(1)} + \lambda^2 \left(i W_{(2)} - \frac{1}{2} W_{(1)}^2 \right) + O(\lambda^3)$$

Indeed by comparison,

$$A_{(0)} = 1$$

$$A_{(1)} = i W_{(1)}$$

$$A_{(2)} = i W_{(2)} - \frac{1}{2} W_{(1)}^2$$

so that

$$W_{(1)} = -i A_{(1)}$$

$$W_{(2)} = -i \left(A_{(2)} - \frac{1}{2} A_{(1)}^2 \right)$$

Explicitly,

$$A_{(1)}^2 = \left[-\int d^Dx \, (G_{xy} J_y)(D_{xx} - (D_{xz} K_z)^2) \right]^2$$

$$= \int d^Dx \int d^Dx' \, (G_{xy} J_y)(G_{x'z} J_z)(D_{xx} - (D_{xu} K_u)^2)$$

$$\times (D_{x'x'} - (D_{x'v} K_v)^2)$$

$$= \int d^Dx \int d^Dx' \, (G_{xy}J_y)(G_{x'z}J_z)\Big(D_{xx} D_{x'x'} - 2 D_{xx}(D_{x'u}K_u) + (D_{xu}K_u)(D_{x'v}K_v)\Big)$$

which cancels the disconnected diagrams precisely! we find

$$W_{int}[J,K] = -i\lambda A_{(1)} - i\lambda^2 \left(A_{(2)} - \frac{i}{2} A_{(1)}^2\right) + O(\lambda^3)$$

$$= i\lambda \int d^Dx \, (G_{xy}J_y)\left[D_{xx} - (D_{xz}K_z)^2\right]$$

$$-i\lambda^2 \int d^Dx \int d^Dx' \Big[2 D_{xx'} G_{xx'} (D_{xu}K_u)(D_{x'v}K_v)$$

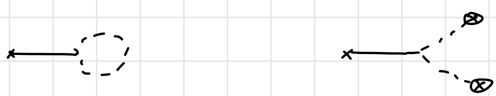
$$+ G_{xx'} D_{x'x'} (D_{xu}K_u)^2 - \frac{i}{2} (D_{xu}K_u) G_{xx'} (D_{x'v}K_v)^2$$

$$+ (G_{xy}J_y)(G_{x'z}J_z)(D_{xx'})^2 - 2 (G_{xy}J_y)(G_{x'z}J_z)$$

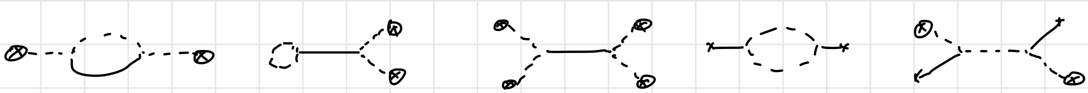
$$\times D_{xx'} (D_{xu}K_u)(D_{x'v}K_v)\Big] + O(\lambda^3)$$

In terms of Feynman diagrams

At $O(\lambda)$:



At $O(\lambda^2)$:



4. Full effective action is

$$W[J,K] = W_0[J,K] + W_{int}[J,K]$$

and

$$W_0[J,K] = -i \cdot -\frac{1}{2} \int d^Dx \int d^Dy \left(J_x G_{xy} J_y + K_x D_{xy} K_y \right)$$

$$= \frac{i}{2} \cdot \int d^Dx \int d^Dy \left(J_x G_{xy} J_y + K_x D_{xy} K_y \right)$$

Since we set $J, K = 0$ after differentiating for correlation functions, the only relevant part of W_{int} is $O(J^2, K^0)$, i.e.

$$W_{int}[J,K] = -i\lambda^2 \int d^Dx \int d^Dy \, (G_{xu} J_u)(G_{yv} J_v)(D_{xy})^2 + O(\lambda^3)$$

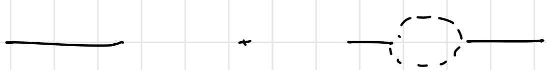
Then

$$\langle \phi(x_1) \phi(x_2) \rangle_c = i \left(\frac{1}{i} \right)^2 \cdot \frac{\delta^2 W}{\delta J(x_1) \delta J(x_2)} \Bigg|_{J,K=0}$$

$$= -i \cdot \Big(i \, G_F(x_1, x_2) - i \, 2\lambda^2 \int d^Dx \int d^Dy \, G_F(x, x_1) \, G_F(y, x_2) \, D_F(x,y)^2 \Big)$$

$$= G_F(x_1, x_2) - 2\lambda^2 \int d^Dx \int d^Dy \, G_F(x_1, x) \, D_F(x,y)^2 \, G_F(y, x_2) + O(\lambda^3)$$

Diagrammatically

= 
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