A Coalgebraic Decision Procedure for NetKAT

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Abstract

NetKAT is a new domain-specific language and logic for specifying and verifying network packet-processing functions. It consists of Kleene algebra with tests (KAT) augmented with specialized primitives for modifying and testing packet headers and encoding network topologies. Previous work developed the design of the NetKAT language and its standard semantics, proved the soundness and completeness of the logic, defined a PSPACE algorithm for deciding equivalence, and presented several practical applications.

This paper develops the coalgebraic theory of NetKAT, including a specialized version of the Brzozowski derivative, and presents a new efficient algorithm for deciding the equational theory using bisimulation. The coalgebraic structure admits an efficient sparse representation that results in a significant reduction in the size of the state space. We discuss the details of our implementation and optimizations that exploit NetKAT's equational axioms and coalgebraic structure to yield significantly improved performance. We present results from experiments demonstrating that our approach is competitive with state-of-the-art tools on several benchmarks.

1. Introduction

Networks have received widespread attention in recent years as a target for domain-specific language design. The emergence of *software-defined networking* (SDN) as a popular paradigm for network programming has led to the appearance of a number of SDN programming languages including Frenetic, Nettle, NetCore, Pyretic, Maple, and PANE, among others [10–12, 26, 27, 37, 38]. The details of these languages differ, but each seeks to provide high-level abstractions to simplify the task of specifying the packet-processing behavior of a network. In addition to SDN languages, a number of verification tools including HSA, VeriFlow, FlowLog, and VeriCon are also being actively developed [2, 16, 17, 28]. As SDN is being actively deployed in production enterprise, data center, and wide-area networks [14, 19, 20], it is becoming clear that SDN is the next major step in the evolution of network technology and is destined to have a significant impact.

Previous work by Anderson et al. [1] introduced NetKAT, a new language and logic for specifying and verifying the packet-processing behavior of networks. NetKAT provides special-purpose primitives for querying and modifying packet headers and encoding network topologies, as well as more general programming constructs such as conditional tests, parallel and sequential composition, and iteration. The language allows desired behavior to be specified equationally. In contrast to competing approaches, NetKAT has a formal mathematical semantics and an equational deductive system that is sound and complete over that semantics, as well as a PSPACE decision procedure. It is based on Kleene algebra with tests (KAT), an algebraic system for propositional program verification that has been studied extensively since 1997 [22]. Several practical applications of NetKAT have been developed including algorithms for testing reachability and non-interference, and a syntactic correctness proof for a compiler that translates programs to hardware instructions for SDN switches.

This paper develops the coalgebraic theory of NetKAT, defines a new algorithm for deciding term equivalence based on this technology, and presents a full implementation in OCaml. The new algorithm is significantly more efficient than the previous naive algorithm, which was PSPACE in the best case and the worst case, as it was based on the determinization of a fundamentally nondeterminstic algorihm [1].

The contributions of this paper are both theoretical and practical. On the theoretical side, we introduce a new coalgebraic model of NetKAT, including a specialized version of the Brzozowski derivative in both semantic and syntactic forms. We prove a version of Kleene's theorem for NetKAT that shows that the coalgebraic model is equivalent to the standard packet-processing and language models introduced by Anderson et al. [1]. A highlight of our theoretical development is a representation theorem showing that the Brzozowski derivative can be concisely representated in a matrix form. On the practical side, we develop a new coalgebraic decision procedure for NetKAT term equivalence based on our theoretical results, along with a full implementation in OCaml. The algorithm constructs a bisimulation between coalgebras built from NetKAT terms via the Brzozowski derivative. The matrix representation enables us to exploit sparseness to give a significant reduction in the size of the state space. The implementation is very efficient in practice—it can verify reachability properties of a real-world campus network in less than a second. The results of benchmarking our implementation demonstrate that our tool compares favorably with the state of the art.

The remainder of this paper is organized as follows. In §2 we briefly review the syntax and semantics of NetKAT from [1]. In §3 we introduce NetKAT coalgebras along with a variant of the Brzozowski derivative. In §4 we prove our main theoretical result on which the correctness of our equivalence algorithm is based: a generalization of Kleene's theorem relating NetKAT expressions and NetKAT coalgebras. In §5 we discuss a streamlined representation of NetKAT coalgebras using matrices, which is needed for our implementation. In §6 we present the details of our implementation, focusing on how we exploit the NetKAT axioms and coalgebraic

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structure to achieve significant performance improvements over the naive algorithm defined previously [1]. In §7 we describe three applications developed from our coalgebraic theory, which are used in the evaluation of our implementation. In §8 we report on the results of experiments. In §9 we discuss related work, and in §10 we present conclusions and identify directions for future research.

2. Overview

This section briefly reviews the syntax and semantics of NetKAT, along with other results that are needed to understand our coalgebraic algorithm described in the following sections [1].

NetKAT is based on Kleene algebra with tests (KAT) [22], a generic equational system for reasoning about partial correctness of programs. KAT is Kleene algebra (KA), the algebra of regular expressions, augmented with Boolean tests. Formally, a KAT is a two-sorted structure $(K, B, +, \cdot, ^*, \overline{}, 0, 1)$, where $B \subseteq K$ and

- $(K, +, \cdot, ^*, 0, 1)$ is a Kleene algebra
- $(B, +, \cdot, \overline{}, 0, 1)$ is a Boolean algebra
- $(B, +, \cdot, 0, 1)$ is a subalgebra of $(K, +, \cdot, 0, 1)$.

The Kleene algebra operators are choice ($+$), sequential composition ($\cdot$, often elided in expressions), iteration (*), fail (0), and skip (1). Elements of B are called *tests*. On tests, $+$ and $\cdot$ behave as Boolean disjunction and conjunction, respectively, and 0 and 1 stand for falsity and truth, respectively. The axioms of Kleene algebra are as follows,

$$\begin{array}{ll}
p + (q + r) = (p + q) + r & p(qr) = (pq)r \\
p + q = q + p & 1 \cdot p = p \cdot 1 = p \\
p + 0 = p + p = p & p \cdot 0 = 0 \cdot p = 0 \\
p(q + r) = pq + pr & (p + q)r = pr + qr \\
1 + pp^* \leq p^* & q + px \leq x \;\Rightarrow\; p^*q \leq x \\
1 + p^*p \leq p^* & q + xp \leq x \;\Rightarrow\; qp^* \leq x
\end{array}$$

where $p \leq q \Leftrightarrow p + q = q$. The axioms of Boolean algebra are

$$\begin{array}{ll}
a + (bc) = (a + b)(a + c) & ab = ba \\
a + 1 = 1 & a + \overline{a} = 1 \\
a \cdot \overline{a} = 0 & aa = a
\end{array}$$

in addition to the axioms of Kleene algebra above. KAT can model standard imperative programming constructs

$$\begin{gathered}
p\,;q = pq \\
\textsf{if } b \textsf{ then } p \textsf{ else } q = bp + \overline{b}q \\
\textsf{while } b \textsf{ do } p = (bp)^*\overline{b}
\end{gathered}$$

as well as Hoare partial correctness assertions

$$\{b\}\,p\,\{c\} \;\Leftrightarrow\; bp \leq pc \;\Leftrightarrow\; bp = bpc \;\Leftrightarrow\; bp\overline{c} = 0.$$

Hoare-style rules become universal Horn sentences in KAT. For example, the Hoare while-rule

$$\frac{\{bc\}\,p\,\{c\}}{\{c\}\,\textsf{while } b \textsf{ do } p\,\{\overline{b}c\}}$$

becomes the universal Horn sentence

$$bcp \leq pc \;\Rightarrow\; c(bp)^*\overline{b} \leq (bp)^*\overline{b}\,\overline{b}c$$

KA and KAT have standard language models consisting of, respectively, the regular sets of finite-length strings over a finite alphabet and the regular sets of *guarded strings* over disjoint finite alphabets of test and action symbols. These language models play an important role in that they are the free models on their generators, which means that they exactly characterize the equational theory. There are other useful models, including binary relation and trace models used in programming language semantics. KAT is complete for the equational theory of binary relation models. The equational theories of KA and KAT are both PSPACE complete.

NetKAT is KAT extended with network-specific primitives for filtering, modifying, and forwarding packets, along with additional axioms for reasoning about programs built using those primitives. More formally, NetKAT is KAT with primitive actions and tests

- $x \leftarrow n$ (assignment)
- $\textsf{dup}$ (duplicate)
- $x = n$ (test)

We also use $\textsf{id}$ and $\textsf{drop}$ for 1 and 0, respectively. Intuitively, the assignment $x \leftarrow n$ assigns the constant value n to the field x in the current packet. The test $x = n$ tests whether the field x of the current packet contains the constant value n. The action $\textsf{dup}$ duplicates the packet in the packet history. For example, the expression

$$\mathit{switch} = 6 \cdot \mathit{port} = 8 \cdot \mathit{dst} \leftarrow 10.0.1.5 \cdot \mathit{port} \leftarrow 5$$

expresses the command: "For all packets located at port 8 of switch 6, set the destination IP address to 10.0.1.5 and forward the packet out on port 5"

The NetKAT axioms consist of the KAT axioms as well as the following axioms:

$$\begin{array}{rcl}
x \leftarrow n \cdot y \leftarrow m & = & y \leftarrow m \cdot x \leftarrow n \quad (\text{if } x \neq y) \\
x \leftarrow n \cdot y = m & = & y = m \cdot x \leftarrow n \quad (\text{if } x \neq y) \\
x = n \cdot \textsf{dup} & = & \textsf{dup} \cdot x = n \\
x \leftarrow n \cdot x = n & = & x \leftarrow n \\
x = n \cdot x \leftarrow n & = & x = n \\
x \leftarrow n \cdot x \leftarrow m & = & x \leftarrow m \\
x = n \cdot x = m & = & 0 \quad (\text{if } n \neq m) \\
\left(\sum_n x = n\right) & = & 1
\end{array}$$

Intuitively, the first axiom states that assignments to distinct fields may be done in either order. The third axiom says that when a packet is duplicated, the field values are preserved in the new packet. The other axioms have similar intuitive interpretations.

The standard model of NetKAT is a packet-forwarding model. A *packet* π is a record whose fields assign constant values n to fields f. A *packet history* is a non-empty sequence of packets

$$\pi_1 :: \pi_2 :: \cdots :: \pi_k,$$

in which the *head packet* is π_1. Operationally, only the head packet exists in the network, but in the logic we keep track of the packet's history to enable precise specification of forwarding behavior involving specific paths through the network. Every NetKAT expression e denotes a function:

$$[\![e]\!] : H \to 2^H$$

where H is the set of all packet histories. Intuitively, the expression e takes an input packet history σ and produces a set of output packet histories $[\![e]\!](\sigma)$. The semantics of the primitive actions and tests are as follows: For a packet history $\pi :: \sigma$ with head packet π,

$$\begin{array}{rcl}
[\![x \leftarrow n]\!](\pi :: \sigma) & = & \{\pi[n/x] :: \sigma\} \\
[\![x = n]\!](\pi :: \sigma) & = & \begin{cases} \{\pi :: \sigma\} & \text{if } \pi(x) = n \\ \varnothing & \text{if } \pi(x) \neq n \end{cases} \\
[\![\textsf{dup}]\!](\pi :: \sigma) & = & \{\pi :: \pi :: \sigma\}
\end{array}$$

where $\pi[n/x]$ denotes packet π with the field x rebound to the value n. Note that the test $x = n$ simply drops the packet if the test is not satisfied and passes it through unaltered if it is satisfied. Thus tests behave as filters on packets. The KAT operations are interpreted as follows:

$$\begin{aligned}
[\![p+q]\!](\sigma) &= [\![p]\!](\sigma) \cup [\![q]\!](\sigma)\\
[\![p\cdot q]\!](\sigma) &= \bigcup_{\tau\in[\![p]\!](\sigma)} [\![q]\!](\tau)\\
[\![p^*]\!](\sigma) &= \bigcup_n [\![p^n]\!](\sigma)\\
[\![1]\!](\sigma) &= [\![\mathsf{pass}]\!](\sigma) \;=\; \{\sigma\}\\
[\![0]\!](\sigma) &= [\![\mathsf{drop}]\!](\sigma) \;=\; \varnothing\\
[\![\neg b]\!](\sigma) &= \begin{cases}\{\sigma\} & \text{if } [\![b]\!](\sigma)=\varnothing\\ \varnothing & \text{if } [\![b]\!](\sigma)=\{\sigma\}\end{cases}
\end{aligned}$$

The interpretation of sequential composition is often called *Kleisli composition*, as it is composition in the Kleisli category of the powerset monad. The operator $+$ accumulates actions. Thus the expression $(\mathit{port} \leftarrow 8) + (\mathit{port} \leftarrow 9)$ describes the behavior of a switch that sends a copy of the packet to ports 8 and 9. Note that this is a departure from the usual Kleene interpretation of $+$ as nondeterministic choice—NetKAT treats $+$ as conjuctive rather than disjunctive. Nevertheless, it is not difficult to show that the axioms of KAT and NetKAT are sound over this interpretation. More difficult is proving completeness, as was done by [1].

The proof of completeness uses a language model for NatKAT that plays a similar role as the regular sets of strings and guarded strings do for KA and KAT respectively. The language model consists of the regular sets of *reduced strings* of the form

$$\alpha p_0 \;\mathsf{dup}\; p_1 \;\mathsf{dup}\; p_2 \cdots p_{n-1} \;\mathsf{dup}\; p_n,\;\; n\geq 0,$$

where α is a *complete test* $x_1 = n_1 \cdot \ldots \cdot x_k = n_k$, the p_i are *complete assignments* $x_1 \leftarrow n_1 \cdot \ldots \cdot x_k \leftarrow n_k$, and $x_1,\ldots,x_k$ are all of the fields in some arbitrary but fixed order. Every string of primitive actions and tests is equivalent to a reduced string modulo the NetKAT axioms. The set of reduced strings is described by the expression $\mathsf{At}\cdot P\cdot(\mathsf{dup}\;\cdot P)^*$, where At is the set of complete tests and P the set of complete assignments. The complete tests are the atoms (minimal nonzero elements) of the Boolean algebra generated by the primitive tests. Complete tests and complete assignments are in one-to-one correspondence determined by the sequence of values $n_1,\ldots,n_k$.

The NetKAT axioms can be expressed in a simpler form for reduced strings. Let α_p be the complete test corresponding to the complete assignment p. Likewise, let p_β be the complete assignment corresponding to the complete test β. The NetKAT axioms for reduced strings are as follows:

$$\alpha\,\mathsf{dup} = \mathsf{dup}\,\alpha \qquad p\alpha_p = p \qquad \alpha p_\alpha = \alpha$$

$$\alpha\alpha=\alpha \qquad \alpha\beta = 0,\; \alpha\neq\beta \qquad qp = p \qquad \textstyle\sum_{\alpha\in\mathsf{At}}\alpha = 1$$

See [1] for a comprehensive treatment.

3. NetKAT Coalgebra

Coalgebra is a general framework for modeling and reasoning about state-based systems [4, 5, 30, 33, 35]. A central aspect of coalgebra is the characterization of equivalence in terms of *bisimulation*. Our work is motivated by recent experiences with bisimulation-based decision procedures for KA and KAT [4, 5, 30]. However, to apply these techniques to NetKAT, we must first develop its coalgebraic theory. This will provide a combinatorial view of NetKAT similar to classical automata theory for KA and automata on guarded strings for KAT. This section develops this theory, which provides the necessary structure for our bisimulation-based decision procedure.

For background on the general theory of coalgebra in modeling state-based systems, see the survey article by Rutten [32]. The only general knowledge needed from this domain are:

1. Coalgebras are typically defined in terms of a set of *states* along with *observation* and *continuation* maps. The observation map gives some observable information about each state and the continuation map specifies transition(s) to the next state(s). The nature of these maps varies widely depending on the type of the system being modeled.
2. Two states are considered *bisimilar* if the observation maps produce identical information for both states and the continuation map leads again to bisimilar states.
3. A *homomorphism* is a map between coalgebras that preserves the structure of observations and continuations.
4. There is often a *final coalgebra* into which there is a unique homomorphism from any other coalgebra of the same type. Two states are bisimilar if and only if this homomorphism maps them to the same value in the final coalgebra.

As an example, a deterministic automaton over a finite alphabet Σ is a coalgebra with an observation map $S\to 2$ that indicates whether a state is an accepting state and a continuation map $S\times\Sigma\to S$ that specifies the transitions of the automaton. The final coalgebra is 2^{Σ^*}, the powerset of the set of all strings over Σ, with observation and continuation maps given by the (semantic) Brzozowski derivative $\varepsilon : 2^{\Sigma^*}\to 2$ such that $\varepsilon(L)=1$ if and only if L contains the null string and $\delta(L,a)=\{w\mid aw\in L\}$ for $a\in\Sigma$ and $w\in\Sigma^*$. The unique homomorphism from an automaton to the final coalgebra takes a state s to the set of strings that would be accepted by the automaton if s were the start state.

There is also a *syntactic* Brzozowski derivative defined inductively on regular expressions Exp over Σ:

$$\begin{aligned}
&E(e_1{+}e_2) = E(e_1)+E(e_2) && D_a(e_1{+}e_2) = D_a(e_1)+D_a(e_2)\\
&E(e_1e_2) = E(e_1)\cdot E(e_2) && D_a(e_1e_2) = \\
& && \quad D_a(e_1)\cdot e_2 + E(e_1)\cdot D_a(e_2)\\
&E(e^*) = 1 && D_a(e^*) = D_a(e)\cdot e^*\\
&E(1) = 1 && D_a(1) = D_a(0) = 0\\
&E(0) = E(a) = 0,\; a\in\Sigma && D_a(b) = \begin{cases}1 & b=a\\ 0 & b\neq a\end{cases}
\end{aligned}$$

The map taking a regular expression e to the set of strings it represents is the unique homomorphism to the final coalgebra.

3.1 Definitions

A NetKAT coalgebra consists of a set of states S along with *continuation* and *observation maps*

$$\delta_{\alpha\beta} : S\to S \qquad\qquad \varepsilon_{\alpha\beta} : S\to 2$$

for $\alpha,\beta\in\mathsf{At}$. A deterministic NetKAT automaton is simply a finite-state NetKAT coalgebra with a distinguished start state $s\in S$. (There are also corresponding notions of nondeterministic automaton and a determinization procedure, but we will not need these for our formal development.) The inputs to the automaton are reduced strings belonging to the set $U = \mathsf{At}\cdot P\cdot(\mathsf{dup}\;\cdot P)^*$. That is, U contains strings of the form

$$\alpha p_0\;\mathsf{dup}\;p_1\;\mathsf{dup}\cdots\mathsf{dup}\;p_n$$

for some $n\geq 0$. Intuitively, $\delta_{\alpha\beta}$ attempts to consume $\alpha p_\beta\;\mathsf{dup}$ from the front of the input string and move to a new state with a residual input string. This succeeds if and only if the reduced string is of the form $\alpha p_\beta\;\mathsf{dup}\;x$ for some $x\in(P\cdot\;\mathsf{dup})^*\cdot P$, in which case the automaton moves to a new state as determined by $\delta_{\alpha\beta}$

with residual input string βx. The observation map $\varepsilon_{\alpha\beta}$ determines whether the string αp_β should be accepted in the current state.

Formally, acceptance is determined by a coinductively defined predicate $\mathsf{Accept} : S \times U \to 2$:

$$\mathsf{Accept}(t, \alpha p_\beta\ \mathsf{dup}\ x) = \mathsf{Accept}(\delta_{\alpha\beta}(t), \beta x)$$
$$\mathsf{Accept}(t, \alpha p_\beta) = \varepsilon_{\alpha\beta}(t).$$

A reduced string $x \in U$ is *accepted* by the automaton if $\mathsf{Accept}(s, x)$, where s is the start state. Thus a NetKAT coalgebra is a coalgebra for the set endofunctor

$$FX = X^{\mathsf{At}\times\mathsf{At}} \times 2^{\mathsf{At}\times\mathsf{At}} \tag{3.1}$$

The continuation and observation maps comprise the structure map of the coalgebra:

$$(\delta, \varepsilon) : X \to FX.$$

One can see immediately from equation (3.1) that $X^{\mathsf{At}\times\mathsf{At}}$ and $2^{\mathsf{At}\times\mathsf{At}}$ are isomorphic to the families of square matrices over X and 2, respectively, with rows and columns indexed by At. Indeed, in §5, we will exploit the one-to-one correspondence between P and At to express δ and ε in matrix form.

3.2 The Brzozowski Derivative

This section develops a variant of the Brzozowski derivative for NetKAT. The derivative comes in two versions: semantic and syntactic. The semantic version is defined on subsets of U and gives rise to a NetKAT coalgebra $(2^U, \delta, \varepsilon)$ that is final for the NetKAT signature (3.1). The syntactic version is defined on NetKAT expressions and alse gives rise to a coalgebra (Exp, D, E). The language interpretation $G : \mathsf{Exp} \to 2^U$ is the unique coalgebra morphism to the final coalgebra.

Final Coalgebra. The final coalgebra for NetKAT is given by the semantic derivative:

$$\delta_{\alpha\beta} : 2^U \to 2^U \qquad \varepsilon_{\alpha\beta} : 2^U \to 2$$
$$\delta_{\alpha\beta}(A) = \{\beta x \mid \alpha p_\beta\ \mathsf{dup}\ x \in A\} \quad \varepsilon_{\alpha\beta}(A) = [\alpha p_\beta \in A].$$

One can show that this is the final coalgebra for the NetKAT signature by showing that bisimilarity implies equality, but we will not need this fact for our development.

Syntactic Coalgebra. There is also a syntactic derivative:

$$D_{\alpha\beta} : \mathsf{Exp} \to \mathsf{Exp} \qquad E_{\alpha\beta} : \mathsf{Exp} \to 2,$$

where Exp is the set of reduced NetKAT expressions. It is defined inductively as follows:

$$D_{\alpha\beta}(p) = 0 \qquad D_{\alpha\beta}(b) = 0 \qquad D_{\alpha\beta}(\mathsf{dup}) = \alpha \cdot [\alpha = \beta]$$
$$D_{\alpha\beta}(e_1 + e_2) = D_{\alpha\beta}(e_1) + D_{\alpha\beta}(e_2)$$
$$D_{\alpha\beta}(e_1 e_2) = D_{\alpha\beta}(e_1) \cdot e_2 + \sum_\gamma E_{\alpha\gamma}(e_1) \cdot D_{\gamma\beta}(e_2)$$
$$D_{\alpha\beta}(e^*) = D_{\alpha\beta}(e) \cdot e^* + \sum_\gamma E_{\alpha\gamma}(e) \cdot D_{\gamma\beta}(e^*)$$

$$E_{\alpha\beta}(p) = [p = p_\beta] \qquad E_{\alpha\beta}(b) = [\alpha = \beta \leq b]$$
$$E_{\alpha\beta}(\mathsf{dup}) = 0 \qquad E_{\alpha\beta}(e_1 + e_2) = E_{\alpha\beta}(e_1) + E_{\alpha\beta}(e_2)$$
$$E_{\alpha\beta}(e_1 e_2) = \sum_\gamma E_{\alpha\gamma}(e_1) \cdot E_{\gamma\beta}(e_2)$$
$$E_{\alpha\beta}(e^*) = [\alpha = \beta] + \sum_\gamma E_{\alpha\gamma}(e) \cdot E_{\gamma\beta}(e^*).$$

Note that the definitions for * are circular, but both are well-defined if we take the least fixpoint of the system of equations.

4. Kleene's Theorem for NetKAT

This section proves that $G(e) \subseteq U$ for some NetKAT expression e if and only if it is the set of strings accepted by some finite NetKAT automaton. This result is the generalization to NetKAT of Kleene's theorem relating regular expressions and automata.

4.1 From Automata to Expressions

Let $M = (S, \delta, \varepsilon, s)$ be a finite NetKAT automaton. Consider a graph H with nodes $(S \times \mathsf{At}) \cup \{\mathsf{halt}\}$ and labeled edges

$$(u, \alpha) \xrightarrow{p_\beta \mathsf{dup}} (v, \beta), \qquad \text{if } \delta_{\alpha\beta}(u) = v$$
$$(u, \alpha) \xrightarrow{p_\beta} \mathsf{halt}, \qquad \text{if } \varepsilon_{\alpha\beta}(u) = 1.$$

We claim that for $x \in (P\cdot\ \mathsf{dup})^* \cdot P$,

$$(t, \alpha) \xrightarrow{x} \mathsf{halt} \ \Leftrightarrow\ \mathsf{Accept}(t, \alpha x). \tag{4.1}$$

This can be proved by induction on the length of x. For the basis,

$$(t, \alpha) \xrightarrow{p_\beta} \mathsf{halt} \ \Leftrightarrow\ \varepsilon_{\alpha\beta}(t) = 1 \ \Leftrightarrow\ \mathsf{Accept}(t, \alpha p_\beta).$$

For the induction step,

$$\begin{aligned}(t, \alpha) \xrightarrow{p_\beta \mathsf{dup} x} \mathsf{halt} &\Leftrightarrow \exists u\ (t, \alpha) \xrightarrow{p_\beta \mathsf{dup}} (u, \beta) \xrightarrow{x} \mathsf{halt}\\ &\Leftrightarrow \exists u\ \delta_{\alpha\beta}(t) = u \wedge \mathsf{Accept}(u, \beta x)\\ &\Leftrightarrow \mathsf{Accept}(\delta_{\alpha\beta}(t), \beta x)\\ &\Leftrightarrow \mathsf{Accept}(t, \alpha p_\beta\ \mathsf{dup}\ x).\end{aligned}$$

The set of labels of paths in H from (t, α) to halt is a regular subset of $(P\cdot\ \mathsf{dup})^* \cdot P$ and is described by a regular expression $e(t, \alpha)$. These expressions can be computed by taking the star of H considered as a square matrix. By (4.1), the set of strings accepted by M is the regular subset of U described by $e = \sum_\alpha \alpha \cdot e(s, \alpha)$.

As shown by [1], if $R(e) \subseteq U$, where R is the canonical interpretation of regular expressions as regular sets of strings, then $R(e) = G(e)$. Hence, we have the following theorem.

Theorem 1. *Let M be a finite NetKAT automaton. The set of strings in U accepted by M is $G(e)$ for some NetKAT term e.*

4.2 From Expressions to Automata

For the other direction, we show how to construct a finite NetKAT automaton M_e from an expression e. The states of the automaton are NetKAT expressions modulo associativity, commutativity, and idempotence (ACI), with e as the start state. The continuation and observation maps are the syntactic derivative introduced in §3.2.

Lemma 1. *The set accepted by M_e is $G(e)$.*

Proof. By Lemma 4, G is a coalgebra homomorphism from the syntactic coalagebra (Exp, D, E) to the set-theoretic coalgebra $(2^U, \delta, \varepsilon)$. Proceeding by induction on the length of the string, we have:

$$\begin{aligned}\mathsf{Accept}(e, \alpha p_\beta) &\Leftrightarrow E_{\alpha\beta}(e) = 1\\ &\Leftrightarrow G(E_{\alpha\beta}(e)) = 1\\ &\Leftrightarrow \varepsilon_{\alpha\beta}(G(e)) = 1\\ &\Leftrightarrow \alpha p_\beta \in G(e),\\ \mathsf{Accept}(e, \alpha p_\beta\ \mathsf{dup}\ x) &\Leftrightarrow \mathsf{Accept}(D_{\alpha\beta}(e), \beta x)\\ &\Leftrightarrow \beta x \in G(D_{\alpha\beta}(e))\\ &\Leftrightarrow \beta x \in \delta_{\alpha\beta}(G(e))\\ &\Leftrightarrow \alpha p_\beta\ \mathsf{dup}\ x \in G(e).\end{aligned}$$ □

It remains to show that M_e is finite. This follows from the fact that e has finitely many derivatives up to ACI. We defer the proof of this fact to Lemma 6 in the next section, as it depends on some details of our data representation.

Theorem 2. *For every NetKAT expression e, there is a deterministic NetKAT automaton M_e with at most $|\mathsf{At}| \cdot 2^\ell$ states accepting the set $G(e)$, where ℓ is the number of occurrences of* dup *in e.*

5. Term and Automata Representations

This section develops a collection of concrete structures that are useful for representing NetKAT automata. These structures lead towards building a practical implementation, and also provide further theoretical insights into the structure of the NetKAT language.

5.1 Matrices

The reader has probably noticed that many of the operations used to define the syntactic derivative in terms of $D_{\alpha\beta}$ and $E_{\alpha\beta}$ maps closely resemble matrix operations. Indeed, if we regard the types of the coalgebra operations as having the following types:

$$\delta : X \to X^{\mathsf{At}\times\mathsf{At}} \qquad \varepsilon : X \to 2^{\mathsf{At}\times\mathsf{At}},$$

then we can view $\delta(t)$ as an $\mathsf{At} \times \mathsf{At}$ matrix over X and $\varepsilon(t)$ as an $\mathsf{At} \times \mathsf{At}$ matrix over 2. Moreover, if X is a KAT, then the family of $\mathsf{At} \times \mathsf{At}$ matrices over X again forms a KAT, denoted $\mathsf{Mat}(\mathsf{At}, X)$, under the standard matrix operations [9]. Thus we have

$$\delta : X \to \mathsf{Mat}(\mathsf{At}, X) \qquad \varepsilon : X \to \mathsf{Mat}(\mathsf{At}, 2).$$

So, the syntactic coalgebra defined in §3.2 takes the following form:

$$\begin{gathered} D(p) = 0 \qquad D(b) = 0 \qquad D(\mathsf{dup}) = J \\ D(e_1 + e_2) = D(e_1) + D(e_2) \\ D(e_1e_2) = D(e_1) \cdot I(e_2) + E(e_1) \cdot D(e_2) \\ D(e^*) = E(e^*) \cdot D(e) \cdot I(e^*), \end{gathered}$$

where $I(e)$ is the matrix with e on the main diagonal and 0 elsewhere, and J is the matrix with α on the main diagonal in position $\alpha\alpha$ and 0 elsewhere; and

$$\begin{gathered} E(\mathsf{dup}) = 0 \qquad E(e_1 + e_2) = E(e_1) + E(e_2) \\ E(e_1e_2) = E(e_1) \cdot E(e_2) \qquad E(e^*) = E(e)^*. \end{gathered}$$

Note that in this form E becomes a KAT homomorphism from Exp to $\mathsf{Mat}(\mathsf{At}, 2)$.

Likewise, we can regard the set-theoretic coalgebra presented in §3.2 as having type:

$$\delta : 2^U \to \mathsf{Mat}(\mathsf{At}, 2^U) \qquad \varepsilon : 2^U \to \mathsf{Mat}(\mathsf{At}, 2).$$

Again, in this form, ε becomes a KAT homomorphism:

Lemma 2.

(i) $\varepsilon(1) = I$
(ii) $\varepsilon(A \cup B) = \varepsilon(A) + \varepsilon(B)$
(iii) $\varepsilon(A \cdot B) = \varepsilon(A) \cdot \varepsilon(B)$
(iv) $\varepsilon(A^*) = \varepsilon(A)^*$

Proof. These properties follow straightforwardly from the definitions in §3.2. For example, for (iii) and (iv), we have

$$\begin{aligned} \varepsilon(AB)_{\alpha\beta} &= [\alpha p_\beta \in AB] \\ &= [\exists\gamma\ \alpha p_\gamma \in A \wedge \gamma p_\beta \in B] \\ &= \sum_\gamma\ [\alpha p_\gamma \in A] \cdot [\gamma p_\beta \in B] \\ &= \sum_\gamma\ \varepsilon(A)_{\alpha\gamma} \cdot \varepsilon(B)_{\gamma\beta} \\ &= (\varepsilon(A) \cdot \varepsilon(B))_{\alpha\beta} \end{aligned}$$

$$\varepsilon(A^*) = \varepsilon(\textstyle\bigcup_n A^n) = \sum_n \varepsilon(A)^n = \varepsilon(A)^*. \qquad \square$$

Lemma 3.

(i) $\delta(\bigcup_n A_n) = \sum_n \delta(A_n)$
(ii) $\delta(AB) = \delta(A) \cdot I(B) + \varepsilon(A) \cdot \delta(B)$
(iii) $\delta(A^*) = \varepsilon(A^*) \cdot \delta(A) \cdot I(A^*)$

where $I(A)$ is the matrix with the set A on the main diagonal and $\varnothing$ elsewhere, and the matrix sum in (i) is componentwise union.

Proof. We argue (ii) and (iii) explicitly; (i) follows from linearity.

(ii) By definition, $\delta_{\alpha\beta}(AB) = \{\beta x \mid \alpha p_\beta\ \mathsf{dup}\ x \in AB\}$. To show that $\alpha p_\beta\ \mathsf{dup}\ x \in AB$, the string must be the product of two reduced strings, one from A and one from B. Depending on which of these strings contains the first occurrence of dup, one of the following must occur: (1) there exists γ such that $\alpha p_\beta\ \mathsf{dup}\ x = \alpha p_\gamma \cdot \gamma p_\beta\ \mathsf{dup}\ x$ with $\alpha p_\gamma \in A$ and $\gamma p_\beta\ \mathsf{dup}\ x \in B$; or (2) there exist γ, y, and z such that $\alpha p_\beta\ \mathsf{dup}\ x = \alpha p_\beta\ \mathsf{dup}\ y p_\gamma \cdot \gamma z$ with $\alpha p_\beta\ \mathsf{dup}\ y p_\gamma \in A$, $\gamma z \in B$, and $x = y p_\gamma \gamma z$.

In the first case, we have $\varepsilon_{\alpha\gamma}(A) = 1$ and $\beta x \in \delta_{\gamma\beta}(B)$, hence $\beta x \in \varepsilon_{\alpha\gamma}(A) \cdot \delta_{\gamma\beta}(B)$. In the second case, we have $\beta y p_\gamma \in \delta_{\alpha\beta}(A)$ and $\gamma z \in B$, hence $\beta x = \beta y_\gamma \gamma z \in \delta_{\alpha\beta}(A) \cdot B$. Thus

$$\delta_{\alpha\beta}(AB) = \delta_{\alpha\beta}(A) \cdot B\ \cup\ \textstyle\bigcup_\gamma \varepsilon_{\alpha\gamma}(A) \cdot \delta_{\gamma\beta}(B).$$

Abstracting over indices, we obtain the matrix equation (ii).

(iii) From (i) and (ii):

$$\begin{aligned} \delta(A^*) &= \delta(1 + AA^*) = 0 + \delta(AA^*) \\ &= \delta(A) \cdot I(A^*) + \varepsilon(A) \cdot \delta(A^*). \end{aligned}$$

The derivative is the least fixpoint of this equation, which by an axiom of KAT is the right-hand side of (iii). $\square$

The following lemma says that G is a coalgebra morphism from the syntactic coalgebra (Exp, D, E) to $(2^U, \delta, \varepsilon)$ (recall that G is the unique homomorphism into the final coalgebra).

Lemma 4.

(i) $G(D(e)) = \delta(G(e))$
(ii) $E(e) = \varepsilon(G(e))$

where G is extended componentwise to matrices.

Proof. By induction on e.

(i) For primitive terms p, b and dup,

$$\begin{aligned} G(D_{\alpha\beta}(p)) &= G(0) = \varnothing \\ &= \{\beta x \mid \alpha p_\beta\ \mathsf{dup}\ x \in \{\gamma p \mid \gamma \in \mathsf{At}\}\} \\ &= \delta_{\alpha\beta}(\{\gamma p \mid \gamma \in \mathsf{At}\}) = \delta_{\alpha\beta}(G(p)) \\ G(D_{\alpha\beta}(b)) &= G(0) = \varnothing \\ &= \{\beta x \mid \alpha p_\beta\ \mathsf{dup}\ x \in \{\beta p_\beta \mid \beta \le b\}\} \\ &= \delta_{\alpha\beta}(\{\beta p_\beta \mid \beta \le b\}) = \delta_{\alpha\beta}(G(b)). \end{aligned}$$

$$\begin{aligned} G(D_{\alpha\beta}(\mathsf{dup})) &= G(\alpha \cdot [\alpha = \beta]) \\ &= \{\beta p_\beta \mid \alpha = \beta\} \\ &= \{\beta x \mid \alpha p_\beta\ \mathsf{dup}\ x \in \{\gamma p_\gamma\ \mathsf{dup}\ p_\gamma \mid \gamma \in \mathsf{At}\}\} \\ &= \delta_{\alpha\beta}(\{\gamma p_\gamma\ \mathsf{dup}\ p_\gamma \mid \gamma \in \mathsf{At}\}) \\ &= \delta_{\alpha\beta}(G(\mathsf{dup})) \end{aligned}$$

$$\begin{aligned}
lrsp(e_1 + e_2) &= lrsp(e_1) \cup lrsp(e_2)\\
lrsp(e_1 e_2) &= \{(l_1, r_1 \cdot e_2) \mid (l_1, r_1) \in lrsp(e_1)\} \cup\\
&\quad \{(e_1 \cdot l_2, r_2) \mid (l_2, r_2) \in lrsp(e_2)\}\\
lrsp(e^*) &= \{(e^* \cdot l, r \cdot e^*) \mid (l, r) \in lrsp(e)\}\\
lrsp(\mathsf{dup}) &= \{(1,1)\}\\
lrsp(b) &= lrsp(p) = \varnothing.
\end{aligned}$$

Figure 1. NetKAT left-right spines.

The case $e_1 + e_2$ is straightforward, since G, δ, and D are linear. For products, using Lemma 3 (ii),

$$\begin{aligned}
G(D(e_1 e_2)) &= G(D(e_1) \cdot I(e_2)) + G(E(e_1) \cdot D(e_2))\\
&= G(D(e_1)) \cdot G(I(e_2)) + G(E(e_1)) \cdot G(D(e_2))\\
&= \delta(G(e_1)) \cdot I(G(e_2)) + \varepsilon(G(e_1)) \cdot \delta(G(e_2))\\
&= \delta(G(e_1) \cdot G(e_2))\\
&= \delta(G(e_1 e_2))
\end{aligned}$$

For star, the system defining $D(e^*)$ is

$$D(e^*) = D(e) \cdot I(e^*) + E(e) \cdot D(e^*)$$

whose least solution is

$$D(e^*) = E(e^*) \cdot D(e) \cdot I(e^*).$$

Using Lemma 3 (iii),

$$\begin{aligned}
G(D(e^*)) &= G(E(e^*) \cdot D(e) \cdot I(e^*))\\
&= G(E(e^*)) \cdot G(D(e)) \cdot G(I(e^*))\\
&= \varepsilon(G(e)^*) \cdot \delta(G(e)) \cdot I(G(e)^*)\\
&= \delta(G(e^*)).
\end{aligned}$$

(ii) For p, b and dup,

$$\begin{aligned}
E_{\alpha\beta}(p) &= [p = p_\beta]\\
&= \varepsilon_{\alpha\beta}(\{\gamma p \mid \gamma \in \mathsf{At}\}) = \varepsilon(G(p)).\\
E_{\alpha\beta}(b) &= [\alpha = \beta \leq b]\\
&= \varepsilon_{\alpha\beta}(\{\alpha p_\alpha \mid \alpha \leq b\})\\
&= \varepsilon_{\alpha\beta}(G(b)).\\
E_{\alpha\beta}(\mathsf{dup}) &= 0\\
&= \varepsilon_{\alpha\beta}(\{\gamma p_\gamma \ \mathsf{dup}\ p_\gamma \mid \gamma \in \mathsf{At}\})\\
&= E_{\alpha\beta}(G(\mathsf{dup}))
\end{aligned}$$

The case $e_1 + e_2$ is straightforward, since G, ε, and E are linear. For products, using Lemma 2 (iii),

$$\begin{aligned}
E_{\alpha\beta}(e_1 e_2) &= \sum_\gamma E_{\alpha\gamma}(e_1) \cdot E_{\gamma\beta}(e_2)\\
&= (E(e_1) \cdot E(e_2))_{\alpha\beta}\\
&= (\varepsilon(G(e_1)) \cdot \varepsilon(G(e_2)))_{\alpha\beta}\\
&= (\varepsilon(G(e_1) \cdot G(e_2)))_{\alpha\beta}\\
&= \varepsilon_{\alpha\beta}(G(e_1 e_2))
\end{aligned}$$

For star, using Lemma 2 (iv),

$$E(e^*) = E(e)^* = \varepsilon(G(e))^* = \varepsilon(G(e^*)). \quad \square$$

5.2 Spines

Matrices provide an elegant way to express and encode NetKAT derivatives. It turns out that the set of possible derivatives of a term is finite, and bounded by the size of term itself. To prove this, we will develop the notion of the *spines* of a term e, and show that its derivatives can always be constructed as sums of spines.

Intuitively, the spines of a term can be obtained by locating the occurences of dup, and forming the pair of the terms appearing to the left and right of the dup. The left component of the pair is called the *left spine* and the right component of the pair is called the right spine. For example, the set of spines of $a \cdot$ dup $\cdot b$ is just $\{(a, b)\}$. The spines are related to the derivative in the following way: the left spine represents the term that must be consumed before the occurrence of dup can be consumed itself, and the right spine indicates the term that remains after doing so.

The inductive definition of the left-right spines of e, denoted $lrsp(e)$, is given in Figure 1. In many situations, just the right spines are useful. They can be defined more simply as follows (to lighten the notation, we write $A \cdot e$ for $\{de \mid d \in A\}$ and $e \cdot A$ for $\{ed \mid d \in A\}$ where $A \subseteq \mathsf{Exp}$ and $e \in Exp$):

$$\begin{aligned}
rsp(e_1 + e_2) &= rsp(e_1) \cup rsp(e_2)\\
rsp(e_1 e_2) &= rsp(e_1) \cdot e_2 \cup rsp(e_2)\\
rsp(e^*) &= rsp(e) \cdot e^*\\
rsp(\mathsf{dup}) &= \{1\}\\
rsp(b) &= rsp(p) = \varnothing.
\end{aligned}$$

It is easy to show that every right spine in $rsp(e)$ has the form $1 \cdot e_1 \cdot e_2 \cdots e_n$, where the e_i are subterms of e, and that there is one spine of e for every occurrence of dup in e.

The next lemma relates the derivative of e and its right spines:

Lemma 5. *For any α, β, the derivative $D_{\alpha\beta}(e)$ is a sum of terms of the form βd, where $d \in rsp(e)$.*

Proof. The proof of this lemma goes by induction on the structure of e. Abusing notation slightly by representing sums of terms as sets,[1] we argue the cases for products and star explicitly.

For products, we have the following equalities:

$$\begin{aligned}
D_{\alpha\beta}(e_1 e_2) &= D_{\alpha\beta}(e_1) \cdot e_2 \cup \textstyle\bigcup_\gamma E_{\alpha\gamma}(e_1) \cdot D_{\gamma\beta}(e_2)\\
&\subseteq \beta \cdot rsp(e_1) \cdot e_2 \cup \beta \cdot rsp(e_2)\\
&= \beta \cdot (rsp(e_1) \cdot e_2 \cup rsp(e_2))\\
&= \beta \cdot rsp(e_1 e_2).
\end{aligned}$$

where we use the induction hypothesis in the second step.

For star, we have the following equalities:

$$\begin{aligned}
D_{\alpha\beta}(e^*) &= \textstyle\bigcup_\gamma E_{\alpha\gamma}(e^*) \cdot D_{\gamma\beta}(e) \cdot e^*\\
&\subseteq \beta \cdot rsp(e) \cdot e^*\\
&= \beta \cdot rsp(e^*).
\end{aligned} \qquad \square$$

The final lemma presented in this section shows that the spines of spines of e are themselves spines of e. Hence, taking repeated derivatives does not introduce new terms.

Lemma 6. *If $d \in rsp(e)$, then $rsp(\beta d) \subseteq rsp(e)$.*

[1] This is a convenient abuse which we take with impunity as we are working modulo ACI. The representation of the Brzozowski derivative in this form is often called the *Antimirov derivative*.

Proof. For example, for products,

$$\begin{aligned}
& d \in rsp(e_1e_2) = rsp(e_1)\cdot e_2 \cup rsp(e_2) \\
& \Rightarrow d \in rsp(e_1)\cdot e_2 \text{ or } d \in rsp(e_2) \\
& \Rightarrow (d = ce_2 \text{ and } c \in rsp(e_1)) \text{ or } d \in rsp(e_2) \\
& \Rightarrow (d = ce_2 \text{ and } rsp(\beta c) \subseteq rsp(e_1)) \\
& \quad \text{or } rsp(\beta d) \subseteq rsp(e_2) \\
& \Rightarrow rsp(\beta d) = rsp(\beta c e_2) = rsp(\beta c)\cdot e_2 \cup rsp(e_2) \\
& \quad \subseteq rsp(e_1)\cdot e_2 \cup rsp(e_2) = rsp(e_1e_2) \\
& \quad \text{or } rsp(\beta d) \subseteq rsp(e_2) \subseteq rsp(e_1e_2) \\
& \Rightarrow rsp(\beta d) \subseteq rsp(e_1e_2).
\end{aligned}$$

For star,

$$\begin{aligned}
& d \in rsp(e^*) = rsp(e)\cdot e^* \\
& \Rightarrow d = ce^* \text{ and } c \in rsp(e) \\
& \Rightarrow rsp(\beta d) = rsp(\beta c e^*) = rsp(\beta c)e^* \cup rsp(e^*) \\
& \quad \subseteq rsp(e)\cdot e^* \cup rsp(e^*) = rsp(e^*).
\end{aligned}$$

For dup,

$$\begin{aligned}
& d \in rsp(\mathsf{dup}) = \{1\} \\
& \Rightarrow d = 1 \\
& \Rightarrow rsp(\beta d) = rsp(\beta) = \varnothing \subseteq rsp(\mathsf{dup}).
\end{aligned}$$

We cannot have $d \in rsp(b)$ or $d \in rsp(p)$, since these sets are empty. □

Taken together, these lemmas show that repeated derivatives of e can all be represented as sums of terms of the form βd, where $d \in rsp(e)$. Thus the number of derivatives of e is at most $|\mathsf{At}| \cdot 2^\ell$, where ℓ is the number of occurrences of dup in e. Moreover, these terms can be represented compactly as a pair of an atom and a subset of $rsp(e)$. Using these representations to build NetKAT automata provides a solid foundation for building an efficient implementation, as is described in the next section.

6. Implementation

We have built a system that decides NetKAT equivalence. Given two NetKAT terms, it first converts these terms into automata using Brzozowski derivatives, and then tests whether the automata are bisimilar using a simple coinductive algorithm. Our implementation consists of 4500 lines of OCaml code and includes a parser, pretty printer, and visualizer. We have also integrated our decision procedure into the Frenetic SDN controller platform. This integration enables automated verification of important properties for real-world topologies and configurations.

Our implementation incorporates a number of important enhancements and optimizations that avoid potential sources of combinatorial blowup. In particular, the derivative and matrix-based algorithms described in the preceding sections are formulated in terms of the NetKAT language model—sets of reduced strings of complete tests and assignments. Building a direct implementation of these algorithms would require constructing square matrices indexed by the universe of possible complete tests and assignments—a huge number that grows exponentially with the number of constants in the terms. Clearly basing an implementation on such a strategy would be impractical, even when only used to test equivalence for tiny terms. Instead, our implementation uses a symbolic representation that exploits symmetry and sparseness and incorporates optimizations that aggressively prune away values that do not contribute to the final outcome. Although the underlying equivalence algorithm remains PSPACE complete, the constrained nature of real-world networks allows our tool to be fast in common cases.

6.1 Data Structures

The foundation of our implementation is built from a collection of data structures that provide symbolic representations for NetKAT automata.

Bases. Bases represent sets of pairs of complete tests and assignments symbolically, often avoiding the enumeratation of every possible packet value in the set. Fix a NetKAT term and let f_1 to f_n be the collection of fields appearing in it. Likewise, let U_i be the universe of all values associated with f_i either by a test $f_i = v$ or an assignment $f_i \leftarrow v$. A *base* b is a pair of sequences $A_1, \ldots, A_n; o_1, \ldots, o_n$, where $A_1 \subseteq U_1$ to $A_n \subseteq U_n$ are sets of values and $o_1 \in U_1$ to $o_n \in U_n$ are optional values. The set represented by b contains all tests where the value of the test for field f_i is drawn from A_i and the value for the assignment to f_i is either v_i if o_i is defined and equal to v_i, or a value drawn from A_i otherwise.

Matrices. Using bases, it is straightforward to build a sparse matrix representation where the indices are complete tests and assignments. To encode a 0-1 matrix, we simply use a set of bases (*base set*). To encode a matrix over a set X, we use finite maps from bases to X. For example, when constructing the E matrix for a term e, tests $x_i = m_i$ are represented by

$$U_1, \ldots, U_{i-1}, \{m_i\}, U_{i+1}, \ldots, U_n; ?, \ldots, ?,$$

where ? denotes a missing optional value, and assignments $x_i \leftarrow m_i$ are represented by

$$U_1, \ldots, U_n; ?, \ldots, ?, m_i, ?, \ldots, ?.$$

Sums and products can be obtained using matrix addition and multiplication as implemented using base sets. The product of bases, $(A_1, \ldots, A_n; v_1, \ldots, v_n)$ and $(B_1, \ldots, B_n; u_1, \ldots, u_n)$, is non-zero if there exists a complete assignment in the left base that matches a complete test in the right for each field. If $v_i = ?$, then the intersection of A_i and B_i must be non-empty, otherwise the tests corresponding to the ith field will drop all packets produced by the left base. On the other hand, if $v_i \neq ?$, then its value must belong to B_i. The resulting product $C_1, \ldots, C_n; w_1, \ldots, w_n$ is determined as follows:

$$C_i = \begin{cases} A_i & \text{if } j_i \neq ? \\ A_i \cap B_i & \text{if } j_i = ? \end{cases} \quad = w_i = \begin{cases} j_i & \text{if } j_i \neq ? \text{ and } k_i = ? \\ k_i & \text{if } j_i = ? \text{ and } k_i \neq ? \\ & \text{or } j_i \neq ? \text{ and } k_i \neq ? \\ ? & \text{if } j_i = k_i = ? \end{cases}$$

Using the product operation on bases, it is easy to build other matrix operations. For example, multiplication can be implemented by folding over the base sets, and fixed points can be computed using an iterative loop that multiplies at each step or repeated squaring.

6.2 Algorithms

The two core pieces of our implementation are (i) an algorithm that computes automata using Brzozowski derivatives, and (ii) another that checks bisimilarity of automata.

Brzozowski derivative. Our implementation of Brzozowski derivatives uses the left-right *spines* introduced in §5.2. Recall that there is one spine for every occurrence of dup in e. If σ denotes an occurrence of dup, let ℓ_σ and r_σ denote the left spine and right spine, respectively, of that occurrence. It is straightforward to show that

$$D_{\alpha\beta}(e) = \bigcup\{\beta r_\sigma \mid \sigma \text{ an occurrence of } \mathsf{dup},\ E_{\alpha\beta}(\ell_\sigma) = 1\}$$

or more succinctly,

$$D(e) = \sum_\sigma E(\ell_\sigma)\cdot J \cdot I(r_\sigma). \tag{6.1}$$

To further streamline the computation of $D(e)$ we can avoid adding the βr_σ term to $D_{\alpha\beta}(e)$ when βr_σ is zero, or equivalently when

$\beta \cdot x \notin r_\sigma$. Let Φ be a function that replaces all occurrences of dup with 1 as follows:

$$\Phi(p) = p \qquad \Phi(b) = b \qquad \Phi(\mathsf{dup}) = 1$$
$$\Phi(e_1 + e_2) = \Phi(e_1) + \Phi(e_2)$$
$$\Phi(e_1 e_2) = \Phi(e_1) \cdot \Phi(e_2) \qquad \Phi(e^*) = \Phi(e)^*.$$

It is easy to show that for any α, the set $\{\alpha \mid \alpha x \in e\}$ is equal to $\{\alpha \mid \alpha x \in \Phi(e)\}$. Hence, $\beta \cdot x \notin r_\sigma$ is equivalent to $\beta \cdot x \notin \Phi(r_\sigma)$. Moreover, beacuse $\Phi(r_\sigma)$ does not contain dup, the set $\{\alpha \mid \alpha x \in e\}$ is described by the lefthand sequences $(A_1, \ldots, A_n)$ in the base set representation of $E(\Phi(r_\sigma))$. Hence the derivative can be described as:

$$D(e) = \sum_\sigma E(l_\sigma) \cdot E'(\Phi(r_\sigma)) \cdot J \cdot I(r_\sigma),$$

This formulation, which is used in our implementation, has a number of advantages. First, it is expressed entirely in terms of simple matrix operations involving the E, I, and J matrices. Second, it aggressively filters away intermediate terms that do not contribute to the overall result. In particular, if the $\alpha\beta$ entry of $E(\ell_\sigma)$ is 0, then the occurrence of dup indicated by σ does not contribute to the "first" dup in any reduced string denoted by e, so the derivative is also 0. Third, since the spines of spines of e are spines of e, we can calculate the left-right spines once when we construct the term, and subsequent derivatives are guaranteed to have the form of (6.1).

Bisimulation The other step in our NetKAT decision procedure tests the bisimilarity of the automata constructed using Brzozowski derivatives. This standard algorithm works as follows: given two NetKAT terms e_1 and e_2, we first compare the matrices $E(e_1)$ and $E(e_2)$ and check whether they are not identical, returning false immediately if they are not. Otherwise, we calculate all derivatives of e_1 and e_2, and recursively check each of the resulting pairs. The algorithm halts when we have tested every possible derivative reachable transitively from the initial terms. Working modulo ACI guarantees that the algorithm terminates. This coinductive algorithm can be implemented in almost linear time in the combined size of the automata using the union-find data structure [13].

6.3 Optimizations

To further improve performance, our implementation incorporates a number of optimizations designed to reduce the overhead of representing and computing with terms, bases, sparse matrices, etc.

Hash consing and memoization. Encoding real-world network topologies and configurations as NetKAT terms often leads to many repetitions. Our implementation exploits this insight by using aggresive normalization and hash consing so that (many) semantically equivalent terms are represented by the same syntactic term. For instance, we represent products as lists which gives associativity for free, and sums as sets which gives associativity, commutativity, and idempotence of sums for free. We also use smart constructors that recognize identities involving 0 and 1 (along with several others) to further optimize the representations of terms. We calculate term metadata such as left-right spines and the E matrix lazily and store the results in a local memoization table.

Sparse multiplication. The straightforward way to represent matrix multiplication in terms of base sets would be to use nested folds over the sets—the product of two base sets is simply their pointwise cross-product. Sadly, this naive algorithm gives poor performance — and much of the effort is wasted multiplying bases whose product is always 0. We instead implement an algorithm which, rather than iterating over all pairs of bases, instead proactively filters the sets, only retaining elements which could produce a non-zero result. To do this, we (i) build an association from assignments to their originating bases from the left matrix. (ii) For each base in the right matrix, intersect all test value sets in this base with the set of all assignments in the left matrix (per field). We will call the result of this operation the set of potential matches. (iii) Multiply each base in the right matrix with the bases corresponding to its potential matches to produce the overall product. These operations allow us to associate each base in the right operand with a small set of left operand bases, significantly limiting the number of pairs we multiply.

Base compaction. There can be many representations of a set of complete tests and assignments, and as base sets are multiplied, those representations tend to grow. Hence, another key optimization for making matrix multiplication (and many other operations) fast is to compact the base sets whenever possible. Two bases can be merged when (i) one is a subset of the other or (ii) they are adjacent, in the following senses. The base $b_1 = A_1, \ldots, A_n; v_1, \ldots, v_n$ is a *subset* of base $b_2 = B_1, \ldots, B_n; u_1, \ldots, u_n$ when for all fields i, we have $A_i \subseteq B_i$ and $v_i = u_i$. In this case, b_1 and b_2 can be replaced with just b_2. The base b_1 is *adjacent* to b_2 if there exists a field i such that $v_i = u_i$ and for all other fields j both $A_j = B_j$ and $v_j = u_j$. The result of merging these two bases is $B_1, \ldots, A_i \cup B_i, \ldots, B_n; u_1, \ldots, u_n$. Although both of these merging optimziations require bases with identical assignments, we can efficiently reduce the number of bases we attempt to merge by sorting base sets by their assignments. This yields a fast optimization that dramatically compacts the base sets that must be maintained in our implementation.

Fast fixpoints. The algorithm for calculating E matrix presented in §5.1 uses a fixpoint, which is a potentially expensive operation. Our implementation incorporates several optimizations that greatly increase the efficiency of calculating this fixpoint by exploiting the structure of terms encoding networks. Generally speaking, network terms are of the form $in \cdot (p \cdot t)^* \cdot p \cdot out$, where in and out are edge policies that describe the packets entering and leaving the network, p is a policy the describes the behavior of the switches, and t encodes the topology. The edge policies are typically very small compared to p or t. Hence, we can use the edge policies to cut down the size of the $(p \cdot t)^*$ term as we take its fixpoint. We first unfold $in \cdot (p \cdot t)^* \cdot p \cdot out$ to $in \cdot out + in \cdot (p \cdot t)^* \cdot out$ and then find the $in \cdot (p \cdot t)^*$ fixpoint by calculating $in \cdot p \cdot t + in \cdot p \cdot t \cdot p \cdot t + \cdots + in \cdot (p \cdot t)^i$, stopping when we have reached at the fixpoint. We have determined empirically that this process converges must faster than other techniques for computing fixed points, such as repeatedly squaring the $(p \cdot t)$ term.

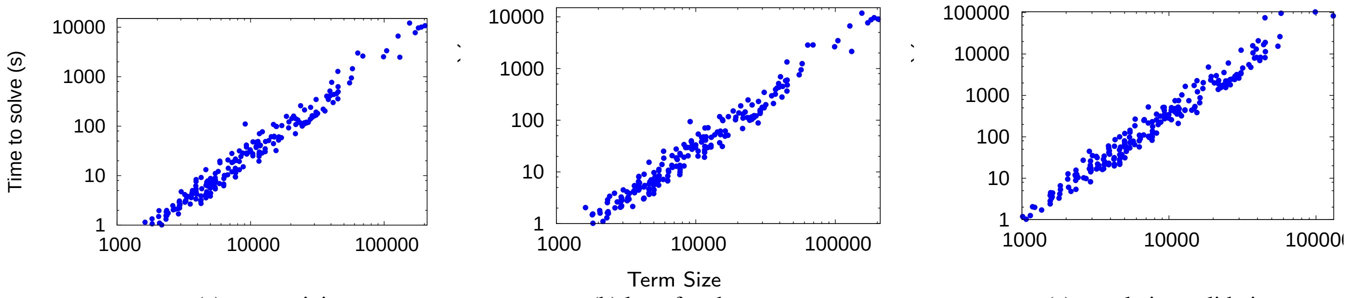
7. Additional Applications

The utility of NetKAT's coalgebraic theory is not limited to checking equivalence via bisimulation. The E and D matrices are also useful for solving many other practical verification problems directly. This section discusses several such applications: all-pairs connectivity, loop freedom, and translation validation.

Connectivity. Reachability—can host A communicate with host B over the network—is a fundamental property of a network. Indeed, there are now many automated tools that can check reachability properties involving individual locations—see §9 for a survey. Connectivity is a slightly stronger property that asks whether *every* host in a network can communicate with every other host. In other words, connectivity tests whether a network provides the functionality of "one big switch" that forwards traffic between all of its ports.

As connectivity cares only about the end-to-end properties of a network, it is agnostic to paths. Hence, it can be modeled in NetKAT by simply setting all occurrences of dup to 1 using Φ from

Topology	Term Size	# Switches	Largest Policy	Parse Time	Loop Freedom	Connectivity	Translation
Airtel	2412	15	112	0.102	3.386	1.755	10.424
Uran	5000	23	264	0.327	5.905	5.757	71.18
GtsSlovakia	10388	34	385	0.99	25.361	25.166	258.83
Uunet	20350	48	539	6.684	138.147	143.387	2007.68
Telcove	41474	72	781	32.826	280.413	300.265	6959.08
Oteglobe	56844	92	838	93.045	944.955	944.555	26292.8
Pern	131092	126	4757	311.644	2140.63	2478.24	83245.7



(a) connectivity (b) loop freedom (c) translation validation

Figure 2. Topology Zoo experimental results.

§6 and checking the following:

$$\Phi(in \cdot (p \cdot t)^* \cdot p \cdot out) = \sum_{(sw, sw', pt, pt')} \begin{pmatrix} switch = sw \cdot port = pt \cdot \\ switch \leftarrow sw' \cdot port \leftarrow pt' \end{pmatrix}$$

where p encodes the switch policy, t encodes the topology, and in the summation, (sw, pt) and (sw, pt') range over all outward-facing switch-port pairs—i.e., those adjacent to a host. Intuitively the left (top) half of the equation encodes the end-to-end forwarding behavior of the network—forwarding that starts from a state matching in and traverses the switch policy and topology any number of times and eventually reaches a state matching out—while the bottom equation represents a specification of a network that forwards directly between every outward-facing switch-port pair. Because connectivity does not involve keeping track of paths, it can be verified simply by calculating the E matrices and checking for equality.

Forwarding loops. A network has a *forwarding loop* if some packets traverse a cycle repeatedly. Forwarding loops are a frequent source of error in networks and have been identified as the cause of outages both in local-area networks, where loops are often produced by protocols for computing broadcast spanning trees, and on the broader Internet, where inter-domain protocols such as BGP can easily produce loops during periods of reconvergence [15]. Making matters worse, forwarding loops are often masked by features such as TTL (time-to-live) fields—a run-time mechanism that enforces an upper bound on the length of any loop by decrementing a counter at each hop and dropping the packet when the counter reaches 0.

To check whether a packet has a loop, we need to determine if there exists a packet that can reach the same location with the same value twice. Unfortunately this property is somewhat difficult to express efficiently as a NetKAT equivalence problem. One possibility is the equation $in \cdot (p \cdot t)^{2^n} = 0$, where p is the switch policy, t is the topology, and n is the number of complete tests which occur in the program. The intuition behind this equation is that there are only 2^n distinct packet values, so any packet that traverses the network more than 2^n times must have been in some state at least twice, and thus will loop forver. However, while this equation is correct, checking it directly is cumbersome—2^n is a large number, and exponentiation is an expensive operation in our implementation.

To make the problem more tractable, we can recast it into an equivalent formulation using a quantifer:

$$\forall \alpha.\ \mathit{prefix}(in \cdot (p \cdot t)^*) \cdot \alpha \cdot p \cdot t \cdot (p \cdot t)^* \cdot \alpha = 0,$$

where *prefix* is the prefix closure. This equation directly encodes a packet that visits the same state twice, allowing us to avoid an expensive exponentiation operation. Moreover, it is not hard to show that the inner term is already prefixed closed as it ends with a star so we can reformulate our test to:

$$\forall \alpha.\ in \cdot (p \cdot t)^* \cdot \alpha \cdot p \cdot t \cdot (p \cdot t)^* \cdot \alpha = 0.$$

After converting to matrices, we obtain:

$$\forall (\alpha, \beta) \in E(in \cdot (p \cdot t)^*).\ \beta \cdot (p \cdot t) \cdot (p \cdot t)^* \cdot \beta = 0.$$

We then observe that for any complete test β and term x, we have $\beta \cdot x \cdot \beta = 0$ if and only if $E_{\beta\beta}(x) = 0$. This yields a fast algorithm for determing loop freedom. We iterate through the sets of possible α and β with non-zero entries in $E(in \cdot (p \cdot t)^*)$ and check the entry in $E(p \cdot t \cdot (p \cdot t)^*)$: if it is also non-zero then the network has a loop. We have used this algorithm to check for loops in networks with topologies containing thousands of switches and configurations with thousands of forwarding rules on each switch.

Translation validation. One way to check the correctness of a compiler is to test whether the instructions it emits have same semantics as the programs provided as input. This technique is called *translation validation* [29]. We can use translation validation to check the NetKAT compiler itself, which is used by the Frenetic controller [11]. Unlike the applications just discussed, which only depend on analyzing E matrices and do not require bisimulation, translation validation uses the full NetKAT decision procedure. This is due to the fact that it must check the equivalence of the paths generated by the compiler rather than just checking end-to-end forwarding.

We have developed a simple application that uses bisimulation to validate the output of the Frenetic compiler. It takes an input policy p and invokes the NetKAT compiler to convert it to a sequence of OpenFlow fowarding rules, one for each switch. As was shown in the original NetKAT paper [1], the language is expressive enough to encode these rules, so we can reflect them back into

NetKAT terms as nested cascades of conditionals,

$$\begin{array}{rcl} c & = & \text{if } pat_1 \text{ then } acts_1 \text{ else} \\ & & \cdots \\ & & \text{if } pat_k \text{ then } acts_k \text{ else } 0 \end{array}$$

where each pat_i is a positive conjunction of tests and each act_i is a sequence of modifications. To verify equivalence, we simply check whether $p = (c \cdot t)^* \cdot c$, where t is the topology. If this succeeds, we know that the forwarding rules emitted by the compiler encode the same paths as those specified in the program.

8. Evaluation

To evaluate our implementation, we conducted a number of experiments on a variety of benchmarks. These experiments were designed to answer the following questions: Can our coalgebraic decision procedure effectively answer practical questions about real-world network topologies and configurations? How does its performance scale as the inputs grow in size? How does its performance compare to other network property checking tools?

Benchmarks. We ran experiments on the following benchmarks:

Topology Zoo: this public dataset contains a collection of 261 actual network topologies from around the world [18]. largely consisting of regional ISPs and carrier networks. The topologies range in size from 5 nodes (the original ARPANET) up to over 196 nodes (Cogentco). We generated forwarding policies by placing hosts into the topology at random and computing paths that forward between all pairs of hosts. This benchmark provides the ability to experiment with a wide variety of topologies, ranging from trees, to stars, to meshes, and even seemingly random structure.

FatTrees: these topologies, which are commonly used in datacenter networks, consist of a tree-structured heirarchy where the switches at each level have multiple redundant connections to the switches one level up. We wrote a simple Python script that generates FatTrees for a given pair of depth and fanout parameters as well as a NetKAT policy that provides connectivity between hosts. This benchmark provides the ability to experiment with the scalability of our tool on commonly used topologies of varying size.

Stanford Backbone: this includes the 16-node topology of the Stanford campus backbone network as well as the actual configurations of each router [16]. This benchmark serves as an example of a complete real-world network and provides a means to compare performance directly against other propety-checking tools.

Properties. Our experiments focused on checking the following properties and used the applications described in §7:

Connectivity: Does the network provide connectivity for all hosts?

Loop-Freedom: Does the network have any forwarding loops?

Translation Validation: Does the NetKAT compiler translate high-level policies into equivalent OpenFlow forwarding rules?

For the Stanford benchmark, to facilitate a head-to-head comparison against HSA, we also performed a simple reachability query from a single source to single destination.

Methodology. We ran our experiments on a small cluster of Dell r720 servers with four eight-core 2.70GHz Intel Xeon CPU E5-2680 processors and 64GB of RAM running Ubuntu 12.04.4 LTS. We restricted each experiment to run on a single core. We collected running times using the Unix system time command for total process times and `Sys.time()` for sub-process times. We excluded the time required to parse inputs and generate policies and only report the amount of time used for actual verification.

Results and Analysis. The results of our experiments on the Topology Zoo and FatTree benchmarks are depicted in Figures 2 and 3. We plot the running times of the benchmarks on inputs of varying size and also provide a sampling of datapoints in a table. All times are reported in seconds.

For the Topology Zoo benchmark, we see that our implementation is able to check small topologies with dozens of switches and policies with hundreds of rules on each switch in tens of seconds, and it scales to topologies with hundreds of switches and policies with thousands of rules without difficulty. The graphs in Figure 2 plot the running time of all three applications against the size of the NetKAT input term. Note that the plots use a semi-logarithmic scale to show the overall trend. The performance of loop detection is similar to connectivity. The performance of translation validation is slightly slower, taking about an order of magnitude longer on large inputs, taking tens of hours for topologies with thousands of switches. This is expected. as translation validation involves invoking the full bisimulation algorithm. However, our tool is still able to complete and produce the correct result.

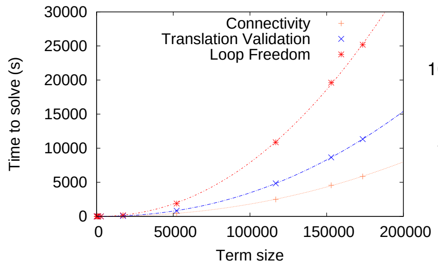
For the FatTree benchmark, we measured the scalabilty of our tool as the size of the network increases from a small number of nodes to several hundred nodes. The graph on the left of Figure 3 plots the performance of all three applications against the size of the NetKAT input term. We observe similar scaling as for the FatTree benchmarks—small networks complete in seconds while larger networks can take up to several hours. The graph on the right of Figure 3 compares the performance of our three applications, giving the relative running time compared to the slowest application—loop freedom—on FatTrees of increasing size. On small inputs, all three applications take roughtly the same amount of time, on larger inputs, connectivity is fastest, loop detection is about half as fast, and translation validation is again half as fast.

Comparison to Header Space Analysis. For our final experiment, we compared the running-time of our tool against the HSA network property-checking tool [16]. HSA works by doing a symbolic ternary simulation of the space of possible packet values through the policy and policy. It incorporates numerous optimizations to prune the space and keep the representation compact. HSA is able to answer simple queries like reachability and loop detection involving a single host or port, but it does not check full equivalence (unless one performed iterated reachability queries over the entire state space). Nevertheless, for many properties of interest, HSA is able to produce an answer in a few seconds.

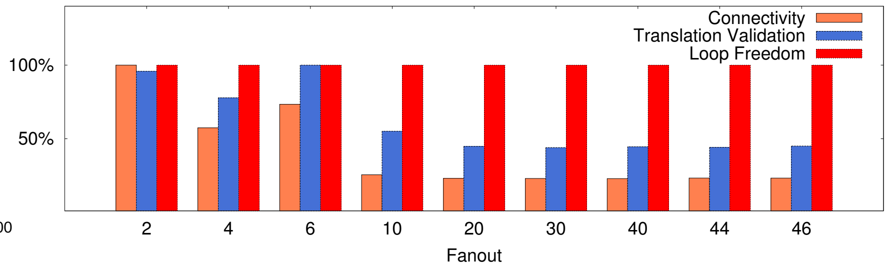
To compare the performance of our tool against HSA, we made several improvements to our tool. First, we wrote a new front-end that parses the router configurations for the Stanford backbone. Second, we wrote a tool that converts configurations based on IP prefix matching into a policies that only tests concrete IP values. This tool works by computing a partitioning of the space of all possible IP addresses that respects the constants mentioned in the program and then replacing each IP prefix with the union of representatives of the equivalence classes it includes. Third, we developed a new optimization that statically anlayzes NetKAT policies to determine which fields are kept constant and uses the NetKAT axioms and partial evaluation to dramatically reduce the size of the search space. For example, if a policy matches on IP destination addresses and never modifies those addresses, then for any particular address we can partially evaluate the policy to obtain a smaller policy that is specialized to that host. This analysis and optimization is integrated into our algorithm for constructing NetKAT automata and applied automatically during the computation of the star.

With these enhancements, our tool is able to answer a reachability query involving a single source host in 0.67 seconds. Note that this query is evaluated on the production Stanford backbone network with 16 routers and thousands of forwarding rules! In the original HSA paper, the authors report 13 seconds for a single reachability query. We were able to replicate this number on our

Fanout	Depth	Term Size	# of Switches	Largest Policy	Parse Time	Connectivity	Translation	Loop Freedom
4	2	311	6	16	0.004000	0.031	0.042	0.054
10	2	2807	15	590	0.052003	0.699997	1.511	2.742
20	2	17207	30	3880	0.636040	38.246	74.17	165.548
30	2	52207	45	8670	6.256391	435.082	832.18	1891.75
46	2	173519	69	21781	126.051878	5858.85	11338.7	25179.1



(a) scalability



(b) relative performance

Figure 3. FatTree experimental results.

own testbed. Since publishing their original paper, the authors have built a hand-optimized version of HSA in C that can answer the same query in 0.5 seconds, but we were not able to replicate this figure at the time of the writing of this paper.

Discussion. Overall, our experiments demonstrate that our NetKAT decision procedure is able to scale to real-world network topologies and configurations and provides good performance on many common properties. Although the algorithm is PSPACE complete, our implementation is still fast enough to be used for offline verification of production networks and can answer simpler questions such as point-to-point reachability in well under a second, making it also suitable for more dynamic situations.

9. Related Work

NetKAT [1] is the latest in a series of domain-specific languages for SDN programming developed as a part of the Frenetic project [11, 12, 26, 27]. NetKAT largely inherits its syntax, semantics, and application methodology from these earlier efforts but adds a complete deductive system and PSPACE decision procedure [1]. These new results in NetKAT build on the strong connection to earlier algebraic work in KA and KAT [21, 22, 24]. The present paper extends work on NetKAT further, developing the coalgebraic theory of the language and engineering an implementation of these ideas in an OCaml prototype. The overall result is the first practical implementation for deciding NetKAT equivalence.

The coalgebraic theories of KA and KAT and related systems have been studied extensively in recent years [8, 23, 33, 35], uncovering strong relationships between the algebraic/logical view of systems and the combinatorial/automata-theoretic view. We have exploited these ideas heavily in the development of NetKAT coalgebra and NetKAT automata. Finally, in our implementation, we have borrowed many ideas and optimizations from the coalgebraic implementations of KA and KAT and other related systems [4, 5, 30] to provide enhanced performance, making automated decision feasible even in the face of PSPACE completeness.

A large number of languages for SDN programming have been proposed in recent years. Nettle [37] applies ideas from functional reactive programming to SDN programming, and focuses on making it easy to express dynamic programs using time-varying signals rather than event loops and callbacks as in most other systems. PANE [10] exposes an interface that allows individual hosts in a network to request explicit functionality such as increased bandwidth for a large bulk transfer or bounded latency for a phone call. Internally PANE uses hierarchical tables to represent and manage the set of requests and a compiler inspired by NetCore [26]. Maple [38] provides a high-level programming interface that enables programmers to express network programs directly in Java, using a special library to match and modify packet headers. Under the hood, the Maple compiler builds up representations of network traffic flows using a tree structure and then compiles these to hardware-level forwarding rules. Several different network programming languages based on logic programming have been proposed including NDLog [25] and FlowLog [28]. The key difference between all of these languages and the system presented in this paper is that NetKAT has a sound and complete deductive theory and supports automated reasoning about program equivalence.

Lastly, there is a growing body of work focused on applications of formal methods ranging from lightweight testing to full-blown verification to SDN. The NICE [7] tool uses a model checker and symbolic execution to find bugs in network programs written in Python. Automatic Test Packet Generation [40] constructs a set of packets that provide coverage for a given network-wide configuration. Retrospective Causal Inference [34] uses techniques based on delta debugging to reduce bugs to minimal input sequences. The VeriCon [2] system uses first-order logic and a notion of admissible topologies to automatically check network-wide properties. It uses the Z3 SMT solver as a back-end decision procedure. Several different systems have proposed techniques for checking network reachability properties including Xie et al. [39], Header Space Analysis [16], and VeriFlow[17]. These tools either translate reachability problems into problem instances for other tools, or they use custom decision procedures that extend basic satisfiability checking or ternary simulation with domain-specific optimizations to obtain improved performance. Compared to these tools, NetKAT is unique in its focus on algebraic and coalgebraic structure of network programs. Moreover, as shown in the original NetKAT paper, many properties including reachability can be reduced to equivalence.

10. Conclusion

This paper develops the coalgebraic theory of NetKAT and a new decision procedure based on bisimulation. The coalgebraic theory includes a definition of NetKAT automata, a variant of the Brzozowski derivative, and a version of Kleene's theorem relating terms and automata. A novel aspect of the theory is the concise representation of the Brzozowski derivative in terms of matrices and spines. Our implementation improves on a previous naive algorithm [1] and initial experimental results are promising. In the future, we intend to continue to make further enhancements and perform extensive testing on practical examples from [1]. A straightforward extension is to incorporate well-studied algorithmic enhancements to the bisimulation construction such as up-to techniques [4, 31]. We also plan to explore extending alternative algorithms for deciding equivalence of KAT expressions [6, 36]. Another possible direction is to study nondeterministic NetKAT automata, which could provide more compact representations, or algorithms for deciding equivalence [3, 4]. We also intend to deploy our tool in Frenetic [11].

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