Street for pedestrians

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Abstract

The singular aim of these notes is to give a less condensed and more clear explanation of Ross Street's significant paper (Street 1974). We highly encourage reader to also refer to (Kock 1995) and (Lack 2000).

1 Pseudo algebras for strict 2-monads

DEFINITION 1.1. Let \mathcal{K} be a 2-category and $(T: \mathcal{K} \to \mathcal{K}, i: 1 \Rightarrow T, m: T^2 \Rightarrow T)$ a strict 2-monad on \mathcal{K} . A **pseudo-algebra** of T consists of

- i. a 0-cell A in K,
- ii. a 1-cell $\mathfrak{a}: TA \to A$ which we call structure map,
- iii. and invertible 2-cells $\zeta: 1_A \Rightarrow \mathfrak{a} \circ i_A$ and $\theta: \mathfrak{a} \circ T\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$,

subject to the following coherence axioms:

$$(\theta \cdot m_{TA}) \circ (\theta \cdot T^2 \mathfrak{a}) = (\theta \cdot Tm_A) \circ (\mathfrak{a} \cdot T\theta)$$

expressed by equality of pasting diagrams:

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$$T^{3}A \xrightarrow{T^{2}\mathfrak{a}} T^{2}A \xrightarrow{T\mathfrak{a}} T^{2}A \xrightarrow{T\mathfrak{a}} T^{2}A \xrightarrow{T^{2}\mathfrak{a}} T^{2}A$$

$$\downarrow^{m_{TA}} \downarrow \qquad \downarrow^{m_{A}} \downarrow \qquad \downarrow^{T\mathfrak{a}} \downarrow \qquad \downarrow^$$

and

$$(\theta \cdot Ti_A) \circ (\mathfrak{a} \cdot T\zeta) = id_{\mathfrak{a}} = (\theta \cdot i_{TA}) \circ (\zeta \cdot \mathfrak{a})$$

expressed by equality of pasting diagrams:

$$TA \xrightarrow{1_{TA}} TA$$

$$T_{i_A} \downarrow T.\zeta \Downarrow \downarrow 1_{TA} \qquad TA \xrightarrow{\mathfrak{a}} A$$

$$T_{i_A} \downarrow T.\zeta \Downarrow \downarrow 1_{TA} \qquad TA \xrightarrow{\mathfrak{a}} A$$

$$T_{i_A} \downarrow T.\zeta \Downarrow \downarrow 1_{TA} \qquad TA \xrightarrow{\mathfrak{a}} A$$

$$T_{i_A} \downarrow T.\zeta \Downarrow \downarrow 1_{TA} \qquad TA \xrightarrow{\mathfrak{a}} TA \Leftrightarrow IA$$

$$T_{i_A} \downarrow T.\zeta \Downarrow \downarrow I_{i_A} \qquad IA$$

$$T_{i_A} \downarrow TA \xrightarrow{T_{i_A}} TA \Leftrightarrow IA$$

$$T_{i_A} \downarrow TA \xrightarrow{T_{i_A}} TA \Leftrightarrow IA$$

$$TA \xrightarrow{T_{i_A}} TA \Leftrightarrow IA$$

DEFINITION 1.2. Suppose $(\mathfrak{a}, \zeta_A, \theta_A) : TA \to A$ and $(\mathfrak{b}, \zeta_B, \theta_B) : TB \to B$ are pseudo-algebras of a 2-monad T. A **lax morphism** from \mathfrak{a} to \mathfrak{b} consists of a 1-cell $f : A \to B$ and a 2-cell \check{f}

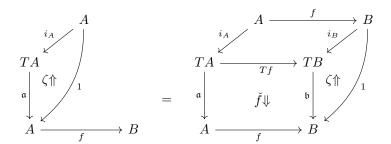
$$TA \xrightarrow{Tf} TB$$

$$\downarrow a \qquad \qquad \downarrow b$$

$$A \xrightarrow{f} B$$

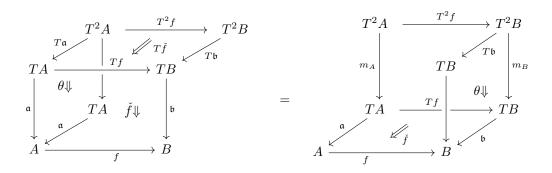
in such a way that

• $f \cdot \zeta_A = (\check{f} \cdot i_A) \circ (\zeta_B \cdot f)$ expressing the following pasting equality



and

• $(f \cdot \theta_A) \circ (\check{f} \cdot T\mathfrak{a}) \circ (\mathfrak{b} \cdot T\check{f}) = (\check{f} \cdot m_A) \circ (\theta_B \cdot T^2 f)$ expressing the following pasting equality



DEFINITION 1.3. A 2-monad $T: \mathcal{K} \to \mathcal{K}$ is said to be **lax idempotent** if given any two (pseudo) T-algebras $\mathfrak{a}: TA \to A$, $\mathfrak{b}: TB \to B$ and a 1-cell $f: A \to B$, there exists a unique 2-cell $\check{f}: \mathfrak{b} \circ Tf \Rightarrow f \circ \mathfrak{a}$ rendering (f, \check{f}) a lax morphism of pseudo T-algebras.

$$TA \xrightarrow{Tf} TB$$

$$\downarrow a \qquad \qquad \downarrow b$$

$$A \xrightarrow{f} B$$

Remark 1.4. Dually, reverse the direction of \check{f} in definition 1.3, then we get the notion of **co-lax idempotent** monad.

2 KZ-monads

DEFINITION 2.1. A 2-monad $T: \mathcal{K} \to \mathcal{K}$ is said to be KZ- $monad^1$ if $m \dashv i \cdot T$ in the 2-category $[\mathcal{K}, \mathcal{K}]$ with identity counit.

REMARK 2.2. Dual to the definition above, we define a monad T to be a **co-KZ-monad** by requiring $i \cdot T \dashv m$ with identity unit.

Suppose T is a co-KZ-monad and $i \cdot T \dashv m$. In particular unit of this adjunction is identity since $m \circ (i \cdot T) = 1$. Moreover, the identity 2-cell

has a mate

$$T \xrightarrow{1} T$$

$$\downarrow \lambda \Downarrow \qquad \downarrow i.T$$

$$T \xrightarrow{T.i.} T^{2}$$

$$(4)$$

with properties $m \cdot \lambda = id_{1_T}$ and $\lambda \cdot i = id_{(T \cdot i) \circ i}$. These identity follow from triangle identities of adjunction $i \cdot T \dashv m$ and $(i \cdot T) \circ i = (T \cdot i) \circ i$ by naturality of i.

Suppose $\mathfrak{a} \colon TA \to A$ is a pseudo algebra for T. We would like to calculate the composite 2-cell

$$TA \xrightarrow[Ti_A]{i_{TA}} T^2A \xrightarrow[\mathfrak{a} \circ m_A]{i_{TA}} TA$$

In the diagram below, since $m_A \circ \lambda_A = id$, the left column of 2-cells collapses to identity, and therefore we have

$$\theta \cdot \lambda_A = \zeta^{-1} \cdot \mathfrak{a}$$

¹KZ: short for 'Kock-Zöberlein'

On the other hand, we can compose row-wise instead, and we get

$$\theta \cdot \lambda_A = (\theta \cdot Ti_A) \circ (\mathfrak{a} \circ T\mathfrak{a} \cdot \lambda_A) = (\mathfrak{a} \cdot T\zeta^{-1}) \circ (\mathfrak{a} \circ T\mathfrak{a} \cdot \lambda_A)$$

Thus, in the end, we have

$$TA \xrightarrow{\lambda_A \downarrow} T^2A \xrightarrow{T\mathfrak{a}} TA \xrightarrow{\mathfrak{a}} A = TA \xrightarrow{\mathfrak{a}} A \xrightarrow{\zeta^{-1} \downarrow} A$$

LEMMA 2.3. Let T be a KZ-monad, and A an object of K. Then any pseudo T-algebra on A is left adjoint to unit i_A . Conversely, if i_A has a left adjoint with invertible counit then this left adjoint is a pseudo T-algebra.

REMARK 2.4. This observation requires a bit of conceptual explanation: for a KZ-monad T, any object admits at most one pseudo T-algebra structure, up to unique isomorphism. So a KZ-monad is a nicely-behaved 2-monad whose algebras are 'property-like' in the sense that the structure is a (reflective) left adjoint to the unit. Similarly, for a co-KZ-monad T the structure $\mathfrak a$ is right adjoint to the unit i_A and the invertible unit of this adjunction is given by $\zeta\colon 1\Rightarrow \mathfrak a i_A$ in diagram (1).

$$TA \stackrel{i_A}{\underbrace{ }} A$$

What about counit of $i_A \dashv \mathfrak{a}$? Here is a calculation² of counit using mate λ_A introduced in diagram 4.

(5)

We prove the triangle identities of adjunction with proposed unit and counit:

$$\begin{split} (\mathfrak{a} \centerdot T\zeta^{-1} \circ (T\mathfrak{a} \centerdot \lambda_A)) \circ (\zeta \centerdot \mathfrak{a}) &= (\zeta^{-1} \centerdot \mathfrak{a}) \circ (\zeta \centerdot \mathfrak{a}) \quad \text{\{by equality of pasting diagrams (5) \}} \\ &= id_{\mathfrak{a}} \qquad \qquad \text{\{factoring out \mathfrak{a}\}} \end{split}$$

²The dual of this situation, i.e. unit in the case of KZ-monad, is calculated in page 112 of (Street 1974).

Also,

$$\begin{split} ((T\zeta^{-1}\circ(T\mathfrak{a}\boldsymbol{.}\lambda_A))\boldsymbol{.}i_A)\circ(i_A\boldsymbol{.}\zeta) &= (T\zeta^{-1}\boldsymbol{.}i_A)\circ(i_A\boldsymbol{.}\zeta) \qquad \{\lambda_A\boldsymbol{.}i_A = id\} \\ &= (i_A\boldsymbol{.}\zeta^{-1})\circ(i_A\boldsymbol{.}\zeta) \qquad \{\text{2-naturality of } i\colon 1\Rightarrow T\} \\ &= id_{i_A} \qquad \qquad \{\text{factoring out } i_A\} \end{split}$$

In (Street 1974), we also see a converse of remark above.

LEMMA 2.5. Suppose $T: \mathcal{K} \to \mathcal{K}$ is a co-KZ 2-monad and suppose a 0-cell A, a 1-cell $\mathfrak{a}: TA \to A$, and an isomorphism 2-cell $\zeta: 1 \Rightarrow \mathfrak{a} \circ i_A$ are given in \mathcal{K} , and furthermore, ζ^{-1} satisfies pasting equality (5). We have:

- 1. ζ is the unit for an adjunction $i_A \dashv a$ whose counit is given by $(T\zeta^{-1}) \circ (T\mathfrak{a} \cdot \lambda_A)$ (composite 2-cell in (6)).
- 2. The 2-cell θ : $\mathfrak{a} \circ T\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$, obtained by taking double mate of $\lambda_A \cdot i_A = id$, is isomorphism.

$$T^{2}A \xleftarrow{Ti_{A}} TA \qquad T^{2}A \xrightarrow{T\mathfrak{a}} TA$$

$$i_{TA} \uparrow id \uparrow \uparrow \downarrow \downarrow \downarrow \mathfrak{a}$$

$$TA \xleftarrow{i_{A}} A \qquad TA \xrightarrow{\mathfrak{a}} A$$

3. 2-cell θ enriches (A, \mathfrak{a}, ζ) with the structure of a pseudo T-algebra.

PROPOSITION 2.6. Any KZ-monad (resp. co-KZ-monad) is lax idempotent (resp. co-lax idempotent). (Street 1974) (Kock 1995)

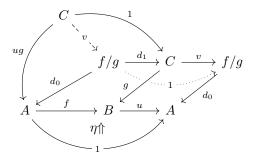
3 A general useful lemma in 2-categories

There is an innocent looking yet quite important proposition in (Street 1974) which may be overlooked in first reading of the paper.³ This is proposition 6 in that paper. We state it here.

PROPOSITION 3.1. Suppose $f: A \to B$ is a 1-cell with right adjoint u, unit η , and counit ϵ in a 2-category \mathcal{K} with comma objects. For any 1-cell $g: C \to B$, the unique filling arrow $v: C \to f/g$ obtained by factoring $\epsilon \cdot g$ through (strict) comma square $\langle f/g, d_0, d_1, \phi \rangle$ is right adjoint to d_1 with counit identity.

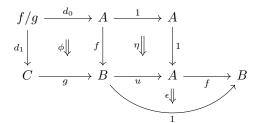
³Unfortunately, this occurred in the case of author.

The 1-cell v in the proposition is uniquely determined by equations $d_1v = 1$, $d_0v = ug$, and $\phi \cdot v = \epsilon \cdot g$. Moreover, the proposition states that we can lift the 2-cell η in the lower part of the diagram to a 2-cell $1 \Rightarrow vd_1$ in the upper part.



Proof. We first construct the would-be unit β of adjunction $d_1 \dashv v$. Using the fact $(\epsilon \cdot f) \circ (f \cdot \eta) = 1$ in chasing the diagram below, we obtain:

$$(\phi \centerdot vd_1) \circ (fu \centerdot \phi) \circ (f \centerdot \eta \centerdot d_0) = (\epsilon \centerdot gd_1) \circ (fu \centerdot \phi) \circ (f \centerdot \eta \centerdot d_0) = \phi$$



We (uniquely) define $\tau_1: 1 \Rightarrow vd_1$ to be the unique 2-cell with

$$d_0 \cdot \tau_1 = (u \cdot \phi) \circ (\eta \cdot d_0)$$

$$d_1 \cdot \tau_1 = 1$$
(7)

One readily verifies that with id and τ_1 , d_1 and v satisfy triangle equations of adjunction. \square

REMARK 3.2. A useful special case of above proposition is when f and g are both identity 1-cells 1: $E \to E$. In that case $f/g \simeq E^{\to}$ and $v = i_E$. The unit $\tau_1 \colon 1_{B^{\to}} \Rightarrow i_E \circ e_1$ is the unit of familiar adjunction $e_1 \dashv i_E$. In the case when 2-category $\mathcal K$ is 2-category of (small) categories, $\tau_1(u) = (u,1)$ for any $u \colon b_0 \to b_1$ in E^{\to} .

$$\begin{array}{ccc}
e_0 & \xrightarrow{u} & e_1 \\
\downarrow u & & \downarrow 1 \\
e_1 & \xrightarrow{1} & e_1
\end{array}$$

similarly, the dual of proposition 3.1 when applied to f = g = 1 gives i_E as left adjoint of $e_0 cdots E^{\rightarrow} \to E$. The unit of this adjunction is identity, making e_0 a retraction. The counit is given by the unique 2-cell $\tau_0 cdots i_E \circ e_0 \Rightarrow 1_{E^{\rightarrow}}$ defined by the equations $e_0 \tau_0 = 1$ and $e_1 \tau_0 = \phi$. In particular, in 2-category of small categories we have $\tau_0(u) = (1, u)$.

4 Fibrations as pseudo-algebras of a co-KZ-monad

Let K be a representable 2-category. Define K/B to be the strict slice 2-category over B, meaning the morphism triangles commute up to equality. (Street 1974) constructs KZ-monads $L, R: K/B \rightrightarrows K/B$. The idea is, for a morphism $p: E \to B$, an algebra $R(p) \to p$ (resp. $L(p) \to p$) if it exist, corresponds to the fibration structure on p (resp. opfibration structure). We will only present explicit construction and calculation for the case of fibration⁴ and thus, we will mainly concern ourselves with 2-monad R. However, when necessary, we will comment on the dual results for the case of opfibrations. We now define 2-monad R: It takes an object (E,p) to (B/p,R(p)) where

$$\begin{array}{ccc}
B/p & \xrightarrow{\hat{d}_1} & E \\
R(p) & \phi_p \uparrow & \downarrow^p \\
B & \xrightarrow{1} & B
\end{array}$$
(8)

is a comma square.

Remark 4.1. 2-cell ϕ_p can be constructed as follows:

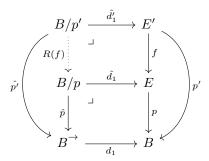
$$B/p \xrightarrow{\hat{d_1}} E$$

$$B/p \xrightarrow{\hat{d_1}} E$$

$$\downarrow^p$$

The action of R on morphisms is given as follows:

If $f: (E', p') \to (E, p)$ is a 1-cell in \mathcal{K}/B , then define R(f) to be the unique 1-cell with $\hat{d}_1 \circ R(f) = f \circ \hat{d}'_1$ and $\hat{p} \circ R(f) = \hat{p}'$.

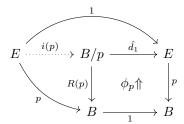


⁴Unlike Street's paper whereby he works with opfibration structures and as a result, he chooses to work with 2-monad L on \mathcal{K}/B which takes p to L(p) := p/B.

Similarly if $\sigma: f \Rightarrow g$ is a 2-cell in \mathcal{K}/B , then we have a unique induced 2-cell $R(\sigma): R(f) \Rightarrow R(g)$ with $\hat{d}_1 \cdot R(\sigma) = \sigma \cdot \hat{d}'_1$ and $\hat{p} \cdot R(\sigma) = id_{\hat{n}'}$.

PROPOSITION 4.2. 2-functor $R: \mathcal{K}/B \to \mathcal{K}/B$ is a 2-monad.

The unit of monad $i: id \Rightarrow R$ at (E,p) is given by the unique arrow $i(p): E \to B/p$ with property that $R(p) \circ i(p) = p$ and $\hat{d}_1 \circ i(p) = 1_E$, and moreover $\phi_p \cdot i(p) = id_p$, all inferred by universal property of comma object B/p.



It also follows that $\hat{d}_1 \dashv i(p)$ with identity counit. Indeed, i(p) is v in proposition 3.1, when f=1 and g=p. From there, we also get the unit $\tau_1(p)$ of adjunction with $R(p) \cdot \tau_1(p) = \phi_p$. The multiplication $m \colon R^2 \Rightarrow R$ of monad at 0-cell (E,p) is given by the unique arrow $m(p) \colon B/R(p) \to B/p$

$$B/R(p) \xrightarrow{\widehat{d_{1}^{\rightarrow}}} B/p \xrightarrow{\widehat{d_{1}}} E$$

$$\downarrow p$$

$$\downarrow q$$

with the property that $R(p) \circ m(p) = R^2(p)$ and $\hat{d_1} \circ m(p) = \hat{d_1} \circ \widehat{d_1}$, and moreover $\phi_p \cdot m(p) = (\phi_p \cdot \widehat{d_1}) \circ (\phi \cdot \widehat{d_0} \hat{p}) = (\phi_p \cdot \widehat{d_1}) \circ \phi_{R(p)}$, all inferred by universal property of comma object B/p.

Proposition 4.3. 2-monad $R: \mathcal{K}/B \to \mathcal{K}/B$ is a co-KZ-monad.

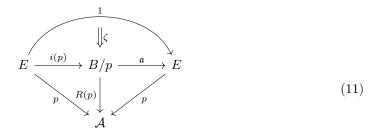
Proof. We have to show that $i \cdot T \dashv m$.

Now, we would like to see what a pseudo algebra $\mathfrak{a}: R(p) \to p$ in \mathcal{K}/B looks like. The

fact that \mathfrak{a} is a morphism in \mathcal{K}/B provides us with a morphism \mathfrak{a} which makes the diagram

$$\begin{array}{ccc}
B/p & \xrightarrow{\mathfrak{a}} & E \\
R(p) & \swarrow p & \\
B & &
\end{array} \tag{10}$$

commute. Moreover, by remark 2.4 R being a co-KZ-monad generates an adjunction $i(p) \dashv a$ whose unit is the invertible 2-cell $\zeta: 1 \Rightarrow \mathfrak{a} \circ i(p)$



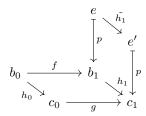
such that $p \cdot \zeta = id_p$.

In the example below we investigate how the construction above look like when we choose 2-category of (locally small) categories as our working 2-category.

EXAMPLE 4.4. Let's take $\mathcal{K} = \mathfrak{Cat}$ to be the strict 2-category of categories, functors, and natural transformations. First and foremost, for a functor $p \colon E \to B$, the comma category B/p is given as a category whose objects are pairs $\langle f \colon b \to p(e), e \rangle$ where f is morphism in $B \colon 5$

$$b_0 \xrightarrow{f} b_1$$

Morphisms of B/p are of the form

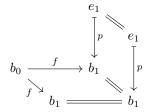


Functor R(p) as in diagram (8) takes pair $\langle f, e \rangle$ to $b_0 = \text{dom}(f)$, and \hat{d}_1 is simply the second projection; it takes $\langle f, e \rangle$ to e. The unit of monad R at (E, p), i.e. $i(p) \colon E \to B/p$, takes an object e of E to the object

 $^{{}^{5}}e \mapsto b_{1}$ indicates that $p(e) = b_{1}$.

$$p(e) = p(e)$$

and $\tau_1(p) \colon 1_{B/p} \Rightarrow i(p) \circ \hat{d}_1$ induces a morphisms $B/p \to B/p^{\to}$ which takes an object of B/p in above to



We also note that $\widehat{d_1^{\rightarrow}}$ (as in diagram 9) is given by the action

and multiplication m(p) given by

$$b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2 \xrightarrow{\qquad \qquad b_0 \xrightarrow{g \circ f} b_2} b_2$$

Now, suppose that $\mathfrak{a}: R(p) \to p$ is a pseudo algebra for 2-monad R. By commutativity of diagram 10 we know that $p(\mathfrak{a}\langle f, e \rangle) = \text{dom}(f)$. So we draw

$$egin{aligned} egin{aligned} egin{aligned\\ egin{aligned} egi$$

As observed in diagram 11 we get an isomorphism lift of identity in the base:

$$\begin{array}{ccc}
e & \xrightarrow{\zeta(e)} & \mathfrak{a}\langle 1_{p(e)}, e \rangle \\
\downarrow^p & & \downarrow^p \\
p(e) & = & p(e)
\end{array}$$

Observe that functors $R(i(p)) \colon B/p \to B/R(p)$ and $i(R(p)) \colon B/p \to B/R(p)$ are given as follows:

$$R(i(p)): \quad \bigcup_{b_0 \xrightarrow{f} b_1}^{e} \qquad \longmapsto \qquad \bigcup_{b_0 \xrightarrow{f} b_1}^{e} \qquad \bigcup_{b_0 \xrightarrow{f} b_1}^{e}$$

and

$$i(R(p)):$$

$$b_0 \xrightarrow{f} b_1 \qquad \longmapsto \qquad b_0 \xrightarrow{g} b_1 \xrightarrow{g} b_1$$

and the mate 2-cell λ as in diagram (4) appears as a natural transformations in this case where $\lambda_p : i(R(p)) \Rightarrow R(i(p))$ can be illustrated as

$$b_0 = b_0 \xrightarrow{f} b_1 \downarrow p$$

$$b_0 \xrightarrow{f} b_1 = b_1$$

We keep in mind that $R(\mathfrak{a}) \circ R \cdot i(p) \langle f, e \rangle = \langle f, \mathfrak{a} \langle 1_{b_1}, e \rangle \rangle$, and hence $R(\zeta) \langle f, e \rangle$ is illustrated in below:

$$b_0 \xrightarrow{f} b_1 \xrightarrow{f} b_1$$

$$b_0 \xrightarrow{f} b_1$$

$$b_1 \xrightarrow{f} b_1$$

$$b_1 \xrightarrow{f} b_1$$

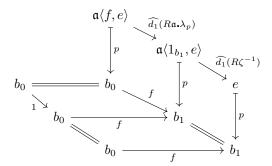
$$(12)$$

In addition, invertible 2-cell $\theta(p)$: $\mathfrak{a} \circ R(\mathfrak{a}) \Rightarrow \mathfrak{a} \circ m(p)$ provides us with an isomorphism $\mathfrak{a}\langle f, \mathfrak{a}\langle g, e \rangle \rangle \to \mathfrak{a}\langle gf, e \rangle$. Now, we study coherence equations (2) and (3) in our case, which state that for any morphism $f: b_0 \to b_1$ together with any object e in E over b_1 , the following diagram (in the fibre over b_0) commute:

$$\begin{split} \mathfrak{a}\langle f,e\rangle & \xrightarrow{\mathfrak{a.}R(\zeta)} \mathfrak{a}\langle f,\mathfrak{a}\langle 1_{b_1},e\rangle\rangle \\ \downarrow & \downarrow \\ \mathfrak{a}\langle 1_{b_0},\mathfrak{a}\langle f,e\rangle\rangle & \xrightarrow{\theta.i(R(p))} \mathfrak{a}\langle f\circ 1_{b_0},e\rangle \end{split}$$

and, for every chain of morphisms $b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2 \xrightarrow{h} b_3$ in B and any object e in E over b_3 , the diagram (in the fibre over b_0)

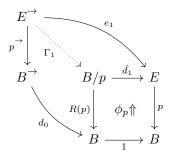
commutes. Finally, the counit of adjunction $i(p) \dashv \mathfrak{a}$, as computed in diagram 6, gives us the lift $\tilde{f} = \hat{d}_1((R\mathfrak{a} \cdot \lambda_p) \circ R\zeta^{-1})$ of f:



It remains to prove that \tilde{f} as defined is cartesian. One can try to prove this directly. However, we prove this in a more general setting in the next section.

5 Chevalley criterion

Suppose p is a 0-cell in \mathcal{K}/B . There is a unique derived 1-cell Γ_1 with properties $R(p)\Gamma_1 = d_0 p^{\rightarrow}$, $\hat{d}_1 \Gamma_1 = e_1$, and $\phi_p \cdot \Gamma_1 = p \cdot \phi_E$.



LEMMA 5.1. We have $\hat{d}_1\Gamma_1 \cdot \tau_0 = \phi_E$ and $R(p)\Gamma_1 \cdot \tau_0 = id_{R(p)\Gamma_1}$ and from these it follows that $(\tau_1(p) \cdot \Gamma_1) \circ (\Gamma_1 \cdot \tau_0) = i(p) \cdot \phi_E$, by 2-dimensional universal property of B/p.

Proof. The first identity holds since $e_1 \cdot \tau_0 = \phi_E$ due to universal property of comma object E^{\rightarrow} . For the second identity observe that $R(p)\Gamma_1 \cdot \tau_0 = pe_0 \cdot \tau_0 = id_{pe_0} = id_{R(p)\Gamma_1}$, by one

of triangle identity of adjunction $i_E \dashv e_0$. Now, notice that

$$\begin{split} \hat{d}_1[(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0)] &= \phi_E = \hat{d}_1[i(p) \centerdot \phi_E] \\ \\ R(p)[(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0)] &= R(p) \centerdot \tau_1(p) \centerdot \Gamma_1 = \phi_p \centerdot \Gamma_1 = p \centerdot \phi_E = R(p)[i(p) \centerdot \phi_E] \end{split}$$

DEFINITION 5.2. We say a 1-cell $p: E \to B$ in \mathcal{K} satisfies **Chevalley criterion** if Γ_1 has a right adjoint $\Lambda_1 \mathcal{K}/B$ with isomorphism counit. Sometimes we call such an adjunction $\Gamma_1 \dashv \Lambda_1$ a Chevalley adjunction.

PROPOSITION 5.3. There is a bijection between collection of 1-cells $p: E \to B$ equipped with an R-pseudo algebra $(\mathfrak{a}, \zeta, \theta)$ and collection of Chevalley adjoints $(\Gamma_1, \Lambda_1, \epsilon, \eta)$. Moreover, if pseudo-algebra is normalized then counit ϵ is identity.

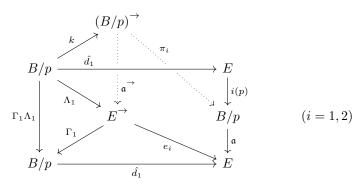
Proof. Given a pseudo algebra \mathfrak{a} : $R(p) \to p$, we construct a right adjoint Λ_1 and show that the counit of adjunction is isomorphism. Hence p satisfies Chevalley criterion. Note that the unit $\tau_1(p)$ of adjunction $\hat{d}_1 \dashv i(p)$ defines a unique 1-cell $k \colon B/p \to (B/p)^{\to}$ obtained by factoring $\tau_1(p)$ through comma square $\langle (B/p)^{\to}, \pi_0, \pi_1, \phi_{B/p} \rangle$. Thus, $\pi_0 k = 1_{B/p}$ and $\pi_1 k = i(p)\hat{d}_1$, and $\phi_{B/p} \cdot k = \tau_1(p)$. Define $\Lambda_1 := \mathfrak{a}^{\to} \circ k$. We note that

$$e_0 \Lambda_1 = e_0 \mathfrak{a}^{\rightarrow} k$$
 {definition of Λ_1 }
 $= \mathfrak{a} \pi_0 k$ {definition of $\mathfrak{a}^{\rightarrow}$ }
 $= \mathfrak{a}$ {definition of k }

This establishes that Λ_1 is indeed a 1-cell in \mathcal{K}/B , since $d_0p^{\rightarrow}\Lambda_1 = pe_0\Lambda_1 = p\mathfrak{a} = R(p)$. Also, a diagram chase shows that the front square in the diagram below commutes:

$$\hat{d}_1 \Gamma_1 \Lambda_1 = e_1 \Lambda_1 \qquad \{\text{definition of } \Gamma_1 \}
= e_1 \mathfrak{a}^{\rightarrow} k \qquad \{\text{definition of } \Lambda_1 \}
= \mathfrak{a} \pi_1 k \qquad \{\text{definition of } \mathfrak{a}^{\rightarrow} \}
= \mathfrak{a} i(p) \hat{d}_1 \qquad \{\text{definition of } k \}$$
(14)

 $^{^6}$ However, the converse of this statement is not true in general as one can observe in the construction of adjunction from



We also note that

$$R(p)\Gamma_1\Lambda_1 = d_0 p^{\rightarrow} \Lambda_1 = R(p)$$

$$\phi_p \cdot (\Gamma_1\Lambda_1) = p \cdot \phi_E \cdot \Lambda_1 = p \mathfrak{a} \cdot \phi_{B/p} \cdot k = p \mathfrak{a} \cdot \tau_1(p) = R(p) \cdot \tau_1(p) = \phi_p$$

$$(15)$$

Equations (14) and (15), and definition of $R(\mathfrak{a}i(p))$ altogether prove that

$$\Gamma_1 \circ \Lambda_1 = R(\mathfrak{a} \circ i(p)) = R(\mathfrak{a}) \circ R(i(p))$$

and we shall show that counit $\epsilon \colon \Gamma_1 \circ \Lambda_1 \Rightarrow 1$ is given by $R(\zeta^{-1})$ which is invertible.⁷ Also notice that $p\hat{d}_1 \cdot \epsilon = p\hat{d}_1 \cdot R(\zeta^{-1}) = p \cdot \zeta^{-1} \cdot \hat{d}_1 = id_{p\hat{d}_1}$, and $R(p) \cdot \epsilon = R(p) \cdot R(\zeta^{-1}) = id_{R(p)}$. This guarantees that the counit lives in \mathcal{K}/B . Moreover, definition of $R(\zeta)$ implies that $\phi_p \cdot \epsilon = \phi_p$. Now, we propose the unit; define the 2-cell $\eta \colon 1 \Rightarrow \Lambda_1 \circ \Gamma_1$ to be the unique 2-cell with

$$e_0 \cdot \eta = (\mathfrak{a}\Gamma_1 \cdot \tau_0) \circ (\zeta \cdot e_0)$$

$$e_1 \cdot \eta = \zeta \cdot e_1$$
(16)

Note that the vertical composition of 2-cells in (16) makes sense since $\mathfrak{a}i(p)e_0 = \mathfrak{a}\Gamma_1 i_E e_0$ which holds as one can easily see that $\Gamma_1 i_E = i(p)$. Furthermore, $e_0 \cdot \eta$ and $e_1 \cdot \eta$ are compatible in the sense that

$$\begin{split} (\phi_E \cdot \Lambda_1 \Gamma_1) \circ (e_0 \eta) &= (\phi_E \cdot \mathfrak{a}^{\rightarrow} k \Gamma_1) \circ (e_0 \eta) & \{\text{definition of } \Lambda_1\} \\ &= (\mathfrak{a} \phi_{B/p} \cdot k \Gamma_1) \circ (e_0 \eta) & \{\text{definition of } \mathfrak{a}^{\rightarrow}\} \\ &= (\mathfrak{a} \tau_1(p) \cdot \Gamma_1) \circ (e_0 \eta) & \{\text{definition of } k\} \\ &= (\mathfrak{a} \tau_1(p) \cdot \Gamma_1) \circ (\mathfrak{a} \Gamma_1 \cdot \tau_0) \circ (\zeta \cdot e_0) & \{\text{substituting } e_0 \cdot \eta\} \\ &= \mathfrak{a}(\tau_1(p) \cdot \Gamma_1 \circ \Gamma_1 \cdot \tau_0) \circ (\zeta \cdot e_0) & \{\text{factoring out } \mathfrak{a}\} \\ &= (\mathfrak{a} i(p) \cdot \phi_E) \circ (\zeta \cdot e_0) & \{\text{Lemma 5.1}\} \\ &= (\zeta \cdot e_1) \circ \phi_E & \{\text{exchange rule}\} \\ &= (e_1 \eta) \circ (\phi_E) & \{\text{substituting } e_1 \cdot \eta\} \end{split}$$

When $K = \mathfrak{Cat}$, $R(\zeta)$ is illustrated in diagram 12.

In the next step, we prove that proposed unit⁸ η and counit ϵ satisfy triangle equations of adjunction. To prove the first identity, we notice that

$$R(p) \cdot [(\epsilon \cdot \Gamma_1) \circ (\Gamma_1 \cdot \eta)] = [R(p) \cdot (\epsilon \cdot \Gamma_1)] \circ [R(p) \cdot (\Gamma_1 \cdot \eta)] = (id_{R(p)}\Gamma_1) \circ (pe_0 \cdot \eta) = id_{R(p)\Gamma_1}$$

where the last identity follows from the fact that $pe_0 \cdot \eta = id_{pe_0} = id_{R(p)\Gamma_1}$. Similarly, we have

$$\hat{d}_1 \cdot [(\epsilon \cdot \Gamma_1) \circ (\Gamma_1 \cdot \eta)] = (\zeta^{-1} \cdot \hat{d}_1 \Gamma_1) \circ (e_1 \cdot \eta) = (\zeta^{-1} \cdot e_1) \circ (\zeta \cdot e_1) = id_{\hat{d}_1 \Gamma_1}$$

Therefore, $(\epsilon \cdot \Gamma_1) \circ (\Gamma_1 \cdot \eta) = id_{\Gamma_1}$. To prove the second identity, $(\Lambda_1 \cdot \epsilon) \circ (\eta \cdot \Lambda_1) = id_{\Lambda_1}$, we first prove the following lemma:

Lemma 5.4.
$$\Gamma_1 \cdot \tau_0 \cdot \Lambda_1 = R(\mathfrak{a}) \cdot \lambda_p$$

Proof. First we verify that the domain and codomain of these 2-cells match.

$$B/p \xrightarrow{\Lambda_1} E^{\xrightarrow{e_0}} E \xrightarrow{i_E} E^{\xrightarrow{}} \xrightarrow{\Gamma_1} B/p$$

Indeed,

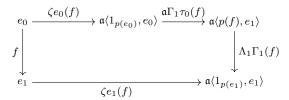
$$\Gamma_1 i_E e_0 \Lambda_1 = i(p) e_0 \Lambda_1 = i(p) \mathfrak{a} = R(\mathfrak{a}) i(R(p))$$

and as we observed earlier $\Gamma_1 \circ \Lambda_1 = R(\mathfrak{a})R(i(p))$. So, the domain and codomain of $\Gamma_1 \cdot \tau_0 \cdot \Lambda_1$ and $R(\mathfrak{a}) \cdot \lambda_p$ agree. The lemma follows from identities in below in conjunction with comma property of B/p for 2-cells.

$$\widehat{d}_1 \centerdot (\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1) = \phi_E \centerdot \Lambda_1 = \mathfrak{a} \tau_1(p) = \widehat{\mathfrak{ad}_1} \lq \lambda_p = \widehat{d}_1 \centerdot R(\mathfrak{a}) \centerdot \lambda_p$$

$$R(p) \centerdot (\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1) = id_{pe_0} \centerdot \lambda_1 = id = R^2(p) \centerdot \lambda_p = R(p)R(\mathfrak{a}) \centerdot \lambda_p$$

⁸Perhaps, it is illuminating to see what this unit look like in the case of $\mathcal{K} = \mathfrak{Cat}$. Indeed, for a morphism $f \colon e_0 \to e_1$ in E^{\to} , $\eta(f)$ is given as follows:



Using lemma above we have,

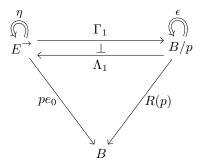
$$e_0 \cdot [(\Lambda_1 \cdot \epsilon) \circ (\eta \cdot \Lambda_1)] = (\mathfrak{a} \cdot \epsilon) \circ ((\mathfrak{a} \Gamma_1 \tau_0) \circ (\zeta e_0)) \cdot \Lambda_1 = (\mathfrak{a} \cdot R(\zeta^{-1})) \circ (\mathfrak{a} R(\mathfrak{a}) \cdot \lambda_p) \circ (\zeta \mathfrak{a}) = (\zeta^{-1} \mathfrak{a}) \circ (\zeta \mathfrak{a}) = id_{e_0 \Lambda_1} \circ (\zeta \mathfrak{a}) = id_{e_0 \Lambda_$$

The penultimate equality comes from equality of pasting diagrams 5. Similarly, using the fact that $e_1\Lambda_1 = ai(p)\hat{d}_1$, we get

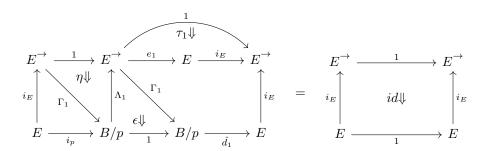
$$e_1 \cdot [(\Lambda_1 \cdot \epsilon) \circ (\eta \cdot \Lambda_1)] = (\mathfrak{a}i(p)\hat{d}_1 \cdot \epsilon) \circ (\zeta \cdot e_1\Lambda_1) = (\mathfrak{a}i(p)\zeta^{-1}\hat{d}_1) \circ (\zeta \cdot \mathfrak{a}i(p)\hat{d}_1) = id_{e_1\Lambda_1}$$

The last identity is by exchange law of horizontal-vertical composition of 2-cells. From these two equations we deduce the second adjunction identity.

Conversely, suppose we are given a Chevalley adjunction, that is to say an adjunction $\Gamma_1 \dashv \Lambda_1$ over B:



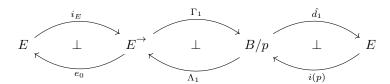
such that the counit ϵ is an isomorphism, $R(p)\Gamma_1 = pe_0$, $pe_0\Lambda_1 = R(p)$, $R(p) \cdot \epsilon = id_{R(p)}$, and $pe_0 \cdot \eta = id_{pe_0}$. We define pseudo-algebra $\mathfrak{a} : B/p \to E$ as composite $e_0\Lambda_1$. Note that $p\mathfrak{a} = pe_0\Lambda_1 = R(p)\Gamma_1\Lambda_1 = R(p)$, since the adjunction $\Gamma_1 \dashv \Lambda_1$ takes place in \mathcal{K}/B . We propose $e_1\eta i_E$ for ζ . First we prove that $\eta \cdot i_E$ is invertible and thence ζ is invertible. Using $\tau_1 \cdot i_E = id$, we have $(i_E\hat{d}_1\epsilon \cdot i(p)) \circ (\tau_1 \cdot \Lambda_1\Gamma_1 i_E) \circ (\eta \cdot i_E) = id_{i_E}$. This is illustrated in the following pasting equality:



Same pasting equality shows that $(i_E \hat{d}_1 \cdot \epsilon \cdot i(p)) \circ (\tau_1 \cdot \Lambda_1 \Gamma_1 i_E)$ is 2-sided inverse of $\eta \cdot i_E$. Whiskering with e_1 reveals inverse of ζ :

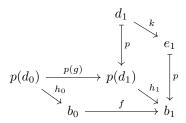
$$\zeta^{-1} = (e_1 i_E \hat{d}_1 \cdot \epsilon \cdot i(p)) \circ (e_1 \cdot \tau_1 \cdot \Lambda_1 \Gamma_1 i_E) = \hat{d}_1 \cdot \epsilon \cdot i(p)$$

since $e_1 \cdot \tau_1 = id_{e_1}$. Indeed, ζ^{-1} is the counit of composite adjunction in below:

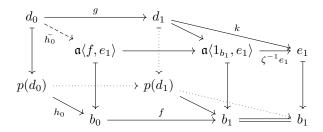


Notice that we have proved that ζ is invertible regardless of invertibility of ϵ . Also, obviously if ϵ is identity then so is ζ . To finish the proof, by lemma 2.5 it suffices to prove that $\zeta^{-1} = \hat{d}_1 \cdot \epsilon \cdot i(p)$ satisfies pasting equality in (5), and moreover, $i(p) \dashv a$ with ζ and $R(\zeta^{-1}) \circ (R(\mathfrak{a}) \cdot \lambda_p)$ as unit and counit of this adjunction respectively.

EXAMPLE 5.5. We now return to prove our promise at the end of example 4.4. We would like to show that \tilde{f} , obtained as whiskering \hat{d}_1 with counit of $i(p) \dashv \mathfrak{a}$, is indeed cartesian. Here, we appeal to the bijection $\operatorname{Hom}_{B/p}(\Gamma_1(g), \langle f, e_1 \rangle) \cong \operatorname{Hom}_{E^{\to}}(g, \Lambda_1 \langle f, e_1 \rangle)$ natural in $g \colon d_0 \to d_1$ in E^{\to} and $\langle f, e_1 \rangle$ in B/p. This bijection states that any diagram of the form



where the square in base commutes and k lies above h_1 can be (uniquely) completed to the diagram below:



Taking g to be identity we obtain the usual condition which expresses cartesian property of lift \tilde{f} . Also, one can easily show that unique morphism h_0 over h_0 is calculated by the expression $(e_0\Lambda_1(h_0,h_1,k)) \circ (\mathfrak{a}\Gamma_1\tau_0(g)) \circ (\zeta e_0(g))$.

EXAMPLE 5.6. Let $p: E \to B$ be a cloven Grothendieck fibration. Note that the data of a cloven Grothendieck fibration includes structure of a cleavage, that is a choice of cartesian lifts:

$$\rho_{a,b}: \prod_{\text{Hom}(a,b)} \prod_{e \in E_b} \sum_{e' \in E_a} \mathcal{C}art_E(e',e)$$

where $Cart_E(e', e)$ denotes the set of cartesian morphisms from e' to e.

We say p is *split* if for all pairs of objects a, b:

$$\operatorname{snd} \rho_{a,c}(g \circ f, e) = \operatorname{snd} \rho_{b,c}(g, e) \circ \operatorname{snd} \rho_{a,b}(f, \operatorname{fst} \rho_{b,c}(g, e))$$

and we say p is *normal* if for all objects e in E:

$$\operatorname{snd} \rho_{pe,pe}(1_{pe},e) = 1_e$$

We have the following correspondence:

$$\left\{ \begin{array}{c} \text{cleavages} \\ \text{of } p \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{pseudo-algebras} \\ (\mathfrak{a}, \zeta, \theta) \text{ of } R \text{ at } p \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{right adjoints of } \Gamma_1 \\ \text{with isomorphism counit} \end{array} \right\}$$

It follows that any two cleavages of p are isomorphic in a unique way.

DEFINITION 5.7. For a category B, define 2-category $\mathbf{Fib}(B)$ of fibrations over B whose 0-cells are Grothendieck fibrations, whose 1-cells are fibred functors over B (i.e. those functors over B which preserve cartesian morphisms), and 2-cells are vertical natural transformations (i.e. transformations over B). Compositions are usual composition of functors and natural transformations.

REMARK 5.8. Example 4.4 can be encapsulated as follows: The forgetful 2-functor $U: \mathbf{Fib}(B) \to \mathfrak{Cat}/B$ is 2-monadic: the *free fibration* of a functor $p: E \to B$ is fibration $R(p): B/p \to B$; cleavage (aka fibration structure) on p is uniquely (in fact unique up to unique isomorphism) determined by a pseudo algebra structure for 2-monad R = UF. Strict algebra structures of R correspond to splitting fibration structures on p.

$$F \stackrel{\frown}{\bigcup} U$$

$$\mathfrak{Cat}/B \stackrel{\longleftarrow}{\bigcup} R$$

We also note that for a category B the domain functor $\operatorname{cod}: B^{\to} \to B$ is the free Grothendieck fibration on identity functor $1: B \to B$; that is $\operatorname{dom} = R(1)$. In more explanatory terms this fact states that

We also note that for a category B with pullbacks the codomain functor cod: $B \to B$ is the free Grothendieck fibration with existential quantifiers on identity functor 1: $B \to B$;

References

Kock, A. (1995), 'Monads for which structures are adjoint to units', *Journal of Pure and Applied Algebra* Vol.104, 41–59.

REFERENCES

- Lack, S. (2000), 'A coherent approach to pseudomonads', Advances in Mathematics Vol.152 (Issue 2), 179–202.
- Street, R. (1974), 'Fibrations and yoneda's lemma in a 2-category', *Lecture Notes in Math.*, Springer, Berlin Vol.420, 104–133.