Integração Numérica

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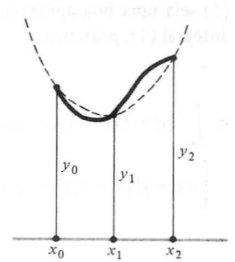
Cálculo Numérico
Curso de Física

- A Regra de Simpson resulta da integração em $[a, b]$ do polinômio de Lagrange de grau dois.



- Considerando os pontos x_0, x_1, x_2 igualmente espaçados, o polinômio interpolante de Lagrange de grau dois é dado por

$$\begin{aligned} P_2(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) \\ &+ \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) \\ &+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \end{aligned}$$

Portanto

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_{x_0}^{x_2}\Big(\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)\\ &+ \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)\\ &+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)\Big)dx\\ &+ \int_{x_0}^{x_2}\frac{(x-x_0)(x-x_1)(x-x_2)}{6}f^{(3)}(\xi(x))\,dx\end{aligned}$$

Uma Estimativa mais precisa...

Suponha que f possa ser expandida em série de Taylor em torno de $x = x_1$.

$$\begin{aligned} f(x) &= f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2}(x - x_1)^2 \\ &+ \frac{f'''(x_1)}{6}(x - x_1)^3 + \frac{f^{(iv)}(\xi(x))}{24}(x - x_1)^4, \end{aligned}$$

com

$$\xi(x) \in (x_0, x_2) \quad \text{e} \quad f \in \mathcal{C}^4(x_0, x_2).$$

Portanto

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)\,dx &= \int_{x_0}^{x_2} \left(f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2}(x - x_1)^2 \right. \\
&+ \left. \frac{f'''(x_1)}{6}(x - x_1)^3 \right) dx + \int_{x_0}^{x_2} \frac{f^{(iv)}(\xi(x))}{24}(x - x_1)^4\,dx, \\
&= f(x_1)(x_2 - x_0) + \frac{f'(x_1)}{2}\left((x_2 - x_1)^2 - (x_0 - x_1)^2 \right) \\
&+ \frac{f''(x_1)}{6}\left((x_2 - x_1)^3 - (x_0 - x_1)^3 \right) \\
&+ \frac{f'''(x_1)}{24}\left((x_2 - x_1)^4 - (x_0 - x_1)^4 \right) \\
&+ \int_{x_0}^{x_2} \frac{f^{(iv)}(\xi(x))}{24}(x - x_1)^4\,dx
\end{aligned}$$

Analisando o termo do “erro”...

O Teorema do Valor Médio para Integrais implica que existe um $\xi_1 \in (x_0, x_2)$ tal que

$$\int_{x_0}^{x_2} \frac{f^{(iv)}(\xi(x))}{24}(x-x_1)^4\,dx = \frac{f^{(iv)}(\xi_1)}{24}\int_{x_0}^{x_2}(x-x_1)^4\,dx$$

$$\boxed{\int_{x_0}^{x_2} \frac{f^{(iv)}(\xi(x))}{24}(x-x_1)^4\,dx = \frac{f^{(iv)}(\xi_1)}{120}(x-x_1)^5\Big]_{x_0}^{x_2}}$$

Agora note que, $x_2 - x_1 = x_1 - x_0 = h$. Logo

$$
\begin{aligned}
(x_2 - x_1)^2 - (x_0 - x_1)^2 &= (x_2 - x_1)^4 - (x_0 - x_1)^4 = 0 \\
(x_2 - x_1)^3 - (x_0 - x_1)^3 &= 2h^3 \\
(x_2 - x_1)^5 - (x_0 - x_1)^5 &= 2h^5
\end{aligned}
$$

Segue que

$$
\int_{x_0}^{x_2} f(x)\, dx = 2hf(x_1) + h^3 \frac{f''(x_1)}{3} + h^5 \frac{f^{(iv)}(\xi_1)}{60}
$$

Vamos analisar o termo associado à derivada de segunda ordem.

De fato,
A expansão em Série de Taylor de f em torno de x_1 é dada por

$$\begin{array}{rcl} f(x) & = & f(x_1) + f'(x_1)(x-x_1) + \dfrac{f''(x_1)}{2}(x-x_1)^2 + \dfrac{f'''(x_1)}{6}(x-x_1)^3 \\ & + & \dfrac{f^{(iv)}(\zeta)}{24}(x-x_1)^4 \text{ para algum } \zeta \in (x_0, x_2) \end{array}$$

Logo, para $\xi_2, \xi_3 \in (x_0, x_2)$ temos

$$\begin{array}{rcl} f(x+h) & = & f(x_1) + f'(x_1)h + \dfrac{1}{2}f''(x_1)h^2 + \dfrac{1}{6}f'''(x_1)h^3 + \dfrac{1}{24}f(\xi_2)h^4 \\ f(x-h) & = & f(x_1) - f'(x_1)h + \dfrac{1}{2}f''(x_1)h^2 - \dfrac{1}{6}f'''(x_1)h^3 + \dfrac{1}{24}f(\xi_3)h^4 \end{array}$$

$$\Longrightarrow f(x_1+h) + f(x_1-h) = 2f(x_1) + f''(x_1)h^2 + \frac{1}{24}h^4\left(f^{(iv)}(\xi_2) + f^{(iv)}(\xi_3)\right)$$

De onde tem-se

$$f''(x_1) = \frac{1}{h^2}\Big(f(x_1+h) + f(x_1-h) - 2f(x_1)\Big) - \frac{1}{24}h^2\Big(f^{(iv)}(\xi_2) + f^{(iv)}(\xi_3)\Big)$$

$$\Longrightarrow f''(x_1) = \frac{1}{h^2}\Big(f(x_2) + f(x_0) - 2f(x_1)\Big) - \frac{1}{24}h^2\Big(f^{(iv)}(\xi_2) + f^{(iv)}(\xi_3)\Big)$$

Assim

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)\,dx &= 2hf(x_1) + h^3\frac{f''(x_1)}{3} + \frac{h^5}{60}f^{(iv)}(\xi_1) \\
&= 2hf(x_1) + \frac{h^3}{3}\left[\frac{1}{h^2}\Big(f(x_2) + f(x_0) - 2f(x_1)\Big)\right. \\
&- \left.\frac{1}{24}h^2\Big(f^{(iv)}(\xi_2) + f^{(iv)}(\xi_3)\Big)\right] + \frac{h^5}{60}f^{(iv)}(\xi_1)
\end{aligned}$$

Logo

$$\begin{aligned}\int_{x_0}^{x_2} f(x)\,dx &= 2hf(x_1) + \frac{h}{3}f(x_0) - 2\frac{h}{3}f(x_1) + \frac{h}{3}f(x_2) \\ &+ h^5\left(\frac{f^{(iv)}(\xi_1)}{60} - \frac{f^{(iv)}(\xi_2)}{24} - \frac{f^{(iv)}(\xi_3)}{24}\right)\end{aligned}$$

Existe $\xi \in (x_0, x_2)$ tal que

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\Big(f(x_0) + 4f(x_1) + f(x_2)\Big) + h^5\left(\frac{f^{(iv)}(\xi)}{60} - 2\frac{f^{(iv)}(\xi)}{24}\right)$$

$$\Longrightarrow \boxed{\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\Big(f(x_0) + 4f(x_1) + f(x_2)\Big) - \frac{h^5}{90}f^{(iv)}(\xi)}$$

Integração Numérica - Regra de Simpson

Portanto

$$\boxed{\int_{x_0}^{x_2} f(x)\, dx \approx \frac{h}{3}\Big(f(x_0) + 4f(x_1) + f(x_2)\Big)}$$

$$\boxed{R(f) = -\frac{h^5}{90} f^{(iv)}(\xi)}$$

Considere uma partição $\mathcal{P} = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$ (n par) de $[a, b]$ com pontos igualmente espaçados.

$$\begin{aligned}\int_{a=x_0}^{b=x_n} f(x)\,dx &= \sum_{i=0}^{\frac{n}{2}-1} \int_{x_{2i}}^{x_{2i+2}} f(x)\,dx \\ &= \sum_{i=0}^{\frac{n}{2}-1} \frac{h}{3}\Big[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\Big] - \sum_{i=0}^{\frac{n}{2}-1} \frac{h^5}{90} f^{(iv)}(\zeta_i)\end{aligned}$$

para $\zeta_i \in (x_{2i}, x_{2i+2})$.

Analisando o termo do erro, temos

$$\sum_{i=0}^{\frac{n}{2}-1} \frac{h^5}{90} f^{(iv)}(\zeta_i) = \frac{h^5}{90} f^{(iv)}(\zeta) \frac{n}{2}, \quad \zeta \in (x_0, x_n)$$

Como $\dfrac{b-a}{n/2} = h \Longrightarrow n = \dfrac{2(b-a)}{h}$ temos

$$\sum_{i=0}^{\frac{n}{2}-1} \frac{h^5}{90} f^{(iv)}(\zeta) = \frac{h^4(b-a)}{90} f^{(iv)}(\zeta)$$

Logo

$$\int_{a=x_0}^{b=x_n} f(x)\, dx = \sum_{i=0}^{\frac{n}{2}-1} \frac{h}{3}\Big[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\Big] - \frac{h^4(b-a)}{90} f^{(iv)}(\zeta),$$

para $\zeta \in (x_0, x_n)$.

Assim,

$$\boxed{\int_{a=x_0}^{b=x_n} f(x)\,dx \approx \sum_{i=0}^{\frac{n}{2}-1} \frac{h}{3}\left[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right]}$$

com

$$\boxed{|E(f)| = \frac{h^4|b-a|}{90}|f^{(iv)}(\zeta)| \leq h^4\frac{|b-a|}{90}\max_{x\in[a,b]}|f^{(iv)}(x)|}$$

Integração Numérica - Regra de Simpson Repetida

Análise erro na aproximação de $\int_0^1 e^x \, dx$ usando Regra do Trapézio e regra de Simpson.

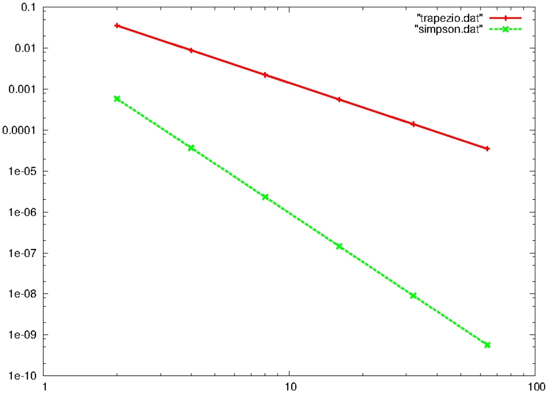


Figura: Trapézio × Simpson