Assignment 3 AQFT

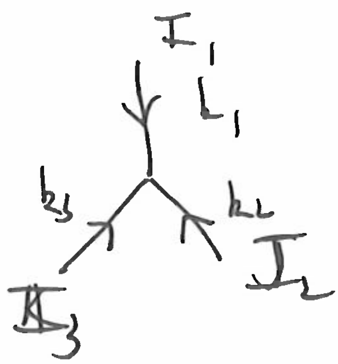
1. Each scalar has propagator

$$\langle 0|T\,\phi_I(x)\,\Phi_J(y)|0\rangle = \delta_{IJ}\,D(x,y) = D_{IJ}(x,y)$$

where

$$D(x,y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\,\frac{i}{k^2-m^2+i\epsilon}$$

The vertex



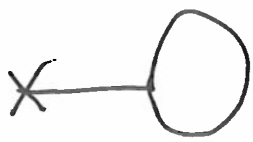
~~is~~ $\cancel{V_{I_1I_2I_3}(k_1,k_2,k_3) = iS_{int}}$

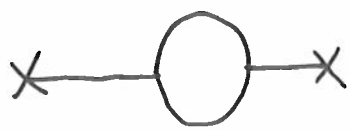
$$V_{I_1I_2I_3}(k_1,k_2,k_3)\,(2\pi)^4\,\delta^4(k_1+k_2+k_3) = \frac{\delta^3}{\delta\tilde\phi_{I_1}(k_1)\,\delta\tilde\phi_{I_2}(k_2)\,\delta\tilde\phi_{I_3}(k_3)}\, i\tilde S_{int}$$

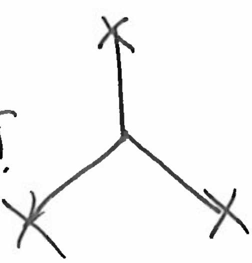
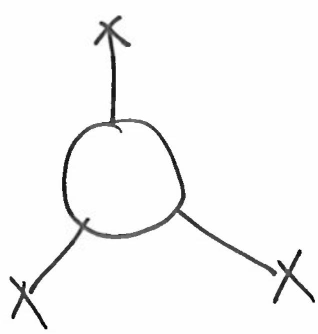
$$= -i\,g_{I_1I_2I_3}\,(2\pi)^4\,\delta^4(k_1+k_2+k_3)$$

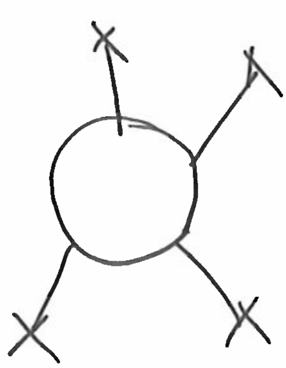
So $V_{I_1 I_2 I_3}(k_1, k_2, k_3) = -i g_{I_1 I_2 I_3}$

2. sum of 1PI diagrams

$i\Gamma(\bar{\phi}) =$ 

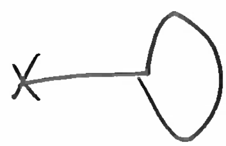
$+\frac{1}{2}$  $+\frac{1}{2^2}$ 

$+\frac{1}{3!}$  $+\frac{1}{3!}$ 

$+\frac{1}{2^3}$  $+$ higher order

'Symmetry factors from Srednicki'.

In DeWitt notation



$$= \quad \phi_I \left(-i g_{IJK}\right) D_{JK}$$

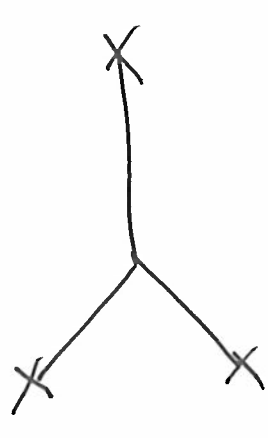
In full this is $\quad -i \int d^4x \sum_{I,J,K} \phi_I(x)\, g_{IJK}\, \delta_{JK}\, D(x,x)$

from computation of externals

$\frac{1}{2}$ 

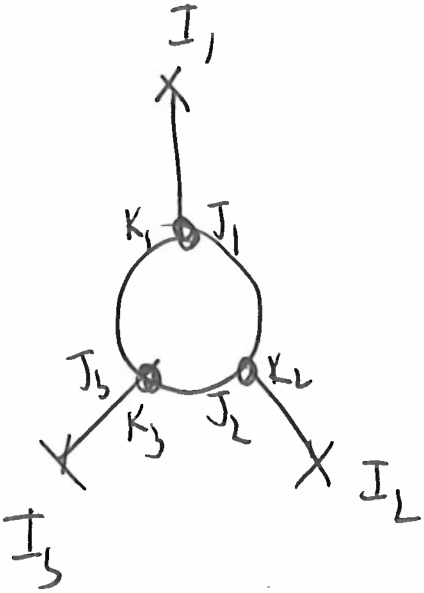
$$= \quad \frac{1}{2} \phi_I \left(D^{-1}\right)^{IJ} \phi_K$$

$$= \quad i \int d^4x \left[\sum_I \frac{1}{2} (\partial \phi_I)^2 - \frac{1}{2}(m^2 - i\epsilon)\phi_I^2 \right]$$

$\frac{1}{3!}$ 

$$= \quad \frac{-i}{3!} g_{IJK}\, \Phi^I \Phi^J \phi^K$$

$$= \quad \frac{-i}{3!} \int d^4x \sum_{IJK} g_{IJK}\, \Phi^I(x) \Phi^J(x) \phi^K(x)$$

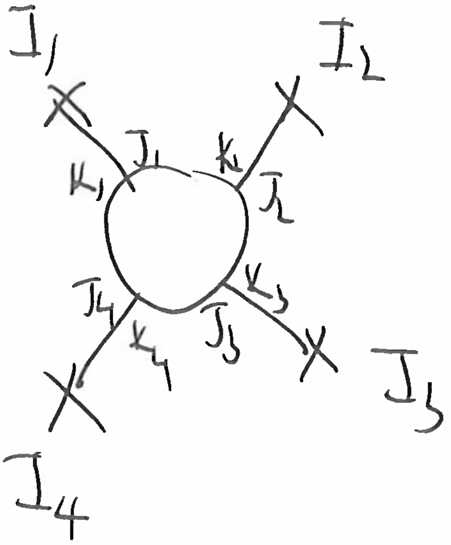
$\frac{1}{3!}$ 

$$= \frac{1}{3!} \phi_{I_1}\phi_{I_2}\phi_{I_3} \left(-i g_{I_1 J_1 K_1}\right)\left(-i g_{I_2 J_2 K_2}\right)\left(-i g_{I_3 J_3 K_3}\right)$$

$$\times \quad D_{J_1 K_2} D_{J_2 K_3} D_{J_3 K}$$

$$= \frac{i}{3!} \int d^4x_1 \int d^4x_2 \int d^4x_3 \sum_{I_1, I_2, I_3, J_1, J_2, K_1} \phi_{I_1}(x_1)\phi_{I_2}(x_2)\phi_{I_3}(x_3)\, g_{I_1 J_1 K_1}\, g_{I_2 J_2 \cancel{J}_1}\, g_{I_3 K J_2}$$

$$\times \quad D(x_1, x_2) D(x_2, x_3) D(x_3, x_1)$$

$\frac{1}{2^3}$ 

$$= \frac{1}{2^3} \phi_{I_1} \phi_{I_2} \phi_{I_3} \phi_{I_4} (-i g_{I_1 J_1 K_1})(-i g_{I_2 J_2 K_2})(-i g_{I_3 J_3 K_3})(-i g_{I_4 J_4 K_4}$$

$$\times \quad D_{J_1 K_2} D_{J_2 K_3} D_{J_3 K_4} D_{J_4 K_1}$$

$$= \frac{1}{2^3} \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 \sum_{I_1 .. I_4, J_1 ... J_4} \phi_{I_1}(x_1) \phi_{I_2}(x_2) \phi_{I_3}(x_3) \phi_{I_4}(x_4)$$

$$\times \quad g_{I_1 J_1 J_4} g_{I_2 J_2 J_1} g_{I_3 J_3 J_2} g_{I_4 J_4 J_3}$$

$$\times \quad D(x_1, x_2) D(x_2, x_3) D(x_3, x_4) D(x_4, x_1)$$

3.

$$Z = \int d^4\eta \left(1 + \frac{1}{2}\sum \eta_i A_{ij} \eta_j + \frac{1}{8}\sum_{ijkl} \eta_i\eta_j\eta_k\eta_l A_{ij}A_{kl} \right.$$

$$\left. + g\sum_{ijkl} \eta_i\eta_j\eta_k\eta_l B_{ijkl} \right)$$

+ higher order stuff vanishes

since $\int d^4\eta$ = 4 derivatives only the terms with 4 η's matter.

Levi-Civita!

Now $\int d^4\eta \, \eta_i\eta_j\eta_k\eta_l = \epsilon_{ijkl}$

$$\therefore \boxed{Z = \sum_{ijkl} \epsilon_{ijkl} \left(B_{ijkl} + \frac{1}{8} A_{ij} A_{kl} \right)}$$

$$Z = \sum_{ijklabcd} \varepsilon_{ijklabcd} \left[g^2 B_{ijkl} B_{abcd} + \frac{g}{8} B_{ijkl} A_{ab} A_{cd} + \frac{1}{2^4 4!} A_{ij} A_{kl} A_{ab} A_{cd} \right]$$

5. $$Z = \int d^{\infty}\eta \, e^{\frac{1}{2} \sum \eta_i A_{ij} \eta_j} \left[1 + \sum_{ijkl} g \, \eta_i \eta_j \eta_k \eta_l B_{ijkl} + \sum_{ijklabcd} g^2 \eta_i \eta_j \eta_k \eta_l \eta_a \eta_b \eta_c \eta_d \times \left(B_{ijkl} B_{abcd} \right) \right]$$

Wicks theorem

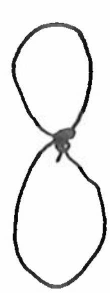
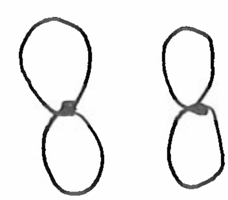
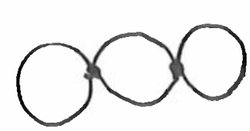
$$\contraction{}{\eta_i}{}{\eta_j}\eta_i\eta_j = (A^{-1})_{ij}$$

For $g=0$ $\quad Z = \int^{\infty} d\eta \; e^{\frac{1}{2}\sum \eta_i A_{ij} \eta_j}$

$$= \sqrt{\det(A)} \qquad \text{(from lecture notes).}$$

$$\frac{Z}{\sqrt{\det(A)}} = 1 + g \sum_{ijkl} \langle \eta_i \eta_j \eta_k \eta_l \rangle B_{ijkl}$$

$$+ g^2 \sum_{ijklabcd} \langle \eta_i \eta_j \eta_k \eta_l \eta_a \eta_b \eta_c \eta_d \rangle B_{ijkl} B_{abcd}$$

$= 1 + g$  $+ g^2 \Big[$  $+$  $\Big]$

Now $\langle \eta_i \eta_j \eta_k \eta_l \rangle = (A^{-1})_{ij} (A^{-1})_{kl}$

$$- (A^{-1})_{ik} (A^{-1})_{jl}$$

$$\langle \eta_i \eta_j \eta_k \eta_l \eta_a \eta_b \eta_c \eta_d \rangle$$

$$= (A^{-1})_{ij} (A^{-1})_{kl} (A^{-1})_{ab} (A^{-1})_{cd}$$

+ antisymmetrized permutations.

6.

$$\int d^N\eta = \int d\eta_N \cdots \int d\eta_1$$

$$\delta(\eta) = \eta_1 \cdots \eta_N$$

$$\int d\eta_1\, \delta(\eta) = \eta_2 \cdots \eta_N$$

$$\int d\eta_2 \int d\eta_1\, \delta(\eta) = \eta_3 \cdots \eta_N$$

repeating $\int d^N\eta\, \delta(\eta) = 1$.

7. Consider

$$\int d^N\eta \; f(\eta)\,\delta(\eta)$$

Write $f(\eta)$ as a Taylor expansion. Since $\delta(\eta)$ has N etas and there are only N etas being integrated over, any more gives a repetition which must vanish.

$$\therefore \int d^N\eta\, f(\eta)\,\delta(\eta) = \int d^N\eta\, f(0)\,\delta(\eta) = f(0).$$

8.

$$\delta(\eta) = C\int d^N J\; e^{-i\eta_N J_N} \cdots e^{-i\eta_1 J_1}$$

$$= C\left(\int dJ_N e^{-i\eta_N J_N}\right) \cdots \left(\int dJ_1 e^{-i\eta_1 J_1}\right)$$

Now $\int dJ_i e^{-i\eta_i J_i} = \int dJ_i \left(1 - i\eta_i J_i\right)$

$$= \int dJ_i \left(1 + i J_i \eta_i\right)$$

$$= i\eta_i$$

$$\therefore \delta(\eta) = i^N C\, \eta_N \ldots \eta_1 = i^N C\, \delta(\eta)$$

$$\therefore \boxed{C = \frac{1}{i^N}}$$