Extra Problems

December 6, 2022

0.1 Extra Problems – Applied Linear Algebra

This document contains the computational component of exercises in Allaire and Kaber, *Numerical Linear Algebra*, Springer. The problems are intended to be done in MATLAB, but I choose to do them using Python. These extra problems are not assigned as homework for the course named above, and are purely for my practice with the computational aspect of numerical linear algebra. I tried these problems out as an exercise in preparation for the final exam and my answers may be subject to errors.

0.1.1 Exercise 2.15

$$P = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 2 \\ -1 & 1 & 2 & -2 \\ 1 & 3 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad A = PDP^{-1}$$

1. $\sigma(A) = \sigma(D) = \{2, 3, 4\}$ ($\because D$: upper triangular)
2.

[1]:
```
import numpy as np
import numpy.linalg as la
import matplotlib.pyplot as plt
from numba import njit
```

[3]:
```
P = np.array([[1, 2, 2, 1], [2, 3, 3, 2], [-1, 1, 2, -2], [1, 3, 2, 1]])
D = np.array([[2, 1, 0, 0], [0, 2, 1, 0], [0, 0, 3, 0], [0, 0, 0, 4]])
A = P.dot(D.dot(la.inv(P)))
```

[4]:
```
lam, U = la.eig(A)
lam
```

[4]:
```
array([4.+0.00000000e+00j, 3.+0.00000000e+00j, 2.+1.65960863e-07j,
       2.-1.65960863e-07j])
```

I find that the function `eig` is subject to some machine precision and thus recognizes two separate values for the eigenvalue of 2.

[5]:
```
la.matrix_power(A, 3)
```

[5]: array([[693., -307., -56., -127.],
       [1256., -568., -112., -216.],
       [-734., 398., 120.,   50.],
       [742., -326., -56., -138.]])

[5]:
```
la.matrix_power(A, 10)
```

[5]: array([[9231335.00000188, -4475213.00000088, -1047552.00000016,
         -1327437.00000031],
       [18112472.00000345, -8833352.00000161, -2095104.00000029,
         -2538824.00000056],
       [-16392202.00000272,   8259246.00000124,   2096128.00000023,
          1968814.00000043],
       [9404386.00000201, -4533238.00000093, -1047552.00000017,
         -1384438.00000033]])

The machine precision errors get inflated as we take higher matrix powers. This is probably due to errors accumulating from each multiplication.

0.1.2 Exercise 2.21

1.

[6]:
```
def SymmetricMat(n):
    """
    returns a symmetric n\times n matrix
    """
    A_ = np.random.randn(n, n)
    A = A_ + A_.T

    return A
```

[7]:
```
n = 5
A = SymmetricMat(n)
A
```

[7]: array([[2.13991674, 1.36084063, -0.97331958, -0.20948179, -0.15519334],
       [1.36084063, -0.1783174 , -1.55205954, 0.44404328, -1.68380325],
       [-0.97331958, -1.55205954, 0.43291873, -1.93496778, 0.52039052],
       [-0.20948179, 0.44404328, -1.93496778, 2.74160215, 1.83700503],
       [-0.15519334, -1.68380325, 0.52039052, 1.83700503, 0.37035364]])

[8]:
```
lam, U = la.eig(A) # eigendecomposition of A

# Sum using decomposition
S = np.zeros(n)
for i in range(n):
    S = S + lam[i]*np.outer(U[:, i], U[:, i])
```

```
S
```

[8]:
```
array([[ 2.13991674,  1.36084063, -0.97331958, -0.20948179, -0.15519334],
       [ 1.36084063, -0.1783174 , -1.55205954,  0.44404328, -1.68380325],
       [-0.97331958, -1.55205954,  0.43291873, -1.93496778,  0.52039052],
       [-0.20948179,  0.44404328, -1.93496778,  2.74160215,  1.83700503],
       [-0.15519334, -1.68380325,  0.52039052,  1.83700503,  0.37035364]])
```

[9]:
```
la.norm(S - A) # check the norm of difference
```

[9]:
```
6.9634419216236965e-15
```

S is equivalent to A up to machine precision.

2.

[10]:
```
D = np.diag(lam)
D[0, 1] = 1
D
```

[10]:
```
array([[ 4.50186036,  1.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  3.80560903,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  1.1056723 ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        , -2.30331375,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  0.        , -1.60335408]])
```

[11]:
```
B = U @ D @ U.T
mu, V = la.eig(B)

S2 = np.zeros(n)
for i in range(n):
    S2 = S2 + mu[i]*np.outer(V[:, i], V[:, i])
S2
```

[11]:
```
array([[-1.88424341, -3.81346609, -0.59098895,  0.45099422,  0.98425252],
       [-3.81346609,  5.97222943, -0.22641825, -0.18647433, -1.90256606],
       [-0.59098895, -0.22641825,  1.16929196,  1.51333317, -1.18568612],
       [ 0.45099422, -0.18647433,  1.51333317, -1.8362864 , -1.13499942],
       [ 0.98425252, -1.90256606, -1.18568612, -1.13499942,  2.29349871]])
```

The matrix B appears to be very slightly different from A. The values look similar but are off by a small amount. I now check the 2-norm of the difference.

[12]:
```
la.norm(B - A, ord = 2)
```

[12]:
```
0.9999999999999999
```

Here are some other norms.

[13]:
```
print("Frobenius norm: {}".format(la.norm(B - A, ord = "fro")))
print("1-norm: {}".format(la.norm(B - A, ord = 1)))
print("Infinity norm: {}".format(la.norm(B - A, ord = np.inf)))
```

```
Frobenius norm: 0.9999999999999998
1-norm: 1.1331729635073913
Infinity norm: 1.6572504065929505
```

The difference in 2-norm between B and A are 1. I now justify this result. First, define

$$E_{12} = (e_{ij}) = \begin{cases} 1 & i = 1, j = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Then, we can write $A = UDU^\top, B = U(D + E_{12})U^\top$. This gives

$$\|B - A\|_2 = \|U(D + E_{12})U^\top - UDU^\top\|_2 = \|UE_{12}U^\top\|_2 = \|E_{12}\|_2 = 1,$$

since U is unitary.

Therefore, we have $\|B - A\|_2 = 1$.

0.1.3 Exercise 2.23

1.

[15]:
```
n = 5
A = np.random.randn(n, n)
lam, U = la.eig(A)
lam # spectrum
```

[15]:
```
array([ 1.13543842,  0.53523625, -3.07739753, -1.98488579, -0.9448822 ])
```

[16]:
```
max(abs(lam)) # spectral radius of A
```

[16]:
```
3.0773975319616436
```

2.

[17]:
```
A
```

[17]:
```
array([[ 0.83398591, -0.52278093,  0.8868695 , -1.25797755, -1.22115511],
       [-0.03385667, -0.59546792, -1.30801879, -1.26162425, -0.42912919],
       [ 2.64197445, -0.77925375, -1.45211983, -0.70251011,  0.3804059 ],
       [ 0.43960925, -0.41377582,  0.06989936, -1.31847039, -0.11599152],
       [ 0.26813113, -0.74085515,  1.37390446, -0.42839547, -1.80441861]])
```

[18]:
```
gamma = []
for i in range(n):
    gamma.append(sum(abs(A[:, i])) - abs(A[i, i]))

gamma # calculated $\gamma_i$s
```

[18]:
```
[3.3835715054031383,
 2.456665654292216,
 3.6386921092677222,
 3.650507379514934,
 2.1466817180952926]
```

[19]:
```python
D = []
for i in range(n):
    D.append((A[i,i] - gamma[i], A[i,i] + gamma[i]))
D
```

[19]:
```
[(-2.5495855946424726, 4.217557416163804),
 (-3.0521335776599945, 1.8611977309244376),
 (-5.090811941029193, 2.1865722775062517),
 (-4.9689777703616205, 2.3320369886682473),
 (-3.9511003314553674, 0.3422631047352176)]
```

[20]:
```python
in_di = []
for i in range(n):
    for_lam = []
    for j in range(n):
        for_lam.append(abs(lam[i] - A[j, j]) <= gamma[j]) # check whether␣
 ↪\lambda_i \in D_j

    in_di.append(max(for_lam))
in_di
```

[20]:
```
[True, True, True, True, True]
```

I find that, indeed, the eigenvalues fall into at least one of the Gershgorin disk (Gershgorin theorem).

To prove this rigorously, let $\lambda \in \sigma(A)$ and $x \in \mathbb{R}^n$ be its corresponding eigenvector. Suppose $|x_i| \geq |x_j|, \forall j = 1, \dots, n$ i.e., the ith element of x has the largest modulus. Since we have $Ax = \lambda x$, looking at the ith element yields

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i \Rightarrow \sum_{j \neq i} a_{ij} x_j = (\lambda - a_{ii}) x_i$$

Then, by triangle inequality,

$$|\lambda - a_{ii}| = \left| \sum_{j \neq i} a_{ij} \frac{x_j}{x_i} \right| \leq \sum_{j \neq i} \left| a_{ij} \frac{x_j}{x_i} \right| = \sum_{j \neq i} |a_{ij}| \left| \frac{x_j}{x_i} \right| \leq \sum_{j \neq i} |a_{ij}|$$

Therefore, for this particular i, we have $\lambda \in D_i$. Since λ was arbitrary, we have the desired.

3.

[21]:
```python
def DiagDomMat(n):
    """
```

```
    returns a diagonally dominant matrix of size n\times n
    """
    A = np.random.randn(n, n)
    for i in range(n):
        A[i, i] = np.random.rand(1) + sum(abs(A[i, :])) - abs(A[i, i])

    return A
```

[22]:
```
A = DiagDomMat(3)
A
```

[22]:
```
array([[ 2.71498062,  1.68936973, -0.23156278],
       [-0.45721152,  1.12730861, -0.05158287],
       [ 1.15115462, -1.61038548,  2.87366852]])
```

[23]:
```
la.det(A)
```

[23]:
```
10.818990412811266
```

[24]:
```
la.det(DiagDomMat(10))
```

[24]:
```
460501452.37675214
```

[25]:
```
la.det(DiagDomMat(2))
```

[25]:
```
0.5313084286764557
```

[26]:
```
la.det(DiagDomMat(3))
```

[26]:
```
6.439479738485481
```

[27]:
```
la.det(DiagDomMat(4))
```

[27]:
```
73.9873022258113
```

I notice that all the determinants are strictly positive. This can be proved with the Gershgorin theorem above.

Suppose $A \in \mathcal{M}_n(\mathbb{R}^n)$ is diagonally dominant. Let $\lambda \in \sigma(A)$. Then, by Gershgorin theorem, we have $\lambda \in D_i$ for some Gershgorin disk D_i. This gives

$$|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| < |a_{ii}|.$$

By triangle inequality, we have

$$||\lambda| - |a_{ii}|| \leq |\lambda - a_{ii}| < |a_{ii}| \Rightarrow -|a_{ii}| < |\lambda| - |a_{ii}| < |a_{ii}|.$$

Therefore, we have

$$0 < |\lambda| < 2|a_{ii}|,$$

which implies $\lambda \neq 0$.

0.1.4 Exercise 3.8

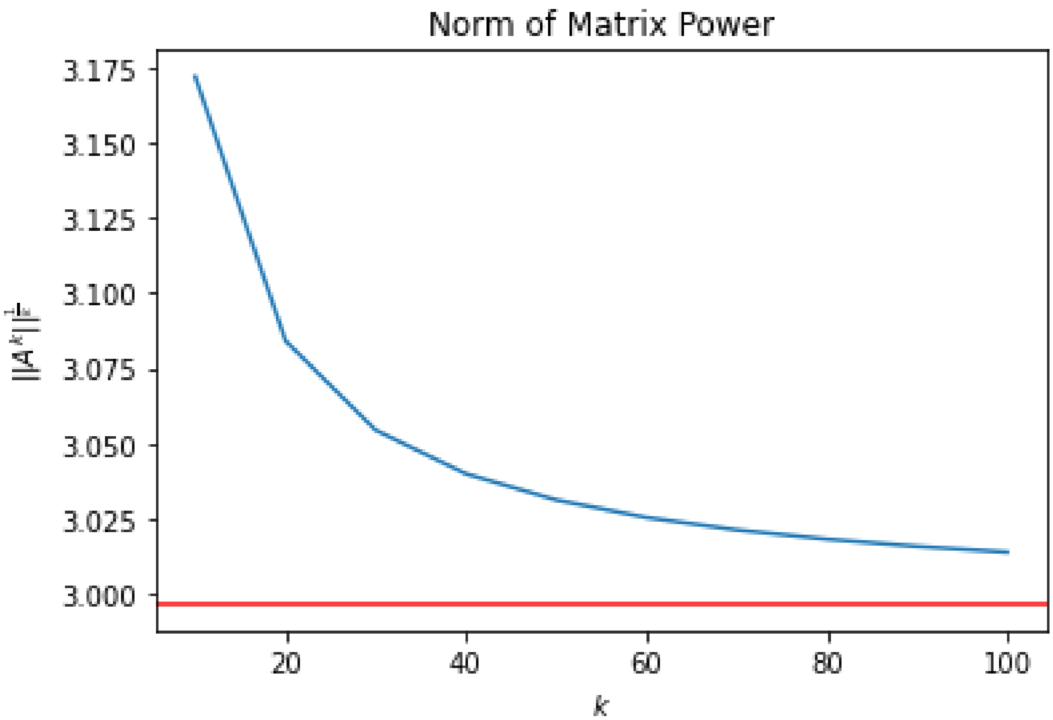
[28]:
```
n = 10
A = np.random.randn(n, n)
rho = max(abs(la.eig(A)[0]))
rho
```

[28]:
```
2.9965717124228335
```

[29]:
```
ks = np.linspace(10, 100, num = 10)
res = []
for k in ks:
    A_k = la.matrix_power(A, int(k))
    res.append(la.norm(A_k, ord = 2)**(1/k))
res
```

[29]:
```
[3.171994189780434,
 3.084045731445063,
 3.054361165218563,
 3.039828496453234,
 3.031124788558357,
 3.0253383035411328,
 3.0212119531664925,
 3.018120871684719,
 3.015718888446027,
 3.013798678160939]
```

[30]:
```
plt.plot(ks, res)
plt.axhline(rho, color = "red")
plt.title("Norm of Matrix Power")
plt.xlabel("$k$")
plt.ylabel("$||A^k||^\\frac{1}{k}$")
plt.show()
```



The plot seems to suggest that the value of the norm converges to $\rho(A)$ as $k \to \infty$.

Suppose $\lambda \in \sigma(A)$ s.t. $|\lambda| = \rho(A)$, i.e., the eigenvalue with the largest modulus.

Let $x \in \mathbb{R}^n$ be a corresponding eigenvector and $y \in \mathbb{R}^n$ be any vector. Then, by definition of a matrix norm, we have

$$\|A^k xy^\top\| \leq \|A^k\|\|xy^\top\|.$$

Since x is an eigenvector,

$$\|A^k xy^\top\| = \|\lambda^k xy^\top\| = |\lambda^k|\|xy^\top\| \leq \|A^k\|\|xy^\top\|$$

$$\Rightarrow |\lambda|^k = |\lambda^k| \leq \|A^k\| \Rightarrow \rho(A) \leq \|A^k\|^{\frac{1}{k}}$$

as desired.

0.1.5 Exercise 4.4

1.

[12]:
```
@njit
def matmul_ikj(B, C):
    """
    returns the matrix product multiplying in i, k, j order
    """
    p = B.shape[0]
    q = C.shape[1]
```

```python
    r = B.shape[1]
    A = np.zeros((p, q))
    for i in range(p):
        for k in range(r):
            for j in range(q):
                A[i,j] = A[i,j] + B[i,k] * C[k,j]
    return A

@njit
def matmul_strassen(B, C):
    """
    returns the matrix product using Strassen's algorithm
    """
    n = B.shape[0]
    A = np.zeros((n, n))
    s1, s2 = slice(0, n//2), slice(n//2, n)

    if n <= 64:
        A = matmul_ikj(B, C)

    else:
        B11, B12, B21, B22 = B[s1,s1], B[s1,s2], B[s2, s1], B[s2, s2]
        C11, C12, C21, C22 = C[s1,s1], C[s1,s2], C[s2, s1], C[s2, s2]
        M1 = matmul_strassen((B11 + B22), (C11 + C22))
        M2 = matmul_strassen((B21 + B22), C11)
        M3 = matmul_strassen(B11, (C12 - C22))
        M4 = matmul_strassen(B22, (C21 - C11))
        M5 = matmul_strassen((B11 + B12), C22)
        M6 = matmul_strassen((B21 - B11), (C11 + C12))
        M7 = matmul_strassen((B12 - B22), (C21 + C22))
        A[s1, s1] = M1 + M4 - M5 + M7
        A[s1, s2] = M3 + M5
        A[s2, s1] = M2 + M4
        A[s2, s2] = M1 - M2 + M3 + M6

    return A
```

[13]:
```python
B = np.random.randn(50, 70)
C = np.random.randn(70, 50)
A1 = B @ C
A2 = matmul_strassen(B, C)
la.norm(A1 - A2)
```

[13]: 0.0

0.1.6 Exercise 5.13 (Hager's Algorithm)

$S = \{x \in \mathbb{R}^n : \|x\|_1 = 1\}$, $f(x) = \|A^{-1}x\|_1$, $A \in \mathcal{M}_n(\mathbb{R})$: nonsingular. Consider the 1-norm conditioning

$$\mathrm{cond}_1(A) = \|A\|_1 \max_{x \in S} f(x)$$

1. $\|A\|_1$ is the largest column sum.

2. f attains maximum at e_j for some $j \in \{1, \dots, n\}$

Proof: Let $A^{-1} = (\tilde{a}_{ij})$. Note that for some j^*, we have

$$\|A^{-1}\|_1 = \sum_{i=1}^{n} |\tilde{a}_{ij^*}| = \|A^{-1}_{j^*}\|,$$

where $A^{-1}_{j^*}$ is the j^*th column of A^{-1} column of A^{-1}. In other words, j^*th column maximizes the column sum of A^{-1}. Then, we have

$$\|A^{-1}e_{j^*}\|_1 = \|A^{-1}_{j^*}\|_1 = \|A^{-1}\|_1,$$

where e_i is the ith canonical basis vector for $\mathbb{R}^n$.

By definition of a subordinate norm, e_{j^*} maxmizes f.

3. Given $x \in \mathbb{R}^n$, $\tilde{x}$ solves $A\tilde{x} = x$ and $\bar{x}$ solves $A^\top \bar{x} = s$, where s is the sign vector of $\tilde{x}$. Then, $f(x) = \langle \tilde{x}, s \rangle$.

Proof: Since A is given to be nonsingular, we have $\tilde{x} = A^{-1}x$. Then, we have

$$\langle \tilde{x}, s \rangle = |\tilde{x}_1| + |\tilde{x}_2| + \cdots + |\tilde{x}_n| = \sum_{i=1}^{n} |\tilde{x}_i| = \|\tilde{x}\|_1$$

Therefore,

$$f(x) = \|A^{-1}x\|_1 = \|\tilde{x}\|_1 = \langle \tilde{x}, s \rangle$$

as desired.

4. $\forall a \in \mathbb{R}^n, f(x) = \bar{x}^\top (a - x) \leq f(a)$.

Proof: Fix $a \in \mathbb{R}^n$. Then, $f(a) = \|A^{-1}a\|_1$. Note that $\bar{x}^\top = s^\top A^{-1}$. Then,

$$\bar{x}^\top (a - x) = s^\top A^{-1}(a - x) = s^\top A^{-1}a - s^\top A^{-1}x \leq \|A^{-1}a\|_1 - \|A^{-1}x\|_1,$$

where the last inequality follows from $a_j \leq |a_j|, \ \ -a_j \leq |a_j|, \ \ a = (a_1, \dots, a_n)$. Therefore,

$$f(x) + \bar{x}^\top (a - x) \leq \|A^{-1}x\|_1 + \|A^{-1}a\|_1 - \|A^{-1}x\|_1 = \|A^{-1}a\|_1 = f(a).$$

Hence, $f(x) + \bar{x}^\top (a - x) \leq f(a)$ as desired. Since $a \in \mathbb{R}^n$ was arbitrary, this holds for any vector a.

5. If $\bar{x}_j > \langle x, \bar{x} \rangle$ for some index j, then $f(e_j) > f(x)$.

Proof: Note that

$$\langle x, \bar{x} \rangle = \langle x, (A^\top)^{-1} s \rangle = \langle x, (A^{-1})^\top s \rangle = \langle A^{-1} x, s \rangle = \langle \tilde{x}, s \rangle = f(x).$$

Also,

$$\bar{x}_j = \langle e_j, \bar{x} \rangle = \langle e_j, (A^\top)^{-1} s \rangle = \langle A^{-1} e_j, s \rangle \leq \|A^{-1} e_j\|_1 = f(e_j).$$

Therefore, if $\bar{x}_j > \langle x, \bar{x} \rangle$, we have

$$f(e_j) > f(x).$$

6.

a) Note that

$$f(x) + s^\top A^{-1}(y - x) = \|A^{-1}x\|_1 + s^\top A^{-1} y - \|A^{-1} x\|_1$$

and $f(y) = \|A^{-1}y\|_1$.

Suppose $y \approx x$ so that $\text{sign}(\tilde{y}) = \text{sign}(\tilde{x})$. Then, $s = \text{sign}(\tilde{y})$, where $\tilde{y} = A^{-1}y$. This gives

$$s^\top A^{-1} y = \|A^{-1} y\|_1.$$

Hence, $f(y) = f(x) + s^\top A^{-1}(y - x)$ as desired.

b) $\|\bar{x}\|_\infty \leq \langle x, \bar{x} \rangle \Rightarrow x$: local maximum of f on S.

Proof: Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$. Suppose the jth element maximizes $|\bar{x}_i|$ for $i \in \{1, \dots, n\}$. Then, $\|\bar{x}\|_\infty = |\bar{x}_j|$.

From 5, we know that $|\tilde{x}_j| = |f(e_j)| \leq f(x)$ for all $j \in \{1, \dots, n\}$.

By the result of 2, we know f is maximized for some $e_{j^*}, j^* \in \{1, \dots, n\}$. Since $f(e_j) \leq f(x)$ for all $j \in \{1, \dots, n\}$, x must maximize f.

7. The algorithm is laid in the function defined for Part 8.

8.

[47]:
```python
def hager(A):
    """
    returns the 1-norm conditioning of matrix A using Hager's algorithm

    Arguments:
    ---------
    A: $n\times n$ array, the matrix to find the 1-norm conditioning for. Must␣
↪be nonsingular.
    """
    n = A.shape[0]

    x0 = np.repeat(1/n, n) # initial vector (x above)

    f = 0 # initial value of f(x)
```

```python
    x_curr = x0

    while(True):
        tilde_x = la.solve(A, x_curr) # solve A\tilde x = x above

        if la.norm(tilde_x, ord =1) <= f:
            break
        else:
            f = la.norm(tilde_x, ord = 1)

        s = np.sign(tilde_x) # s
        bar_x = la.solve(A.T, s) # solve A^\top \bar x = s above

        if max(abs(bar_x)) <= np.dot(bar_x, x_curr): # check for \lvert␣
↪x_j\rvert \le \langle x, \bar x\rangle (6b)
            break
        else:
            x_curr = np.zeros(n)
            x_curr[np.argmax(bar_x)] = 1 # set new x as e_j

    return f
```

[48]:
```python
# Example
n = 5
A_ = np.random.randn(n, n)
A = A_ + A_.T # to ensure A is nonsingular
hager(A)
```

[48]: 3.1566276430133953

[49]:
```python
la.norm(la.inv(A), ord = 1)
```

[49]: 3.1566276430133953

I find that my formulation of Hager's algorithm works correctly and gives the same result as the 1-norm of the inverse using the default functions in `numpy`. This is valuable in the sense that we do not have to directly compute the inverse to obtain the 1-norm conditioning, as calculating inverses are numerically unstable.