Report 1.6

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Work problem 1.7, p. 29 of Agresti. As a "part f", provide a Bayesian 95% credible set for π, using a uniform prior. Compare your answer to the confidence intervals obtained in the rest of the problem.

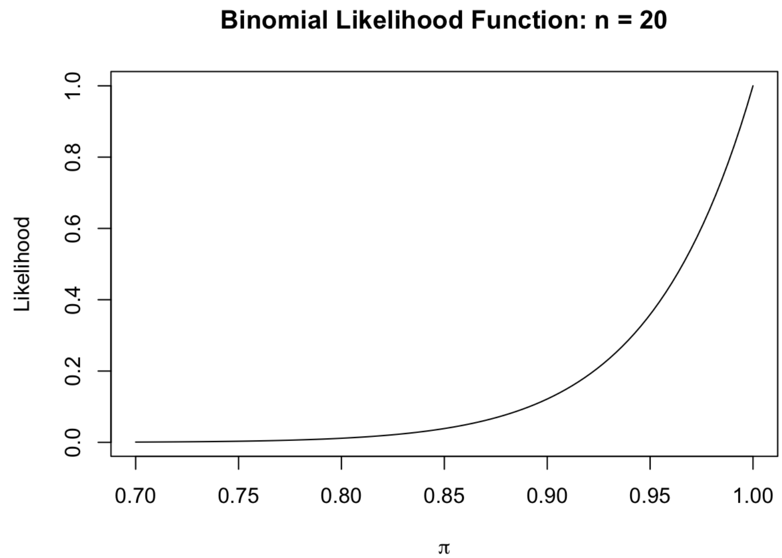
1.7 In a crossover trial comparing a new drug to a standard, π denotes the probability that the new one is judged better. It is desired to estimate π and test $H_0 : \pi = 0.50$ against $H_a : \pi \neq 0.50$. In 20 independent observations, the new drug is better each time.

a. Find and sketch the likelihood function. Is it close to the quadratic shape that large-sample normal approximations utilize?

   Since the new drug is better each time, we have 20 "successes" in 20 total trials, so the likelihood function is

$$\begin{aligned} L(\pi|n,x) &= \binom{20}{20}\pi^{20}(1-\pi)^{20-20} \\ &= \pi^{20} \end{aligned}$$

   which is shown in the plot below for different values of π. The shape is not close at all to the quadratic shape that large-sample normal approximations utilize.



b. Give the ML estimate of π. Conduct a Wald test and construct a 95% Wald confidence interval for π. Are these sensible?

The ML estimate of π is $\hat{\pi} = \frac{x}{n} = 20/20 = 1$.

The Wald test statistic is $\frac{\hat{\pi}-\pi_0}{SE} = \frac{\hat{\pi}-\pi_0}{\sqrt{\hat{\pi}(1-\hat{\pi})/n}} = \frac{0.5}{\sqrt{1(0)/20}}$, which is undefined ($\infty$) and not sensible.

A 95% Wald confidence interval for π is

$$\hat{\pi} \pm z_{\alpha/2} * SE = 1 \pm 1.96(0) = (1,1)$$

which is not sensible.

c. Conduct a score test, reporting the P-value. Construct a 95% score confidence interval. Interpret.

The score test statistic is

$$\begin{aligned} z_s &= \frac{u(\pi_0)}{[l(\pi_0)]^{1/2}}, \text{ where } u(\pi_0) = \frac{y}{\pi_0} - \frac{n-y}{1-\pi_0}, \quad l(\pi_0) = \frac{n}{\pi_0(1-\pi_0)} \\ &= \frac{y - n\pi_0}{\sqrt{n\pi_0(1-\pi_0)}} \\ &= \frac{\hat{\pi} - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}} \\ &= \frac{1-.5}{\sqrt{.5(.5)/20}} \\ &= \frac{.5}{\sqrt{.0125}} = 4.472. \end{aligned}$$

A 95% score confidence interval is the π_0 solutions to

$$\begin{aligned}(\hat{\pi} - \pi_0)/\sqrt{\pi_0(1-\pi_0)/n} &= \pm z_{\alpha/2} \\ |1 - \pi_0| &= 1.96\sqrt{\pi_0(1-\pi_0)/20}\end{aligned}$$

which yields an interval of (0.839, 1). So, we are 95% confident that the true proportion of observations in which the new drug is better than the standard drug is between 0.839 and 1.

d. Conduct a likelihood-ratio test and construct a likelihood-based 95% confidence interval. Interpret.

The likelihood-ratio test statistic, which is approximately χ^2_1, is

$$\begin{aligned}-2(L_0 - L_1) &= 2\left[y \log \frac{\hat{\pi}}{\pi_0} + (n-y)\log\frac{1-\hat{\pi}}{1-\pi_0}\right] \\ &= 2\left[20\log(2) + 0\right] \\ &= 2(20)\log(2) \\ &= 27.7.\end{aligned}$$

A 95% confidence interval is the π_0 solutions to

$$2\left[y \log \frac{\hat{\pi}}{\pi_0} + (n-y)\log\frac{1-\hat{\pi}}{1-\pi_0}\right] = \chi^2_1(0.05) = 3.84,$$

which yields an interval of (0.908, 1), meaning that we are 95% confident that the true proportion of observations in which the new drug is better than the standard drug is between 0.908 and 1.

e. Construct an exact binomial test. Interpret.

We can use 'binom.test()' in 'R' to obtain the results of an exact binomial test, which are

p-value: 1.9e-06

95% CI: (0.832, 1)

Based on these results, we conclude that $\pi \neq 0.5$ since the confidence interval does not contain 0.5.

We are 95% confident that the true proportion of observations in which the new drug is better than the standard drug is between 0.832 and 1.

f. Provide a Bayesian 95% credible set for π, using a uniform prior. Compare your answer to the confidence intervals obtained in the rest of the problem.

Using a uniform prior for π, a 95% Bayesian HPD credible set is (.867, .999), so there is a 95% probability that the true value of π is between .867 and .999.

The Bayesian interval is contained in the score and exact binomial intervals, but is slightly wider than the likelihood-ratio interval.