Derivative of Least Squares Objective Function

Computational Photography, Exercise 5/6

Raphael Braun and Jieen Chen

Eberhard Karls Universität Tübingen, Computergrafik

Derivative of Least Squares Objective Function

Start by expanding the Least Squares Objective Function $E(x)$:

$$\begin{aligned} E(x) &= \|b - Ax\|_2^2 \\ &= (b - Ax)^T(b - Ax) \\ &= b^Tb - b^TAx - (Ax)^Tb + (Ax)^TAx \end{aligned}$$

Use the identity $(Ax)^T = x^TA^T$:

$$E(x) = b^Tb - b^TAx - x^TA^Tb + x^TA^TAx$$

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Please note that $E(x)$ is a scalar function, so all summands are also scalars. This means that $b^T Ax$ and $x^T A^T b$ are exactly the same numbers. Therefore we can write:

$$E(x) = b^T b - 2b^T Ax + x^T A^T Ax$$

For the derivative we use the identities from `https://en.wikipedia.org/wiki/Matrix_calculus` (Scalar-by-vector identities denominator layout) to compute the derivative.

First we use $\frac{\partial(u+v)}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$ for $u = u(x)$ and $v = v(x)$:

$$\frac{\partial}{\partial x} E(x) = \frac{\partial}{\partial x} b^T b - \frac{\partial}{\partial x} 2b^T Ax + \frac{\partial}{\partial x} x^T A^T Ax$$

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Then lets use $\frac{\partial a}{\partial x} = 0$ if a is not a function of x:

$$\frac{\partial}{\partial x}E(x) = 0 - \frac{\partial}{\partial x}2b^T Ax + \frac{\partial}{\partial x}x^T A^T Ax$$

Now use $\frac{\partial b^T Ax}{\partial x} = A^T b$ with A and b not functions of x:

$$\frac{\partial}{\partial x}E(x) = 0 - 2A^T b + \frac{\partial}{\partial x}x^T A^T Ax$$

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Finally use $\frac{\partial x^T A x}{\partial x} = (A + A^T)x$ with A not a function of x. Be careful, that the A in the identity corresponds to $A^T A$ in our equation:

$$\frac{\partial}{\partial x} E(x) = 0 - 2A^T b + (A^T A + (A^T A)^T)x$$

If you utilize $(A^T A)^T = A^T A$ you get.

$$\frac{\partial}{\partial x} E(x) = 0 - 2A^T b + (A^T A + A^T A)x$$

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Now just simplify the result

$$\begin{aligned}\frac{\partial}{\partial x}E(x) &= 0 - 2A^Tb + (A^TA + A^TA)x \\ &= -2A^Tb + 2A^TAx \\ &= 2A^T(Ax - b)\end{aligned}$$

And this is it. This derivative is used in the code of the gradient descent algorithm.