11.

解 因 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{e^{2(x^2+y^2)}-1}{x^2+y^2}=2$ ，$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^2y}{x^2+y^2}=0$ [redacted]

原式$=\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{e^{2(x^2+y^2)}-1}{x^2+y^2}+\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x^2y}{x^2+y^2}=2$ [redacted]

12

解 令 $z=u^v$，其中 $u=x+y, v=x-y$ [redacted]

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$ [redacted]

$$=vu^{v-1}-u^v\ln u=(x-y)(x+y)^{x-y-1}-(x+y)^{x-y}\ln(x+y)\cdots$$

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}(vu^{v-1}-u^v\ln u)=u^{v-1}+v[(v-1)u^{v-2}+u^{v-1}\ln u]$$

$$-[(vu^{v-1}+u^v\ln u)\ln u+u^{v-1}]=v(v-1)u^{v-2}-u^v\ln^2 u\ \cdot$$

$$=(x-y)(x-y-1)(x+y)^{x-y-2}-(x+y)^{x-y}\ln^2(x+y)\ \ldots$$

13.

解 因区域 D 分割成如图的 D_1 和 D_2 $\cdots$

$xyf(x^2+y^2)$ 关于 x 和 y 为奇函数，

故 $\iint\limits_D yxf(x^2+y^2))d\sigma = 0$ ， [redacted]

从而 $\iint\limits_D y(|y-x|-xf(x^2+y^2))d\sigma = \iint\limits_D y|y-x|d\sigma$

$= \iint\limits_{D_1} y(y-x)d\sigma + \iint\limits_{D_2} y(x-y)d\sigma$ [redacted]

$= \int_0^1 dy\int_{-y}^{y} y^2dx - \int_0^1 dx\int_{-x}^{x} y^2dy$. [redacted]

$= \frac{1}{3}$ [redacted]

14

解 作辅助线 $l: y=0$ 如图，使 $L+l$ 构成一封闭曲线，方向为逆时针方向，所围成的区域记作 D .

记 $P=(y+1)(\sin x - x^2), Q = xy^2 + \cos y$,显然， P,Q 在 D 内满足格林公式的条件，

于是场力所做的功为

$$W = \int\limits_{L+l} Pdx + Qdy - $$
$$-\int\limits_l Pdx + Qdy$$ [redacted]

$$= \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy - \int\limits_l Pdx + Qdy$$ [redacted]

$$= \iint\limits_D (y^2+x^2-\sin x)dxdy - \int_{-a}^{a}(\sin x - x^2)dx$$ [redacted]

$$= (\frac{\pi a}{4} + \frac{2}{3})a^3$$ [redacted]

15

解 作辅助曲面$\Sigma_1 : z=0$,其正向与z轴正向相反 ,于是$\Sigma+\Sigma_1$作成一简单闭曲面.

记其围成的立体为Ω [redacted]

记$P=xy^2+t, Q=y(z^2+t^2), R=x^2z+yz+2$,显然，它们在$\Omega$内具有连续的偏导数，于是所求流量为

$$\Phi=\int_0^2(\iint_{\Sigma}(xy^2+t)dydz+y(z^2+t^2)dzdx+(x^2z+yz+2)dxdy)dt$$ [redacted]

根据高斯公式有

$$\begin{aligned}\Phi=&\int_0^2[\iint_{\Sigma+\Sigma_1}(xy^2+t)dydz+y(z^2+t^2)dzdx+(x^2z+yz+2)dxdy \quad\cdots\\&-\iint_{\Sigma_1}(xy^2+t)dydz+y(z^2+t^2)dzdx+(x^2z+yz+2)dxdy]dt\\&=\int_0^2(\iiint_{\Omega}(x^2+y^2+z^2+y+t^2)dxdydz)dt-\\&-\int_0^2(\iint_{\Sigma_1}(xy^2+t)dydz+y(z^2+t^2)dzdx+(x^2z+yz+2)dxdy)dt\end{aligned}$$

$$=2\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^R r^4\sin\varphi dr+\frac{16}{9}\pi R^3+4\pi R^2$$

$$=4\pi R^2(\frac{1}{5}R^3+\frac{4}{9}R+1). \qquad\cdots$$

16

解 $a_n = n^2+1, \lim\limits_{n\to+\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to+\infty}\frac{n^2+2n+2}{n^2+1} = 1$,所以 ,原幂级数的收敛半径 $R=1$.

又因 $\sum\limits_{n=1}^{+\infty}(n^2+1)$ 和 $\sum\limits_{n=1}^{+\infty}(-1)^n(n^2+1)$ 均发散，故原幂级数的收敛区间为

$(-1,1)$ [redacted]

令 $\sum\limits_{n=1}^{+\infty} n^2x^{n-1} = s(x)$ ，两边从 0 到 x 积分得 $\sum\limits_{n=1}^{+\infty} nx^n = \int_0^x s(x)dx$

$$\int_0^x s(x)dx = x\sum_{n=1}^{+\infty} nx^{n-1} = x(\sum_{n=1}^{+\infty} x^n)' = x(\frac{x}{1-x})' = \frac{x}{(1-x)^2}$$

于是 $s(x) = \frac{1+x}{(1-x)^3}$ ，

从而 $\sum\limits_{n=1}^{+\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}$ [redacted]

$$\sum_{n=1}^{+\infty}(n^2+1)x^n = \frac{x(1+x)}{(1-x)^3} + \frac{x}{1-x}$$ [redacted]

$$\sum_{n=1}^{+\infty}\frac{n^2-1}{2^{n-1}} = \sum_{n=1}^{+\infty}\frac{n^2}{2^{n-1}} - \sum_{n=1}^{+\infty}\frac{1}{2^{n-1}} = 2\sum_{n=1}^{+\infty}\frac{n^2}{2^n} - 2 = 10$$ [redacted]

17

解 设所求点为 $M(x_0, y_0, z_0)$ ，球面在该点的切平面方程为

$x_0x + y_0y + z_0z = 1$ ………2 分

切平面与三坐标平面围成立体的体积是 $\frac{1}{6x_0y_0z_0}$. [redacted]

求最小体积问题归结为求函数 $u = f(x, y, z) = xyz$ 在条件 $x^2 + y^2 + z^2 = 1$ 下的最大值问题.于是作拉格朗日函数 $L(x, y, z, \lambda) = xyz + \lambda(x^2 + y^2 + z^2 - 1)$ · [redacted]

令 $L_x'(x, y, z, \lambda) = yz + 2\lambda x = 0$ ，

$L_y'(x, y, z, \lambda) = xz + 2\lambda y = 0$ ，

$L_z'(x, y, z, \lambda) = xy + 2\lambda z = 0$ ，

$x^2 + y^2 + z^2 = 1$. [redacted]

联立求解得 $x = y = z = \frac{1}{\sqrt{3}}$

于是点 $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ 为所求. [redacted]

18．解

$$\int_0^1 dx \int_0^x f(x+y)dy = \frac{1}{2}\int_0^1 dx \int_0^1 f(x+y)dy$$ [redacted]

对于积分 $\int_0^1 dx \int_0^1 f(x+y)dy$ ，令 $x + y = u, y = v$ ，

于是 $\int_0^1 dx \int_0^1 f(x+y)dy = \int_0^1 du \int_0^u f(u)|J|dv + \int_1^2 du \int_{u-1}^1 f(u)|J|dv$

$= \int_0^1 uf(u)du + \int_1^2 (2-u)f(u)du$ [redacted]

$$\int_1^2 (2-u)f(u)du \xlongequal{令u=1+t} \int_0^1 (1-t)f(t+1)dt$$

$$= \int_0^1 (1-t)f(t)dt = \int_0^1 f(t)dt - \int_0^1 tf(t)dt$$

所以 $\int_0^1 dx \int_0^1 f(x+y)dy = \int_0^1 f(t)dt = a \Rightarrow \int_0^1 dx \int_0^x f(x+y)dy = \frac{a}{2}$.