Notes on the 2D aggregation model

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February 3, 2021

0.1 Introduction

These are notes on the aggregation model in two spatial dimensins. In particular, I wrote down my thoughts in the introduction of saturation effects without compromising the interaction mechanisms.

0.2 The linear model

Our starting point is Fetecau's model:

$$\partial_t u + \gamma\, \boldsymbol{e}_\phi \cdot \nabla_x u = -\lambda(x,\phi)u + \int\limits_{\phi'=-\pi}^{\pi} T(x,\phi',\phi)u(x,\phi',t)\,\mathrm{d}\phi' \tag{1}$$

We can interpret $\lambda(x,\phi)$ as *the rate at which individuals turn away from angle ϕ at location x* and $T(x,\phi',\phi)$ as *the rate at which individuals facing a direction between ϕ' and $\phi'+d\phi'$ turn to direction ϕ at location x*. The functions λ and T also depend on time, but I will suppress that dependence in my notation.
In this model, the turning rate T is linear in the population density u. We would like to include saturation by replacing the T in the second term on the right-hand side by $S(T)$, where S is some sigmoidal function like $S(z) = M(1+\tanh(z-t))/2$ (where M, the maximum and t, the threshold, are new parameters). The difficulty is to do this in such a way that the "conservation of mass" is satisfied. For model (1) we can derive the conservation law as follows. Let $U(t)$ be the total population ("mass"), and let the animals graze on some compact domain $D \in \mathbb{R}^2$. On the left hand side:

$$\begin{aligned}
\int\limits_{x\in D}\int\limits_{\phi=\pi}^{\pi} (\partial_t u + \gamma\, \boldsymbol{e}_\phi\cdot\nabla_x u)\ \mathrm{d}\phi\mathrm{d}x &= \frac{\mathrm{d}U}{\mathrm{d}t} + \int\limits_{x\in D}\int\limits_{\phi=-\pi}^{\pi} (\cos(\phi)\partial_{x_1}u + \sin(\phi)\partial_{x_2}u)\ \mathrm{d}\phi\mathrm{d}x && (2)\\
&= \frac{\mathrm{d}U}{\mathrm{d}t} + \int\limits_{x\in D} \nabla\cdot F\,\mathrm{d}x = \frac{\mathrm{d}U}{\mathrm{d}t} + \int\limits_{\partial D} F\cdot\,\mathrm{d}s \\
&= \frac{\mathrm{d}U}{\mathrm{d}t} && (3)
\end{aligned}$$

where

$$F = \begin{pmatrix} \int \cos(\phi)u\,\mathrm{d}\phi \\ \int \sin(\phi)u\,\mathrm{d}\phi \end{pmatrix}$$

and we have assumed that the boundary integral is zero because of theboundary conditions. This works, for instance, for periodic boundary conditions on a rectangle and for homogeneous Dirichlet boundary conditions. Meanwhile, on the right hand side (changing the name of the angle in the firs term to ϕ'):

$$-\int\limits_{x\in D}\int\limits_{\phi'=-\pi}^{\pi} \lambda(x,\phi')u(x,\phi',t)\,\mathrm{d}\phi'\mathrm{d}x + \int\limits_{x\in D}\int\limits_{\phi=-\pi}^{\pi}\int\limits_{\phi'=-\pi}^{\pi} T(x,\phi',\phi)u(x,\phi',t)\,\mathrm{d}\phi'\,\mathrm{d}\phi\mathrm{d}x \tag{4}$$

$$= \int\limits_{x\in D}\int\limits_{\phi'=-\pi}^{\pi}\left(-\lambda(x,\phi') + \int\limits_{\phi=-\pi}^{\pi} T(x,\phi',\phi)\,\mathrm{d}\phi\right)u(x,\phi',t)\,\mathrm{d}\phi'\mathrm{d}x \tag{5}$$

If the total mass in conserved then the right hand side must evaluate to zero *for all u*, so we have

$$\lambda(x,\phi') = \int\limits_{\phi=-\pi}^{\pi} T(x,\phi',\phi)\,\mathrm{d}\phi \tag{6}$$

as a condition. Fetecau then defines λ and T in terms of convolutions of attraction, repulsion and alignment kernels with u while making sure this relation holds. His definitions are given in equations (2.1.3) and (2.1.4) in Eryn's thesis. Both are given by the sum of alignment (al), attraction (a) and repulsion (r) forces, and each of those is computed as the convolution of a kernel with the desity u. For T, the convolution kernel has two components:

1. The kernels K^{d} (distance) and K^{o} (orientation) can be though of as the *inherent sensitivity* of the individual. For instance, if the individual takes visual cues, the distance kernel should be large for distances at which the individual's eyes work well and decay beyond that range. The orientation kernel should then be maximal for small angles between the direction the individual is moving in and the vector pointing from the individual to the influencer - i.e. the influence of individuals *ahead* of the individual at x. Of course, the form of these kernels depends very much on the communication mechanism.

2. The weight functions w_{al}, w_{a} and w_{r}. We can loosely think of these functions as modelling the influence an individual has if it is observed. For the alignment force, we have $w_{\mathrm{al}}(\phi-\phi', \phi'-\theta)$, meaning that the weight function depends on the *turning angle* $\phi - \phi'$ and on the difference between the angle the individual at x is turning to and the angle the influencing individual in moving in. For instance, we could pick a function that is maximal for $\phi - \phi' = \phi' - \theta = 0$ and decays with both arguments. This would model that the individual prefers small turning angles and prefers to align precisely with the "influencer". For attraction and repulsion, we have $w_{\mathrm{a,r}}(\phi - \phi', \phi' - \psi)$, where ψ is the angle between the positive x-axis and the vector pointing from the individual at x to the "influencer". Suppose we pick w_{a} so that it is maximal for $\phi' - \psi = 0$. This means that the individual at x tends to turn towards the "influencer". Conversely, for repulsion, a natural choice is a function that is maximal for $\phi' - \psi = \pi$.

The weight functions are absent in the definitions of λ_{al}, λ_{a} and λ_{r}. Otherwise, the definitions are the same. With these definitions, conservation law (6) does indeed hold:

$$
\begin{aligned}
\int_{-\pi}^{\pi} T(x,\phi,\phi')\,\mathrm{d}\phi' = \int_{-\pi}^{\pi} T_{\mathrm{al}}(x,\phi,\phi')\,\mathrm{d}\phi' + \int_{-\pi}^{\pi} T_{\mathrm{a}}(x,\phi,\phi')\,\mathrm{d}\phi' + \int_{-\pi}^{\pi} T_{\,r}(x,\phi,\phi')\,\mathrm{d}\phi' = \qquad (7)\\
q_{\mathrm{al}} \int_{s\in D} \int_{-\pi}^{\pi} K^{\mathrm{d}}_{\mathrm{al}}(x-s) K^{\mathrm{o}}_{\mathrm{al}}(\theta,\phi) \left\{ \int_{-\pi}^{\pi} w_{\mathrm{al}}(\phi-\phi', \phi'-\theta)\,\mathrm{d}\phi' \right\} u(s,\theta,t)\,\mathrm{d}\phi \mathrm{d}s \\
+ q_{\mathrm{a}} \ldots + q_{\mathrm{r}} \ldots \\
q_{\mathrm{al}}\lambda_{\mathrm{al}}(x,\phi) + q_{\mathrm{a}}\lambda_{\mathrm{a}}(x,\phi) + q_{\mathrm{r}}\lambda_{\mathrm{r}}(x,\phi) = \lambda(x,\phi)
\end{aligned}
$$

where I have used that fact that each of the weight functions integrates to unity.

0.3 Approach in Eryn's work

Our first, naive attempt to introduce saturation effects followed this line of though. Let us define

$$
\Gamma(x,\phi,\phi') = \sum_{k=\mathrm{al,a,r}} q_k \int_{x\in D} \int_{-\pi}^{\pi} K^{\mathrm{d}}_k K^{\mathrm{o}}_k w_k u(s,\theta,t)\,\mathrm{d}\theta \mathrm{d}s \qquad (8)
$$

where I have omitted the arguments of the distance and orientation kernels and weight functions for simplicity. For Fetecau's model, T and Γ coincide. We also define

$$
\gamma(x,\phi) = \sum_{k=\mathrm{al,a,r}} q_k \int_{x\in D} \int_{-\pi}^{\pi} K^{\mathrm{d}}_k K^{\mathrm{o}}_k u(s,\theta,t)\,\mathrm{d}\theta \mathrm{d}s \qquad (9)
$$

and see that, in Fetecau's model, γ coincides with λ. Now we try $\lambda = S(\gamma)$ and $T = S(\Gamma)$ in model (1). This fails because the conservation of mass now requires that

$$S(\gamma(x, \phi')) = \int\limits_{\phi=-\pi}^{\pi} S(\Gamma(x, \phi', \phi))\, \mathrm{d}\phi$$

which should hold identically, for all densities u and some nonlinear function S.
To avoid this difficulty, we simplified the model by

- assuming the weight functions only depend on the turning angle,
- assuming all three weight functions are identical and
- taking the weight function out of the nonlinearity.

We then defined

$$\lambda(x, \phi) = S(\gamma) \tag{10}$$

$$T(x, \phi, \phi') = w(\phi - \phi')\lambda(x, \phi) \tag{11}$$

and the conservation of mass holds again. However, I now think this is too strong a simplification. How can we model excuded volume for bacteria, for instance, if we cannot encode in the weight function for repulsion that the organism tends to move away from others at close range?

0.4 A better approach – I think

Let us see what happens when we change the model to

$$\partial_t u + \gamma\, \boldsymbol{e}_\phi \cdot \nabla_x u = -\Lambda(x, \phi)u + \int\limits_{\phi'=-\pi}^{\pi} S(T(x, \phi', \phi))u(x, \phi', t)\, \mathrm{d}\phi' \tag{12}$$

where T is defined as in Fetecau's paper and Λ is the new turning rate. Again, we integrate over space and angle to find

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\int\limits_{x\in D} \int\limits_{\phi'=-\pi}^{\pi} \Lambda(x, \phi')u(x, \phi', t)\, \mathrm{d}\phi' \mathrm{d}x + \int\limits_{x\in D} \int\limits_{\phi=-\pi}^{\pi} \int\limits_{\phi'=-\pi}^{\pi} S(T(x, \phi', \phi))u(x, \phi', t)\, \mathrm{d}\phi'\, \mathrm{d}\phi \mathrm{d}x \tag{13}$$

and we find that, in order to have conservation of mass, we need

$$\Lambda(x, \phi') = \int\limits_{\phi=-\pi}^{\pi} S(T(x, \phi', \phi))\, \mathrm{d}\phi \tag{14}$$