Introduction Draft

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1 Introduction

In the nature, many organisms prefer to live and move in groups. Animal grouping behaviour is also an important topic in both biological and mathematical fields.

2 Aggregation Model with Saturated Turning Term

In this section, the main content is how to make an adaptation of a saturation function to a two-dimensional aggregation model conducted by Fetecau. I will started with a brief description on Fetecau's model, and followed by our new adaptation to the non-linear part of the model.

2.1 Introduction to Fetecau's Model

In Fetecau's paper, the model is described as:

$$\partial_t u(\phi, X, t)+\gamma e_\phi\cdot\nabla_X u(\phi, X, t) = -\lambda(\phi, X)u(\phi, X, t)+\int_{-\pi}^{\pi} T\left(\phi, \phi', X\right) u\left(\phi', X, t\right) \mathrm{d}\phi' \quad (2.1)$$

In this model, there are four terms which represent different actions of the density u at the location $X = (x_1, x_2)$ within the range $[-\pi, \pi]$ facing at angle $\phi \in [-\pi, \pi]$ at time t.

We first look at the first term, $\partial_t u(\phi, X, t)$, which represents the change of density u over time t. Then if we look at the second term, $\gamma e_\phi \cdot \nabla_X u(\phi, X, t)$, where $e_\phi = (\cos(\phi), \sin(\phi))$, represents the how the density changes as the individuals at the location X moves in the direction ϕ with speed γ. These two terms forms the linear part of the model.

The right hand side of the model are non-linear which is also called the interaction terms. These two terms are the result of the density change from the interaction with neighbours. The third term $-\lambda(X, \phi)u(\phi, X, t)$represents the density move out of the position X at angle ϕ, where $\lambda(X, \phi)$ is the reorientation function. The last term $\int_{-\pi}^{\pi} T\left(\phi, \phi', X\right) u\left(\phi', X, t\right) \mathrm{d}\phi'$is the density which moves into X at ϕ. The function $T(\phi, \phi', X)$ is a turning function which represents the turning rate of changing from direction ϕ' to ϕ at X.

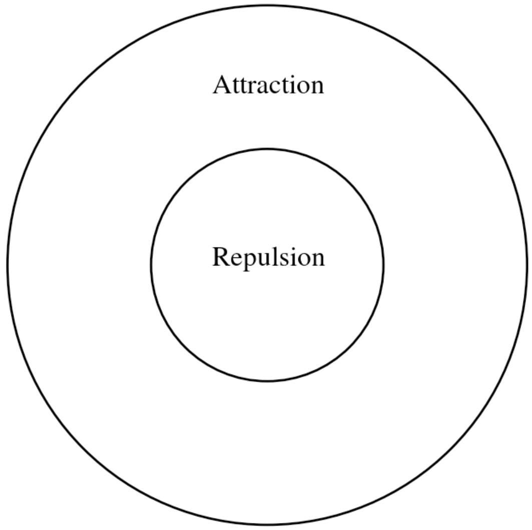


Figure 1: The distance kernel indicating the neighbour area which an individual will be attracted or repelled with an individual centered in the middle

The turning function is represented as:

$$T(\phi, \phi', X) = T_a(\phi, \phi', X) + T_r(\phi, \phi', X) \tag{2.2}$$

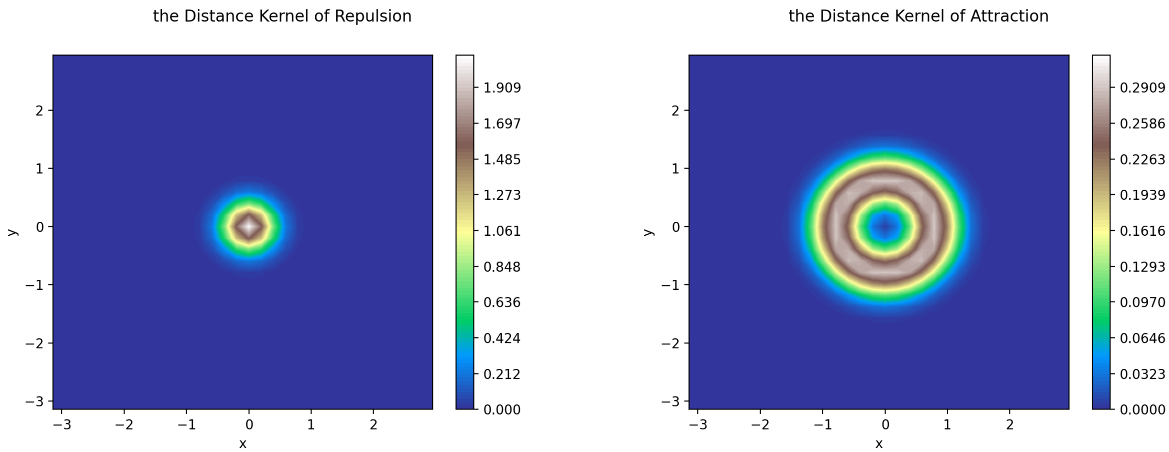
$$T_j(\phi, \phi', X) = q_j \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} K_j^d(X - S) K_j^o(\phi, X - S) w_j \left(\phi, \phi', X - S\right) u(S, \theta, t) \mathrm{d}\theta \mathrm{d}S \tag{2.3}$$

where q_j is a constant stands for the strength of the turning function.

To explain the turning function $T_j(\phi, \phi', X)$and the reorientation function $\lambda(X, \phi)$, we first define the distance kernel K_j^d and the orientation kernel K_j^o for both attractive or repulsive interactions ($j = a, r$). The distance kernel K_j^d represents how the sights of individuals, which is represented in Figure 1. Usually, individuals do not like their neighbours to be to close to them, thus for the repulsion distance kernel, it is the area directly surrounding the individual. The attraction distance kernel is the direct opposite, where an individual does not want to stay to far away with their neighbour, thus the attraction distance kernel is a little bit far away from the individual. Let X be the individual's position and S be the neighbour's position and $X - S = (s_1, s_2)$, then the distance kernel can be written as:

$$K_j^d(X - S) = \frac{1}{A_j} e^{-\left(\sqrt{s_1^2 + s_2^2} - d_j\right)^2 / m_j^2} \tag{2.4}$$

where $j = a, r$ represents whether the distance kernel is for attraction or repulsion, and $A_j = \pi m_j \left(m_j e^{-d_j^2/m_j^2} + \sqrt{\pi} d_j + \sqrt{\pi} d_j \operatorname{erf}\left(d_j/m_j\right)\right)$ which makes $\iint K_j^d \mathrm{d}(X - S) = 1$. In (2.4), d_j is the interaction ranges and the m_j is the width of the interaction ranges. In Figure 2 it shows the K_a^d and K_r^d with the same m_j but different d_j.



(a) the Repulsion Distance Kernel when $d_r = 0$ and $m_r = \frac{\pi}{8}$

(b) the Attraction Distance Kernel when $d_a = \frac{\pi}{4}$ and $m_a = \frac{\pi}{8}$

Figure 2: the Distance Kernel for Attraction and Repulsion

The other kernel is the orientation kernel, K_j^o, which depends on the individual's angle ϕ and the angle between the individual's and neighbour's position ψ. The angle ψ can be calculated as

$$\cos\psi = \frac{s_1}{\sqrt{s_1^2 + s_2^2}}, \quad \sin\psi = \frac{s_2}{\sqrt{s_1^2 + s_2^2}} \tag{2.5}$$

Then the orientation kernel for attraction and repulsion is:

$$K_a^o(\phi - \psi) = \frac{1}{2\pi}(-\cos(\phi - \psi) + 1) \tag{2.6}$$

$$K_r^o(\phi - \psi) = \frac{1}{2\pi}(\cos(\phi - \psi) + 1) \tag{2.7}$$

where for attraction, the orientation kernel maximizes when ψ is at the opposite of ϕ ($\phi - \psi = \pi$); for repulsion, the orientation kernel maximizes when ψ is close to the angle ϕ ($\phi - \psi = 0$). Then the total kernel for attraction and repulsion is:

$$K_j(\phi, X - S) = K_j^d(X - S)K_j^o(\phi, X - S) \tag{2.8}$$

Another component of the Turning function is the probability function $w_j(\phi, \phi', X - S)$. This gives the probability of the individuals moving from ϕ' to ϕ with the interaction between their neighbours.

$$w_j(\phi, \phi', X - S) = g_\sigma\left(\phi' - \phi - \kappa_j\left(\sin(\phi' - \psi)\right)\right) \tag{2.9}$$

where

$$g_\sigma(\theta) = \frac{1}{\sqrt{\pi}\sigma}\sum_{z\in\mathbb{Z}} e^{-\left(\frac{\theta + 2\pi z}{\sigma}\right)^2}, \quad \theta \in (-\pi, \pi) \tag{2.10}$$

is an approximation of periodic Gaussian function with $0 < \kappa_a \leq 1$ and $-1 \leq \kappa_r < 0$. We can combine the Kernel and probability function to form a new function

$$Kw_j(\phi, \phi'), X - S) = K_j(\phi, X - S)w_j\left(\phi, \phi', X - S\right) \tag{2.11}$$

Thus, the turning function is a convolution over the spacial parameter $X = (x_1, x_2)$. Let $\int_{-\pi}^{\pi} u(S, \theta, t)\mathrm{d}\theta = U(S, t)$, the turning function can be rewritten as:

$$T_j\left(\phi, \phi', X\right) = q_j \int_{\mathbb{R}^2} Kw_j\left(\phi, \phi', X - S\right) U(S, t)\mathrm{d}S \tag{2.12}$$

with q_j represents the strength of the specific attractive or repulsive turning functions. The total turning function is

$$T\left(\phi, \phi', X\right) = T_a\left(\phi, \phi', X\right) + T_r\left(\phi, \phi', X\right) \tag{2.13}$$

For the reorientation term $\lambda(\phi, X)$, it can also be written in terms of the turning function, where

$$\lambda(\phi, X) = \lambda_a(\phi, X) + \lambda_r(\phi, X) \tag{2.14}$$

$$\lambda_j(\phi, X) = \int_{-\pi}^{\pi} T_j\left(\phi, \phi', X\right) \mathrm{d}\phi \tag{2.15}$$

which represents the total volume changing away from the original angle ϕ'.

2.2 Adaptation to Saturation in the Interaction Term

In addition to the original model, there is a new adaptation, a new Saturation function $S(x)$ added to the Turning function. Therefore, the non-linear interaction term $N(\phi, X)$ becomes

$$\begin{aligned} N(\phi, X) &= -\lambda'(\phi, X)u(\phi, X, t) + \int_{-\pi}^{\pi} S\Big(T\left(\phi, \phi', X\right)\Big) u\left(\phi', X, t\right) \mathrm{d}\phi' \\ &= -\int_{-\pi}^{\pi} S\Big(T_j\left(\phi, \phi', X\right)\Big) \mathrm{d}\phi u(\phi, X, t) + \int_{-\pi}^{\pi} S\Big(T\left(\phi, \phi', X\right)\Big) u\left(\phi', X, t\right) \mathrm{d}\phi' \end{aligned} \tag{2.16}$$

In this paper, the saturation function is calculated in the form of

$$S(x) = \frac{H}{2}\left(1 + \tanh\left(\frac{T - \mu}{\sigma}\right)\right) \tag{2.17}$$

There are three parameters in the saturation function, which are H, μ and σ. H represents the maximum value of the function, μ relates the midpoint of the function, and σ stands for the width of the function. By applying the saturation function, it simulates the scenario where sometimes an individual does not prefer to interact with its neighbours and limits the influence of the interaction term or there is a stimuli during the interaction.

Therefore, the new Model with saturation term is

$$\partial_t u(\phi, X, t) + \gamma e_\phi \cdot \nabla_X u(\phi, X, t) = -\lambda'(X, \phi)u(\phi, X, t) + \int_{-\pi}^{\pi} S\Big(T\left(\phi, \phi', X\right)\Big) u\left(\phi', X, t\right) \mathrm{d}\phi' \tag{2.18}$$