Result Draft

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1 Result

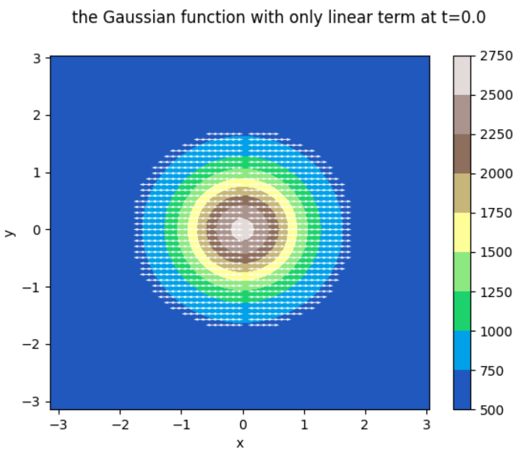
In this section, we are going to show the results with the saturated interaction term. There are total of two initial cases we have tested on. The first one is one single Gaussian bump in the center and the other one is two Gaussian bumps. Also different saturation functions are applied to different initial conditions. The plots shown are the total density $U(x_1, x_2, t)$, which is calculated by $U(x_1, x_2, t) = \frac{2\pi}{N} \sum_{\phi_{j=0}}^{j=64} u(\phi_j, x_1, x_2, t)$ where $N = 64$ is the number of grid points during the simulation.

1.1 Initial condition 1: One Gaussian bump

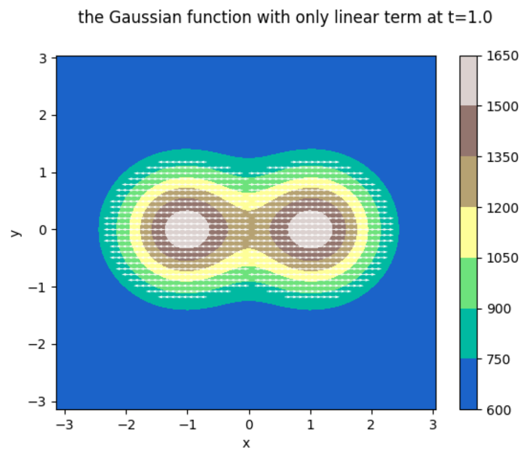
The first initial condition is only one Gaussian bump at the center, and the rest area has an evenly distributed density

$$u_0(\phi, X) = \begin{cases} 1000 \cdot \exp(-(x_1^2 + x_2^2)) & \phi = -\pi, 0 \\ 100 & \text{otherwise} \end{cases} \tag{1.1}$$

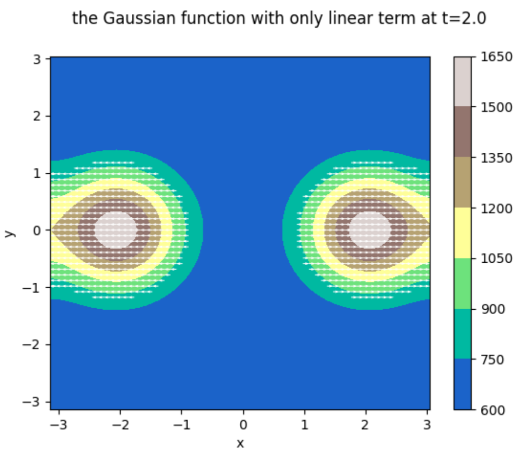
If there is no interaction term in effect, the model is linear and the Gaussian bump is moving constantly toward the angle ϕ at speed $\gamma = 1$, as shown in Figure 1.



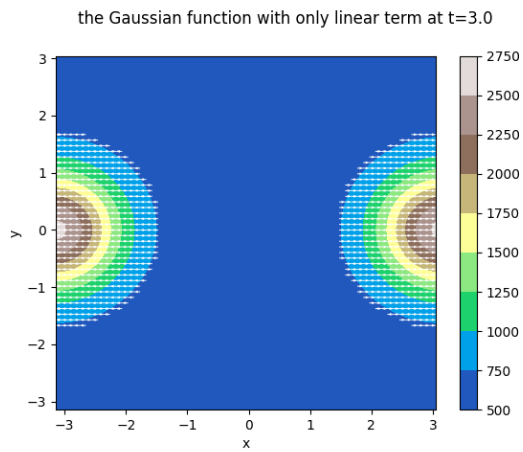
(a) when $t = 0$



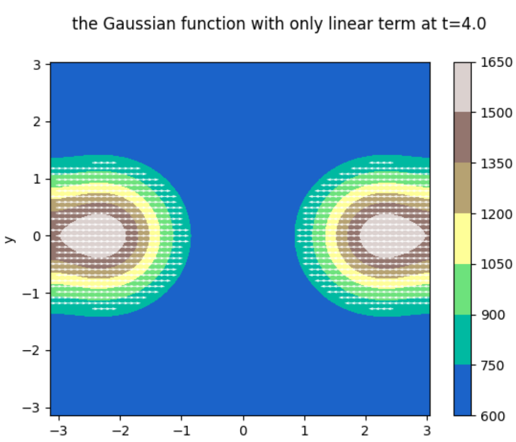
(b) when $t = 1$



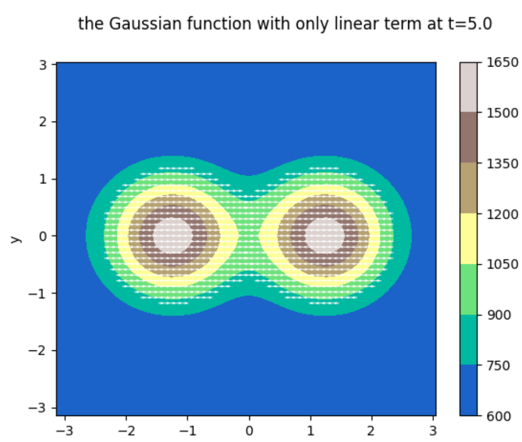
(c) when $t = 2$



(d) when $t = 3$



(e) when $t = 4$



(f) when $t = 5$

Figure 1: the Total Population Density U at different time t with $q_a = 2, q_r = 0.5$

Then, the interaction term with saturation function is added to the model. For the interaction parameters, we set attraction to be dominant, where $q_a = 2, m_a = 0.3, d_a = 0.5$; for the repulsive kernels, we set $q_r = 0.5, m_r = 0.2, d_r = 0$. Then we apply saturation function to the interaction term. In this case, we applied

$$S(T) = \frac{5}{2}(1 + \tanh(\frac{T-5}{1.2})) \tag{1.2}$$

where the maximum value $H = 5$, and centred at $\mu = 5$ with the width $\sigma = 1.2$. The plot of the saturation function is shown on Figure 3.

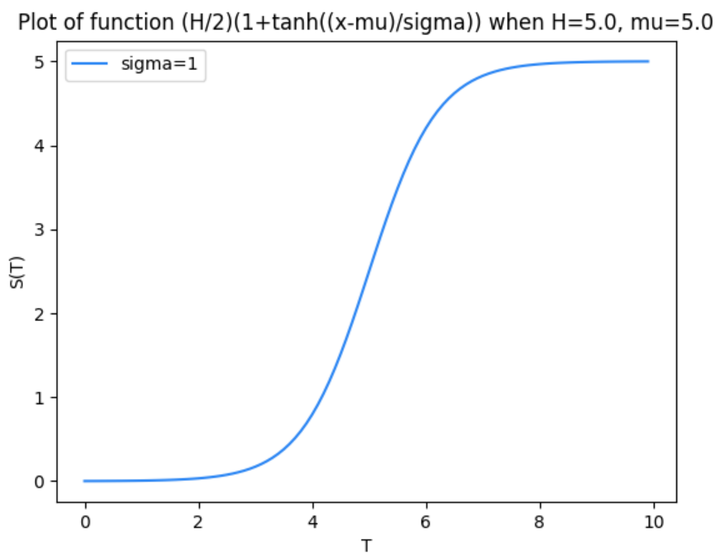
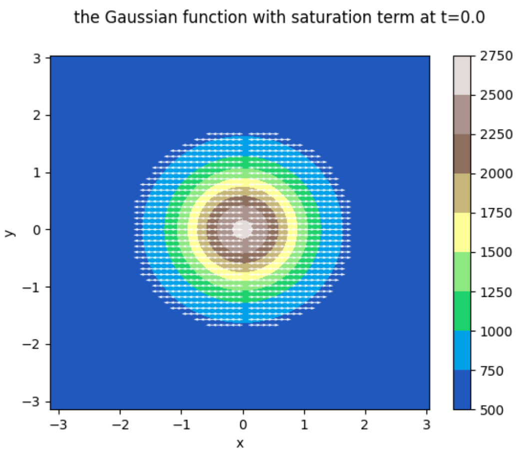
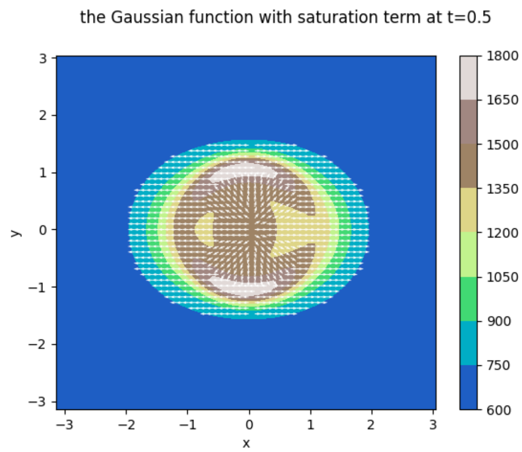


Figure 2: The Saturation Function applied to the Model

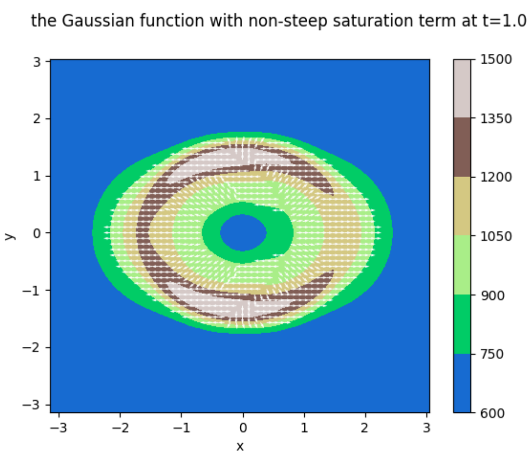
And then the model with saturated interaction term is shown below. From the figures, we can see that at the beginning when t is small, the model is interacting with its neighbours and changes its directions. But as t increased, the interaction term start to decrease due to the effect of saturation term.



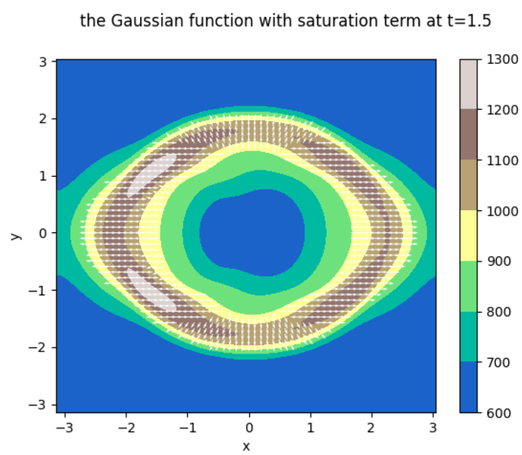
(a) when $t = 0$



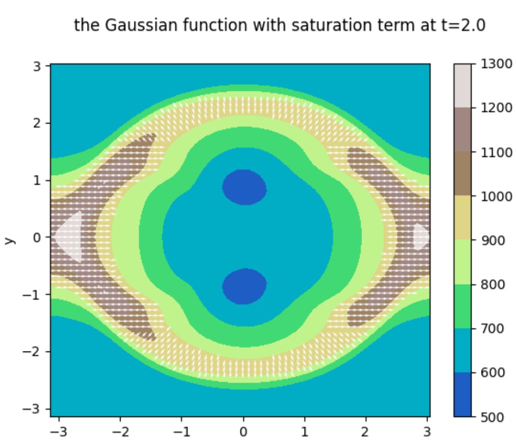
(b) when $t = 0.5$



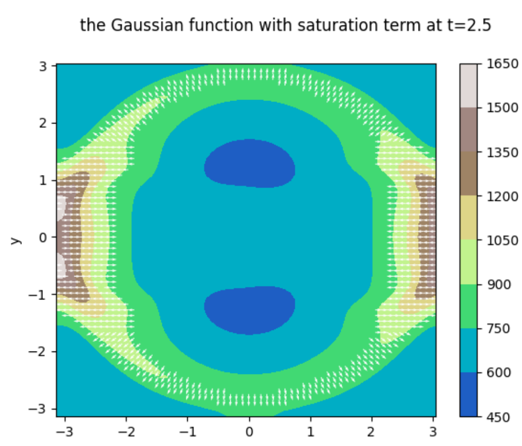
(c) when $t = 1$



(d) when $t = 1.5$



(e) when $t = 2$



(f) when $t = 2.5$

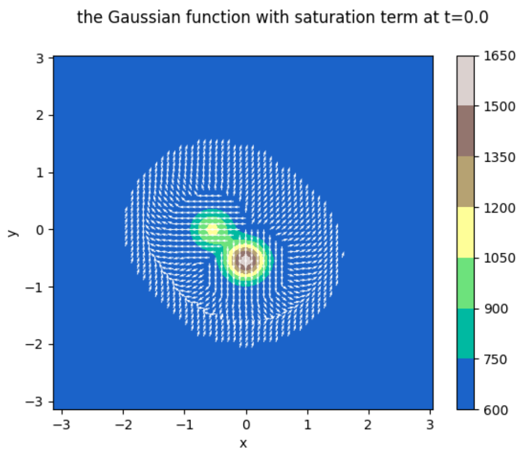
Figure 3: the Total Population Density U at different time t with $q_a = 2, q_r = 0.5$

1.2 Initial condition 2: Two Gaussian bumps

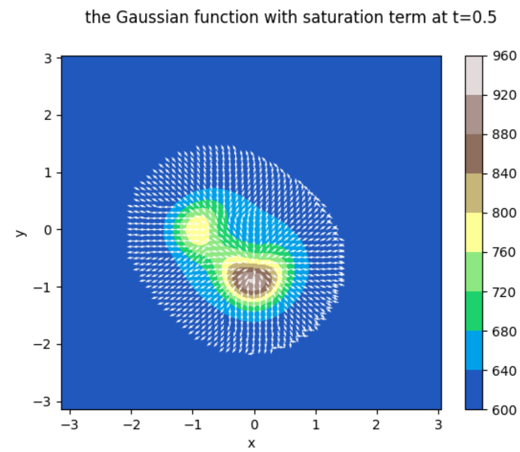
In this section, the initial condition has two Gaussian bumps with different mass at each bump, where

$$u_0(\phi, X) = u_0(\phi, X) = \begin{cases} 10000 \cdot \exp(-(10 \cdot ((x_2 + 0.5)^2 + x_1^2))) & \phi = -\frac{\pi}{2} \\ 5000 \cdot \exp(-(10 \cdot ((x_1 + 0.5)^2 + x_2^2))) & \phi = -\pi \\ 100 & \text{otherwise} \end{cases} \tag{1.3}$$

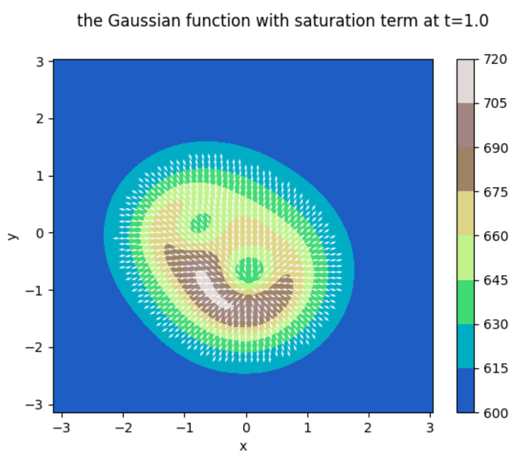
For the interaction parameters, we set the values to be the same as the previous example. Instead of applying the saturation function to the model, we first run the model without any saturation function. From Figure 4, we can see that at first, the repulsion term is dominant and the two bumps started to spread out in different angles. However, as time increases, although the bumps keep spreading out, there are still attraction force between the individuals such that they finally form one big bump instead of two bumps at the beginning.



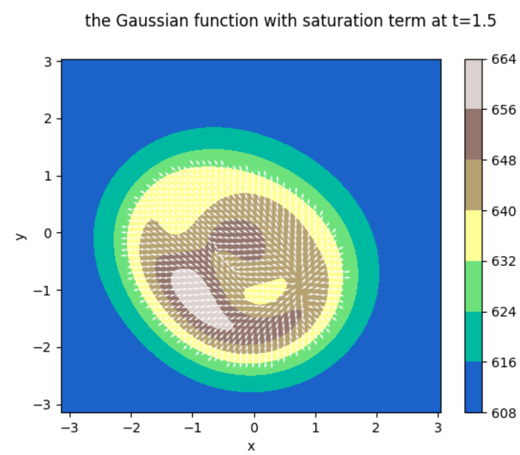
(a) when $t = 0$



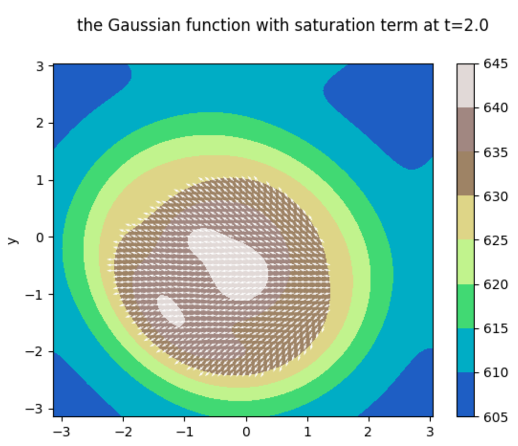
(b) when $t = 0.5$



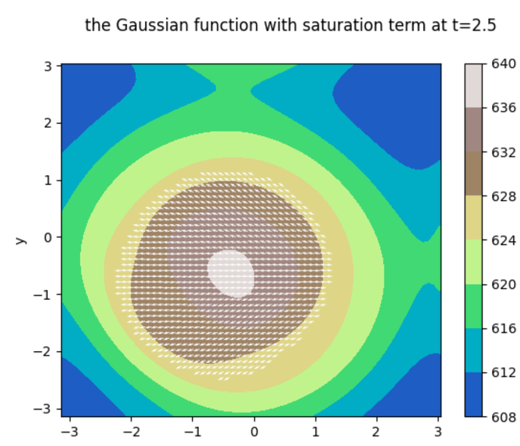
(c) when $t = 1$



(d) when $t = 1.5$



(e) when $t = 2$

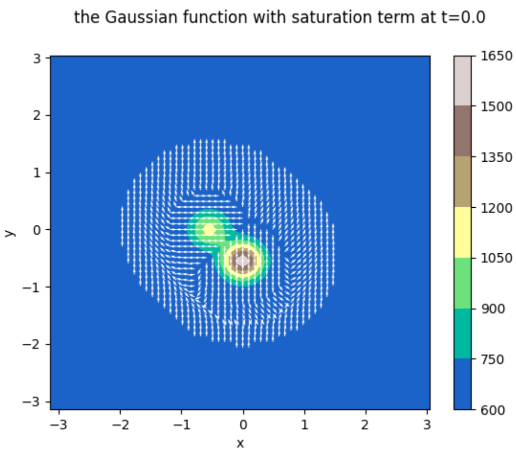


(f) when $t = 2.5$

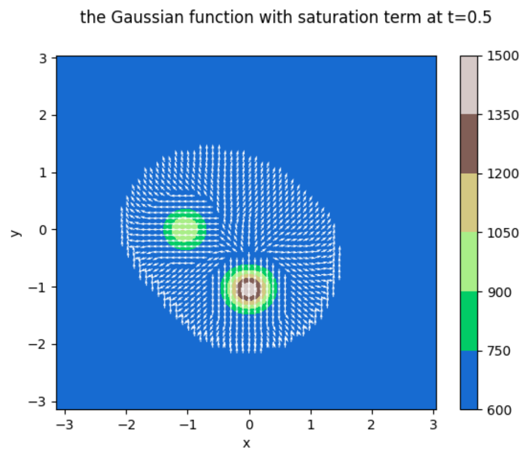
Figure 4: the Total Population Density U at different time t with no Saturation function with $q_a = 2, q_r = 0.5$

Now if we apply the same saturation function in (1.2), the result shows in Figure 5, which seems like there is only transportation term in effect with very little repulsive interaction force. The reason is that for this initial condition, the range of the turning function is $[0, 3.59]$.When we apply saturation function (1.2), most of the turning function values are in the minimum section and finally transform into values very close to zero.

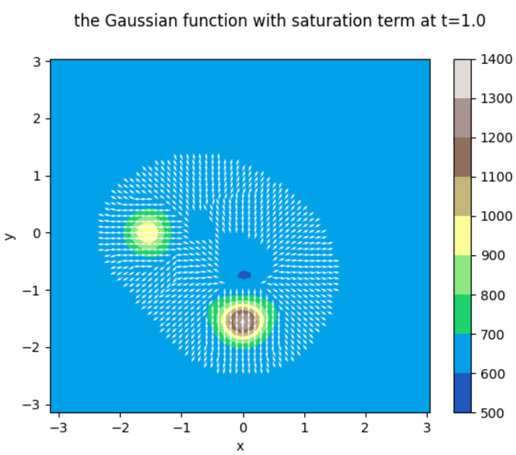
If we would like to establish a model with a similar behaviour shown in Figure 3, we can apply another saturation function such that the turning function still effects the model to interact with its neighbours.



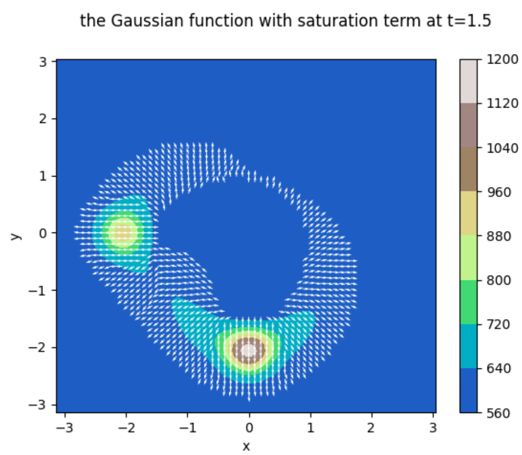
(a) when $t = 0$



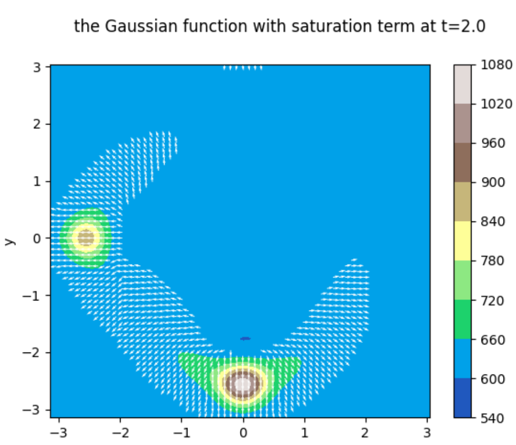
(b) when $t = 0.5$



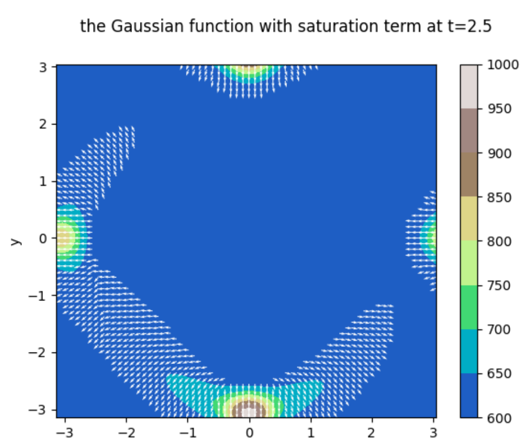
(c) when $t = 1$



(d) when $t = 1.5$



(e) when $t = 2$



(f) when $t = 2.5$

Figure 5: the Total Population Density U at different time t with Saturation function (1.2) with $q_a = 2, q_r = 0.5$

Therefore, we applied a new saturation function

$$S(T) = \frac{2}{2}(1 + \tanh(\frac{T-2}{1})) \tag{1.4}$$

which shown in Figure 6 with $H = 2, \mu = 2$ and $\sigma = 1$.

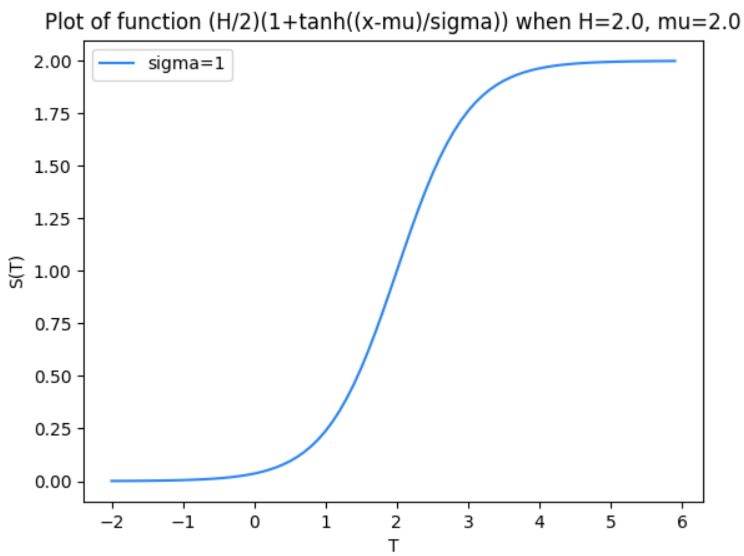
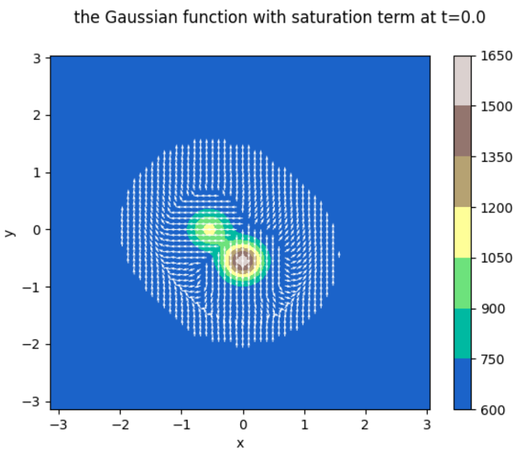
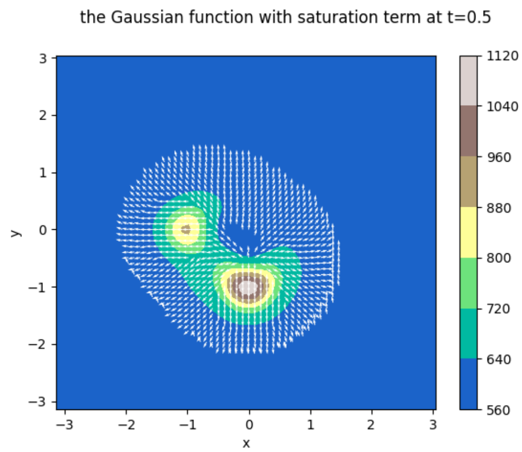


Figure 6: The Saturation Function applied to the Model

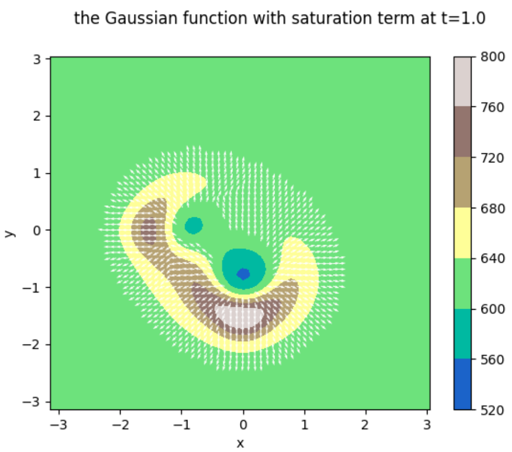
From the picture, we can see that for most values of the Turning function, it falls into the slope section in the saturation function. Also, with some small values of the turning functions, they trasform into values close to 0. If we apply the new saturation function to the model, the result is shown in Figure 7. From the plots, we can see that at the beginning, the two bumps are interacting with each other and tending to form one big bump. However, as t increase, the value of the turning function falls into the minimum section of the saturation function, and the individuals stopped interacting with each other but just move in their own direction.



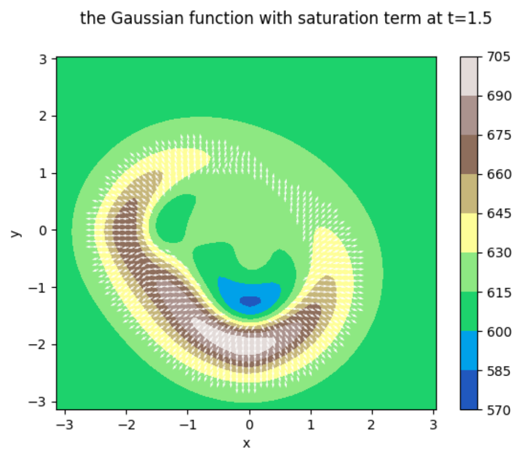
(a) when $t = 0$



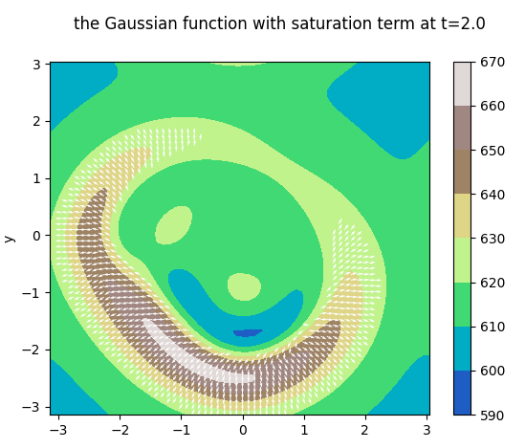
(b) when $t = 0.5$



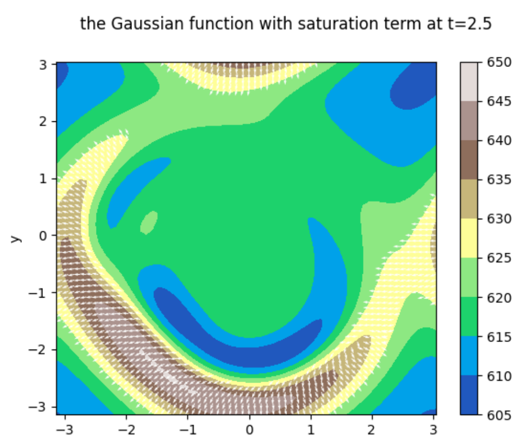
(c) when $t = 1$



(d) when $t = 1.5$



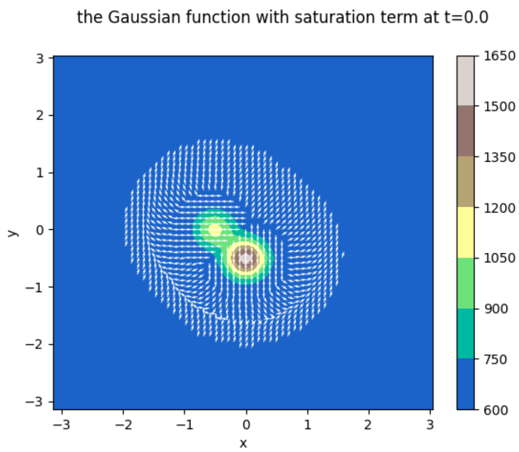
(e) when $t = 2$



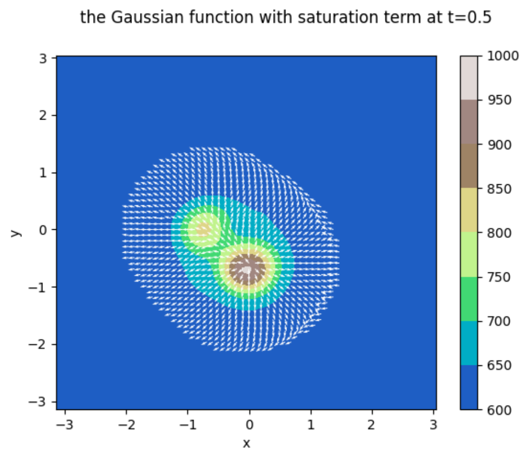
(f) when $t = 2.5$

Figure 7: the Total Population Density U at different time t with Saturation function (1.4) with $q_a = 2, q_r = 0.5$

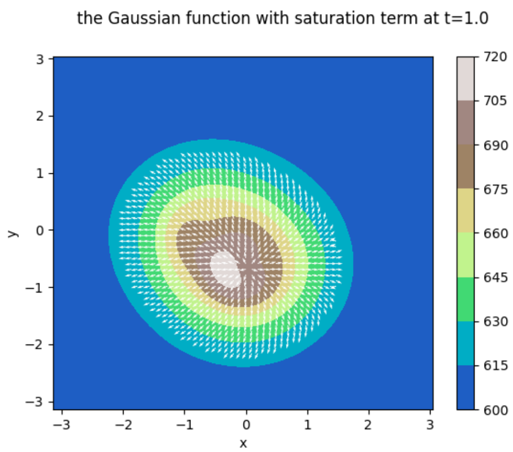
Changing the saturation function is not the only way to change the interaction in the system. Similarly, we can change the value of q_a, q_r to simulate the aggregation under different scenario. In the following test, we set $q_a = 4, q_r = 1$ which is proportional to the original q_a, q_r with the same initial condition. The results shown in Figure 8 states that the model starts with a similar spreading out motion as the results in Figure 4. Many individuals first changes their direction and then starts to aggregate and form a big bump between the two bumps. However, since the strength of the interaction forces is increased, the model does not only aggregate faster but also disperses more slowly. This can be shown by comparing Figure 4(c),(d) with Figure 8(c),(d). In Figure 4(c) and (d), there is an aggregation motion but we can still see some unevenly distributed area in the bump. Whereas in Figure 8(c) and (d), the model quickly forms a bump similar to a Gaussian bump and disperses more slowly.



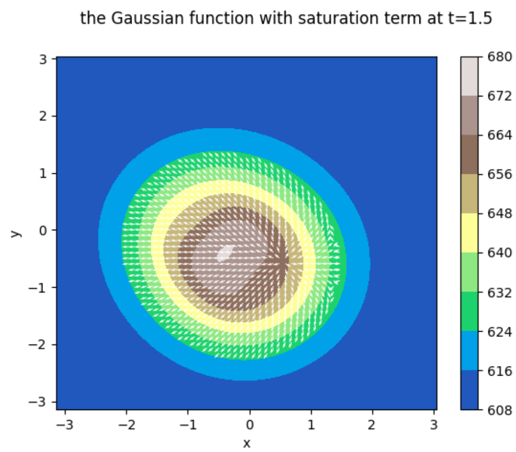
(a) when $t = 0$



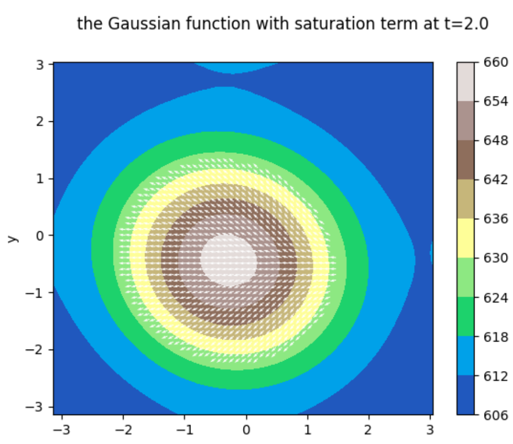
(b) when $t = 0.5$



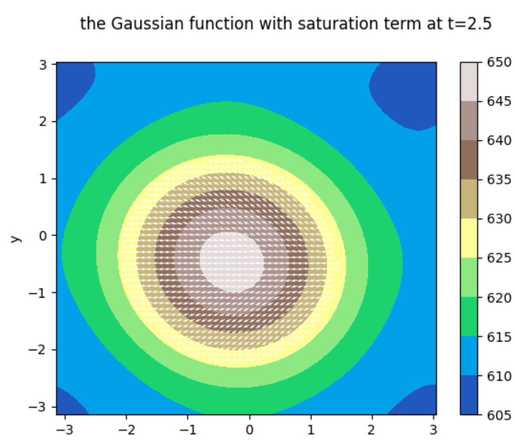
(c) when $t = 1$



(d) when $t = 1.5$



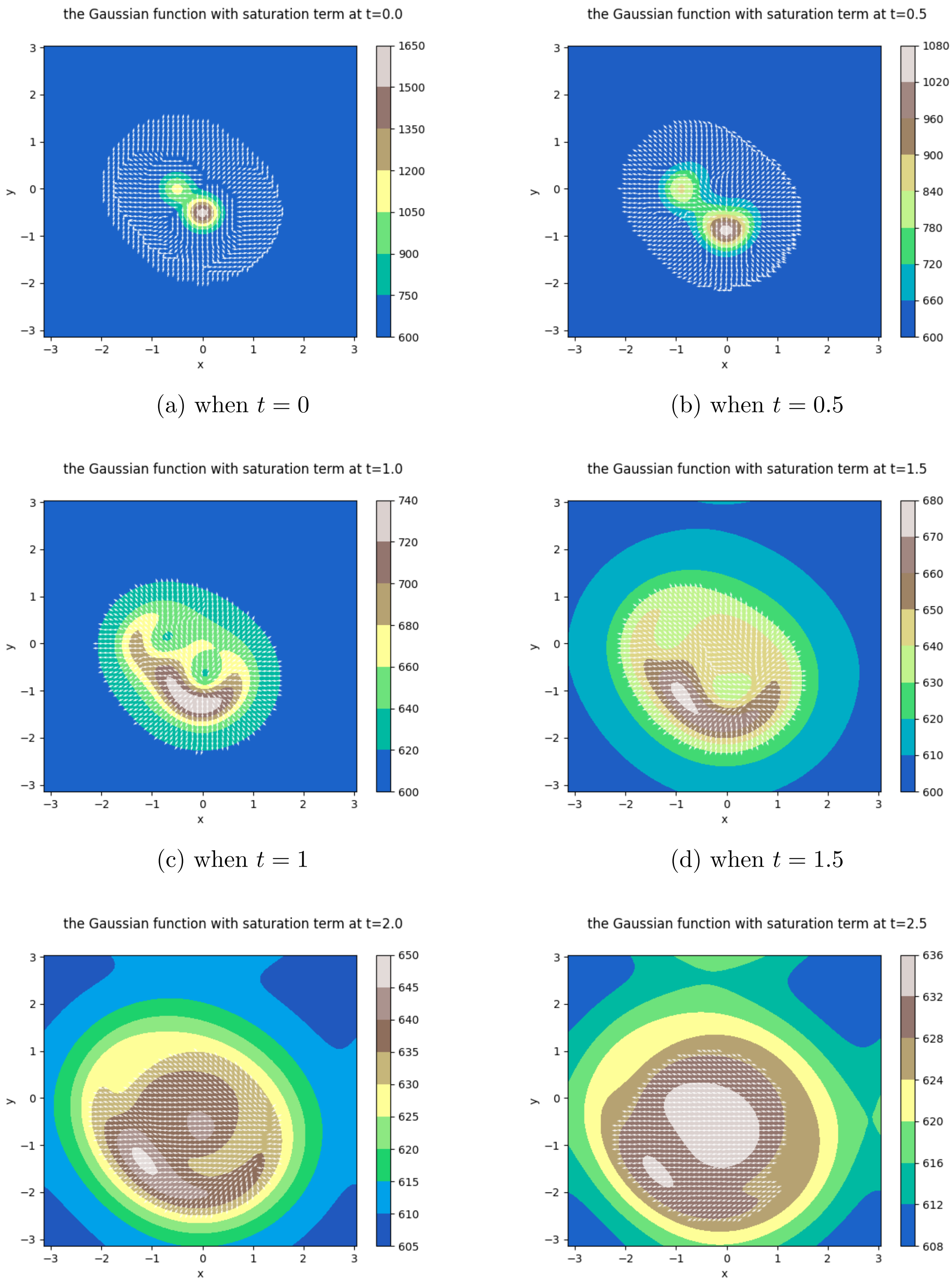
(e) when $t = 2$



(f) when $t = 2.5$

Figure 8: the Total Population Density U at different time t with no Saturation function with $q_a = 4, q_r = 1$

Now if we apply the same saturation function in (1.4) to this model, the result is shown in Figure 9. From the plots, it is interesting to see that with the saturation term applied, the model behaves similar to the model in Figure 4, where no saturation function is applied with smaller q_a, q_r. One important reason is that with the increase of the q_a, q_r, the magnitude of turning function also increases. In this example, the turning function is in the range of $[0, 7.17]$. Therefore, if (1.4) is applied, most of the turning function values falls into the maximun section of the saturation function. Then the maximum magnitude of the turning function would be decreased to 2, which is a value closer to the turning function magnitude when $q_a = 2, q_r = 0.5$.



(a) when $t = 0$

(b) when $t = 0.5$

(c) when $t = 1$

(d) when $t = 1.5$

(e) when $t = 2$

(f) when $t = 2.5$

Figure 9: the Total Population Density U at different time t with Saturation function (1.4) with $q_a = 4, q_r = 1$

However, if we compare Figure 9 and Figure 7, where both cases have the same saturation function and initial function applied, the outcome is distinct. In Figure 9, the interaction term still has relatively significant influences on the model as t increases, but in Figure 7, especially in subplots (e) and (f), the influence of the interaction term has been largely decreased and the transportation term has the main effects on the model. The main reason is that at $t = 2$ and $t = 2.5$, the maximum of the tuning function magnitude is still greater than 2 in the case shown in Figure 9. Therefore, we can apply another saturation function (1.5) to the model such that when the values of the turning function decreases, it will be transformed into values closer to 0 by the saturation function.

$$S(T) = \frac{6}{2}(1 + \tanh(\frac{T-6}{1})) \tag{1.5}$$

where $H = 6$, $\mu = 6$ and $\sigma = 1$ with the plot shown in Figure 10.

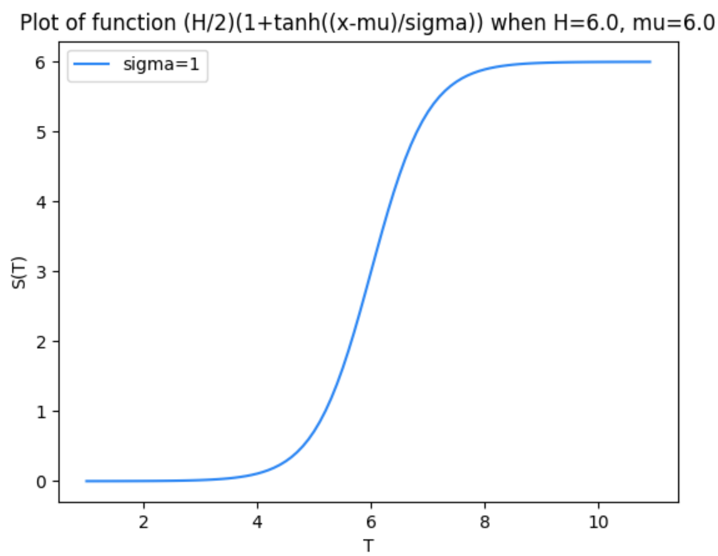
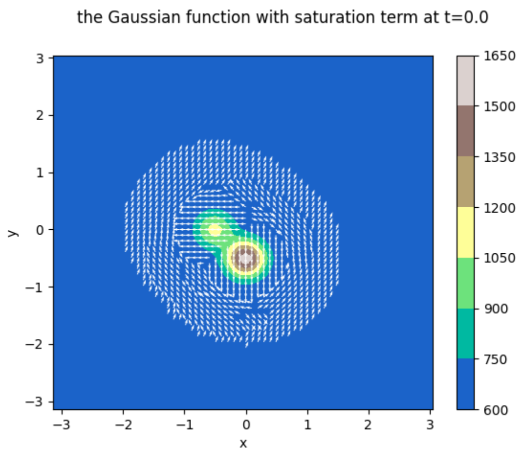
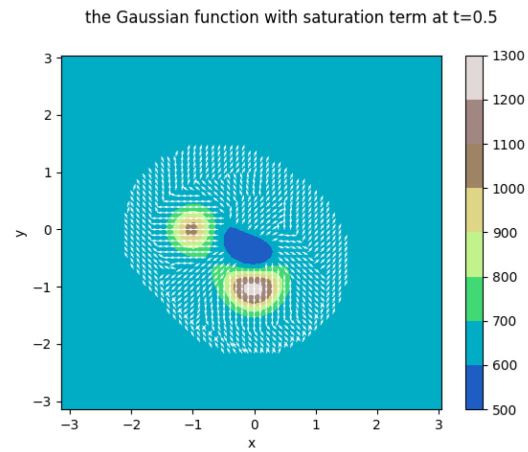


Figure 10: The Saturation Function applied to the Model

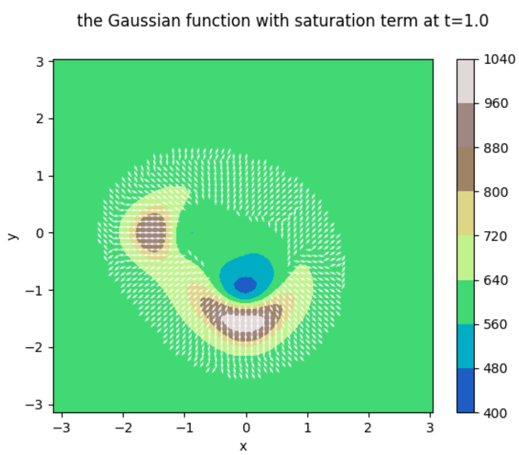
In Figure 11, we can see that in subfigure (a), (b) and (c), there are still some interaction between the two bumps, but as t increases, the interaction force decreases and the transportation term starts to dominate the motion. This behaviour is similar to Figure 7 but with different saturation function applied due to different interaction force strength.



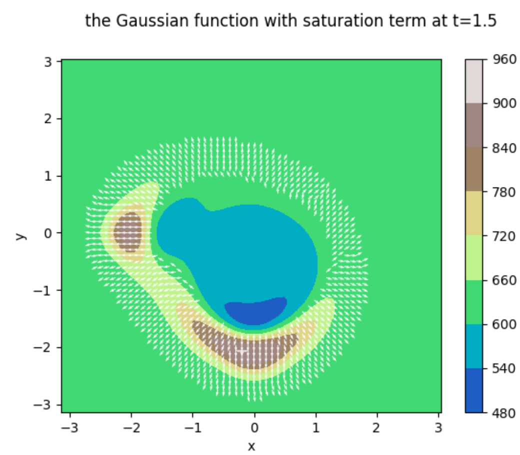
(a) when $t = 0$



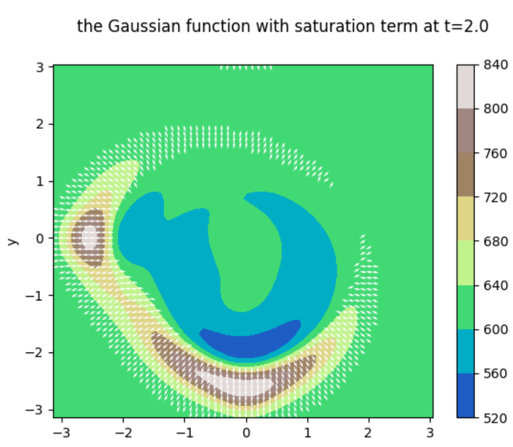
(b) when $t = 0.5$



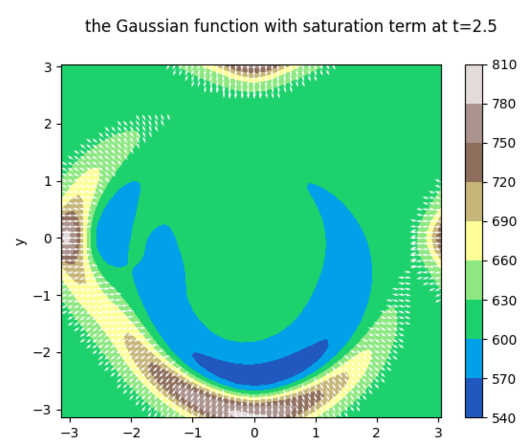
(c) when $t = 1$



(d) when $t = 1.5$



(e) when $t = 2$



(f) when $t = 2.5$

Figure 11: the Total Population Density U at different time t with Saturation function (1.5) with $q_a = 4, q_r = 1$

We can also apply another saturation function to the system with the same value of $H = 6$ and $\mu = 6$, but different width $\sigma = 2.5$.

$$S(T) = \frac{6}{2}(1 + \tanh(\frac{T-6}{2.5})) \tag{1.6}$$

As shown in Figure 12, the slope section is less steep than the previous saturation function in (1.5).

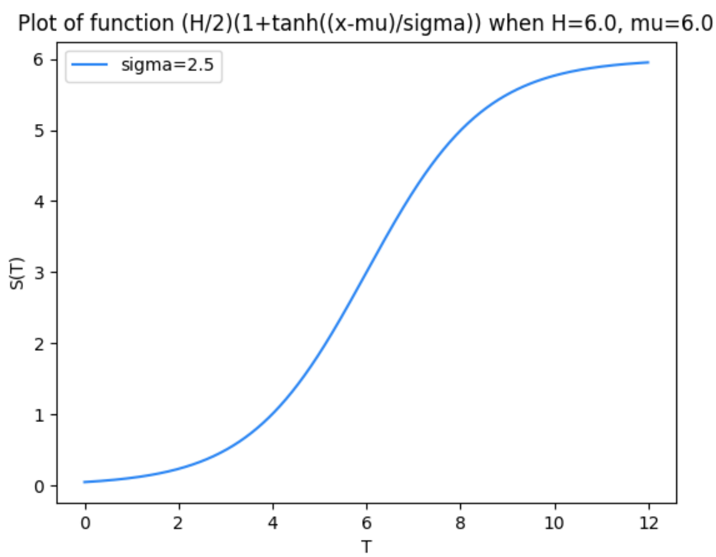
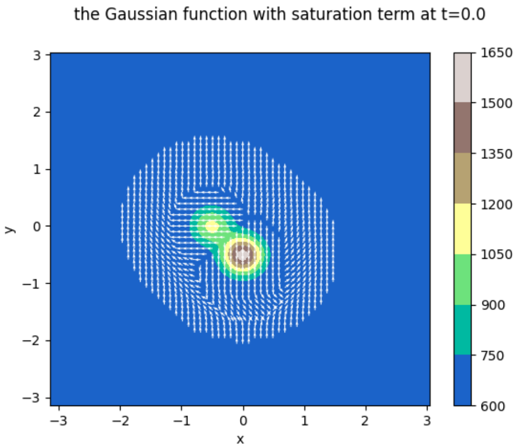
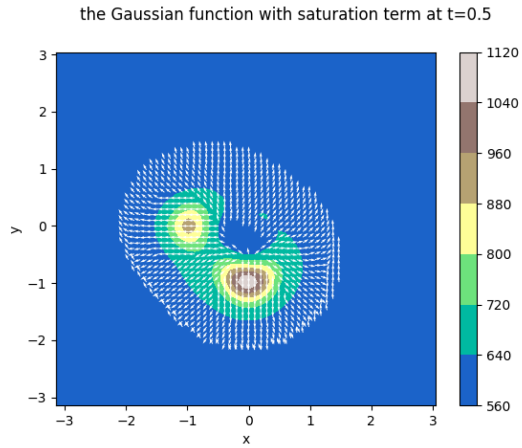


Figure 12: The Saturation Function applied to the Model

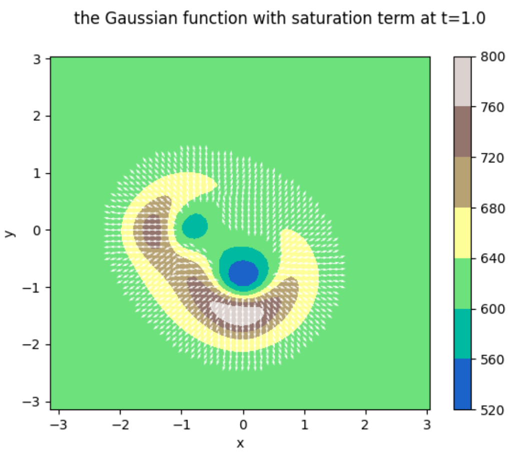
As the new saturation function is applied to the model as shown in Figure 13, it demonstrates that as t increases, although the transportation term dominates the model, the interaction forces still have some observable influences to the model. In the subplot (e) and (f), it forms one peak between the original two bumps whereas in Figure 11, there are always two peaks at $-\pi$ and $-\frac{\pi}{2}$ direction. The main reason is that since the width μ in (1.6) is greater than the μ in (1.5), then the minimum section of the saturation function is thinner, which transform less values in the Turning function to a value that is closer to zero. Therefore, as t increases, the turning function is still in effect and some individuals are still interacting with its neighbours.



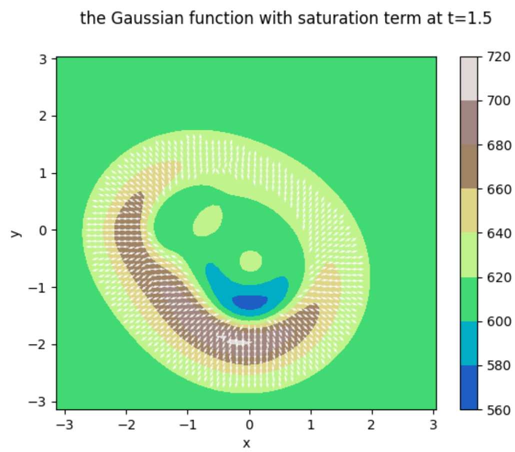
(a) when $t = 0$



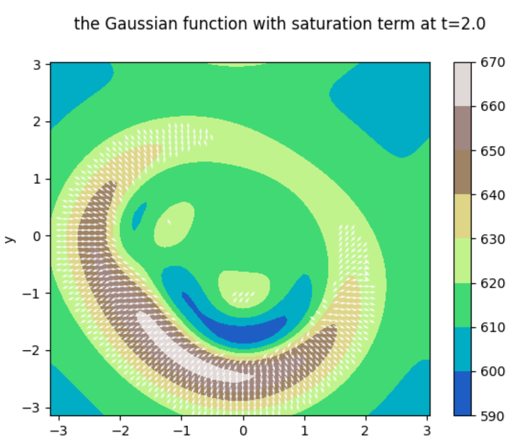
(b) when $t = 0.5$



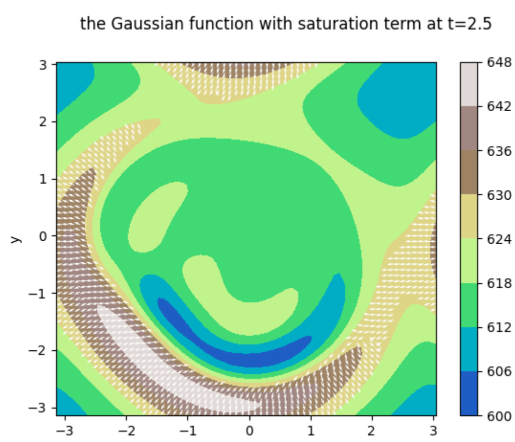
(c) when $t = 1$



(d) when $t = 1.5$



(e) when $t = 2$



(f) when $t = 2.5$

Figure 13: the Total Population Density U at different time t with Saturation function (1.6) with $q_a = 4, q_r = 1$

2 Conclusion

In a 2-dimensional aggregation model, the non-linear interaction part plays a significant role to the behaviour of the aggregation motion. The interaction part mainly states the interaction between the individual and the position of their neighbours for attractive and repulsive motions. In Fetecau's model, they changes many different parameters to show the different cases of aggregation model. For instance, they had adjusted the κ_j value in the probability function $w_j(\phi, \phi', X - S)$ to change weight of the kernel for an individual to change from it's original orientation ϕ' to an new angle ϕ.

Another way is to directly change the q_j term which represents the strength of attractive or repulsive interaction. In the same system, even if the proportion between q_a and q_r remains the same, the outcome is still different. When the values of q_a and q_r are different, the behaviour of the systems might be similar to each other, but the strength of attractive or repulsive force would vary depends on q_j. If a system is attraction dominated, greater q_j values would result in a faster aggregation behaviour of the model.

In our system, we added another factor, which is the saturation function $S(x)$, to the turning function in the interaction term, to introduce another way to change the behaviour of our model. The saturation function $S(x)$ has three parameters: the maximal height H, the midpoint μ and width σ.

There are three cases that would happen if we adapt a saturation function to the model, and all three cases are related to the range of turning function values. The first case is when most values of the turning function are placed in the maximum section of the saturation term. Then the system behaviour are highly dependent on the maximum value of the saturation function. The second case is when most of the range of the turning function falls into the minimum section, where most of turning function values are transferred to values that are really close to zero. In this case, many individuals in the system do not interact with their neighbours but maintain the same angle over time. The third case is when most of the turning function values are in the slope section of the saturation function. In this case, the behaviour of the system would depend on the value of width μ in the saturation function. If μ is relatively small, it means that more turning values would transform into the maximum value H and minimum value 0. If μ becomes larger, then the interaction term will have a greater influence to the model.

For this study, there are also some parts that can be further developed. For instance, in some species, there are also alignment interaction force which let each individual to turn to the same angle as their neighbours. In that case, not only the position of the neighbours, but the angles of neighbours would be considered in the system.

The saturation function can also be further developed. In this study, we used hyperbolic tangent function as the saturation function. There are also other increasing and bounded functions which can be applied to the system to simulate other scenarios for the interaction between different organisms.