Fetecau Model Part 1

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1 Introduction

1D model:

$$\partial_t u^+ + \partial_x(\gamma u^+) = -\lambda^+[u^+, u^-] + \lambda^-[u^+, u^-]u^-$$

$$\partial_t u^- + \partial_x(\gamma u^-) = -\lambda^+[u^+, u^-]\lambda^-[u^+, u^-]u^-$$

$\mathbf{u}^+$: right moving density of individual $\quad$ $\mathbf{u}_-$: left moving density of individual

$\boldsymbol{\gamma}$: constant speed $\quad$ $\boldsymbol{\lambda^+(\lambda_-)}$:turning rate of original at right(left) to the left(right)

2 Model Description

2D model:

$$\partial_t u + \gamma \boldsymbol{e}_\phi \cdot \nabla_x u = -\lambda(\boldsymbol{x}, \phi)u + \int_{-\pi}^{\pi} T(\boldsymbol{x}, \phi)u(\boldsymbol{x}, \phi^{'}, t)d\phi^{'}$$

$u(\boldsymbol{x}, \phi^{'}, t)$ is the density of individuals at $\boldsymbol{x} = (x, y)$moving in the direction $\phi(-\pi, \pi])$

$\boldsymbol{\gamma}$: constant speed in the direction $\boldsymbol{e}_{/phi} = (cos\phi, sin\phi)$

$\lambda(\boldsymbol{x}, \phi)$: reorientation rate of individuals at $(\boldsymbol{x}, \phi)$

$T(\boldsymbol{x}, \phi)$: the rate of individuals at x that changes their direction from $\phi^{'}$to ϕ

$$\lambda(\boldsymbol{x}, \phi) = \int_{-\pi}^{\pi} T(\boldsymbol{x}, \phi)u(\boldsymbol{x}, \phi^{'}, t)d\phi^{'}$$

Assumptions on $\lambda(\boldsymbol{x}, \phi)$ and $T(\boldsymbol{x}, \phi^{'}, \phi)$:

1. Distance from neighbour

Distance kernel: what is distance kernel?

$$K_j^d(\boldsymbol{x}) = \frac{1}{A_j} e^{-(\sqrt{x^2+y^2}-d_j)^2/m_j^2}$$

with $j = r, al, a$ and d_j represents the ranges.

A_j ensures that each kernel integrates to 1:

$$A_j = \pi m_j (m_j e^{-d_j^2/m_j^2} + \sqrt{\pi} d_j + \sqrt{\pi} d_j erf(d_j/m_j))$$

where erf is error rate function.

Figure 1b: superposition?

2. Neighbour's orientation

Can be described using rotation kernels dependent on the:

reference individual: movement direction ϕ and position $\boldsymbol{x}$

reference individual's neighbour: movement direction θ and position $\boldsymbol{s}$

Modeling the turning rate λ:

q_j with $j = al, a, r$is a constant represents the strength of alignment, attraction and repulsion.

Alignment:

Less likely to change direction if their neighbours are going with the **same direction**.

More likely to turn if their neighbours are going with **opposite direction**

$$K_{al}^o(\theta; \phi) = \frac{1}{2\pi}(-cos(\phi - \theta) + 1)$$

If θ is very close to ϕ, then $cos(\phi - \theta)$ will be closer to 1, and $(-cos(\phi - \theta) + 1)$ will become 0. $K_{al}^o(\theta; \phi) = 0$

If θ is completely opposite to ϕ, then $cos(\phi - \theta)$ will be closer to -1, and $(-cos(\phi - \theta) + 1)$ will become 2. $K_{al}^o(\theta; \phi) = 1/\pi$

o and d represents distance kernel and rotation kernel.

$$\lambda_{al}(\boldsymbol{x}, \phi) = q_{al} \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} K_{al}^d(\boldsymbol{x} - \boldsymbol{s}) K_{al}^o(\theta; \phi) u(\boldsymbol{s}, \theta, t) d\theta d\boldsymbol{s}$$

Attraction:

Rotation mechanism depend on $\boldsymbol{x}$and $\boldsymbol{s}$ positions. Let $\boldsymbol{s}$ - $\boldsymbol{x}$=(s_x, s_y), then we can get ψ by

$$cos(\psi) = \frac{s_x}{\sqrt{s_x^2 + s_y^2}}, sin(\psi) = \frac{s_y}{\sqrt{s_x^2 + s_y^2}}$$

then we get

$$K_a^o(\boldsymbol{s};\boldsymbol{x},\phi)=\frac{1}{2\pi}(-cos(\phi-\psi)+1)$$

If ψ is very close to ϕ, then $cos(\phi-\psi)$ will be closer to 1, and $(-cos(\phi-\psi)+1)$ will become 0. $K_a^o(\boldsymbol{s};\boldsymbol{x},\phi)=0$

If ψ is completely opposite to ϕ, then $cos(\phi-\psi)$ will be closer to -1, and $(-cos(\phi-\psi)+1)$ will become 2. $K_a^o(\boldsymbol{s};\boldsymbol{x},\phi)=1/\pi$

$$\lambda_a(\boldsymbol{x},\phi)=q_a\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}K_a^d(\boldsymbol{x}-\boldsymbol{s})K_a^o(\boldsymbol{s};\boldsymbol{x},\phi)u(\boldsymbol{s},\theta,t)d\theta d\boldsymbol{s}$$

Repulsion:

Totally opposite to the attraction pattern where

$$K_r^o(\boldsymbol{s};\boldsymbol{x},\phi)=\frac{1}{2\pi}(cos(\phi-\psi)+1)$$

where when the angle is closer, the more likely they are going to turn, an vice versa.

$$\lambda_r(\boldsymbol{x},\phi)=q_r\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}K_r^d(\boldsymbol{x}-\boldsymbol{s})K_r^o(\boldsymbol{s};\boldsymbol{x},\phi)u(\boldsymbol{s},\theta,t)d\theta d\boldsymbol{s}$$

Sum All Together

$$\lambda(\boldsymbol{x},\phi)=\lambda_{al}(\boldsymbol{x},\phi)+\lambda_a(\boldsymbol{x},\phi)+\lambda_r(\boldsymbol{x},\phi)$$

Modeling the reorientation terms

$T(\boldsymbol{x},\phi^{'},\phi)$:

Turning probability function: $\omega(\phi^{'}-\phi,\phi^{'}-\theta)$.

ω represents the possibility of turning rate from direction $\phi^{'}$ to ϕ as a result of interactions with individuals located(moving) at direction θ. And we get reorientation terms similar to the turning rate.

$$T_{al}(\boldsymbol{x},\phi^{'},\phi)=q_{al}\int_{\mathbb{R}^2}\int_{\pi}^{\pi}K_{al}^d(\boldsymbol{x}-\boldsymbol{s})K_{al}^o(\theta;\phi)\omega_{al}(\phi^{'}-\phi,\phi^{'}-\theta)u(\boldsymbol{s},\theta,t)d\theta d\boldsymbol{s}$$

$$T_a(\boldsymbol{x},\phi^{'},\phi)=q_a\int_{\mathbb{R}^2}\int_{\pi}^{\pi}K_a^d(\boldsymbol{x}-\boldsymbol{s})K_a^o(\boldsymbol{s};\boldsymbol{x},\phi^{'})\omega_a(\phi^{'}-\phi,\phi^{'}-\theta)u(\boldsymbol{s},\theta,t)d\theta d\boldsymbol{s}$$

$$T_r(\boldsymbol{x}, \phi', \phi) = q_r \int_{\mathbb{R}^2} \int_{\pi}^{\pi} K_r^d(\boldsymbol{x} - \boldsymbol{s}) K_r^o(\boldsymbol{s}; \boldsymbol{x}, \phi') \omega_r(\phi' - \phi, \phi' - \theta) u(\boldsymbol{s}, \theta, t) d\theta d\boldsymbol{s}$$

And similar to the turning rate λ, the total term of T is:

$$T(\boldsymbol{x}, \phi', \phi) = T_{al}(\boldsymbol{x}, \phi', \phi) + T_a(\boldsymbol{x}, \phi', \phi) + T_r(\boldsymbol{x}, \phi', \phi)$$

What is the reorientation term? What does this term represent?

Modeling the probability function ω_{al}, ω_a and ω_r

$$\omega(\phi' - \phi, \phi' - \theta) = g_\sigma(\phi' - \phi - v(\phi' - \theta))$$

g_σ is an approximation of the delta function with width σ

v is the turning function with highest possibility of an individual moving in the direction ϕ' turns to $\phi = \phi' - v(\phi' - \theta)$ **What is turning function?**

$\sigma > 0$ is the uncertainty of turning.

Possible expressions for g_σ are a **periodic Gaussian function** or a **Normalized step function**:

$$g_\sigma(\theta) = \frac{1}{\sqrt{\pi}\sigma} \sum_{z \in \mathbb{Z}} e^{-\left(\frac{\theta + 2\pi z}{\sigma}\right)^2}, \theta \in (-\pi, \pi)$$

what is z above?

$$g_\sigma = \begin{cases} \frac{1}{2\sigma}, |\theta| < \sigma \\ 0\sigma < |\theta| <= \pi \end{cases} \tag{1}$$

Turning function v:

$$v(\theta) = \kappa sin(\theta)$$

$$v(\theta) = \kappa\theta), -1 <= \kappa <= 1$$

Sign of κ is very important. if κ is negative, then it is **repulsive turning**, if κ is positive, then it is **attracting-like turning**.

3 Numerical results

Use Fourier Methods to calculate the convolution integral for space and angle.

What does it mean by "performs a multiplication in the discrete Fourier transform"?

$$\widehat{K * u(l)} = \hat{K}(l)\hat{u}(l)$$

Apply 2D Fourier transform to $\gamma \boldsymbol{e}_{\phi} \cdot \nabla_x u$ and then obtain

$$\gamma(cos\phi l_1 + sin\phi l_2)\hat{u}$$

where $l_1 and l_2$ are horizontal and vertical components of the wave number.

Use the **4th Runge-Kutta Method** to solve for the numerical solution.

Space discretization: rectangular grid on $[-L/2, L/2) \times [-L/2, L/2)$ with N^2 points, with $\Delta x = \Delta y = L/N$

Angle discretization: equidistant grid on $[-\pi, \pi)$ with M points, $\Delta\phi = 2 * \pi/M$

To avoid aliasing, all multiplications of Fourier modes are done on an extended spatial grid of size $(\frac{3}{2}N)^2$ and an angular gird of size $\frac{3}{2}M$.

Discrete Convolutions to compute turning rates

The integral defining λ_{al} is trivial to compute its discrete Fourier Spectrum.

The integrals that define λ_a and λ_r are similar and can represent a convolution in space only. What we get is

$$\lambda_r(\boldsymbol{x}, \phi) = q_r \int_{\mathbb{R}^2} K_r^d(\boldsymbol{x} - \boldsymbol{s}) K_r^o(\boldsymbol{s}; \boldsymbol{x}, \phi) \int_{-\pi}^{\pi} u(\boldsymbol{s}, \theta, t) d\theta d\boldsymbol{s}$$

The θ integral is the zero mode of u, and the remaining space integral is a convolution.

Calculation of the reorientation terms included in T

We can change $T_a(\boldsymbol{x}, \phi', \phi)$ and $T_r(\boldsymbol{x}, \phi', \phi)$ in space only. For every ϕ and ϕ' fixed, then we get

$$\omega_a(\phi - \phi', \phi' - \psi) = g\sigma_a(\phi' - \phi - \kappa_a sin(\phi' - \psi))$$

$$= g\sigma_a(\phi' - \phi - \kappa_a(sin\phi' cos\psi - cos\phi' sin\psi)$$

$$= g\sigma_a(\phi' - \phi - \kappa_a(sin\phi' \frac{s_x}{s_x^2 + s_y^2} - cos\phi' \frac{s_y}{s_y^2 + s_x^2}))$$

which is a function of $\boldsymbol{x} - \boldsymbol{s}$, which means it is a convolution is the space only.