Note on the Turning function and Non-linear term

Yuelin Qi

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1 What is the Non-linear Term

$$\int_{\phi'=-\pi}^{\pi} S\left(T\left(x, \phi', \phi\right)\right) u\left(x, \phi', t\right) \mathrm{d}\phi' \tag{1.1}$$

S: saturation function

$T(x, \phi, \phi')$ is the rate of individuals at position $\vec{x}$ changes their direction from ϕ' to ϕ.

The turning rate function

$$T(x, \phi, \phi') = T_{al}(x, \phi, \phi') + T_r(x, \phi, \phi') + T_a(x, \phi, \phi') \tag{1.2}$$

The turning rate function under **Alignment, Repulsion and Attraction** will be listed in the following subsection.

The functions inside of each turning function is:

1. Distance Kernel

$$K_j^d(\boldsymbol{x}) = \frac{1}{A_j} e^{-\left(\sqrt{x^2+y^2}-d_j\right)^2 / m_j^2}$$

with $j = r, al, a$ and d_j describe the repulsion (j = r), alignment (j = al) and attraction (j = a) interaction ranges, and m_j gives the width of these ranges. A_j ensures that each kernel integrates to 1 :

$$A_j = \pi m_j \left(m_j e^{-d_j^2/m_j^2} + \sqrt{\pi} d_j + \sqrt{\pi} d_j erf\left(d_j/m_j\right)\right)$$

where erf is error rate function.

2. Orientation Kernel

For alignment, the K_{al}^o is:

$$K_{al}^o(\theta;\phi) = \frac{1}{2\pi}(-\cos(\phi-\theta)+1)$$

For the attraction and repulsion, the Kernel is:

$$K_a^o(\mathbf{s};\mathbf{x},\phi) = \frac{1}{2\pi}(-\cos(\phi-\psi)+1)$$
$$K_r^o(\mathbf{s};\mathbf{x},\phi) = \frac{1}{2\pi}(\cos(\phi-\psi)+1)$$

3. Possibility function

$$w\left(\phi'-\phi,\phi'-\theta\right) = g_\sigma\left(\phi'-\phi-v\left(\phi'-\theta\right)\right)$$

where g_σ is a periodic Gaussian or a normalized step function:

$$g_\sigma(\theta) = \frac{1}{\sqrt{\pi}\sigma}\sum_{z\in\mathbb{Z}} e^{-\left(\frac{\theta+2\pi z}{\sigma}\right)^2}, \quad \theta\in(-\pi,\pi),$$

$$g_\sigma(\theta) = \begin{cases} \frac{1}{2\sigma} & |\theta|<\sigma \\ 0 & \sigma<|\theta|\leq\pi \end{cases}$$

with the turning function v equals to:

$$v(\theta) = \kappa\sin\theta$$
$$v(\theta) = \kappa\theta$$

1.1 Alignment

The equation for T_{al} is:

$$T_{al}(x,\phi,\phi') = q_{al}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi} K_{al}^d(\vec{x}-\vec{s})K_{al}^o(\phi-\theta)w_{al}(\phi'-\phi,\phi'-\theta)u(\vec{s},\theta)d\theta d\vec{s} \tag{1.1.1}$$

Where

θ : neighbour's moving direction $\qquad$ s : neighbour's position

ϕ : original angle $\qquad$ ϕ' : new angle

ω : possibility function of turning rate from direction ϕ' to ϕ as a result of interactions with individuals moving at direction θ

Before start apply Fourier Transform to the function, the wave number are explained below:

1. k_1: wave number for ϕ.
2. k_2: wave number for ϕ'.
3. k_3: wave number for x_1.
4. k_4: wave number for x_2.

Let's started with direction kernel first:

$$T_{al}(x,\phi,\phi') = q_{al}\int_{\mathbb{R}^2} K^d_{al}(\vec{x}-\vec{s})\int_{-\pi}^{\pi} K^o_{al}(\phi-\theta)w_{al}(\phi'-\phi,\phi'-\theta)u(\vec{s},\theta)d\theta d\vec{s} \tag{1.1.2}$$

Then take out the part:

$$\int_{-\pi}^{\pi} K^o_{al}(\phi-\theta)w_{al}(\phi'-\phi,\phi'-\theta)u(\vec{s},\theta)d\theta \tag{1.1.3}$$

Then if we would like to find the Fourier transform of 1.1.3, over ϕ and ϕ' we can get:

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K^o_{al}(\phi-\theta)w_{al}(\phi'-\phi,\phi'-\theta)u(\vec{x},\theta)e^{-ik_1\phi}e^{-ik_2\phi'}d\theta d\phi d\phi' \tag{1.1.4}$$

Let $p_1 = \phi-\theta$, $p_2 = \phi'-\theta$, and $p_3 = \theta$.The Jacobian is

$$\begin{vmatrix} & \phi & \phi' & \theta \\ p_1 & 1 & 0 & -1 \\ p_2 & 0 & 1 & -1 \\ p_3 & 0 & 0 & 1 \end{vmatrix} = 1$$

Then the equation becomes

$$\int_{\theta=-\pi}^{\pi}\int_{p_1=-\pi-\theta}^{\pi-\theta}\int_{p_2=-\pi-\theta}^{\pi-\theta} K^o_{al}(p_1)w_{al}(p_2-p_1,p_2)u(\vec{x},p_3)e^{-ik_1(p_1+p_3)}e^{-ik_2(p_2+p_3)}dp_2dp_1dp_3 \tag{1.1.5}$$

Then if we take the fourier transform of the equation over the orientation, we get:

$$\int_{\theta=-\pi}^{\pi}\int_{p_1=-\pi-\theta}^{\pi-\theta}\int_{p_2=-\pi-\theta}^{\pi-\theta} K^o_{al}(p_1)w_{al}(p_2-p_1,p_2)u(\vec{x},p_3)e^{-ik_1p_1}e^{-ik_1p_3}e^{-ik_2p_2}e^{-ik_2p_3}dp_2dp_1dp_3 \tag{1.1.6}$$

Therefore, we can conclude that:

$$\int_{\theta=-\pi}^{\pi} u(\vec{x},p_3)e^{-i(k_1+k_2)p_3}dp_3\int_{p_1=-\pi-\theta}^{\pi-\theta}\int_{p_2=-\pi-\theta}^{\pi-\theta} K^o_{al}(p_1)w_{al}(p_2-p_1,p_2)e^{-ik_1p_1}e^{-ik_2p_2}dp_2dp_1 \tag{1.1.7}$$

Which finally we get:

$$\widehat{K^o_{al}(k_1)}\widehat{w_{al}\widehat{(k_1,k_2)}}u(\widehat{k_1+k_2},\vec{s}) \tag{1.1.8}$$

where the hat means Fourier transform.

Then start Fourier transform on the distance $\vec{x}$,

$$\int_{\mathbb{R}^2} K^d_{al}(\vec{x}-\vec{s})u(\widehat{k_1+k_2},\vec{s})d\vec{s} \tag{1.1.9}$$

Similarly we perform Fourier transform on 1.1.9, we get:

$$\int_{s_1}\int_{s_2}\int_{x_2}\int_{x_1} K^d_{al}(x_1-s_1,x_2-s_2)u(\widehat{k_1+k_2,s_1},s_2)e^{-ik_3x_1}e^{-ik_4x_2}dx_1dx_2ds_2ds_1 \tag{1.1.10}$$

which finally ends up with:

$$K^d_{al}\widehat{\widehat{(k_3,k_4)}}u(\widehat{\widehat{\widehat{k_1+k_2,k_3}}},k_4) \tag{1.1.11}$$

Thus the Fourier form of the turning function is:

$$q_{al}\widehat{\widehat{K^d_{al}(k_3,k_4)}}\widehat{K^o_{al}(k_1)}w_{al}\widehat{\widehat{(k_1,k_2)}}u(\widehat{\widehat{\widehat{k_1+k_2,k_3}}},k_4)(t) \tag{1.1.12}$$

1.2 Attraction

The equation for T_a is:

$$T_a\left(\boldsymbol{x},\phi',\phi\right)=q_a\int_{\mathbb{R}^2}\int_{\pi}^{\pi} K^d_a(\boldsymbol{x}-\boldsymbol{s})K^o_a\left(\boldsymbol{s};\boldsymbol{x},\phi'\right)\omega_a\left(\phi'-\phi,\phi'-\psi\right)u(\boldsymbol{s},\theta,t)d\theta d\boldsymbol{s} \tag{1.2.1}$$

where ψ is the direction toward a direction, and can be calculated as:

$\boldsymbol{s}-\boldsymbol{x}=\left(s_x,s_y\right)$,

$$\cos(\psi)=\frac{s_x}{\sqrt{s_x^2+s_y^2}},\sin(\psi)=\frac{s_y}{\sqrt{s_x^2+s_y^2}} \tag{1.2.2}$$

Then the Orientation Kernel K^o_a and ω_a is a convolution in the space variable, then we can let $K_a(\boldsymbol{x}-\boldsymbol{s},\phi')=K^d_a(\boldsymbol{x}-\boldsymbol{s})K^o_a(\boldsymbol{x}-\boldsymbol{s},\phi')$. Then the Fourier transform over the real space is:

$$q_aK_a\left(\widehat{\widehat{k_3,k_4}},\phi'\right)\omega_a\left(\phi'-\widehat{\widehat{\phi,\phi'}},k_3,k_4\right)\int_{\pi}^{\pi}u(\widehat{\widehat{k_3,k_4}},\theta,t)d\theta \tag{1.2.3}$$

Then apply the Fourier transform over the angle ϕ and ϕ', we get

$$q_aK_a\left(\widehat{\widehat{\widehat{k_3,k_4}}},k_2\right)\omega_a\left(k_1,\widehat{\widehat{\widehat{\widehat{k_1+k_2}}}},k_3,k_4\right)\int_{\pi}^{\pi}u(\widehat{\widehat{k_3,k_4}},\theta,t)d\theta \tag{1.2.4}$$

which is equal to

$$K_a\,\widehat{\widehat{\widehat{(k_3, k_4}}}, k_2)\omega_a\,(k_1, \widehat{\widehat{\widehat{\widehat{k_1 + k_2}}}}, k_3, k_4)u(\widehat{\widehat{k_3, k_4, 0}})(t) \tag{1.2.5}$$

1.3 Repulsion

The repulsion part should be similar to the attraction part, but the orientation kernel has the opposite sign.

$$K_r\,\widehat{\widehat{\widehat{(k_3, k_4}}}, k_2)\omega_r\,(k_1, \widehat{\widehat{\widehat{\widehat{k_1 + k_2}}}}, k_3, k_4)u(\widehat{\widehat{k_3, k_4, 0}})(t) \tag{1.3.1}$$