MATH 146 – Things every MATH 144 student should know (a review)

Differentiation: definition

The *instantaneous rate of change* of f at $x = c$ is called the *derivative* of f at $x = c$ and is denoted by $f'(c)$ or $\frac{df}{dx}\Big|_{x=c}$ or $\frac{df(c)}{dx}$. It is given by the limit

$$\frac{df(c)}{dx} = f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Defining $x_2 = c + h$, the *slope of the tangent line to the curve* $y = f(x)$ *at* $x = c$ is obtained by taking the limit $x_2 \to c$ or, equivalently, $h \to 0$:

$$\begin{aligned} m &= \lim_{x_2 \to c} \frac{f(x_2) - f(c)}{x_2 - c} \\ &= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}. \end{aligned}$$

Thus, m is precisely equal to the derivative of $f(x)$ at $x = c$; that is, $m = f'(c)$.

Integration: definition

If $v(t)$ is the velocity function of an object, then the distance covered by the object between t_1 and t_2 can be obtained by calculating the area under the graph of $v(t)$ between $t = t_1$ and $t = t_2$.
An approximation can be obtained by slicing the area into rectangles of width Δt and summing over the areas of the rectangles.
By taking the limit where $\Delta t \to 0$ (and equivalently the number of rectangles goes to infinity), we obtain an exact expression for the area under the graph – and the distance covered. This process is called *integrating the function* $v(t)$ *between* $t = t_1$ *and* $t = t_2$, and has notation

$$\int_{t_1}^{t_2} v(t)\ dt.$$

It is the inverse operation of differentiation, as we saw (Week 8) when we studied the Fundamental Theorem of Calculus.

Limits: (informal) definition

We write

$$\lim_{x \to a} f(x) = L,$$

and say that *the limit of* $f(x)$*, as* x *approaches* a*, is equal to* L, if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a (on either side of a) but not equal to a.

Note that

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to a^+} f(x) = L.$$

Vertical asymptotes

The line $x = a$ is called a *vertical asymptote* of the curve $y = f(x)$ if either

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or both.}$$

Note that the function $f(x)$ does not have to blow up on both sides of $x = a$ for it to be a vertical asymptote; as long as the limit is infinite on one side of $x = a$ it is a vertical asymptote.

We say that $\lim_{x \to a} f(x)$ *exists* if $\lim_{x \to a} f(x) = L$, with L a *finite number* ($L = 0$ is perfectly fine).
It may *not exist* if the left-sided and right-sided limits are not equal, or if the limit is infinite (since $\pm\infty$ is not a finite number).

Continuity

A function $f(x)$ is *continuous* at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Thus, for a function to be continuous, it must satisfy the three conditions:

1. $f(a)$ exists;
2. $\lim_{x \to a} f(x)$ exists;
3. $\lim_{x \to a} f(x) = f(a)$.

Polynomials, rational functions, root functions, trigonometric functions, exponential functions, and logarithmic functions are examples of functions that are continuous at every point in their domain.

Limit laws

Assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then:

1. *Limit of a constant:* $\lim_{x \to a} c = c$ for any $c \in \mathbb{R}$.
2. *Limit of x:* $\lim_{x \to a} x = a$.
3. *Sum rule:* $\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$.
4. *Product rule:* $\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$.
5. *Quotient rule:* $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$.
6. *Root rule:* $\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x)\right)^{1/n}$, where n is a positive integer. For n even we require that $\lim_{x \to a} f(x) > 0$ so that the root is real and the limit is well defined.

More on the derivative of a function

- A function f is *differentiable* at $x = a$ if $f'(a)$ exists. It is differentiable on an open interval (a, b) if it is differentiable at every point in the interval.
- A function f that is not continuous at $x = a$ is also not differentiable at $x = a$. Equivalently, a function f that is differentiable at $x = a$ must be continuous at $x = a$.
- The converse is *not true*: a function f that is continuous at $x = a$ may not be differentiable at $x = a$.
- A function can fail to be differentiable at $x = a$ in three different ways:
  - It is *not continuous* at $x = a$;
  - The left-sided limits and right-sided limits of the difference quotient are not the same, in which case f has a *corner* or a *kink* at $x = a$;
  - The limit of the difference quotient is infinite, in which case f has a *vertical tangent* at $x = a$.
- In general, the *n'th derivative* of f is denoted by $f^{(n)}(x)$ or $\dfrac{d^n f}{dx^n}$. It is the derivative of $f^{(n-1)}$, that is, it is obtained from f by differentiating n times.

Differentiation rules

- *Power rule*: For any real number $a \in \mathbb{R}$,
$$\frac{d}{dx}\left(x^a\right) = ax^{a-1}.$$
- *Constant multiple rule:* For any constant c and differentiable function $f(x)$,
$$\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}f(x).$$
- *Sum and difference rules:* For any two differentiable functions $f(x)$ and $g(x)$,
$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$
- *Product rule:* For any two differentiable functions $f(x)$ and $g(x)$,
$$\frac{d}{dx}\left(f(x)g(x)\right) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x).$$
- *Chain rule:* Let f and g be two functions such that g is differentiable at x and f is differentiable at $g(x)$. Then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x, and its derivative is given by
$$F'(x) = f'(g(x)) \cdot g'(x).$$
- *Quotient rule:* Use the product rule + chain rule.

Derivatives of trig functions

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\csc x = -\csc x \cot x,$$
$$\frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}\sec x = \sec x \tan x,$$
$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\cot x = -\csc^2 x.$$

Note the useful trig limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

Implicit differentiation

Given a relation

$$H(x, y) = 0,$$

we say that f is a function defined implicitly by this relation if

$$H(x, f(x)) = 0$$

for all x in the domain of f. For implicit differentiation, differentiate both sides of the equation with respect to x, treating y as a differentiable function of x. Collect the terms involving y' on one side of the equation and solve for y'.

One-to-one functions

- A function is *one-to-one* if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2.$$

- *Horizontal line test:* A function is one-to-one if no horizontal line intersects its graph more than once.

Inverse functions

- Let f be a one-to-one function with domain A and range B. Then its *inverse function* f^{-1} has domain B and range A and is defined by

$$y = f(x) \qquad \text{if and only if} \qquad f^{-1}(y) = x.$$

- The domain of f is the range of f^{-1}, while the range of f is the domain of f^{-1}.
- Inverse functions satisfy:

$$f^{-1}(f(x)) = x \qquad \text{for } x \text{ in the domain of } f,$$
$$f(f^{-1}(x)) = x \qquad \text{for } x \text{ in the domain of } f^{-1}.$$

  This means the inverse of a function "undoes" what the function did to its argument, and restores the argument.

- The graph of $y = f^{-1}(x)$ can be obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

More rules of differentiation

$$\begin{aligned}
\frac{d}{dx}(e^x) =& e^x, \\
\frac{d}{dx}(a^x) =& a^x \ln(a), \\
\frac{d}{dx}(\ln(x)) =& \frac{1}{x}, \\
\frac{d}{dx}(\log_a(x)) =& \frac{1}{x\ln(a)}.
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\sin^{-1}(x) =& \frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx}\cos^{-1}(x) &= -\frac{1}{\sqrt{1-x^2}}, \\
\frac{d}{dx}\tan^{-1}(x) =& \frac{1}{1+x^2}, & \frac{d}{dx}\cot^{-1}(x) &= -\frac{1}{1+x^2}, \\
\frac{d}{dx}\sec^{-1}(x) =& \frac{1}{x\sqrt{x^2-1}}, & \frac{d}{dx}\csc^{-1}(x) &= -\frac{1}{x\sqrt{x^2-1}}.
\end{aligned}$$

Antiderivatives

- A function F is called an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.
- If F is an antiderivative of f on I, then the most general antiderivative of f on I is

  $$F(x) + C,$$

  where $C \in \mathbb{R}$ is an arbitrary constant.

Indefinite integrals

- We use the notation

  $$\int f(x)\ dx = F(x) + C$$

  to denote the most general antiderivative of f; this expression is called the *indefinite integral of f*.
- The notation $\int f(x)\ dx$ means "finding the general antiderivative of $f(x)$," just as $\frac{d}{dx}f(x)$ means "finding the derivative of $f(x)$."

Table of indefinite integrals

$\int c\ f(x)\ dx = c\int f(x)\ dx$	$\int (f(x)+g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx$
$\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{x}\ dx = \ln\lvert x\rvert + C$
$\int e^x\ dx = e^x + C$	$\int a^x\ dx = \frac{a^x}{\ln(a)} + C$
$\int \sin x\ dx = -\cos x + C$	$\int \cos x\ dx = \sin x + C$
$\int \sec^2 x\ dx = \tan x + C$	$\int \csc^2 x\ dx = -\cot x + C$
$\int \sec x \tan x\ dx = \sec x + C$	$\int \csc x \cot x\ dx = -\csc x + C$
$\int \frac{1}{x^2+1}\ dx = \tan^{-1} x + C$	$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + C$
$\int \frac{1}{x\sqrt{x^2-1}}\ dx = \sec^{-1} x + C$	

Summation Notation

- For $\{a_i\}$ a set of numbers indexed by the integers, and for integers $k \leq n$ we have

$$\sum_{i=k}^{n} a_i = a_k + a_{k+1} + \cdots + a_{n-1} + a_n.$$

- *Properties of sums:*

$$\sum_{i=k}^{n} ca_i = c\sum_{i=k}^{n} a_i, \qquad \sum_{i=k}^{n}(a_i \pm b_i) = \sum_{i=k}^{n} a_i \pm \sum_{i=k}^{n} b_i.$$

For any integer k with $a < k < b$,

$$\sum_{i=a}^{b} c_i = \sum_{i=a}^{k} c_i + \sum_{i=k+1}^{b} c_i.$$

Definite integrals

- Let $f(x)$ be a function defined for $x \in [a,b]$. We divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a$, $x_1 = a + \Delta x$, $\ldots$, $x_n = b$ be the right endpoints of these intervals. The Riemann sum R_n is defined by

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

The *definite integral of f from a to b*, denoted by $\int_a^b f(x)\,dx$, is the $n \to \infty$ limit of R_n:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, \qquad \text{with } \Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x,$$

provided that the limit exists. If it exists, we say that f is *integrable* on $[a, b]$.

- In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the *integrand*, and a and b are called the *limits of integration*: a is the *lower limit* while b is the *upper limit*.

Properties of definite integrals

1. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$
2. $\displaystyle\int_a^a f(x)\,dx = 0,$
3. $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx,$
4. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx,$
5. $\displaystyle\int_a^b dx = b - a,$
6. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$
7. If $f(x)$ is even, then $\displaystyle\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx,$
8. If $f(x)$ is odd, then $\displaystyle\int_{-a}^a f(x)\,dx = 0,$
9. If $f(x) \geq 0$ for $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geq 0,$
10. If $f(x) \geq g(x)$ for $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx,$
11. If $m \leq f(x) \leq M$ for $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a),$$

12.

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

The Fundamental Theorem of Calculus (FTC)

Let $f(x)$ be a continuous function on $[a, b]$. Then:

1. If $g(x) = \int_a^x f(t)\ dt$, for $a \leq x \leq b$, then $g'(x) = f(x)$;
2. $\int_a^b f(x)\ dx = F(b) - F(a)$, where F is an arbitrary antiderivative of f.

The substitution rule

If $u = g(x)$ is a differentiable function and $f(x)$ is continuous over the range of $g(x)$, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

In other words, the substitution $u = g(x)$, with $du = g'(x)dx$, "undoes" the chain rule.
For definite integrals, one needs to transform the limits of integration from x-values to u-values as you perform a substitution:

$$\int_a^b f((g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du, \qquad \text{with } u = g(x), du = g'(x)dx.$$

Then you can evaluate the resulting definite integral in u directly using the Fundamental Theorem of Calculus (FTC).

Intermediate Value Theorem (IVT)

- Let f be a continuous function over the interval $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, with $f(a) \neq f(b)$. Then there exists a $c \in (a, b)$ such that $f(c) = N$.
- Equivalently, for any N between $f(a)$ and $f(b)$, the horizontal line $y = N$ must intersect the graph of f at least once.

Mean Value Theorem (MVT)

- Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then the *Mean Value Theorem* states that there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- Geometrically, the Mean Value Theorem is saying that there must be a $c \in (a, b)$ such that the tangent line to $y = f(x)$ at $x = c$ is parallel to the secant line between $(a, f(a))$ and $(b, f(b))$.
- It can also be understood as saying that there must be a $c \in (a, b)$ at which the instantaneous rate of change of f is equal to its average rate of change between a and b.
- *Rolle's Theorem* (where $f(a) = f(b)$) is a special case of the MVT.

Min and max of a function

Let $c \in D$ where D is the domain of $f(x)$. Then $f(c)$ is:

- The *absolute maximum* of $f(x)$ on D if $f(c) \geq f(x)$ for all $x \in D$;
- The *absolute minimum* of $f(x)$ on D if $f(c) \leq f(x)$ for all $x \in D$;
- The *local maximum* of $f(x)$ on D if $f(c) \geq f(x)$ for all x near c;
- The *local minimum* of $f(x)$ on D if $f(c) \leq f(x)$ for all x near c.

Near c means "for all x in some open interval containing c".

In general, absolute minima and maxima can occur either at local minima or maxima, or, if D is a closed interval, at the endpoints of the interval.

Extrema and critical numbers

- A *critical number* of f is a number c in the domain D of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.
- If f has a local minimum or maximum at c, then c is a critical number of f (Fermat's Theorem).
- However, the converse is not true: not all critical numbers are local minima or maxima.

Extreme Value Theorem

- If f is continuous on $[a, b]$, then f must attain an absolute maximum $f(c)$ and an absolute minimum $f(d)$ at some numbers $c, d \in [a, b]$.

How to find the absolute extrema of a continuous function f on $[a, b]$

1. Find the critical numbers of f in (a, b), and evaluate f at the critical numbers;
2. Evaluate f at the endpoints a and b;
3. Compare the values. The largest of these values is the absolute maximum, while the smallest is the absolute minimum.

Asymptotes

Vertical asymptotes

- The vertical line $x = a$ is a *vertical asymptote* of $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

Horizontal asymptotes

- The horizontal line $y = L$ is a *horizontal asymptote* of $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L, \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L.$$

Summary of curve sketching

Information from $f(x)$:

A. **Domain:** find the domain of f.

B. **Intercepts:** Find the y-intercepts $(0, f(0))$ and x-intercepts (the points where $f(x) = 0$). Find where f is positive and negative.

C. **Symmetry:** Is f even or odd? Is f periodic?

D. **Asymptotes:** Find the vertical, horizontal and slant asymptotes of f if any exist.

Information from $f'(x)$:

E. **Intervals of increase and decrease:** Find $f'(x)$ and determine when $f'(x) > 0$ (f is increasing) and when $f'(x) < 0$ (f is decreasing).

F. **Local max and min points:** Find the critical points of f, if any, and identify the local max and min.

Information from $f''(x)$:

G. **Concavity:** Find $f''(x)$, determine when $f''(x) > 0$ (f is concave up) and when $f''(x) < 0$ (f is concave down).

H. **Inflection points:** Find the inflection points of f, if any.

And finally...

I. **Sketch the graph:** Plot key points (intercepts, critical points, local max and min, points of inflection), and sketch the curve together with its asymptotes.

Taylor polynomials

The idea of Taylor polynomials is to approximate a function $f(x)$ at a point $x = a$ by higher degree polynomials.

- The polynomial

$$T_d(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(d)}(a)}{d!}(x-a)^d,$$

  where $k! = 1 \cdot 2 \cdot 3 \dots k$, is called the *degree d Taylor polynomial* of $f(x)$ at $x = a$.

- It satisfies the properties:

$$T_d(a) = f(a) \qquad \text{and} \qquad T_d^{(k)}(a) = f^{(k)}(a)$$

  for all $k = 1, 2, \dots, d$.