Final Report

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Project Title	Entanglement Transformation with Implementation in Qiskit

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1 INTRODUCTION

Quantum technologies have proven themselves to be very useful in different areas for a few years. Quantum information is one of the key features of quantum technologies to develop itself even further, and entanglement is very useful in quantum information processes. Entanglement has many areas to be useful; however, in this project, the focus is the entanglement transformation. Namely, if Alice and Bob share an entangled state $|\psi\rangle$, can they transform this entangle state to another entangled state $|\phi\rangle$? They can do operations in their local system like measurement, and they will only be allowed to communicate with local communication.

Entangled particles cannot be described independently even if they are at a large distance from each other, and an entangled state cannot be divided into products of any other states. The creation of entanglement is not necessarily a hard task; a particle decaying can create an entanglement. Entanglement is used in many areas such as quantum cryptography[**1**], teleportation[**2**], machine learning[**3**], dense coding[**4**] has a very important place. These tasks require specific types of entanglement for their purposes, and entanglement transformation can be a powerful tool to achieve these particular entangled states.

Entanglement transformation has been widely studied in quantum computing to reduce noise. Entanglement distillation where noisy entangled quantum channels are turned into pure states[**5**] is an example of entanglement transformation. The Schmidt decomposition of states is used with a correlation of entropy on the transformation of n pairs of identical partly entangled pure states[**6**]. The connection between majorization and entanglement transformation has been one of the work areas. The relation is made with the density matrix of the entangled system[**7**]. Later, the rules for entanglement transformation has studied by using the majorization relation.[**8**]

To understand the entanglement transformation, first a majorization connection is needed to be made. Majorization is used to understand the if one vector is more disorded than oher. Generally two transformation cases will be worked in this project. Firstly we will examine the case of transformation of maximally entangle state to partially entangled state. The bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ will be transformed into a partially entangled $|\psi\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$ state. In the second part, the inverse of this transformation will be examined. The $|\psi\rangle$ state will be transformed into $|\Phi^\pm\rangle$ state. Each Alice and Bob will have a part of the entangled states. The first step will be the general measurement on the Alice's part, depending on the result, Alice will communicate with Bob in classical communication methods. Bob, after learning Alice's result, will apply a unitary gate depending on Alice's result. With Bob's operation, the transformation will be finished.

The implementation of this process will be made in Qiskit. One of the challenges for the project is, the measurement Alice does is not a projective measurement, but a general measurement. For the state to not collapse, we need to perform a general measurement. To perform a general measurement, an ancilla qubit used by Alice to get the necessary information and tell Bob without doing a projective measurement.

2 METHOD

2.1 Majorization

At the beginning of the project, majorization plays an important role. The main purpose of the majorization is to understand the comparison of disordering between probability distributions. A definition of majorization is that, between two d-dimensional vectors r and s, to be able to say that r is majorized by s, first we need to put these vectors in decreasing order. Then r is majorized by s if:

$$\sum_{j=1}^{k} r_j^{\downarrow} \leq \sum_{j=1}^{k} s_j^{\downarrow}. \tag{1}$$

The equation 1 says r is majorized by s (or s majorizes r) and can be shown as $r \prec s$. To make the connection with the entanglement and majorization, the reduced density matrix of the entangled system should be defined. For an entangled system between A and B, if we suppose $|\psi\rangle$ state is entangled, the reduced density matrix of the system is $\rho_\psi = tr_B(|\psi\rangle\langle\psi|)$. Reduced density matrix can be used to calculate the probabilities of local measurements and has all the properties of the density matrix. Eigenvalues of the ρ_ψ can be defined as λ_ψ. For a state $|\psi\rangle$ to be able to transform to state $|\phi\rangle$, the necessary condition is[**9**]:

$$|\psi\rangle \longrightarrow |\phi\rangle \iff \lambda_\psi \prec \lambda_\phi \tag{2}$$

For an entangled state between A and B parties to transform to an another entangled state with probability p by using local operators and classical communication, majorization necessity slightly changes to:

$$|\psi\rangle \longrightarrow |\phi\rangle \iff \lambda_\psi \prec p\lambda_\phi \tag{3}$$

The transformation of partial entanglement to maximal entanglement happens with some probaility p[**10**]. The most general form of the entanglement transformation is formed with Schmidt coefficients of the initial state and the average Schmidt coefficients of the final states[**10**]:

$$|\psi\rangle \longrightarrow |\phi\rangle \iff \lambda_\psi \prec \sum_{j=1}^{k} p_j\lambda_\phi \tag{4}$$

2.2 General Measurement

The first step of the transformation will be the general measurement on Alice's part of the entangled state, depending on the result Alice will communicate with Bob in classical communication methods. To be able to do general measurement, we can use projective measurement and unitary operations. The way to perform a general measurement is Alice adds an ancilla qubit to her system with space M, that has ortonormal basis $|m\rangle$. Letting M_m the measurement operators on Alice's system and a unitary operator U[**11**]:

$$U|\psi\rangle|0\rangle \equiv \sum_m M_m|\psi\rangle|m\rangle \tag{5}$$

The equation 5 is the implementation of general measurement by using projective measurement and unitary operation where $\sum_m M_m^\dagger M_m = I$.

As an example of general measurement on Qiskit, let's say Alice has a qubit and wants to do an general measurement on her system. She first adds an ancilla qubit initally in the $|0\rangle$ state. So her state at the start is in the form of $|\psi\rangle = |00\rangle$. Alice applies a Hadamard gate on his qubit and the state gets in the form of $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. From equation 5, we can write the general measurement as:

$$U|\psi\rangle \otimes |0\rangle = M_0|\psi\rangle \otimes |0\rangle + M_1|\psi\rangle \otimes |1\rangle \tag{6}$$

By letting

$$M_0 = \begin{bmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{bmatrix}, M_1 = \begin{bmatrix} \sin\theta & 0 \\ 0 & \cos\theta \end{bmatrix}$$

Equation 6 can be written in the form of:

$$U\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle = \frac{1}{\sqrt{2}}(\cos\theta|0\rangle+\sin\theta|1\rangle)\otimes|0\rangle+\frac{1}{\sqrt{2}}(\sin\theta|0\rangle+\cos\theta|1\rangle)\otimes|1\rangle \tag{7}$$

Now, let's our unitary U operation to be a R_y on the ancilla qubit.

$$(I\otimes R_y)\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle = \frac{1}{\sqrt{2}}(\cos\theta|00\rangle+\sin\theta|01\rangle)+\frac{1}{\sqrt{2}}(\cos\theta|10\rangle+\sin\theta|11\rangle) \tag{8}$$

We can see that equation **??** is similar to the general measurement. We are only missing a Cnot operation between Alice's and ancilla qubit. After applying Cnot operation where Alice's qubit is the control qubit and ancilla qubit is the target qubit we get:

$$\frac{1}{\sqrt{2}}(\cos\theta|00\rangle + \sin\theta|01\rangle) + \frac{1}{\sqrt{2}}(\cos\theta|11\rangle + \sin\theta|10\rangle) \tag{9}$$

This is the exact equation in 7. So we were able to apply a general measurement in Alice's qubit by using unitary operations and an ancilla qubit.

3 RESULTS AND DISCUSSION

3.1 Transformation of Maximally Entangled State to Partially Entangled State

In this case, the transformation of maximally entangled $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ bell state to partially entangled $|\psi\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$ will be examined. We can create our initial bell states state in qiskit by applying a hadamard gate and a cnot gate to Alice and Bob's qubits. After creating the state, the first step will be the general measurement on the Alice's state, depending on the result, Alice will communicate with Bob in classical communication methods. Bob, after learning Alice's result, will apply a unitary gate depending on Alice's result. With Bob's operation, the transformation of the state $|\psi\rangle$ will be finished. To check the results, by applying gates in reverse order, we should get the initial states we started which are 0s.

Our circuit in qiskit looks like this:

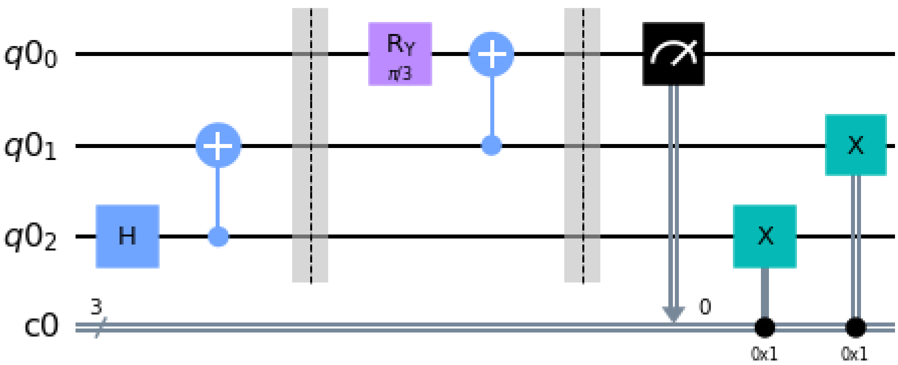

Figure 1: The circuit that transforms two-qubit maximally entangled state to partially entangled state in Qiskit

In the figure 1, from top to bottom, the qubits are ancilla qubit, Alice's qubit and Bob's qubit. Until the first barier we are creating our initial Bell state by applying a Hadamard and Cnot gate to Alice and Bob's qubits. We then get the state:

$$|0\rangle \otimes |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle) \tag{10}$$

Between first and second barrier, we are applying general measurement. In this case, general measurement corresponds to applying a R_y gate and a Cnot gate as in the example given in section 2.2. Our state becomes:

$$(R_y \otimes I \otimes I)\frac{1}{\sqrt{2}}(|000\rangle + |011\rangle) = \frac{1}{\sqrt{2}}((\cos\theta|0\rangle + \sin\theta|1\rangle) \otimes |00\rangle + (\cos\theta|0\rangle + \sin\theta|1\rangle) \otimes |11\rangle) \tag{11}$$

By applying the Cnot operation that comes after the R_y gate the new state is in the form:

$$\frac{1}{\sqrt{2}}(\cos\theta|000\rangle + \sin\theta|011\rangle + \cos\theta|111\rangle + \sin\theta)|100\rangle) \tag{12}$$

Rearranging equation we get:

$$|0\rangle \otimes (\cos\theta|00\rangle + \sin\theta|11\rangle) + |1\rangle \otimes (\sin(\theta)|00\rangle + \cos(\theta)|11\rangle) \tag{13}$$

We can see from 13 that when the ancilla qubit is in the $|0\rangle$ state, we get the desired partially entangled state; however when the ancilla qubit is $|1\rangle$ we get the inverse of the our desired state. Then, when the measurement of ancilla qubit is $|1\rangle$ both Alice and Bob need to apply X gate to their qubits to achive the wanted partially entangled state. After measuring the ancilla qubit, Alice tells the result to Bob. As seen in the figure, they are applying X gates after the second barrier, if the result of ancilla qubit is $|1\rangle$. Then by adding measurement to Alice and Bob's qubits and simulating our results in Qiskit simulator we get the following results:

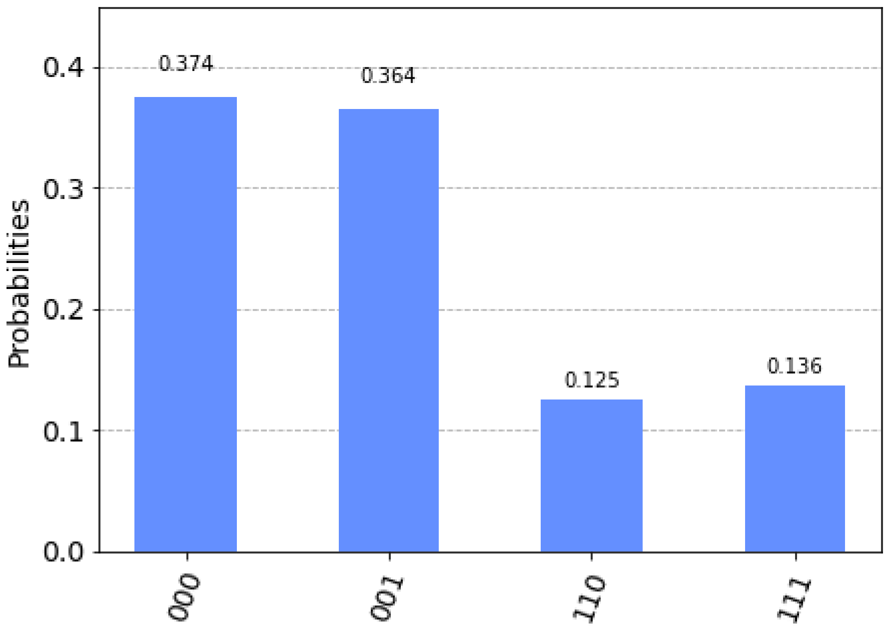

Figure 2: Results of the circuit in Figure 1 on a Qiskit simulator when $\theta = \frac{\pi}{3}$

The results shows the states and their probabilities to give that state in the case of a measurement. In the figure 2, in the $x - axis$ the most left value corresponds to the bottom qubit while the most right value corresponds to the top qubit. From the results we can deduce that, our state is in the $\cos(\theta)|00\rangle + \sin(\theta)|11\rangle$ form whether the ancilla qubit is $|0\rangle$ or $|1\rangle$.

Now to test our results, we can simply apply our operations done until now in reverse order. By this we should get the initial state $|00\rangle$. Our test circuit is then:

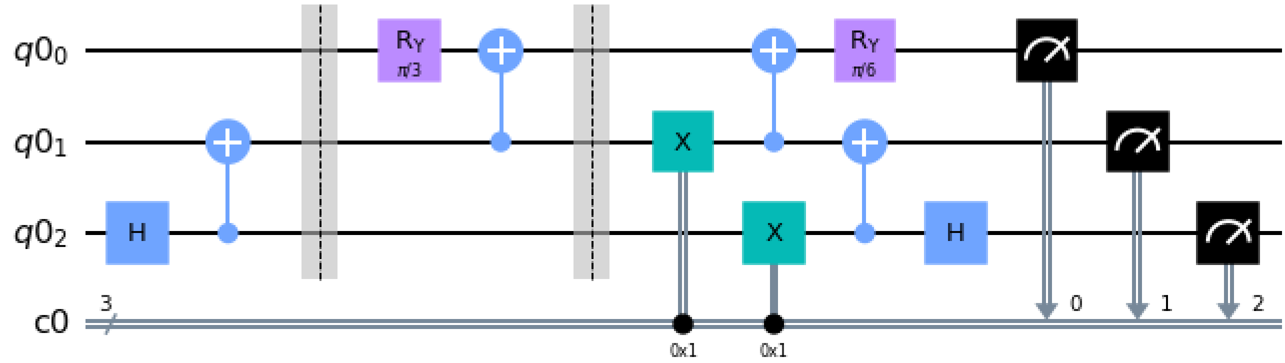

Figure 3: Test circuit of the results in Figure 2 on a Qiskit simulator when $\theta = \frac{\pi}{3}$ and reverse $\theta = \frac{\pi}{6}$

Note that we have to remove the measurement on the ancilla qubit after the general measurement. Otherwise the results would be different since the ancilla qubit state would collapse. The result of the figure 3 on the qiskit simulator gives:

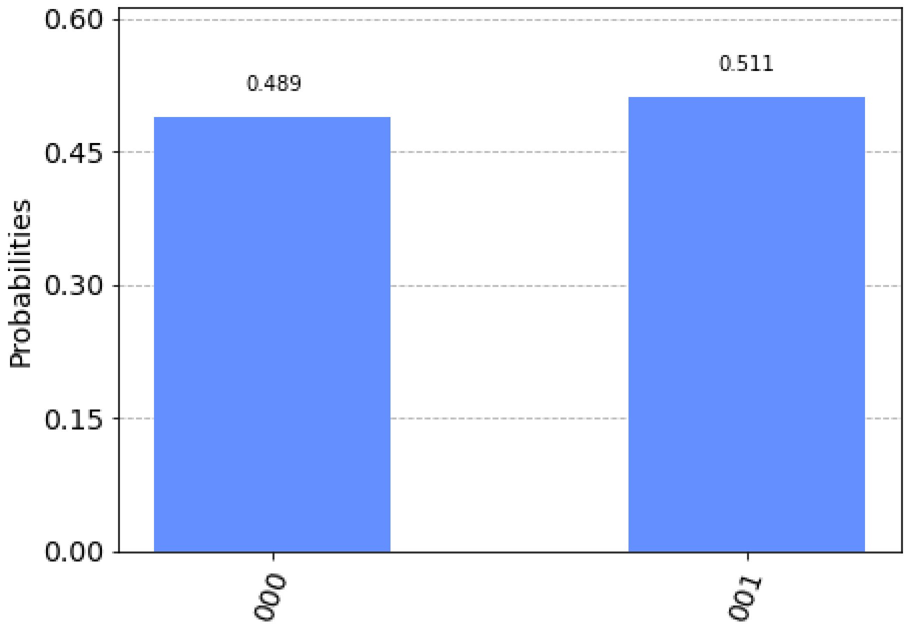

Figure 4: Results of the test circuit in Figure 4 on a Qiskit simulator when $\theta = \frac{\pi}{3}$ and reverse $\theta = \frac{\pi}{6}$

The results tell us that Alice and Bob's qubit are in the $|0\rangle$ state in the end. With this result we conclude the maximally entangled to partially entanglement process.

3.2 Transformation of Partially Entangled State to Maximally Entangled State

In this case, the transformation of partially entangled $|\psi\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$ state to maximally entangled $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ bell state will be examined. We can create our initial partially entangled state in qiskit by applying a rotation around y-axis and a cnot gate to Alice and Bob's qubits. After creating the state, the first step will be the general measurement on the Alice's state, in this case our general measurement takes the form of a controlled-U operation.

An unitary operation U can be defined as $U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$. Then we can define a controlled-U operation as:

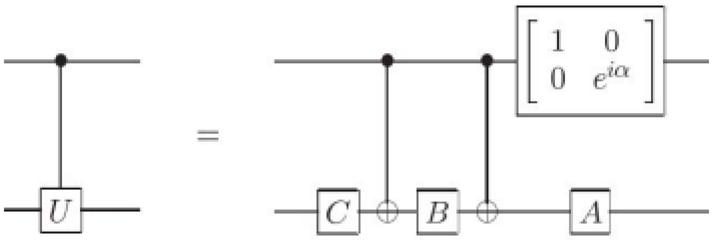

Figure 5: Controled-U operation on two-qubit system.

where $A = R_z(\beta)R_y(\gamma/2)$, $B = R_y(-\gamma/2)R_z(-(\delta+\beta)/2)$, and $C = R_z((\delta-\beta)/2)$. According to this values $ABC = I$[**11**]. In our case, the $\delta = \beta = \alpha = 0$. The control-U operation corresponds to a Cnot operation + R_y gate + Cnot operation + R_y gate, where the degree of this R_y gates correlated to the R_y gate used to create our initial state.

After applying general measurement, our circuit in Qiskit becomes:

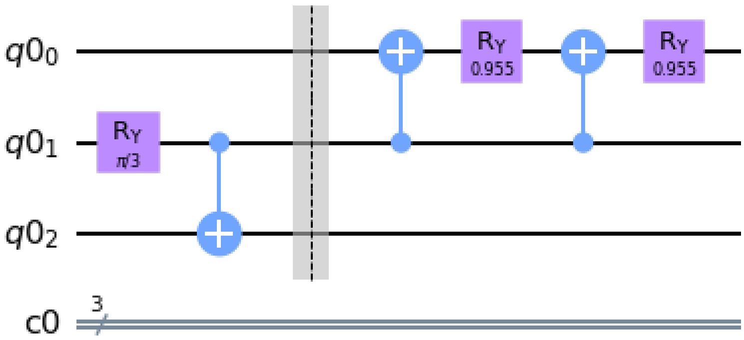

Figure 6: The circuit that transforms two-qubit partially entangled state to maximally entangled state in Qiskit.

Until the first barrier we are creating our initial partially entangled state by applying a R_y gate and Cnot operation to Alice and Bob's qubits. We then get the state:

$$|0\rangle \otimes |\Psi\rangle = |0\rangle \otimes (\cos(\theta)\,|00\rangle + \sin(\theta)\,|11\rangle) \tag{14}$$

After the first barrier, our control-U operation comes. We can see why we used this control-U operation as the general measurement by looking at equation 6 for this case. By letting

$$M_{\mathrm{s}} = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}, M_{\mathrm{f}} = \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}$$

where $x = \tan\theta < 1$ and $y = 1 - x^2$. In this case we will have succesful and failed results. M_{s} represents the succesful results while M_{f} represents the failed ones. The equation 6 becomes:

$$U|\psi\rangle \otimes |0\rangle = M_s|\psi\rangle \otimes |0\rangle + M_f|\psi\rangle \otimes |1\rangle \tag{15}$$

By expanding equation 16 we can write it as:

$$U|\psi\rangle \otimes |0\rangle = (x|0\rangle\langle 0| + |1\rangle\langle 1|)|\psi\rangle \otimes |0\rangle + (y|0\rangle\langle 0|)|\psi\rangle \otimes |1\rangle \tag{16}$$

In the case where Alice's qubit is $|0\rangle$ we get:

$$U|0\rangle \otimes |0\rangle = x|00\rangle + y|01\rangle \tag{17}$$

In the case where Alice's qubit is $|1\rangle$ we get:

$$U|1\rangle \otimes |0\rangle = |10\rangle \tag{18}$$

From equations 17 and 18 we understand that if the Alice's qubit is $|0\rangle$ unitary operation changes the state and if the Alice's qubit is $|1\rangle$ the state is unchanged. This corresponds to a controled-U operation in figure 5.

From the results in the figure 7, we can say that about half of the time the ancilla qubit result is 0, and we successfully get the desired result, about half of the time ancilla qubit is 1 and we failed to get maximal entanglement. This results also contains some additional noises differently than the first part since real quantum computers are noisy systems, and simulators neglects that noise.

Our final state is in the form of $\sin\theta|0\rangle \otimes (\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle))) + \cos\theta \sin\gamma|100\rangle$. This circuit in a real IBM quantum computer gives the results:

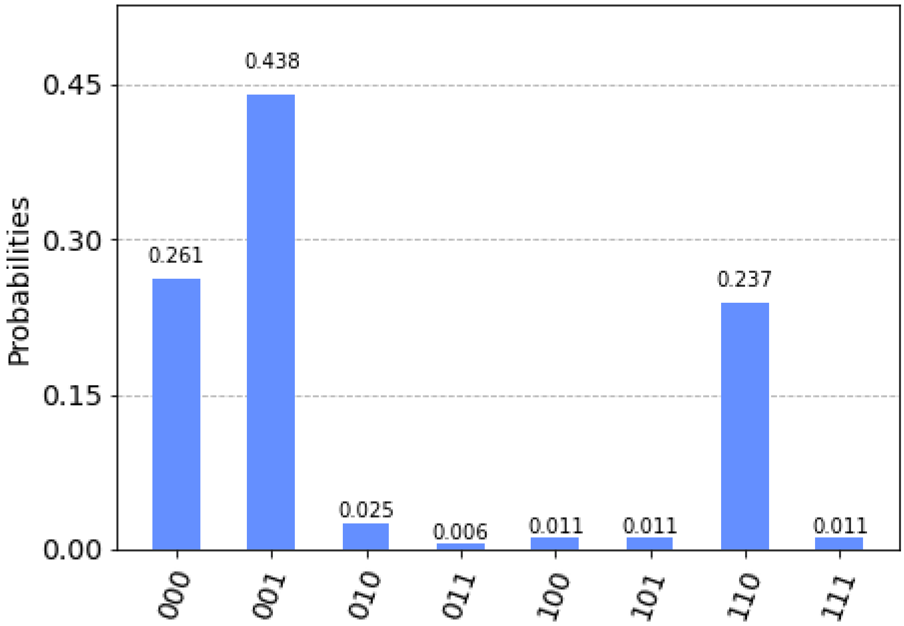

Figure 7: Results of the circuit in Figure 6 on a Qiskit simulator when $\theta = \frac{\pi}{3}$ and $\gamma/2 = 0.955$.

4 CONCLUSION

The initial states were able to transformed into the desired states, Qiskit used as Quantum Computation platform to implement the transformation. In the first part we transformed the maximally entangled state to partially entangled state. In the second part, we successfully transformed the partially entangled state to maximally entangled state with some probability. Majorization showed to be useful tool to state the condition to do the transformation. How to apply general measurement as unitary operations is also showed as a projective measurement with unitary operations.

Overall, entanglement is used in many areas such as quantum cryptography, teleportation, machine learning, dense coding has a very important place. These tasks require specific types of entanglement for their purposes, and entanglement transformation can be a powerful tool to achieve these particular entangled states.

For the future of the project, state tomography can be added to test the results with more accuracy. By using state tomography we can reconstruct the density matrices of the final states and check the fidelities of the states. This could be a more reliable way to test our results.

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