DEPARTMENT OF SCIENCE AND HUMANITIES

SUBJECT CODE : MA2102

SUBJECT NAME : MATRICES AND CALCULUS

DEPARTMENT : COMMON TO ALL BRANCES

YEAR/SEM : I/I

ACEDEMIC YEAR : 2022-2023

BATCH : XIII

HAND WRITTEN MATERIAL

UNIT IV INTEGRAL CALCULUS

Course outcomes	Content	Program outcome
CO 102.4	Definite and Indefinite integrals	PO 1
	Substitution rule	PO 2
	Techniques of Integration	PO 2
	Integration by parts	PO 2
	Trigonometric integrals	PO 2
	Trigonometric substitutions	PO 2
	Integration of rational functions by partial fraction	PO 1
	Integration of irrational functions - Improper integrals	PO 1
	Applications: Hydrostatic force and pressure, moments and centres of mass	PO 3

- **PO 1: Engineering Knowledge**
- **PO 2: Problem analysis**
- **PO 3: Design/Development of Solution**

CHENNAI INSTITUTE OF TECHNOLOGY

VISION

To be an eminent centre for Academia, Industry and Research by imparting knowledge, relevant practices and inculcating human values to address global challenges through novelty and sustainability.

MISSION

IM1. To create next generation leaders by effective teaching learning methodologies and instil scientific spark in them to meet the global challenges.

IM2. To transform lives through deployment of emerging technology, novelty and sustainability.

IM3. To inculcate human values and ethical principles to cater to the societal needs.

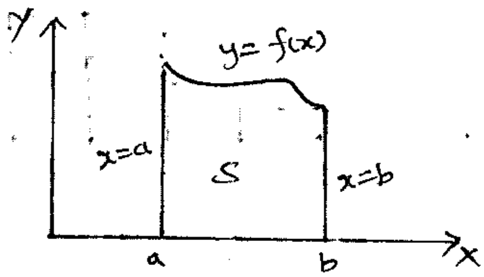
IM4. To contribute towards the research ecosystem by providing a suitable, effective platform for interaction between industry, academia and R & D establishments.

IM5. To nurture incubation centres enabling structured entrepreneurship and start-ups.

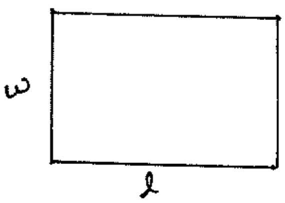
Introduction

In Mathematics, integral assign numbers to functions in a way that can describe displacement, area, volume and open concepts that arise by combining infinesitial data. Integral calculus plays a vital role in Mathematics, Engineering, Science and Economics.

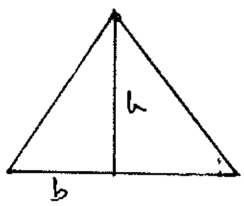
First let us concentrate to solve the area problem. Given a function f which is continuous and non-negative on an interval $[a,b]$, find the areas between the graph of f and the interval $[a,b]$ on the x-axis.



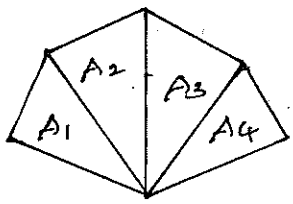
From this diagram, S is bounded by the graph of a continuous function f [where $f(x) \geq 0$], the vertical lines $x=a$ and $x=b$ and the x axis.



$A = lw$



$A = \frac{1}{2}bh$



$A = A_1 + A_2 + A_3 + A_4$

For a rectangle, the area is defined as the product of the length and the width. The area of triangle is half of the base times the height. The are of a polygon is found by dividing it into triangles and adding the areas of the triangles. It is not easy to find the area of a region with curved sides. In defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations.

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We follow a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

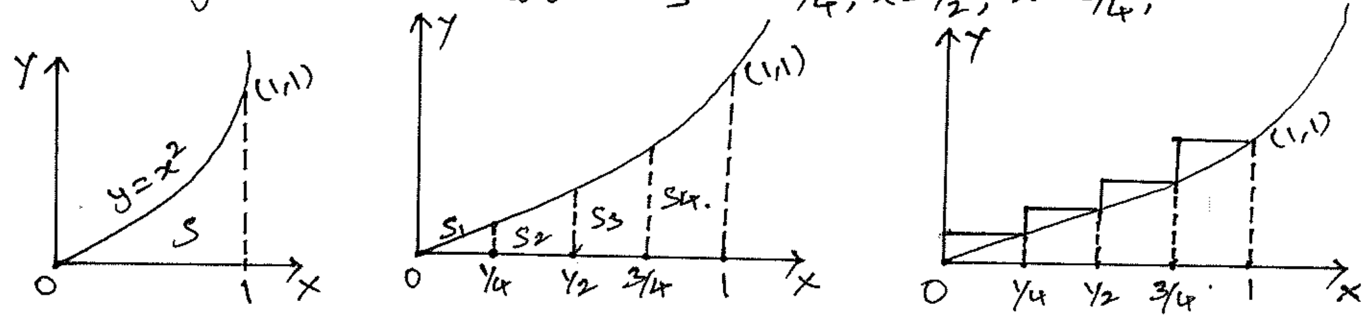
The area problem.

① Use rectangles to estimate the area under the parabola $y=x^2$ from 0 to 1.

Solution:

Given that, the area S is between 0 and 1

We divide S into 4 strips S_1, S_2, S_3 and S_4 by drawing the vertical lines $x=1/4, x=1/2, x=3/4$,



We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip.

The height of these rectangles are the values of the function $f(x)=x^2$ at the right end points of the of the sub-intervals. $[0,1/4], [1/4,1/2], [1/2,3/4]$ & $[3/4,1]$

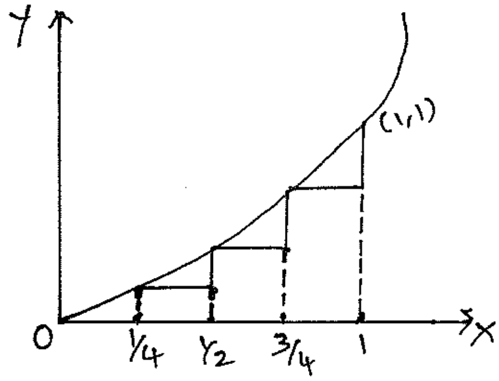
Each rectangle has width $1/4$ and the height are $(1/4)^2, (1/2)^2, (3/4)^2$ and 1. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(3/4\right)^2 + \frac{1}{4}(1)^2 = \frac{15}{32} = 0.46875$$

∴ From the above diagram we see that the area A of S is less than R_4.

$$\therefore A < 0.46875$$

Instead of using the above rectangles, we can use the smaller rectangles from the following graph.



From the graph the heights are the values of f at the left end points of the sub intervals. The sum of the areas of these approximating rectangle is

$$L_4 = \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2$$

$$L_4 = 7/32 = 0.21875$$

We see that the area of S is larger than L_4, so we've lower and upper estimates for A.

$$\therefore 0.21875 < A < 0.46875$$

We should repeat this procedure with a larger number of strips. Now the given region is subdivided into 8 strips of equal width.

$$L_8 = \frac{1}{8}(0)^2 + \frac{1}{8}\left(\frac{1}{8}\right)^2 + \frac{1}{8}\left(\frac{2}{8}\right)^2 + \frac{1}{8}\left(3/8\right)^2 + 1/8\left(4/8\right)^2 + \frac{1}{8}\left(5/8\right)^2 + \frac{1}{8}\left(6/8\right)^2 + \frac{1}{8}\left(7/8\right)^2$$

$$= \frac{1}{8}\left[\frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4} + \frac{25}{64} + \frac{9}{16} + \frac{49}{64}\right]$$

$$\therefore L_4 = 0.273434$$

$$R_8 = \frac{1}{8}\left(\frac{1}{8}\right)^2 + \frac{1}{8}\left(\frac{2}{8}\right)^2 + \frac{1}{8}\left(3/8\right)^2 + \frac{1}{8}\left(4/8\right)^2 + \frac{1}{8}\left(5/8\right)^2 + \frac{1}{8}\left(\frac{6}{8}\right)^2 + \frac{1}{8}\left(7/8\right)^2 + \frac{1}{8}(1)^2$$

$$R_8 = 0.3984375$$

$$\therefore 0.2734375 < A < 0.3984375$$

We can obtain better estimates by increasing the no. of strips

n	L_n	R_n
10	0.285000	0.385000
20	0.3087500	0.3587500
30	0.3168579	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

$\therefore$ A good estimate is obtained by averaging these numbers

$$A = 0.333335$$

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② For the region S in $y=x^2$ from 0 to 1, show that the sum of the areas of the upper approximating rectangles, approaches $\frac{1}{3}$. ie: $\lim_{n\to\infty} R_n = 1/3$.

Solution: Let R_n be the sum of the areas of the n rectangles.

Each rectangle has width $1/n$ and the heights are the values of the function $f(x)=x^2$ at the points $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, n/n$. ie: the heights are $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \dots \left(\frac{n}{n}\right)^2$.

Then $$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n}\cdot\frac{1}{n^2}(1^2+2^2+3^2+\dots+n^2)$$

$$R_n = \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty}\frac{1}{6}\,n\frac{\left(1+\frac{1}{n}\right)}{n}\cdot n\frac{\left(2+\frac{1}{n}\right)}{n}$$

$$\therefore \lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6}\cdot 1\cdot 2 = 1/3.$$

ie: the sum of the areas of the upper apposimating rectangles approaches $1/3$.

Definition:

The area of a region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \dots + f(x_n)\Delta x\right]$$

We can get the same value for left end points.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty}\left[f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_{n-1})\Delta x\right]$$

$$\therefore A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

(3) Find the area under the curve $y=x^3$ on the interval $[0,1]$.

Solution:

Dividing $[0,1]$ into n strips of equal length

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$

$$x_0 = 0, \quad x_1 = 0+\frac{1}{n} = \frac{1}{n} \quad x_2 = 0+2\cdot\frac{1}{n} = 2/n, \ \ldots, \ x_n = 1.$$

If R_n is the right end point approximation using n approximating rectangles, then

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

$$A = \lim_{n\to\infty}\left[\left(\frac{1}{n}\right)^3\frac{1}{n} + \left(\frac{2}{n}\right)^3\frac{1}{n} + \left(\frac{3}{n}\right)^3\frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^3\frac{1}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}(1^3+2^3+\cdots+n^3)$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{1}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2$$

$$\therefore A = \frac{1}{4}.$$

Similarly, if L_n is the ~~rig~~ left end point approximation using n approximating rectangles, then.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

$$= \lim_{n\to\infty}\left[\left(\frac{0}{n}\right)^3\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)^3\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^3\left(\frac{1}{n}\right) + \cdots + \left(\frac{n-1}{n}\right)^3\left(\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^4}\left(1^3+2^3+3^3+\cdots+(n-1)^3\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left(\frac{(n-1)n}{2}\right)^2$$

$$= \frac{1}{4}\lim_{n\to\infty}\frac{n^4\left(1-\frac{1}{n}\right)^2}{n^4}$$

$$\therefore A = \frac{1}{4}.$$

The Definite Integral

The limit of the form

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{x\to\infty}\left[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x\right]$$

If f is a function defined for $a \le x \le b$, we devide $[a,b]$ into n sub-intervals of equal width $\Delta x = \frac{b-a}{n}$. Let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the end points of these sub-intervals and let $x_1^*, x_2^*, x_3^*, \ldots, x_n^*$ be any sample points in these sub-intervals. So x_i^* lies in the i^{th} sub-interval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{\infty} f(x_i^*)\Delta x.$$

provided that this limit exists and gives the same value for all possible points. If it exists, then f is integrable on $[a,b]$.

Theorem 1

If f is continuous on $[a,b]$ or if f has only a finite number of discontinuities, then f is integrable on (a,b). ie: The definite integral $\int_a^b f(x)\,dx$. exists.

Theorem 2

If f is integrable on $[a,b]$ then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

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Properties of Definite integral

Consider the integral $\int_a^b f(x)dx$.

Let $a<b$. Then $\Delta x = \frac{a-b}{n}$

$$\therefore \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

If $a=b$, then $\Delta x=0$, & $\int_a^b f(x)\,dx=0$

Let us assume that f and g are continuous functions

Then (i) $\int_a^b c\,dx = c(b-a)$, where c is any constant

(ii) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

(iii) $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant.

(iv) $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$

(v) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

(vi) If $f(x)\geq 0$ for $a\leq x\leq b$, then $\int_a^b f(x)\,dx \geq 0.$

(vii) If $f(x)\geq g(x)$ for $a\leq x\leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$

(viii) If $m\leq f(x)\leq M$ for $a\leq x\leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

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(4) Let A be the area of the region that lies under the graph of $f(x)=e^{-x}$ between $x=0$ and $x=2$. (a) Using right end points, Find an expression for A as a limit. (b) Estimate the area by taking the sample points to be mid-points and using four sub-intervals and then ten sub-intervals.

Solution:

(a) Since $a=0$ and $b=2$, then $\Delta x = \frac{2-0}{n} = 2/n$.

$$\therefore x_1 = 2/n,\ x_2 = 4/n,\ x_3 = 6/n, \ldots\ldots,\ x_i = \frac{2i}{n}\ \&\ x_n = \frac{2n}{n}$$

$$\therefore R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$= e^{-2/n}\left(\frac{2}{n}\right) + e^{-4/n}\left(\frac{2}{n}\right) + \cdots + e^{-2n/n}(2/n)$$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\left[e^{-2/n} + e^{-4/n} + \cdots + e^{-2n/n}\right]$$

$$A = \lim_{n\to 0} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

(b) When $n=4$, ~~...~~ $\Delta x = 0.5$. The sub intervals are $[0, 0.5]$, $[0.5, 1.0]$, $[1.0, 1.5]$ and $[1.5, 2]$.

The mid-points are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$ and $x_4^* = 1.75$

$$M_4 = \sum_{i=1}^{4} f(x_i^*)\Delta x = f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x.$$

$$= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5)$$

$$= \frac{1}{2}\left(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}\right)$$

$$M_4 = 0.8557$$

When $n=10$ the sub-intervals are $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, $[0.8, 1.0]$, $[1.0, 1.2]$, $[1.2, 1.4]$, $[1.4, 1.6]$, $[1.6, 1.8]$, $[1.8, 2.0]$

The mid points are 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5, 1.7 and 1.9.

$$A = M_{10} = f(0.1)\Delta x + f(0.3)\Delta x + f(0.5)\Delta x + f(0.7)\Delta x + \cdots + f(1.9)\Delta x$$

$$= 0.2\left[e^{-0.1} + e^{-0.3} + e^{-0.5} + e^{-0.7} + e^{-0.9} + e^{-1.1} + e^{-1.3} + e^{-1.5} + e^{-1.7} + e^{-1.9}\right]$$

$$A = 0.8632$$

(5) Evaluate the Riemann sum for $f(x)=x^3-6x$, taking the sample points to be right end points and $a=0, b=3$ and $n=6$. Also evaluate $\int_0^3 (x^3-6x)\,dx$.

Solution:-

(a) When $n=6$, $\Delta x = \frac{b-a}{n} = 1/2$

$\therefore$ the right end points are $x_1=0.5$, $x_2=1.0$, $x_3=1.5$, $x_4=2.0$, $x_5=2.5$ and $x_6=3.0$.

The Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x. = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$$

$$= f(0.5)\Delta x + f(1.0)\Delta x + f(1.5)\Delta x + f(2.0)\Delta x + f(2.5)\Delta x + f(3.0)\Delta x.$$

$$= \frac{1}{2}\left[-2.875 - 5 - 5.625 - 4 + 0.625 + 9\right]$$

$$R_6 = -3.9375$$

(b) With n sub-intervals, we've $\Delta x = \frac{b-a}{n} = 3/n$.

$\Rightarrow x_0 = 0$, $x_1 = 3/n$, $x_2 = 6/n$, $x_3 = 9/n$ & in general $x_i = \frac{3i}{n}$

$$\int_0^3 (x^3-6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(3i/n)\, 3/n.$$

$$= \lim_{n\to\infty} 3/n \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27$$

$$\left\{ \because \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right\}$$

$$\int_0^3 (x^3-6x)\,dx = -6.75$$

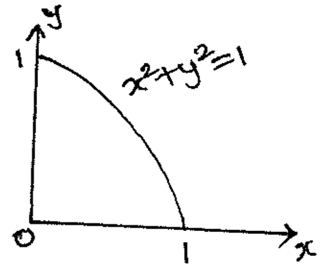
⑥ Evaluate the following integrals by interpreting each in terms of areas. ⓐ $\int_0^1 \sqrt{1-x^2}\,dx$ ⓑ $\int_0^3 (x-1)\,dx$

Solution:

ⓐ Let $f(x)=\sqrt{1-x^2}$. This integral as the area under the curve $y=\sqrt{1-x^2}$. from 0 to 1.

Since $y^2=1-x^2 \Rightarrow x^2+y^2=1$. (ie: a quadratic circle with radius 1.)

$\therefore \int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1) = \pi/4.$



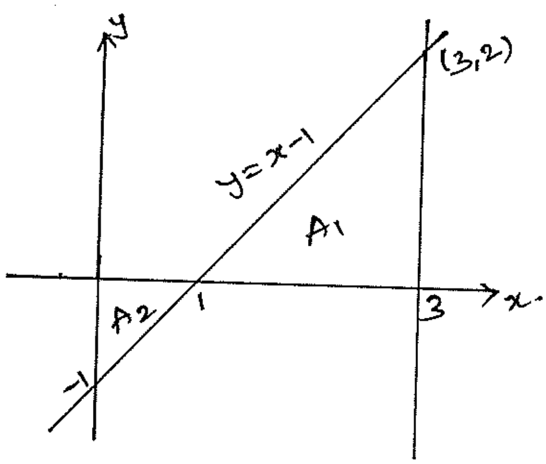
ⓑ The graph of $y=1-x$ is the line with slope 1.

we compute the integral as the difference of the areas of the two triangles.

$\int_0^3 (x-1)\,dx = A_1 - A_2$

$= \frac{1}{2}(2\times2) - \frac{1}{2}(1\times1)$

$\int_0^3 (x-1)\,dx = 1.5$



⑦ Use mid point rule with $n=5$, to approximate $\int_1^2 \frac{1}{x}\,dx$.

Solution:

Let $a=1, b=2, n=5$

Then $\Delta x = \frac{b-a}{n} = \frac{1}{5}$.

$\therefore$ the end points of the sub-intervals are 1, 1.2, 1.4, 1.6, 1.8 and 2.

Also the mid points are 1.1, 1.3, 1.5, 1.7 & 1.9

$\therefore$ The mid point rule gives

$\int_1^2 \frac{1}{x}\,dx = \Delta x[f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)]$

$= \frac{1}{5}\left[\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{1.5}+\frac{1}{1.7}+\frac{1}{1.9}\right]$

$= 0.691908$

Since $f(x)=\frac{1}{x}>0$ for $1\le x\le 2$ the integral represents an area, and the approximation given by the mid-point rule is the sum of the area of the rectangles.

(8) Prove that (a) $\int_a^b x\,dx = \frac{b^2-a^2}{2}$ & (b) $\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$

Solution:

(a) With n sub-intervals, we have $\Delta x = \frac{b-a}{n}$

$\Rightarrow x_i = a + \frac{(b-a)}{n} i$. To evaluate the integral, we use Riemann Sum,

$$\int_a^b x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right)\left(a + \frac{(b-a)}{n} i\right)$$

$$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} \left[a + \left(\frac{b-a}{n}\right) i\right]$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\left[\sum_{i=1}^{n} a + \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) i\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{b-a}{n}\right) a \sum_{i=1}^{n} 1 + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[a\left(\frac{b-a}{n}\right)\cdot n + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[a(b-a) + \frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)\right]$$

$$= ab - a^2 + \frac{1}{2}(a^2 - 2ab + b^2)\cdot 1$$

$$= \frac{1}{2}\left[2ab - 2a^2 + a^2 - 2ab + b^2\right]$$

$$\int_a^b x\,dx. = \frac{b^2-a^2}{2.}$$

(b) With n sub-intervals, we've $\Delta x = \frac{b-a}{n} \Rightarrow x_i = a + \left(\frac{b-a}{n}\right) i$

$$\therefore \int_a^b x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right)\left[a + \left(\frac{b-a}{n}\right) i\right]^2$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \left[a + \left(\frac{b-a}{n}\right) i\right]^2$$

$$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} \left[a^2 + 2a\left(\frac{b-a}{n}\right) i + \left(\frac{b-a}{n}\right)^2 i^2\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{a^2(b-a)}{n} + 2a\left(\frac{b-a}{n}\right)^2 i + \left(\frac{b-a}{n}\right)^3 i^2\right]$$

CHENNAI INSTITUTE OF TECHNOLOGY DEPARTMENT OF SCIENCE AND HUMANITIES

$$\int_a^b x^2\,dx = \lim_{n\to\infty}\left[\frac{a^2(b-a)}{n}\sum_{i=1}^{n} 1 + 2a\left(\frac{b-a}{n}\right)^2\sum_{i=1}^{n} i + \left(\frac{b-a}{n}\right)^3\sum_{i=1}^{n} i^2\right]$$

$$= a^2(b-a) + 2a(b-a)^2 \lim_{n\to\infty}\frac{1}{n^2}\frac{n(n+1)}{2} + (b-a)^3 \lim_{n\to\infty}\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3}$$

$$= a^2b - a^3 + ab^2 - 2a^2b - a^3 + \frac{b^3}{3} - ab^2 + a^2b - \frac{a^3}{3}.$$

$$\int_a^b x^2\,dx. = \frac{b^3-a^3}{3}$$

---

(9) Find the Riemann sum for $f(x) = \sin x$, $0 \le x \le \frac{3\pi}{2}$ with six terms, taking the sample points to be right end-points correct to six decimal places. Repeat the problem with mid-points as the sample points.

Solution:

given that $f(x) = \sin x$, $0 \le x \le 3\pi/2$

$\Delta x = \frac{b-a}{2} = \pi/4$. Since we are using right end-points, $x_i^* = x_i$

Now $R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x. = \Delta x\, f(x_1) + \Delta x f(x_2) + \cdots + \Delta x\, f(x_6)$

$$= \frac{\pi}{4}\left[f(\pi/4) + f(2\pi/4) + f(3\pi/4) + f(\pi) + f\left(\frac{5\pi}{4}\right) + f\left(\frac{6\pi}{4}\right)\right]$$

$$= \frac{\pi}{4}\left[\sin \pi/4 + \sin \pi/2 + \sin 3\pi/4 + \sin\pi + \sin 5\pi/4 + \sin\left(\frac{3\pi}{2}\right)\right]$$

$$= \frac{\pi}{4}\left[\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}} - 1\right]$$

$$= \frac{\pi}{4}\,\frac{1}{\sqrt{2}}$$

$R_6 = 0.555360$

Since we are using the mid points $x_i^* = \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

$$M_6 = \sum_{i=1}^{6} f(\bar{x}_i)\Delta x = \Delta x\left[f(\bar{x}_1)+f(\bar{x}_2)+\cdots+f(\bar{x}_6)\right]$$

$$= \frac{\pi}{4}\left[f(\pi/8)+f(3\pi/8)+f(5\pi/8)+f(7\pi/8)+f(9\pi/8)+f(11\pi/8)\right]$$

$$= \frac{\pi}{4}\left[\sin\frac{\pi}{8}+\sin\frac{3\pi}{8}+\sin\frac{5\pi}{8}+\sin\frac{7\pi}{8}+\sin\frac{9\pi}{8}+\sin\frac{11\pi}{8}\right]$$

$$= \frac{\pi}{4}[1.306563]$$

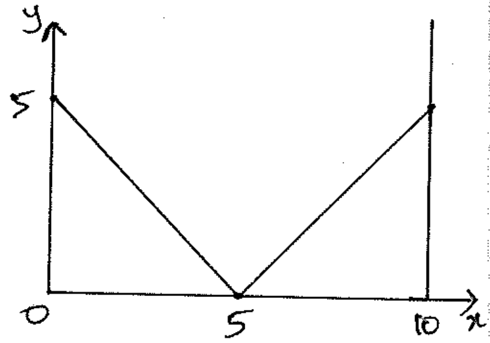
$\therefore M_6 = 1.026172$

(10) Evaluate the integral by interpreting it interms of axes.

$$\int_0^{10} |x-5|\,dx$$

Solution: Let $f(x) = |x-5|$ between 0 & 10.

The value of the integral can be interpreted as the sum of the areas of the two triangles of base length 5 and height 5.



$$\int_0^{10} |x-5|\,dx = 2\cdot\left(\tfrac{1}{2}\right)\cdot 5\cdot 5 = 25.$$

Theorem (1): Fundamental theorem of calculus - part-1.

If f is continuous on $[a,b]$ then the function g is defined by $g(x) = \int_a^x f(t)\,dt$, $a \le x \le b$, is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

Theorem (2): Fundamental theorem of calculus - part-2.

If f is continuous on $[a,b]$ then $\int_a^b f(x)\,dx = F(b)-F(a)$ where F is any anti-derivative of f, (ie) a function such that $F' = f$.

① Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

Sol.

Since $f(t) = \sqrt{1+t^2}$ is continuous, part 1 of the fundamental theorem of calculus gives

$g'(x) = \sqrt{1+x^2}$

② Evaluate the integral $\int_1^3 e^x dx$.

Sol.

Since $f(x) = e^x$ is continuous every where

WKT anti-derivative of $f(x)$ is $F(x) = e^x$.

So part 2 of fundamental theorem gives

$\int_1^3 e^x dx = F(3) - F(1) = e^3 - e^1$.

③ What is wrong with the following calculation.

$$\int_{-1}^{3} \left(\frac{1}{x^2}\right) dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^{3} = \frac{-4}{3}.$$

Sol.

Since $f(x) = \frac{1}{x^2} \geq 0$, $\int_a^b f(x)dx \geq 0$ when $f(x) \geq 0$.

Also $\frac{1}{x^2}$ is discontinuous on $[-1, 3]$, $f(x) = \frac{1}{x^2}$ has an infinite discontinuity at $x = 0$.

$\therefore \int_{-1}^{3} \left(\frac{1}{x^2}\right) dx$ does not exists.

④ Find the derivative of $y = \int_0^{\tan x} \sqrt{t + \sqrt{t}}\, dt$.

Sol

Let $u = \tan x$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Since $\frac{du}{dx} = \sec^2 x$

Then $\frac{d}{dx}\left\{\int_0^{\tan x} \sqrt{t+\sqrt{t}}\, dt\right\} = \frac{d}{dx}\left\{\int_0^u \sqrt{t+\sqrt{t}}\, dt\right\}$

$= \frac{d}{du}\left\{\int_0^u \sqrt{t+\sqrt{t}}\, dt\right\}\frac{du}{dx} = \sqrt{u+\sqrt{u}}\ \sec^2 x.$

$= \sqrt{\tan x + \sqrt{\tan x}}\ \sec^2 x.$

Indefinite Integral

In calculus, an indefinite integral of a function $f(x)$ is a differentiable function F whose derivative is equal to the original function. $f(x)$. i.e., $F' = f(x)$.

Remark:

From the definite and indefinite integrals, we note that a definite integral $\int_a^b f(x)dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is a function. The relation between these two integrals is given by part 2 of the fundamental theorem.

Formulae.

1. $\int cf(x)dx = c\int f(x)dx$
   c is constant.
2. $\int k\,dx = kx + c$
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
4. $\int \frac{1}{x} dx = \log|x| + c$
5. $\int e^x dx = e^x + c$
6. $\int a^x dx = a^x \frac{1}{\log a} + c$
7. $\int \sin x\,dx = -\cos x + c$
8. $\int \cos x\,dx = \sin x + c$
9. $\int \sec^2 x\,dx = \tan x + c$
10. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
11. $\int \sec x \tan x\,dx = \sec x + c$.
12. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$
13. $\int \frac{dx}{x^2+1} = \tan^{-1} x \text{ (or) } -\cot^{-1}(x) + c$
14. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + c \text{ or } -\cos^{-1}(x) + c$
15. $\int \sinh x\,dx = \cosh x + c$
16. $\int \cosh x\,dx = \sinh x + c$
17. $\int \operatorname{sech}^2 x\,dx = \tanh x + c$
18. $\int \operatorname{cosech}^2 x\,dx = -\coth x + c$
19. $\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + c$
20. $\int \operatorname{cosech} x \coth x\,dx = -\operatorname{cosech} x + c$
21. $\int \frac{dx}{\sqrt{x(x^2-1)}} = \sec^{-1}(x) + c \text{ or } \operatorname{cosec}^{-1}(x) + c$
22. $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x) + c$
23. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + c$
24. $\int \frac{dx}{x^2-1} = \tanh^{-1} x + c \text{ (or) } \coth^{-1}(x) + c$.

DEPARTMENT OF SCIENCE AND HUMANITIES

① Evaluate $\int(10x^4-2\sec^2 x)\,dx$.

Sol Let $I=\int(10x^4-2\sec^2 x)\,dx$.

$=10\frac{x^5}{5}-2\tan x+c$.

$\therefore I=2x^5-2\tan x+c$.

② $\int\frac{\cos\theta}{\sin^2\theta}\,d\theta=?$

Sol: Let $I=\int\frac{\cos\theta}{\sin^2\theta}\,d\theta$.

$I=\int\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\,d\theta$.

$I=\int\cot\theta\cdot\operatorname{cosec}\theta\,d\theta$

$\therefore I=-\operatorname{cosec}\theta+c$.

③ Evaluate $\int\left[\sqrt{x^3}+\sqrt[3]{x^2}\right]dx$

Sol. Let $I=\int\left[\sqrt{x^3}+\sqrt[3]{x^2}\right]dx$.

Then $I=\int(x^{3/2}+x^{2/3})\,dx$.

$I=\frac{x^{5/2}}{5/2}+\frac{x^{5/3}}{5/3}+c$.

$=\frac{1}{5}\left[2x^{5/2}+3x^{5/3}\right]+c$.

④ $\int\frac{\sin 2x}{\sin x}\,dx=?$

Sol $\int\frac{\sin 2x}{\sin x}\,dx=\int\frac{2\sin x\cos x}{\sin x}\,dx$

$=\int 2\cos x\,dx$.

$=2\sin x+c$.

⑤ $\int\left[(x^2+1)+\frac{1}{x^2+1}\right]dx=?$

Sol $\int\left[x^2+1+\frac{1}{x^2+1}\right]dx$.

$=\frac{x^3}{3}+x+\tan^{-1}x+c$.

⑥ Evaluate $\int_1^2\left(\frac{1}{x^2}-\frac{4}{x^3}\right)dx$.

Sol

$I=\int_1^2\left(\frac{1}{x^2}-\frac{4}{x^3}\right)dx=\int_1^2\left(x^{-2}-4x^{-3}\right)dx$

$I=\left[\frac{x^{-2+1}}{-2+1}-4\,\frac{x^{-3+1}}{-3+1}\right]_1^2\cancel{dx}$.

$\therefore I=\left[\frac{x^{-1}}{-1}+\frac{4}{3}\,x^{-2}\right]_1^2=\frac{-1}{2}+\frac{1}{2}\,1-2$

$\therefore I=-1$.

⑦ Evaluate $\int_0^1(5x-5^x)\,dx$.

Sol $I=\int_0^1(5x-5^x)\,dx$.

$I=5\left[\frac{x^2}{2}\right]_0^1-\left[\frac{5^x}{\log 5}\right]_0^1$

$=\frac{5}{2}[1-0]-\frac{1}{\log 5}\left[5^1-5^0\right]$

$I=\frac{5}{2}-\frac{4}{\log 5}$

⑧ Evaluate $\int_{\pi/4}^{\pi/3}\operatorname{cosec}^2\theta\,d\theta$.

Sol

$\int_{\pi/4}^{\pi/3}\operatorname{cosec}^2\theta\,d\theta=\left[-\cot\theta\right]_{\pi/4}^{\pi/3}$

$=-\left[\cot\pi/3-\cot\pi/4\right]$

$=\cot\pi/4-\cot\pi/3$.

—.

① Find $\int \frac{dx}{\sin^2 x \cos^2 x}$

Sol

$\int \frac{1}{\sin^2 x \cos^2 x} dx$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x * \cos^2 x} dx.$

$= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx.$

$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx.$

$= \tan x - \cot x + c.$

② $\int \sqrt{1+\sin 2x} \, dx.$

Sol $\int \sqrt{1+\sin 2x} \, dx.$

$= \int \sqrt{\cos^2 x + \sin^2 x + 2 \sin x \cos x} \, dx$

$= \int \sqrt{(\cos x + \sin x)^2} \, dx$

$= \int (\cos x + \sin x) \, dx.$

$= \sin x - \cos x + c.$

③ $\int \frac{1}{1-\cos x} dx.$

Sol

$\int \frac{1}{1-\cos x} dx$

$= \int \frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x} dx$

$= \int \frac{1+\cos x}{1^2 - \cos^2 x} dx.$

$= \int \frac{1+\cos x}{(1-\cos x)(1+\cos x)} dx.$

$= \int \frac{1+\cos x}{\sin^2 x} dx.$

$= \int \frac{1}{\sin^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx.$

$= \int \operatorname{cosec}^2 x \, dx + \int \cot x \operatorname{cosec} x \, dx$

$= -\cot x - \operatorname{cosec} x + c.$

Properties of Definite Integrals

We assume that f and g are continuous functions

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

3. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

   $a < b < c.$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

6. (i) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ iff $f(2a-x) = f(x)$

   (ii) $\int_0^{2a} f(x)\,dx = 0$ iff $f(2a-x) = -f(x)$

7. (i) If $f(x)$ is an even func. Then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

   (ii) If $f(x)$ is an odd func. then $\int_{-a}^{a} f(x)\,dx = 0.$

① Evaluate $\int_{-1}^{1}(2-|x|)\,dx$.

Sol Let $I=\int_{-1}^{1}(2-|x|)\,dx$

Then $I=\int_{-1}^{1}2\,dx-\int_{-1}^{1}|x|\,dx$.

$= 4 - 2\int_0^1 x\,dx$ {∵ $|x|$ is even}

$= 4 - 2\left[\frac{x^2}{2}\right]_0^1 \Rightarrow I=3$

② $\int_0^{\pi/2}\frac{1}{1+\tan x}\,dx$.

Let $I=\int_0^{\pi/2}\frac{1}{1+\tan x}\,dx$.

$I=\int_0^{\pi/2}\frac{\cos x}{\sin x+\cos x}\,dx$ —①

{∵ $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$}

$\therefore I=\int_0^{\pi/2}\frac{\cos(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}\,dx$

$\Rightarrow I=\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}\,dx$ —②

①+② ⇒

$2I=\int_0^{\pi/2}\frac{\sin x+\cos x}{\sin x+\cos x}\,dx$.

$2I=\int_0^{\pi/2}dx \Rightarrow 2I=[x]_0^{\pi/2}$

$\therefore I=\pi/4$.

③ $\int_0^{\pi}\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$.

Sol Let $I=\int_0^{\pi}\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$ —①

Then we've $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$.

$\therefore I=\int_0^{\pi}\frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)}+e^{-\cos(\pi-x)}}\,dx$

$I=\int_0^{\pi}\frac{e^{-\cos x}}{e^{-\cos x}+e^{\cos x}}\,dx$ —②

①+② ⇒ $2I=\int_0^{\pi}\frac{e^{\cos x}+e^{-\cos x}}{e^{-\cos x}+e^{\cos x}}\,dx$

$\Rightarrow 2I=\int_0^{\pi}dx \Rightarrow 2I=[x]_0^{\pi}=\pi$

$\Rightarrow I=\pi/2$

④ $\int_{-2}^{2}|x+1|\,dx$.

Sol Since $|x+1|=\begin{cases}-(x+1), & -2<x<-1\\ (x+1), & -1<x<2\end{cases}$

$\therefore I=\int_{-1}^{1}|x+1|\,dx=-\int_{-2}^{-1}(x+1)\,dx+\int_{-1}^{2}(x+1)\,d$

$\therefore I=-\left[\frac{x^2}{2}+x\right]_{-2}^{-1}+\left[\frac{x^2}{2}+x\right]_{-1}^{2}$

$=-\left[\frac{-1}{2}-0\right]+\left[4-\left(\frac{-1}{2}\right)\right]$

$I=5$

(5) Evaluate $\int_{-\pi/2}^{\pi/2} \sin^{199} x \, dx$.

Sol

Let $I = \int_{-\pi/2}^{\pi/2} \sin^{199} x \, dx$

Let $f(x) = \sin^{199} x$

Then $f(-x) = \sin^{199}(-x)$

$= -\sin^{199} x$.

$= -f(x)$

$\therefore f(x)$ is an odd fun.

$\therefore \int_{-\pi/2}^{\pi/2} \sin^{199} x \, dx = 0$.

$\left\{ \because \int_{-a}^{a} f(x)\,dx = 0, \text{ where } f(x) \text{ is odd} \right\}$

Integration by Substitution rule.

Type I (A).

$\int [f(x)]^n f'(x)\,dx$ (or)

$\int \phi[f(x)]\, f'(x)\,dx$. —(1)

Substitute $u = f(x)$,

$du = f'(x)\,dx$

$(1) \Rightarrow \int u^n \, du$ or $\int \phi(u)\, du$

and then proceed.

(1) Solve $\int \frac{1}{x^2} \sqrt{2 - \frac{1}{x}} \, dx$.

Sol

Let $I = \int \frac{1}{x^2} \sqrt{2 - \frac{1}{x}} \, dx$

$\left\{ \because \int f'(x)\, \phi[f(x)]\,dx \right\}$

$\therefore u = 2 - 1/x \qquad du = \frac{1}{x^2} dx$

$\therefore I = \int \sqrt{u}\, du = \frac{2}{3} u^{3/2} + c$

Hence $I = \frac{2}{3}\left(2 - \frac{1}{x}\right)^{3/2} + c$.

(2) Evaluate $\int x^2 / \sqrt{x+5} \, dx$.

Sol let $u = \sqrt{x+5}$

$du = \frac{1}{2\sqrt{x+5}} dx. \Rightarrow 2du = \frac{dx}{\sqrt{x+5}}$.

Now $u^2 = x+5 \Rightarrow x^2 = (u^2-5)^2$

$\therefore x^2 = u^4 - 10u^2 + 25$

$\therefore I = \int (u^4 - 10u^2 + 25)\, 2du$

$= 2\left(\frac{u^5}{5} - \frac{10u^3}{3} + 25u\right) + c$

$= \frac{2}{5}(x+5)^{5/2} - \frac{20}{3}(x+5)^{3/2} + 50(x+5)^{1/2} + c$

(3) Evaluate $\int \frac{1}{(3x-4)^{3/2}} dx$.

Sol Let $I = \int \frac{1}{(3x-4)^{3/2}} dx$.

Let $u = 3x - 4 \qquad du = 3dx$.

$\Rightarrow \frac{du}{3} = dx$.

$\therefore I = \int \frac{du/3}{u^{3/2}} = \frac{-2}{3} \frac{1}{\sqrt{u}} + c$

$\Rightarrow I = \frac{-2}{3} \frac{1}{\sqrt{3x-4}} + c$.

Type I (B) [logarithmic fun.]

① Evaluate $\int_1^e \frac{\log x}{x}\,dx$.

Sol

Let $I = \int_1^e \frac{\log x}{x}\,dx$.

Let $u = \log x \quad du = \frac{dx}{x}$.

x	1	e
u	0	1

$\therefore I = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

② Evaluate $\int \frac{\sec^2(\log x)}{x}\,dx$.

Sol

Let $I = \int \frac{\sec^2(\log x)}{x}\,dx$.

Let $u = \log x \quad du = \frac{dx}{x}$.

$\therefore I = \int \sec^2 u\,du = \tan u + c$

$\therefore I = \tan(\log x) + c$.

Type I (C) [Exponential funs.]

① Evaluate $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$.

Sol

Let $I = \int_1^2 \frac{e^{1/x}}{x^2}\,dx$.

Let $u = e^{1/x}$; $du = e^{1/x}\left(\frac{-1}{x^2}\right)dx$

$\Rightarrow -du = \frac{e^{1/x}}{x^2}\,dx$

x	1	2
u	e	$e^{1/2}$

$\therefore I = \int_e^{e^{1/2}} (-du) = -[u]_e^{e^{1/2}} = e - \sqrt{e}$

② Evaluate $\int e^{\cos x}\sin x\,dx$.

Sol

Let $I = \int e^{\cos x}\sin x\,dx$

$u = e^{\cos x} \quad du = e^{\cos x}(-\sin x)dx$

$\therefore I = \int(-du) = -u + c = e^{-\cos x} + c$.

③ Evaluate $\int (\log a)^x\,dx$.

Sol Let $I = \int (\log a)^x\,dx$.

Then

$I = \int e^{\log[\log a]^x}\,dx$.

$= \int e^{x[\log(\log a)]}\,dx$.

$= \frac{e^{[\log(\log a)]x}}{\log(\log a)} + c$.

$\therefore I = \frac{(\log a)^x}{\log(\log a)} + c$.

Type I (D) [Trigonometric funs.]

① Evaluate $\int_0^{\pi/2} \cos x \sin(\sin x)\,dx$

Sol

Let $I = \int_0^{\pi/2} \cos x \sin(\sin x)\,dx$

put $u = \sin x \quad du = \cos x\,dx$

x	0	$\pi/2$
u	0	1

$\therefore I = \int_0^1 \sin u\,du = [-\cos u]_0^1$

$\therefore I = 1 - \cos 1$

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(2) $\int \sec x \, dx = ?$

Sol: Let $I = \int \sec x \, dx$

$\cancel{\text{let}}\ I = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$

let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\, dx$

$\therefore I = \int \frac{1}{u}\, du = \log u + c.$

$\Rightarrow I = \log(\sec x + \tan x) + c.$

(3) Evaluate $\int e^{\tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right] dx$

Let $I = \int e^{\tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right] dx$

let $u = \tan^{-1}x$ $\quad du = \frac{dx}{1+x^2}$

$\Rightarrow \tan u = x$

$1+x+x^2 = \tan u + \sec^2 u$

$\therefore I = \int e^u (\tan u + \sec^2 u)\, du$

put $t = e^u \tan u$

$dt = [e^u \sec^2 u + \tan u\, e^u]\, du$

$\therefore I = \int dt = t + c$

$\Rightarrow I = x\, e^{\tan^{-1}x} + c.$

TECHNIQUES OF INTEGRATION

Integration by parts method

$\int u\, dv = uv - \int v\, du.$

(1) Solve $\int_0^1 \frac{y}{e^{2y}}\, dy$

let $I = \int_0^1 y\, e^{-2y}\, dy$

let $u = y$ $\quad dv = e^{-2y} dy$

$du = dy$ $\quad v = e^{-2y}/-2$

$\therefore I = y\left(\frac{e^{-2y}}{-2}\right)\Big|_0^1 + \int_0^1 \frac{e^{-2y}}{2}\, dy$

$= \frac{-1}{2} e^{-2} + \frac{1}{2}\left[\frac{e^{-2y}}{-2}\right]_0^1$

$= \frac{-1}{2} e^{-2} - \frac{1}{4}\left[e^{-2} - 1\right]$

$\Rightarrow I = \frac{1}{4} - \frac{3}{4} e^{-2}$

(2) Evaluate $\int \left(\frac{\log x}{x}\right)^2 dx.$

Sol: Let $I = \int (\log x)^2 / x^2 \, dx.$

$\cancel{\text{let}}\ u = (\log x)^2 \quad dv = \frac{1}{x^2} dx.$

$du = 2 \log x \,(1/x)\, dx \quad v = \frac{-1}{x}.$

$\therefore I = \frac{-1}{x}(\log x)^2 + 2\underbrace{\int (\log x)\left(\frac{1}{x^2}\right) dx}_{(1)}$

Consider $\int (\log x)\left(\frac{1}{x^2}\right) dx$

let $u = \log x \quad dv = dx/x^2$

$du = dx/x \quad v = -1/x.$

$\therefore \int (\log x)\left(\frac{1}{x^2}\right) dx$

$= \frac{-1}{x}\log x + \int \frac{1}{x^2}\, dx$

$= \frac{-1}{x}\log x - \frac{1}{x} + c.$

$\therefore I = \frac{-1}{x}(\log x)^2 - 2\frac{1}{x}\log x - 2\left(\frac{1}{x}\right) + 2c$

DEPARTMENT OF SCIENCE AND HUMANITIES

(3) Evaluate $\int \frac{x}{1+\sin x}\,dx$.

Sol

Let $I = \int \frac{x}{1+\sin x}\,dx$

Then $I = \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$.

$= \int \frac{(x - x\sin x)}{\cos^2 x}\,dx$

$I = \int (x\sec^2 x - x\sec x\tan x)\,dx$ — (1)

Take $\int x\sec^2 x\,dx$.

Let $u = x$ $\quad dv = \sec^2 x\,dx$.

$du = dx$ $\quad v = \int \sec^2 x\,dx$

$\therefore \int x\sec^2 x\,dx$

$= x\tan x - \int \tan x\,dx$

$= x\tan x - \log(\sec x)$

Take $\int x\sec x\tan x\,dx$.

$u = x$, $\quad dv = \sec x\tan x\,dx$

$du = dx$ $\quad v = \sec x$

$\therefore \int x\sec x\tan x\,dx$

$= x\sec x - \log(\sec x + \tan x)$

$\therefore I = \int \frac{x}{1+\sin x}\,dx$

$= x\tan x - \log(\sec x)$

$- x\sec x + \log(\sec x + \tan x) + c$

(4) Evaluate $\int_0^1 \tan^{-1} x\,dx$

Let $I = \int_0^1 \tan^{-1} x\,dx$.

$u = \tan^{-1} x$ $\quad dv = dx$.

$du = dx/1+x^2$ $\quad v = x$.

$\therefore I = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x\,dx}{1+x^2}$

$= \frac{\pi}{4} - \left[\frac{1}{2}\log(1+x^2)\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}\log 2$.

(5) Evaluate $\int_0^{1/2} \cos^{-1} x\,dx$.

Sol

Let $I = \int_0^{1/2} \cos^{-1} x\,dx$.

$u = \cos^{-1} x$ $\quad dv = dx$:

$du = \frac{-dx}{\sqrt{1-x^2}}$ $\quad v = x$.

$\therefore I = x\cos^{-1}(x)\Big|_0^{1/2} + \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}}$

$= \frac{1}{2}\left(\frac{\pi}{3}\right) + \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}}$ — (1)

Let $t = 1-x^2$ $\quad dt = -2x\,dx$

$\Rightarrow -\frac{dt}{2} = x\,dx$.

x	0	1/2
t	1	3/4

$\therefore \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} = \int_1^{3/4} \frac{-dt/2}{\sqrt{t}}$

$= \frac{-1}{2}\left[2\sqrt{t}\right]_0^{3/4} = -\frac{\sqrt{3}}{2} + 1$

$\therefore I = \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$

(6) Evaluate $\int e^{ax}\cos bx\,dx$.

Sol

Let $I = \int e^{ax}\cos bx\,dx$

$u = \cos bx \qquad dv = e^{ax}\,dx$

$du = -\sin bx\,(b) \qquad v = \frac{e^{ax}}{a}$

$\therefore I = \frac{e^{ax}}{a}\cos bx + \int \frac{e^{ax}}{a}\sin bx\,(b)\,dx$

$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx \quad \text{—(1)}$

Take $\int e^{ax}\sin bx\,dx$

$u = \sin bx \qquad dv = e^{ax}\,dx$

$du = b\cos bx\,dx \qquad v = \frac{e^{ax}}{a}$

$\therefore \int e^{ax}\sin bx\,dx$

$= \frac{e^{ax}}{a}\sin bx - \int \frac{e^{ax}}{a}\, b\cos bx\,dx$

$= \frac{e^{ax}}{a}\sin bx - \frac{b}{a} I$

$\therefore (1) \Rightarrow I = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\,\frac{e^{ax}}{a}\sin bx - \frac{b^2}{a^2} I$

$\Rightarrow I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a}\cos bx + b\,\frac{e^{ax}}{a^2}\sin bx.$

$\Rightarrow I\left(\frac{a^2+b^2}{a^2}\right) = \frac{e^{ax}}{a^2}\left[a\cos bx + b\sin bx\right]$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + c.$

(7) Evaluate $\int e^{ax}\sin bx\,dx$.

(8) Evaluate $\int e^{-ax}\cos bx\,dx$.

(9) Evaluate $\int e^{-ax}\sin bx\,dx$.

(10) Find the reduction formula for $\int \sin^n x\,dx$ and also find $\int_0^{\pi/2}\sin^n x\,dx$, $\int_0^{\pi/2}\sin^5 x\,dx$

Sol Let $I_n = \int \sin^n x\,dx$.

Then $I_n = \int \sin^{n-1}x\,\sin x\,dx$.

$u = \sin^{n-1}x \qquad dv = \sin x\,dx$

$du = (n-1)\sin^{n-2}x\cos x\,dx$

$v = -\cos x$

$\therefore I_n = -\cos x\sin^{n-1}x + (n-1)\int \cos x\left(\sin^{n-2}x\cos x\,dx\right)$

$= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$

$= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$

$= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$

$\therefore I_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2} - (n-1)I_n.$

$\Rightarrow I_n = \frac{-1}{n} \cos x \sin^{n-1}x + \frac{n-1}{n} I_{n-2}.$

Now $I_n = \int_0^{\pi/2} \sin^n x\, dx.$

$= \frac{-1}{n} \cos x \sin^{n-1}x \Big|_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}.$

$I_{n-2} = \frac{n-3}{n-2} I_{n-4}.$

$I_{n-4} = \frac{n-5}{n-4} I_{n-6}.$

In general,

$$\int_0^{\pi/2} \sin^n x\, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} I_0 & (n \text{ is even}) \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} I_1 & (n \text{ is odd}) \end{cases} \quad \text{--- (1)}$$

Since $I_n = \int_0^{\pi/2} \sin^n x\, dx.$

$I_0 = \int_0^{\pi/2} dx = \pi/2$

$I_1 = \int_0^{\pi/2} \sin x\, dx = -\cos x \Big|_0^{\pi/2}$

$\therefore I_1 = 1$

$\therefore (1) \Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx.$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & n \text{ is even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1, & n \text{ is odd.} \end{cases}$$

Also $\int_0^{\pi/2} \sin^5 x\, dx.$

$= \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = 8/15$

(11) Reduction formula for $\int \cos^n x\, dx$ and hence find $\int_0^{\pi/2} \cos^n x\, dx$, $\int_0^{\pi/2} \cos^9 x\, dx$

Sol: Let $I_n = \int \cos^n x\, dx$

Then $I_n = \int \cos^{n-1}x \cos x\, dx.$

$u = \cos^{n-1}x \qquad dv = \cos x\, dx$

$du = (n-1)\cos^{n-2}x(-\sin x)dx \qquad v = \sin x.$

$\therefore I_n = \sin x \cos^{n-1}x + (n-1)\int \sin^2 x \cos^{n-2}x\, dx.$

$= \sin x \cos^{n-1}x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\, dx.$

$= \sin x \cos^{n-1}x + (n-1)\int (\cos^{n-2}x\, dx - \cos^n x\, dx)$

$= \sin x \cos^{n-1}x + (n-1) I_{n-2} - (n-1)I_n$

$\Rightarrow nI_n = \sin x \cos^{n-1}x + (n-1) I_{n-2}$

$\therefore I_n = \frac{1}{n} \sin x \cos^{n-1}x + \frac{n-1}{n} I_{n-2}$

now $I_n = \int_0^{\pi/2} \cos^n x \, dx.$

$= \frac{1}{n} \sin x \cos^{n-1} x \Big|_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$; $I_{n-2} = \frac{n-3}{n-2} I_{n-4}$

In general.

$$I_n = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{1}{2} I_0 (n \text{ is even}) \\ \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{2}{3} I_1 (n \text{ is odd}) \end{cases} \quad \text{— (c)}$$

$I_0 = \int_0^{\pi/2} \cos^0 x \, dx = \pi/2$

$I_1 = \int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2}$

$\therefore I_1 = 1$

$$(c) \Rightarrow I_n = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & n \text{ is even} \\ \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1, & n \text{ is odd} \end{cases}$$

also $\int_0^{\pi/2} \cos^9 x \, dx = \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$

$\therefore \int_0^{\pi/2} \cos^9 x \, dx = \frac{128}{315}$

(12) Evaluate $\int \sec^n x \, dx$

let $I_n = \int \sec^n x \, dx.$ — (1)

Then $I_n = \int \sec^{n-2} x \sec^2 x \, dx.$

~~$u = \sec^{n-2} x$~~ ~~$dv = \sec^2 x\,dx$~~ ~~$du = (n-2)$~~

$\therefore I_n = \int \sec^{n-2} x \; d(\tan x) \, dx.$

$u = \sec^{n-2} x \qquad dv = d(\tan x)$

$du = (n-2) \sec^{n-3} x (\sec x \tan x) dx$

& $V = \tan x.$

$\therefore I_n = \tan x \sec^{n-2} x$

$- \int (n-2) \sec^{n-2} x \tan^2 x \, dx.$

$I_n = \tan x \sec^{n-2} x$

$- (n-2) \int \sec^{n-2} x (-1 + \sec^2 x) dx$

$I_n = \tan x \sec^{n-2} x.$

$-(n-2) \left[\int \sec^n x \, dx - \int \sec^{n-2} x \, dx \right]$

$I_n = \tan x \sec^{n-2} x$

$-(n-2) I_n + (n-2) I_{n-2}.$

$\Rightarrow I_n + (n-2) I_n$

$= \tan x \sec^{n-2} x + (n-2) I_{n-2}$

$\Rightarrow (n-1) I_n = \tan x \sec^{n-2} x$

$+ (n-2) I_{n-2}$

$\Rightarrow I_n = \frac{1}{n-1} \tan x \sec^{n-2} x$

$+ \frac{n-2}{n-1} I_{n-2}. \; , n \geq 2.$

$I_0 = \int dx = x + c$ (from (1))

$I_1 = \int \sec x \, dx = \log(\sec x + \tan x) + c$

(13) Evaluate $\int (\log x)^n dx$. also find $\int (\log x)^2 dx$.

Sol Let $I_n = \int (\log x)^n dx$.

$u = (\log x)^n \qquad dv = dx$

$du = n(\log x)^{n-1} \frac{1}{x} dx \qquad v = x$

$\therefore I_n = x(\log x)^n - \int x\, n(\log x)^{n-1} \frac{1}{x} dx$

$= x(\log x)^n - n\int (\log x)^{n-1} dx$.

$\therefore I_n = x(\log x)^n - n I_{n-1}$

$I_0 = \int dx = x + c$

To find $\int (\log x)^2 dx$, put $n=2$ in I_n.

$\therefore I_2 = \int (\log x)^2 dx$

$= x(\log x)^2 - 2I_1$

$= x(\log x)^2 - 2I_1 \quad$ (C)

But $I_1 = x\log x - \cancel{2} I_0$

ie: $I_1 = x\log x - \cancel{2}x \cancel{+c}$

$\therefore$ (C) $\Rightarrow I_2 = x(\log x)^2 - 2(x\log x - x) + c$

$I_2 = x(\log x)^2 - 2x\log x + 2x + c$.

(14) Find the reduction formula for $\int x^n \sin mx\, dx$.

Sol Let $I_n = \int x^n \sin mx\, dx$.

$= \int x^n d\left(-\frac{\cos mx}{m}\right)$

$= -x^n \frac{\cos mx}{m} + \int \frac{\cos mx}{m} d(x^n)$

$= -x^n \frac{\cos mx}{m} + \frac{1}{m}\int \cos mx\,(n x^{n-1}) dx$.

$= -x^n \frac{\cos mx}{m} + \frac{n}{m}\int x^{n-1} d\left(\frac{\sin mx}{m}\right)$

$= -x^n \frac{\cos mx}{m} + \frac{n}{m}\left[x^{n-1}\frac{\sin mx}{m} - \int \frac{\sin mx}{m}(n-1)x^{n-2} dx\right]$

$= -x^n \frac{\cos mx}{m} + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2}\int x^{n-2} \sin mx\, dx$.

$\therefore I_n = -x^n \frac{\cos mx}{m} + \frac{n}{m^2} x^{n-1} \sin mx - \frac{(n-1)n}{m^2} I_{n-2}$.

(14) Evaluate $\int x^n e^x dx$

Let $u = x^n \qquad dv = e^x dx$

$du = n x^{n-1} dx \qquad v = e^x$

$\therefore \int x^n e^x dx = x^n e^x - \int e^x n x^{n-1} dx$

$\therefore I_n = x^n e^x - n I_{n-1}$

Problems based on trigonometric integrals

① Evaluate $\int \sin^6 x \cos^3 x\,dx$

Sol Let $I = \int \sin^6 x \cos^3 x\,dx$.

Then $I = \int \sin^6 x \cos^2 x \cos x\,dx$.

$= \int \sin^6 x (1-\sin^2 x)\cos x\,dx$

put $u = \sin x$ $\quad du = \cos x\,dx$

$I = \int u^6(1-u^2)\,du = \int (u^6 - u^8)\,du$

$\therefore I = \frac{u^7}{7} - \frac{u^9}{9} + c$.

$\therefore I = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + c$

② Solve $I = \int_0^{\pi/3} \tan^5 x \sec^4 x\,dx$

Sol

$I = \int_0^{\pi/3} \tan^4 x \sec^3 x (\sec x \tan x)\,dx$

$= \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^3 x (\sec x \tan x)\,dx$.

let $u = \sec x$ $\quad du = \sec x \tan x\,dx$

x	0	$\pi/3$
u	1	2

$\therefore I = \int_1^2 (u^2-1)^2 u^3\,du$

$= \int_1^2 (u^4 - 2u^2 + 1)u^3\,du$

$= \int_1^2 (u^3 - 2u^5 + u^7)\,du$.

$= \left[\frac{u^4}{4} - \frac{2u^6}{6} + \frac{u^8}{8}\right]_1^2 = \frac{117}{8}$.

Trigonometric Substitution

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$	$1-\sin^2\theta = \cos^2\theta$.
$\sqrt{a^2+x^2}$	$x = a\tan\theta$, $-\pi/2 < \theta < \pi/2$	$1+\tan^2\theta = \sec^2\theta$.
$\sqrt{x^2-a^2}$	$x = a\sec\theta$, $0 \le \theta < \pi/2$, $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2 x$.

① Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Sol

$I = \int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

put $x = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \pi/2$

$dx = 3\cos\theta\,d\theta$

$\therefore I = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}(3\cos x)\,d\theta$.

$= \int \frac{3\sqrt{1-\sin^2\theta}}{9\sin^2\theta}(3\cos\theta)\,d\theta$.

$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int (\csc^2\theta - 1)\,d\theta$

$= -\cot\theta - \theta + c$.

$\therefore I = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + c$.

② Evaluate $I = \int \frac{1}{\sqrt{a^2-x^2}}$

$x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \pi/2$

$dx = a\cos\theta\,d\theta$.

CHENNAI INSTITUTE OF TECHNOLOGY DEPARTMENT OF SCIENCE AND HUMANITIES

$$I = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} a\cos\theta\, d\theta$$

$$= \int \frac{1}{a\cos\theta} a\cos\theta\, d\theta$$

$$\therefore I = \int d\theta = \theta + c = \sin^{-1}\left(\frac{x}{a}\right) + c$$

(3) Let $I = \int \frac{1}{x^2\sqrt{x^2-1}}\, dx,\ x>1$

put $x = \sec\theta$

$dx = \sec\theta\tan\theta$

$$\therefore I = \int \frac{1}{\sec^2\theta\tan\theta} \sec\theta\tan\theta\, d\theta$$

$$= \int \cos\theta\, d\theta$$

$$= \sin\theta + c$$

$$= \frac{\sqrt{x^2-1}}{x} + c$$

Partial Fraction method.

(c) Solve $\int \frac{dx}{(x+1)(x+2)}$

Sol

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$\Rightarrow A(x+2) + B(x+1) = 1$

put $x=-2$, $-B = 1 \Rightarrow \boxed{B=-1}$

put $x=-1$, $\boxed{A=1}$

$$\therefore \int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \log\left(\frac{x+1}{x+2}\right) + c$$

(2) Solve $\int \frac{x^2+1}{(x-3)(x-2)^2}\, dx$

Sol

$$\frac{x^2+1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \quad (1)$$

$$A(x-2)^2 + B(x-2)(x-3) + C(x-3) = x^2+1.$$

put $x=3$, $\boxed{A=10}$

put $x=2$, $\boxed{C=-5}$

equating coefft of x^2 on both sides. $1 = A+B$

$\therefore \boxed{B=-9}$

$$\therefore (1) \Rightarrow \frac{x^2+1}{(x-3)(x-2)^2} = \frac{10}{x-3} - \frac{9}{x-2} - \frac{5}{(x-2)^2}$$

$$\therefore \int \frac{x^2+1}{(x-3)(x-2)^2}dx = 10\int\frac{dx}{x-3} - 9\int\frac{dx}{x-2} - 5\int\frac{dx}{(x-2)^2}$$

$$I = 10\log(x-3) - 9\log(x-2) + 5\left(\frac{1}{x-2}\right) + c$$

(3) Solve $\int \frac{10\, dx}{(x-1)(x^2+9)}$

Sol

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$\Rightarrow A(x^2+9) + (Bx+C)(x-1) = 10$

put $x=1$, $\boxed{A=1}$

put $x=0$, $9A - C = 10$

$\Rightarrow \boxed{C=-1}$

Equating the coefficient of x^2, we get, $A+B=0$

$\Rightarrow \boxed{B=-1}$

$$\therefore \frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{(-x-1)}{x^2+9}$$

$$\text{Now} \int \frac{10\,dx}{(x-1)(x^2+9)} = \int \frac{dx}{x-1} - \int \frac{x\,dx}{x^2+9} - \int \frac{dx}{x^2+9}$$

$$\therefore I = \log(x-1) - \frac{1}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c.$$

(4) Integrate the following.

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$$

Sol Since deg (Nr) > deg (Dr), it is improper

So, $f(x) = q(x) + \frac{r(x)}{p(x)}$

ie:

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4-0x^3-2x^2+4x+1 \\ & x^4-x^3-x^2+x \\ \hline & x^3-x^2+3x+1 \\ & x^3-x^2-x+1 \\ (-) & \\ \hline & 4x. \end{array}$$

$$\therefore \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{4x}{x^3-x^2-x+1}. \quad \text{— (1)}$$

$$\text{Now } \frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)}$$

$$= \frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\Rightarrow 4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

put $x=1$, $\boxed{B=2}$

put $x=-1$, $\boxed{C=-1}$

equating the coefft. of x^2,

$A+C=0 \Rightarrow \boxed{A=1}$

$$\therefore (1) \Rightarrow \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}$$

$$= (x+1) + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

$$\therefore I = \int (x+1)\,dx + \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} - \int \frac{dx}{x+1}$$

$$I = \frac{x^2}{2} + x + \log(x-1) + 2\frac{(-1)}{(x-1)} - \log(x+1) + c$$

$$\therefore I = \frac{x^2}{2} + x - \frac{2}{x-1} + \log\left(\frac{x-1}{x+1}\right) + c$$

—.

The given integral is of the form $\int \frac{dx}{ax^2+bx+c}$

① Evaluate $\int \frac{dx}{3x^2-4x-5}$

Sol

Let $I = \int \frac{dx}{3x^2-4x-5}$

Then $I = \frac{1}{3}\int \frac{dx}{x^2-\frac{4}{3}x-\frac{5}{3}}$

Now

$$x^2-\frac{4}{3}x-\frac{5}{3}$$

$$= x^2-\frac{4}{3}x+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2-\frac{5}{3}$$

$$= \left(x-\frac{2}{3}\right)^2-\frac{19}{9}$$

$$= \left(x-\frac{2}{3}\right)^2-\left(\frac{\sqrt{19}}{3}\right)^2$$

$$\therefore I = \frac{1}{3}\int \frac{dx}{\left(x-\frac{2}{3}\right)^2-\left(\frac{\sqrt{19}}{3}\right)^2}$$

$$\left\{ \because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c \right\}$$

$$\therefore I = \frac{1}{3}\frac{1}{2\left(\frac{\sqrt{19}}{3}\right)}\log\left[\frac{\left(x-\frac{2}{3}\right)-\frac{\sqrt{19}}{3}}{\left(x-\frac{2}{3}\right)+\frac{\sqrt{19}}{3}}\right]+c$$

$$\Rightarrow I = \frac{1}{2\sqrt{19}}\log\left[\frac{3x-2-\sqrt{19}}{3x-2+\sqrt{19}}\right]+c.$$

The given integral is of the form $\int \frac{px+q}{ax^2+bx+c}$

$$Nr = A\frac{d}{dx}(Dr)+B.$$

i.e. $$\int \frac{px+q}{ax^2+bx+c} = \int \frac{A\frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c} + \int \frac{B}{ax^2+bx+c}\,dx.$$

① $\int \frac{2x+3}{x^2+2x+5}\,dx = ?$

Sol Let $2x+3 = A\frac{d}{dx}(x^2+2x+5)+B.$

$$\Rightarrow 2x+3 = A(2x+2)+B \quad \text{— ①}$$

Equating the coefficients of x we get

$2A = 2 \Rightarrow \boxed{A=1}$ & $\cancel{B=3}$

$2A+B=3 \Rightarrow \boxed{B=1}$

$$\therefore 2x+3 = (2x+2)+1.$$

$$\therefore I = \int \frac{2x+3}{x^2+2x+5}\,dx.$$

$$= \int \frac{(2x+2)+1}{x^2+2x+5}\,dx$$

$$= \int \frac{(2x+2)\,dx}{x^2+2x+5}+\int \frac{dx}{x^2+2x+5}$$

$$= \log(x^2+2x+5)+\int \frac{dx}{x^2+2x+5} \quad \text{— ②}$$

Take $\int \frac{dx}{x^2+2x+5}$

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{x^2+2x+1^2-1^2+5}$$

$$= \int \frac{dx}{(x+1)^2+4}.$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c.$$

(2) $\Rightarrow I = \log(x^2+2x+5) + \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$

The given integral is of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

$$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$$

(1) Solve $\int \frac{2x-1}{\sqrt{x^2+5x+6}}\,dx$

Sol

$$2x-1 = A\frac{d}{dx}(x^2+5x+6)+B$$

$$\Rightarrow 2x-1 = (2x+5)A+B$$

Equating the coefficients of x we get

$$2A = 2 \Rightarrow \boxed{A=1}$$

~~put~~ Also, $-1 = 5A+B$

$$\Rightarrow -1 = 5+B \Rightarrow \boxed{B=-6}$$

$$\therefore I = \int \frac{2x-1}{\sqrt{x^2+5x+6}}\,dx$$

$$= \int \frac{(2x+5)}{\sqrt{x^2+5x+6}}\,dx - 6\int \frac{dx}{\sqrt{x^2+5x+6}}$$

$$\therefore I = 2\sqrt{x^2+5x+6} - 6\int \frac{dx}{\left(x+\frac{5}{2}\right)^2+6-\frac{25}{4}}$$

$$= 2\sqrt{x^2+5x+6} - 6\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$= 2\sqrt{x^2+5x+6} - 6\log\left[\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right]+c$$

The given integral is of the form $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

put $px+q = A\,d(ax^2+bx+c)+B$

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx.$$

$$= A\int \sqrt{ax^2+bx+c}\; d(ax^2+bx+c) + B\int \sqrt{ax^2+bx+c}\,dx$$

$$= A\sqrt{u}\,du + B\sqrt{ax^2+bx+c}\,dx$$

(1) Solve $\int (3x-2)\sqrt{x^2+x+1}\,dx$

Sol

$$3x-2 = A\frac{d}{dx}(x^2+x+1)+B$$

$$3x-2 = A(2x+1)+B$$

Equating the coefficients of x, we get

$$2A = 3 \Rightarrow \boxed{A=\frac{3}{2}}$$

also, $A+B=-2 \Rightarrow \boxed{B=\frac{-7}{2}}$

$$\therefore I = \int (3x-2)\sqrt{x^2+x+1}\,dx.$$

$$= \frac{3}{2}\int \sqrt{x^2+x+1}\;d(x^2+x+1)$$

$$-\frac{7}{2}\int\sqrt{x^2+x+1}\;dx$$

$$= \frac{3}{2}\,\frac{(x^2+x+1)^{3/2}}{3/2}$$

$$-\frac{7}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\;dx.$$

$$= \cancel{\tfrac{3}{4}}\,(x^2+x+1)^{3/2}$$

$$-\frac{7}{2}\left[\left(\frac{2x+1}{4}\right)\sqrt{x^2+x+1}\right.$$

$$\left.+\frac{3}{8}\log\left[x+\frac{1}{2}+\sqrt{x^2+x+1}\right]\right] + c$$

Improper Integrals

In a regular (proper) definite integral $\int_a^b f(x)\,dx$ it is assumed that the limits of integration are finite and that the integrand $f(x)$ is continuous for every value of x in the interval $a \le x \le b$.

If atleast one of these conditions is violated, then the integral is known as an improper integral. (Singular or indefinite integral)

Defn. of an improper integral (Type I)

① If $\int_0^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx.$$

② If $\int_t^b f(x)\,dx$ exists for every number $t \le b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided this limit exists. (as a finite no.)

③ If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, if the corresponding limit exists and divergent if the limit does not exists.

③ If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

'a' is any real number.

Defn. of improper Integral

Type:2

① If f is continuous on $[a,b)$ and is discontinuous at b, then $\int_a^b f(x)\,dx$

$= \lim_{t\to b^-} \int_a^t f(x)\,dx$. If this limit exists.

② If f is continuous on $(a,b]$ and is discontinuous at a, then

$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$

If this limit exist

The improper integral $\int_a^b f(x)\,dx$ is convergent if the corresponding limit exists and divergent if the limit does not exist.

③ If f has a discontinuity at c, where $a<c<b$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

① Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.

Sol given $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$.

$= \lim_{t\to\infty} [\log x]_1^t$

$= \lim_{t\to\infty} [\log t - \log 1]$.

$= \lim_{t\to\infty} \log t = \infty$ (not finite)

$\therefore \int_1^\infty \frac{1}{x}\,dx$ is divergent.

② Determine whether the integral $\int_1^\infty \frac{\log x}{x^2}\,dx$ is convergent or divergent.

Sol Take $\int \frac{\log x}{x^2}\,dx$

$u = \log x$ $\quad dv = \frac{1}{x^2}\,dx$

$du = \frac{dx}{x}$ $\quad \Rightarrow v = -1/x$

$\therefore \int \log x/x^2\,dx$

$= \frac{-1}{x}\log x + \int \frac{1}{x^2}\,dx$

$= -\frac{1}{x}\log x - \frac{1}{x}$

$= \frac{-1}{x}[\log x + 1]$

$$\therefore \int_1^\infty \frac{\log x}{x^2}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{\log x}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{-1}{x}\,(\log x+1)\right]_1^t$$

$$= \lim_{t\to\infty} \left[\frac{-1}{t}(\log t+1) + \log 1 + 1\right]$$

$$= \lim_{t\to\infty} \left(\frac{-\log t}{t}\right) = \left(\frac{\infty}{\infty}\text{ form}\right)$$

Apply L'Hospital rule,

$$\cancel{\therefore I = \lim_{t\to\infty}\left[\frac{\log t}{t}\right]}$$

$$I = \lim_{t\to\infty} \left(\frac{-1/t}{1}\right) = 0$$

$$\therefore I = \int_1^\infty \frac{\log x}{x^2}\,dx = 1 \text{ (finite)}$$

$\therefore \displaystyle\int_1^\infty \frac{\log x}{x^2}$ is Convergent.

(3) Evaluate $\displaystyle\int_{-\infty}^0 x e^x\,dx$

Sol

$$I = \lim_{t\to-\infty} \int_t^0 x e^x\,dx.$$

$$= \lim_{t\to-\infty} x\, d(e^x)$$

$$= \lim_{t\to-\infty} \left\{ \left[x e^x\right]_t^0 - \int_t^0 e^x\,dx \right\}$$

$$= \lim_{t\to-\infty} \left[-t e^t - (1-e^t)\right]$$

$$= \lim_{t\to-\infty} \left[-t e^t - 1 + e^t\right] = -1 \text{ (finite)}$$

$\therefore \displaystyle\int_{-\infty}^0 x e^x\,dx$ is Convergent.

(4) For what values of p in the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$. Convergent?

Sol

If $p \neq 1$, $\displaystyle\lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx$

$$= \lim_{t\to\infty} \int_1^t x^{-p}\,dx.$$

$$= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right]$$

$$= \lim_{t\to\infty} \frac{1}{p-1}\left[1 - \frac{1}{t^{p-1}}\right]$$

$$= \begin{cases} \dfrac{1}{p-1}, & p>1, \text{ Converges} \\ \infty, & p\le 1, \text{ diverges} \end{cases}$$

(5) $\displaystyle\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx = ?$

Sol Take $\displaystyle\int \frac{1}{(x-2)^{3/2}}\,dx$ —(1)

put $u = x-2$: $du = dx$

$$(1) \Rightarrow \int \frac{dx}{(x-2)^{3/2}} = \int \frac{du}{u^{3/2}}$$

$$= \int u^{-3/2}\,du$$

$$= \frac{u^{-3/2+1}}{-\frac{3}{2}+1}$$

$$= \frac{u^{-1/2}}{-1/2} = \frac{-2}{\sqrt{u}} = \frac{-2}{\sqrt{x-2}}$$

$$\therefore \int_3^\infty \frac{1}{(x-2)^{3/2}} dx = \lim_{t \to \infty} \int_3^t \frac{1}{(x-2)^{3/2}} dx$$

$$= \lim_{t \to \infty} \left[\frac{-2}{\sqrt{x-2}} \right]_3^t$$

$$= \lim_{t \to \infty} \left[\frac{-2}{\sqrt{t-2}} - \left(\frac{-2}{\sqrt{1}}\right) \right]$$

$$= \lim_{t \to \infty} \frac{-2}{\sqrt{t-2}} + 2$$

$$= 2 \text{ (finite)}$$

$\therefore \int_3^\infty \frac{1}{(x-2)^{3/2}} dx$ is Convergent

⑥ Evaluate $\int_2^3 \frac{1}{\sqrt{3-x}} dx$.

Sol at $x=3$ $f(x)$ is discontinuity

$$\therefore \int_2^3 \frac{dx}{\sqrt{3-x}} = \lim_{t \to 3^-} \int_2^t \frac{dx}{\sqrt{3-x}}$$

$$= \lim_{t \to 3^-} \left[-2\sqrt{3-x} \right]_2^t$$

$$= \lim_{t \to 3^-} \left[-2\sqrt{3-t} + 2 \right]$$

$$= 2 \text{ (finite)}$$

$\therefore \int_2^3 \frac{dx}{\sqrt{3-x}}$ is convergent

⑦ Evaluate $\int_0^3 \frac{1}{\sqrt{x}} dx$.

Sol

Let $f(x) = \frac{1}{\sqrt{x}}$.

at $x=0$, $f(x)$ is discontinuous

$$\therefore \int_0^3 \frac{1}{\sqrt{x}} dx = \lim_{t \to 0^+} \int_t^3 x^{-1/2} dx.$$

$$= \lim_{t \to 0^+} \left[\frac{x^{1/2}}{1/2} \right]_t^3$$

$$= \lim_{t \to 0^+} \left[2\sqrt{x} \right]_t^3$$

$$= \lim_{t \to 0^+} \left[2\sqrt{3} - 2\sqrt{t} \right]$$

$$= 2\sqrt{3} \text{ (finite)}$$

$\therefore \int_0^3 \frac{1}{\sqrt{x}}$ is Convergent.

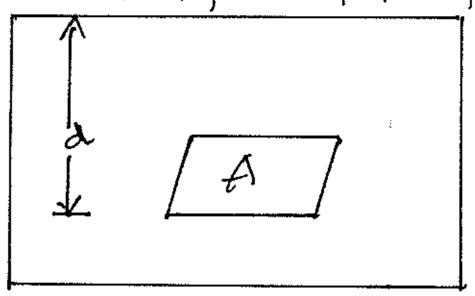
Applications of Integral calculus.

Hydrostatic force and force.

Sea divers realize that water pressure increases as they dive deeper into the sea. This significant feature happens because the weight of the water above them increases.

Let us assume that a thin horizontal plate with area A square meters is submerged in a fluid of density 'ρ' kilograms per cubic meter at a depth 'd' meters below the surface of the fluid.

Surface of the fluid.



The fluid directly above the plate has volume $V = Ad$.

Then its mass is $m = \rho V = \rho A d$.

The force exerted by the fluid on the plate is

$$F = mg = \rho g A d.$$

where 'g' is the acceleration due to gravity

$\therefore$ The pressure on the plate is

$$P = \frac{F}{A} = \rho g d.$$

The SI unit for measuring pressure is Newton per square meter is called pascal.

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We know that the density of water is given by $\rho = 1000\,kg/m^3$

The pressure at the bottom of a swimming pool 2 meter 2 meter deep is given by

$$p = \rho g d = 1000\,kg/m^3 \times 9.8\,m/s^2 \times 2 \text{ meter.}$$

$$\therefore p = 19,600\,pa = 19.6\,kpa$$

The principle of fluid pressure is that, at any point in a liquid the pressure is the same in all directions. Therefore, the divers has the same pressure on nose and ~~both~~ ears.

Hence the pressure in any direction at a depth "d" in a fluid with mass density "ρ" is given by $p = \rho g d = \delta d$.

This relation helps to determine the hydrostatic force against a vertical plate or wall or dam in a fluid.

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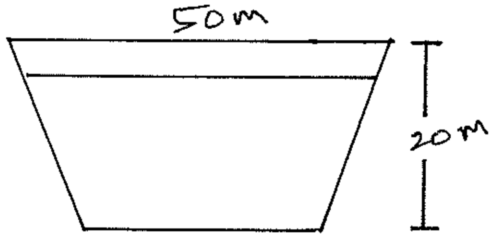
① A dam has the shape of the trapezoid as shown in figure.



The height is 20m and the width is 50m at the top and 30m at the bottom. Find the force on the dam due hydrostatic pressure if the water level is 4m from the top of the dam.

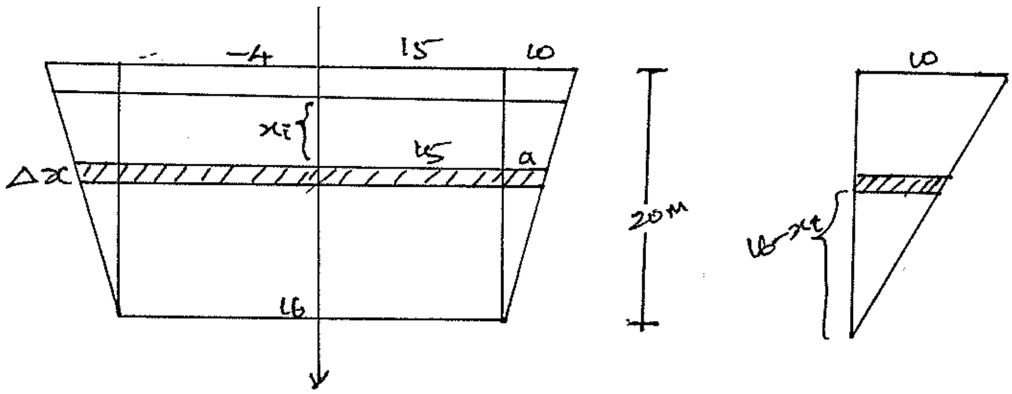
Sol: Let us consider a vertical x axis with origin at the surface of the water and directed downward.

Then the depth of the water is 16 m.

$\therefore$ the interval $[0,16]$ can be divided into subintervals of equal length with end points x_i, $x_i \in [x_{i-1}, x_i]$

Let us consider the i^{th} horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i

$$\therefore \frac{a}{16-x_i} = \frac{10}{20} \Rightarrow a = \frac{16-x_i}{2} = 8 - \frac{x_i}{2}$$



$$w_i = 2(15+a) = 2\left(15+8-\frac{x_i}{2}\right)$$

$$\therefore w_i = 46 - x_i$$

If A_i is the area of the i^{th} strip, then

$$A_i = w_i \Delta x = (46-x_i)\Delta x$$

If Δx is very small, then the pressure P_i on the ~~ith~~ i^{th} strip is almost constant.

$$\therefore P = \rho g d = \delta d.$$

$$P_i = 1000 g x_i$$

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Now the hydrostatic force F_i is

$$F_i = P_i A_i = 1000 g x_i (46 - x_i) \Delta x.$$

As $n \to \infty$,

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} 1000 g x_i (46 - x_i) \Delta x.$$

$$= \int_0^{16} 1000 g x (46 - x) \, dx$$

$$= 1000 g \int_0^{16} (46x - x^2) \, dx$$

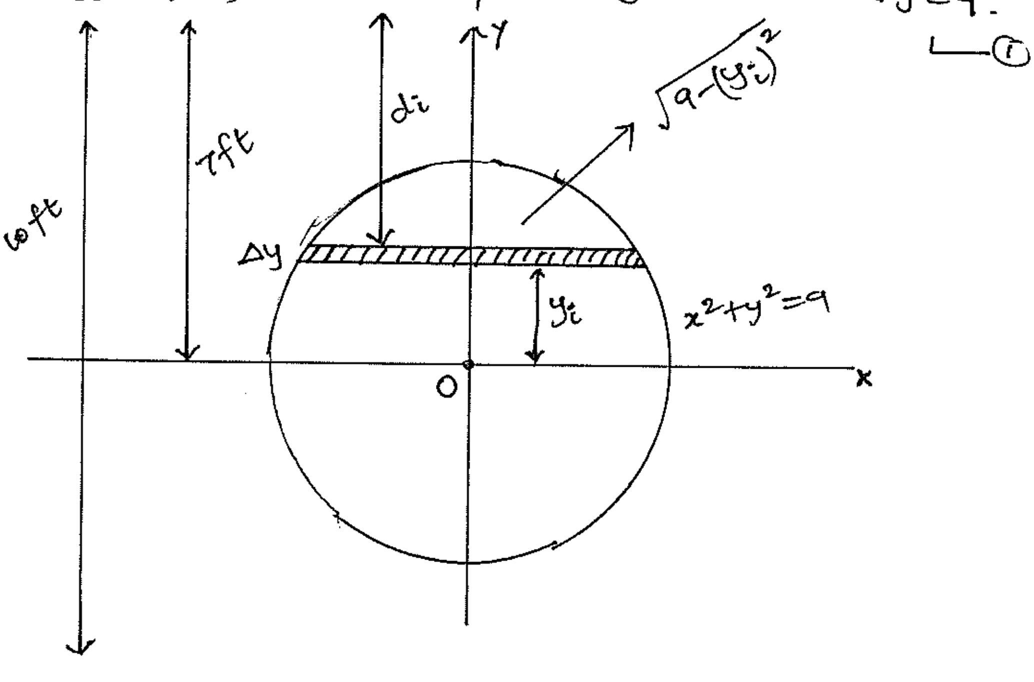
$$= 1000 (9.8) \left[\frac{46x^2}{2} - \frac{x^3}{3} \right]_0^{16}$$

$$= 9800 \left[23 (16)^2 - \frac{1}{3} (16)^3 \right]$$

$$= 4.43 \times 10^7 \text{ N}.$$

② Find the hydrostatic force on one end of a cylindrical drum with radius 3 feet if the drum is submerged in water 10 feet deep.

Sol The circle has a simple equation $x^2 + y^2 = 9$. — ①



we divide the circular region into horizontal strips of equal width.

From (i), the length of the i^{th} strip is given by

$2\sqrt{9-(y_i)^2}$

$\therefore$ Its area (strip) is $A_i = 2\sqrt{9-(y_i)^2}\,\Delta y$.

The pressure on the strip is $\delta d_i = 62.7(7-y_i)$

$\therefore$ the force on the strip is

$$\delta d_i A_i = 62.7(7-y_i)\,2\sqrt{9-(y_i)^2}\,\Delta y$$

$\therefore$ The total force is

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 62.7(7-y_i)\,2\sqrt{9-(y_i)^2}\,\Delta y$$

$$= 125\int_{-3}^{3}(7-y)\sqrt{9-y^2}\,dy$$

$$= 125\times 7\times\int_{-3}^{3}\sqrt{9-y^2}\,dy - 125\int_{-3}^{3} y\sqrt{9-y^2}\,dy$$

$$= 875\int_{-3}^{3}\sqrt{9-y^2}\,dy - 0$$

$$= 875\int_{-3}^{3}\sqrt{3^2-y^2}\,dy$$

$\left\{\because \int\sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}(x/a)\right\}$

$$= 875\left[\frac{1}{2}y\sqrt{9-y^2} + \frac{1}{2}3^2\sin^{-1}(y/3)\right]_{-3}^{3}$$

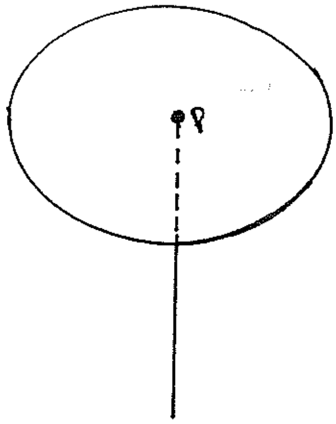
$$= 875\left[\left(\frac{3}{2}(0) + \frac{9}{2}\sin^{-1}(1)\right) - \left(-\frac{3}{2}(0) - \frac{9}{2}\sin^{-1}(1)\right)\right]$$

$$= 875\left[9\sin^{-1}(1)\right]$$

$$F = 12,375 \text{ lb}$$

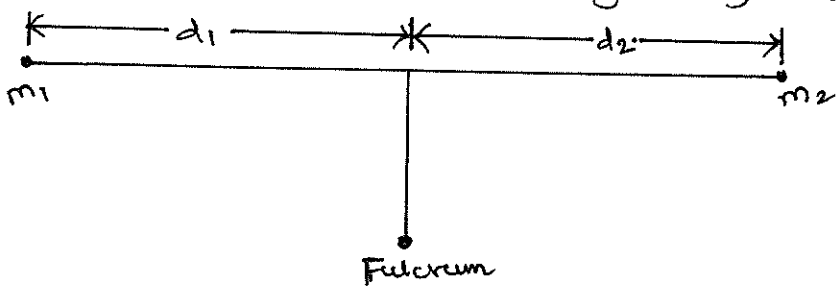
Moments and Centre of Mass.

Let us consider the diagram.



From the diagram, to find the point 'P' on which a thin plate of any given shape balances horizontally, is called the centre of mass or centre of gravity of the plate.

Ex



In the above diagram, the two masses m_1 and m_2 are fixed to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum.

The rod will be a balanced one, if

$$m_1 d_1 = m_2 d_2 \quad \text{——} (1)$$

Let us assume the rod lies along the x axis with m_1 at the point x_1 and m_2 at the point x_2 and the centre of mass at the point $\bar{x}$.

$$\therefore \quad d_1 = \bar{x} - x_1 \quad \& \quad d_2 = x_2 - \bar{x}$$

$$(1) \Rightarrow m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$\Rightarrow (m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$$

$$\Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \text{——} (2)$$

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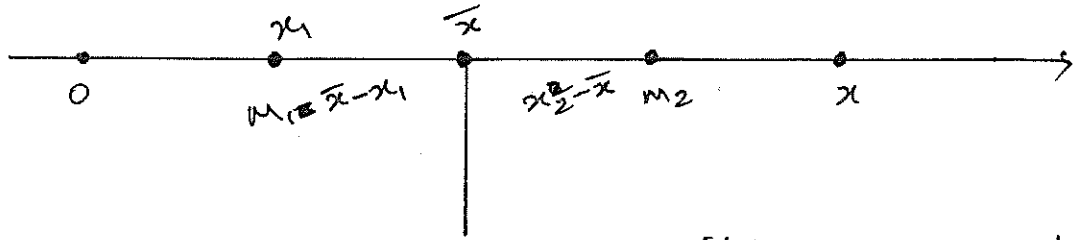
where m_1x_1 & m_2x_2 are moments of m_1 & m_2

and $m = m_1 + m_2$



Suppose we've 'n' particles with corresponding masses $m_1, m_2, m_3, \ldots, m_n$ located at the points $x_1, x_2, \ldots, x_n$ on the x axis, then the centre of mass of the entire system is

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

where $m = \Sigma m_i$ is the total mass of the system and the sum of the individual moments

$M = \sum_{i=1}^{n} m_i x_i$ is called the moment of the system about the origin.

$\therefore M = m\bar{x}$.

Let us consider a system of n particles with masses $m_1, m_2, m_3, \ldots, m_n$ located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane.

$\therefore$ the moment of the system about y-axis, x axis is given by $M_y = \sum_{i=1}^{n} m_i x_i$, $M_x = \sum_{i=1}^{n} m_i y_i$

Then M_y measures the tendency of the system to rotate about the y-axis and M_x measures the tendency to rotate about the x-axis.

$\therefore$ $(\bar{x}, \bar{y})$ of the centre of mass are given by

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

Where $m = \sum_{i=1}^{n} m_i$ is the total mass.

Since $m\bar{x} = M_y$ & $m\bar{y} = M_x$, the centre of mass $(\bar{x}, \bar{y})$ is the point where a single particle of mass m will have the same moments.

Centroid and Symmetry

Consider a flat plate which is called as a lamina with uniform density ρ that occupies a region R of the plane. We wish to locate the centre of mass of the plate which is called as the centroid of the region R.

If R is symmetric about a line l, then the centroid of R lies on l is called principle of symmetry. Hence the centroid of a rectangle is its centre.

Moments should be defined so that if the entire mass of a region is concentrated at the centre of mass, then its moments remains unchanged. The moments of the union of two non-overlapping regions should be the sum of the moments of the individual regions.

The moment of the region R above y-axis is given by

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i) \Delta x = \rho \int_a^b x f(x)\, dx$$

The moment of the region R about the x-axis is

$$M_x = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \, \frac{1}{2}\left[f(\bar{x}_i)\right]^2 \Delta x = \rho \int_a^b \frac{1}{2}\left[f(x)\right]^2 dx$$

For system of particles, the centre of mass of the plate is defined as $m\bar{x} = M_y$ & $m\bar{y} = M_x$.

The mass of the plate is the product of its density and area.

$$m = \rho A = \rho \int_a^b f(x)\, dx.$$

Then

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}.$$

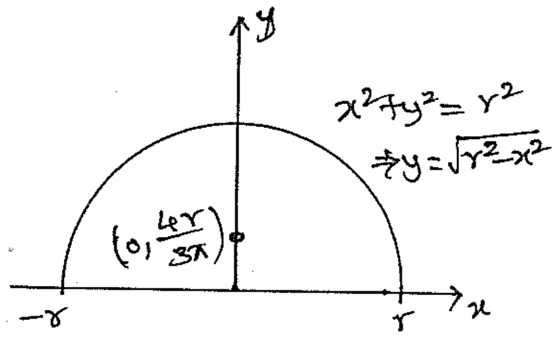
$$\bar{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{\int_a^b f(x)\,dx}.$$

$\therefore$ The centre of mass of the plate or the centroid of R is located at the point $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx$

and

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx.$$

① Find the centre of mass of a semicircular plate of radius r.



Sol

Equation of the circle is $x^2+y^2=r^2 \Rightarrow y^2=r^2-x^2$

$\therefore y = f(x) = \sqrt{r^2-x^2}$ and

$a=-r,\ b=r$.

Since the centre of mass must lie on the y-axis, $\bar{x}=0$

$\therefore$ Area $A = \frac{1}{2}\pi r^2$

Now $\bar{y} = \frac{1}{A}\int_{-r}^{r} \frac{1}{2}[f(x)]^2 dx = \frac{1}{\frac{1}{2}\pi r^2}\int_{-r}^{r}\frac{1}{2}\left[\sqrt{r^2-x^2}\right]^2 dx$

$\therefore \bar{y} = \frac{2}{\pi r^2}\int_0^r (r^2-x^2)\,dx. = \frac{2}{\pi r^2}\left[r^2x - \frac{x^3}{3}\right]_0^r$

$\bar{y} = \frac{2}{\pi r^2}\left[r^3 - \frac{r^3}{3}\right] \Rightarrow \bar{y} = \frac{4r}{3\pi}$

$\therefore (\bar{x},\bar{y}) : (0, \frac{4r}{3\pi})$

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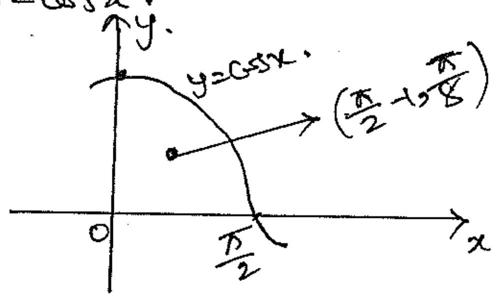
② Find the centroid of the region bounded by the curves $x=0$, $x=\pi/2$, $y=0$ and $y=\cos x$.



Sol

The area of the region

$A = \int_0^{\pi/2} \cos x\, dx.$

$= \sin x\Big]_0^{\pi/2} \Rightarrow \boxed{A=1}$

We know that the centroid of the region is given by $(\bar{x},\bar{y})$, where

$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx$ & $\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2 dx.$

Now $\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx = \int_0^{\pi/2} x\cos x\,dx.$

$u = x \qquad u' = 1 \qquad u'' = 0.$

$v = \cos x \qquad v_1 = \sin x \qquad v_2 = -\cos x.$

$\therefore \bar{x} = \left[x\sin x + \cos x\right]_0^{\pi/2} = \left(\frac{\pi}{2} + 0\right) - (0 - 1)$

$\therefore \bar{x} = \frac{\pi}{2} - 1$

$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx. = \frac{1}{2}\int_a^{\pi/2} \cos^2 x\,dx$

$\bar{y} = \frac{1}{4}\int_0^{\pi/2} (1+\cos 2x)\,dx = \frac{1}{4}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$

$\bar{y} = \frac{\pi}{8}$

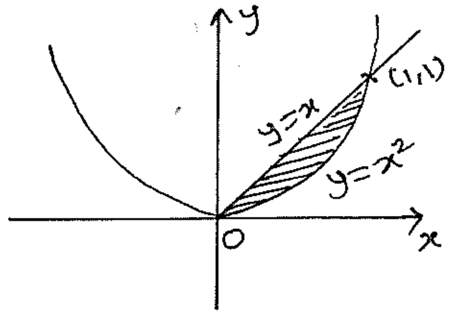
$\therefore$ the centroid is $(\bar{x}, \bar{y}) : \left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)$

③ Note If the region R lies between $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ then the centroid of R is $(\bar{x}, \bar{y})$, where $\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$ and

$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[[f(x)]^2 - [g(x)]^2\right]dx.$

③ Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.

Sol



let $f(x) = x$ & $g(x) = x^2$

$a = 0$ & $b = 1$

Now Area $A = \int_0^1 (x - x^2)\,dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \Rightarrow \boxed{A = \frac{1}{6}}$

We know that $\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx.$

$$\bar{x} = \frac{1}{(1/6)}\int_0^1 x(x - x^2)\,dx$$

$$\bar{x} = 6\int_0^1 (x^2 - x^3)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$\boxed{\bar{x} = \frac{1}{2}}$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[[f(x)]^2 - [g(x)]^2\right]dx.$$

$$= \frac{1}{1/6}\int_0^1 \frac{1}{2}\left[x^2 - x^4\right]dx$$

$$= 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$\boxed{\bar{y} = \frac{2}{5}}.$$

$\therefore$ the centroid is $\left(\frac{1}{2}, \frac{2}{5}\right)$

① Evaluate $\int \frac{dx}{\sqrt{3x-x^2-2}}$

Sol Let $I = \int \frac{dx}{\sqrt{3x-x^2-2}}$

$$3x-x^2-2 = -[x^2-3x+2]$$

$$= -\left[x^2-3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+2\right]$$

$$= -\left[(x-3/2)^2-(1/2)^2\right]$$

$$3x-x^2-2 = \left(\frac{1}{2}\right)^2-(x-3/2)^2$$

$$\therefore \sqrt{3x-x^2-2} = \sqrt{\left(\frac{1}{2}\right)^2-(x-3/2)^2}$$

Now $I = \int \frac{dx}{\sqrt{3x-x^2-2}}$

$$= \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-(x-3/2)^2}} \quad \left\{\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)+c\right\}$$

where $a \to 1/2$ $x \to x-3/2$.

$$\therefore I = \sin^{-1}\left(\frac{(x-3/2)}{1/2}\right)+c.$$

$$I = \sin^{-1} 2(x-3/2)+c.$$

$$I = \sin^{-1}(2x-3)+c.$$

Type your text

② Evaluate $\int \cos^n x \, dx$

Sol Let $I_n = \int \cos^n x \, dx$

Then $I_n = \int \cos^{n-1} x \cos x \, dx$.

Let $u = \cos^{n-1} x$ $\qquad dv = \cos x$

$du = (n-1)\cos^{n-2} x(-\sin x)dx$ $\qquad v = \int \cos x dx = \sin x$

We've $\int u dv = uv - \int v du$

$\therefore I_n = \cos^{n-1} x \sin x - \int \sin x (n-1) \cos^{n-2} x(-\sin x)dx$

$= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x \, dx$

$= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x) \cos^{n-2} x \, dx$

$= \cos^{n-1} x \sin x + (n-1)\int (\cos^{n-2} x - \cos^n x) \, dx$.

$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$.

$\Rightarrow I_n + (n-1)I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

$n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

$\Rightarrow I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$

ie: $\int \cos^n x \, dx = \frac{1}{n}[\cos^{n-1} x \sin x] + \frac{n-1}{n}\int \cos^{n-2} x \, dx$

The ultimate integral is I_0 and I_1.

if n is even ; $I_0 = \int dx = x + c$

if n is odd ; $I_1 = \int \cos x dx = \sin x + c$.

③ Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$ by using the method of partial fractions.

Sol

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x(2x-1)(x+2)}$$

$$= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \quad \text{———— ①}$$

$\Rightarrow x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$

put $x=0$ we get

$-1 = -2A$

$\Rightarrow A = \frac{1}{2}$

put $x=\frac{1}{2}$, we get

$\frac{1}{4}+1-1 = B\left(\frac{1}{2}\right)\left(\frac{5}{2}\right)$

$\frac{1}{4} = \frac{5B}{4}$

$\Rightarrow B = \frac{1}{5}$

put $x=-2$, we get

$4-4-1 = C(-2)(-5)$

$-1 = 10C$

$\Rightarrow C = -\frac{1}{10}$

$$① \Rightarrow \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2}\left(\frac{1}{x}\right) + \frac{1}{5}\left(\frac{1}{2x-1}\right) - \frac{1}{10}\left(\frac{1}{x+2}\right)$$

$$\therefore \int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{dx}{2x-1} - \frac{1}{10}\int \frac{dx}{x+2}$$

$$= \frac{1}{2}\log x + \frac{1}{10}\log(2x-1) - \frac{1}{10}\log(x+2) + c.$$

$$= \frac{1}{2}\log x + \frac{1}{10}\log\left(\frac{2x-1}{x+2}\right) + c$$

(4) Evaluate $\int \frac{2x+3}{x^2+x+1}\,dx$

Sol Let $2x+3 = A\frac{d}{dx}(x^2+x+1)+B$.

Then $2x+3 = A(2x+1)+B$. ——— (1)

Equating the coefficients of x we get

$2 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$

put $x=0$, we get

$3 = A+B \Rightarrow 3 = 1+B \Rightarrow \boxed{B=2}$

(1) $\Rightarrow$ $2x+3 = (2x+1)+2$

$$\int \frac{2x+3}{x^2+x+1}\,dx = \int \frac{2x+1}{x^2+x+1} + \int \frac{2\,dx}{x^2+x+1}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+1-\frac{1}{4}}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+3/4}$$

$$= \log(x^2+x+1) + 2\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right)+c$$

$$\int \frac{(2x+3)}{x^2+x+1}\,dx = \log(x^2+x+1) + \frac{4}{\sqrt{3}}\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]+c.$$

⑤ Determine whether the integral $\int_0^\infty \frac{dx}{x^2+4}$ is convergent or divergent.

Sol

$$\int_0^\infty \frac{1}{x^2+4}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{x^2+4}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^t$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}(t/2) - 0\right]$$

$$= \frac{1}{2}(\pi/2)$$

$$= \pi/4 \text{ (finite)}$$

$\therefore \int_0^\infty \frac{dx}{x^2+4}$ is convergent.

⑥ If $\int_0^{10} f(x)\,dx = 17$, $\int_0^{8} f(x)\,dx = 12$ then find $\int_8^{10} f(x)\,dx$.

Sol

By property, $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$, $a<c<b$

$$\int_0^{10} f(x)\,dx = \int_0^{8} f(x)\,dx + \int_8^{10} f(x)\,dx$$

$$17 = 12 + \int_8^{10} f(x)\,dx$$

$$\Rightarrow \int_8^{10} f(x)\,dx = 5$$