1. Introduction

The project computes all pure and mixed Nash Equilibria of a bi-matrix game.

Supported precision : 0.00001

Main steps:

- Delete strictly dominated actions.
- Find all PNE based on column max and row max.
- Do support numeration, and calculate mixed NE based on each support using gurobi.

2. File structure

- NESolver
  - NESolver5.py
- SDADeleter
  - SDADeleter1.py
- Test
  - Test3.py

3. Software Dependencies

```
python          3.8.12
gurobipy        9.5.0
numpy           1.21.2
pandas          1.3.4
itertools
os
sys
```

4. Example Usage

```
from NESolver.NESolver5 import NESolver
import numpy as np

A = np.array([[8, 2, 2], [3, 9, 3], [-2, -2, 4]])
B = np.array([[1, 2, 4], [-2, 5, 4], [-2, 2, 7]])
n1 = ['I', 'J', 'F']
n2 = ['X','Y','Z']

NESol = NESolver(A=A, B=B, action_name_1=n1,
action_name_2=n2)
NESol.analyze()
info = NESol.find()
print(info)
```

4.1 Input

class NESolver(A, B, action_name_1=None, action_name_2=None)

- A : 2-d numpy array, payoff matrix of player 1

  B : 2-d numpy array, payoff matrix of player 2

  ```
  A = np.array([[8, 2, 2], [3, 9, 3], [-2, -2, 4]])
  B = np.array([[1, 2, 4], [-2, 5, 4], [-2, 2, 7]])
  ```

- action_name_1 : list (optional), action names of player1

  action_name_2 : list (optional), action names of player2

  (Actions will be **indexed from 0** if action_name_1, action_name_2 are None.)

  ```
  n1 = ['I', 'J', 'F']
  n2 = ['X', 'Y', 'Z']
  ```

- A, B should have the same size. The length of a, b should match the shape of A or B.
- Will raise TypeError if input is of wrong datatype :

```
TypeError: The input two payoff matrices should be
numpy array.
TypeError: The input action names should be two list.
```

Will raise ValueError if shapes of inputs do not match :

```
ValueError: The input two payoff matrices should have
same size.
ValueError: The length of input action names should
match the number of actions.
```

4.2 Output

```
NESol = NESolver(A=A, B=B, action_name_1=n1,
action_name_2=n2)
```

4.2.1 NESolver.analyze()

```
NESol.analyze()
```

This method will print out the information of strictly dominated actions(SDA), pure Nash Equilibria(PNE) and mixed Nash Equilibria(MNE) of a given game.

Example output:

```
======= Analyze NE =======
n_SDA:  2
1_SDA:  ['I']
2_SDA:  ['X']
---------------
n_PNE:  2
PNE:
    (('J',), ('Y',))
    (('F',), ('Z',))
---------------
n_MNE:  1
MNE:
            support  NE_count
                                NE_value
```

```
0  ((J, F), (Y, Z))          1  [(0.000000, 0.833333,
0.166667), (0.000000, 0.083333, 0.916667)]
```

- Strictly dominated actions
  - `n_SDA` : The total number of strictly dominated actions of the two players.
  - `1_SDA` : Strictly dominated actions of player 1. (action `I` in this example)
  - `2_SDA` : Strictly dominated actions of player 2. (action `X` in this example)
- Pure Nash Equilibria
  - `n_PNE` : Number of pure Nash Equilibria of the given game. (2 PNE in this example)
  - `PNE` : All pure Nash Equilibria of the given game. ((J, Y) and (F, Z) are the only two PNE in this example)
- Mixed Nash Equilibria
  - `n_MNE` : Number of mixed Nash Equilibria of the given game. (1 MNE in this example)
  - `MNE` : All mixed Nash Equilibria of the given game.
    - Index column : Start from 0.
    - `support` : Actions with positive possibility.
    - `NE_count` : The number of Nash Equilibria based on the support : `1` or `inf`.
    - `NE_value` : If `NE_count==1`, it is the value of the MNE. If `NE_count==inf`, it shows an example of the MNE based on the support.

4.2.2 NESolver.find()

```
info = NESol.find()
print(info)
```

This methods returns a Pandas Dataframe that contains detailed information on calculating the Nash Equilibrium based on each support.

Example output:

```
                    support NE_type  NE_count
                                           NE_value
w1         w2                        x_value  x_count
                     y_value  y_count
0              ((I,), (X,))      -1         0
                                               None
8.00000       NaN                              None      0.0
(1.000000, 0.000000, 0.000000)      1.0
1              ((I,), (Y,))      -1         0
                                               None
NaN       NaN                              None      0.0
                        None      0.0
2              ((I,), (Z,))      -1         0
                                               None
NaN  4.000000  (1.000000, 0.000000, 0.000000)      1.0
                        None      0.0
3            ((I,), (X, Y))      -1         0
                                               None
7.99994       NaN                              None      0.0
(0.999990, 0.000010, 0.000000)      inf
4            ((I,), (X, Z))      -1         0
                                               None
7.99994       NaN                              None      0.0
(0.999990, 0.000000, 0.000010)      inf
5            ((I,), (Y, Z))      -1         0
                                               None
NaN       NaN                              None      0.0
                        None      0.0
6         ((I,), (X, Y, Z))      -1         0
                                               None
7.99988       NaN                              None      0.0
(0.999980, 0.000010, 0.000010)      inf
7              ((J,), (X,))      -1         0
                                               None
NaN       NaN                              None      0.0
                        None      0.0
```

```
8              ((J,), (Y,))     PNE         1  [(0.000000,
1.000000, 0.000000), (0.000000, 1.000000, 0.000000)]
9.00000  5.000000  (0.000000, 1.000000, 0.000000)      1.0
(0.000000, 1.000000, 0.000000)      1.0
9              ((J,), (Z,))      -1         0
                                               None
NaN       NaN                            None      0.0
                          None      0.0
10           ((J,), (X, Y))      -1         0
                                               None
8.99994       NaN                            None      0.0
(0.000010, 0.999990, 0.000000)      inf
11           ((J,), (X, Z))      -1         0
                                               None
3.00000       NaN                            None      0.0
(0.166667, 0.000000, 0.833333)      inf
12           ((J,), (Y, Z))      -1         0
                                               None
8.99994       NaN                            None      0.0
(0.000000, 0.999990, 0.000010)      inf
13        ((J,), (X, Y, Z))      -1         0
                                               None
5.49997       NaN                            None      0.0
(0.583328, 0.416662, 0.000010)      inf
14             ((F,), (X,))      -1         0
                                               None
NaN       NaN                            None      0.0
                          None      0.0
15             ((F,), (Y,))      -1         0
                                               None
NaN       NaN                            None      0.0
                          None      0.0
16             ((F,), (Z,))     PNE         1  [(0.000000,
0.000000, 1.000000), (0.000000, 0.000000, 1.000000)]
4.00000  7.000000  (0.000000, 0.000000, 1.000000)      1.0
(0.000000, 0.000000, 1.000000)      1.0
17           ((F,), (X, Y))      -1         0
                                               None
NaN       NaN                            None      0.0
                          None      0.0
```

```
18          ((F,), (X, Z))      -1         0
                                                   None
3.99994       NaN                                None      0.0
(0.000010, 0.000000, 0.999990)      inf
19          ((F,), (Y, Z))      -1         0
                                                   None
3.99994       NaN                                None      0.0
(0.000000, 0.000010, 0.999990)      inf
20       ((F,), (X, Y, Z))      -1         0
                                                   None
3.99988       NaN                                None      0.0
(0.000010, 0.000010, 0.999980)      inf
21          ((I, J), (X,))      -1         0
                                                   None
NaN       NaN                                None      0.0
                       None      0.0
22          ((I, J), (Y,))      -1         0
                                                   None
NaN  4.999970  (0.000010, 0.999990, 0.000000)      inf
                       None      0.0
23          ((I, J), (Z,))      -1         0
                                                   None
NaN  4.000000  (0.999990, 0.000010, 0.000000)      inf
                       None      0.0
24        ((I, J), (X, Y))      -1         0
                                                   None
5.50000       NaN                                None      0.0
(0.583333, 0.416667, 0.000000)      1.0
25        ((I, J), (X, Z))      -1         0
                                                   None
3.00000       NaN                                None      0.0
(0.166667, 0.000000, 0.833333)      1.0
26        ((I, J), (Y, Z))      -1         0
                                                   None
NaN  4.000000  (0.333333, 0.666667, 0.000000)      1.0
                       None      0.0
27     ((I, J), (X, Y, Z))      -1         0
                                                   None
5.49997       NaN                                None      0.0
(0.583328, 0.416662, 0.000010)      inf
```

```
28          ((I, F), (X,))      -1         0
                                                 None
NaN       NaN                             None      0.0
                        None      0.0
29          ((I, F), (Y,))      -1         0
                                                 None
NaN       NaN                             None      0.0
                        None      0.0
30          ((I, F), (Z,))      -1         0
                                                 None
NaN  6.999970  (0.000010, 0.000000, 0.999990)      inf
                        None      0.0
31        ((I, F), (X, Y))      -1         0
                                                 None
NaN       NaN                             None      0.0
                        None      0.0
32        ((I, F), (X, Z))      -1         0
                                                 None
3.00000       NaN                             None      0.0
(0.166667, 0.000000, 0.833333)      1.0
33        ((I, F), (Y, Z))      -1         0
                                                 None
NaN       NaN                             None      0.0
                        None      0.0
34     ((I, F), (X, Y, Z))      -1         0
                                                 None
NaN       NaN                             None      0.0
                        None      0.0
35          ((J, F), (X,))      -1         0
                                                 None
NaN       NaN                             None      0.0
                        None      0.0
36          ((J, F), (Y,))      -1         0
                                                 None
NaN  4.999970  (0.000000, 0.999990, 0.000010)      inf
                        None      0.0
37          ((J, F), (Z,))      -1         0
                                                 None
NaN  6.999970  (0.000000, 0.000010, 0.999990)      inf
                        None      0.0
```

```
38        ((J, F), (X, Y))      -1         0
                                               None
NaN       NaN                           None      0.0
                        None      0.0
39        ((J, F), (X, Z))      -1         0
                                               None
3.00000       NaN                           None      0.0
(0.166667, 0.000000, 0.833333)      1.0
40        ((J, F), (Y, Z))     MNE         1  [(0.000000,
0.833333, 0.166667), (0.000000, 0.083333, 0.916667)]
3.50000  4.500000  (0.000000, 0.833333, 0.166667)      1.0
(0.000000, 0.083333, 0.916667)      1.0
41     ((J, F), (X, Y, Z))      -1         0
                                               None
3.49997       NaN                           None      0.0
(0.000010, 0.083328, 0.916662)      inf
42       ((I, J, F), (X,))      -1         0
                                               None
NaN       NaN                           None      0.0
                        None      0.0
43       ((I, J, F), (Y,))      -1         0
                                               None
NaN  4.999940  (0.000010, 0.999980, 0.000010)      inf
                        None      0.0
44       ((I, J, F), (Z,))      -1         0
                                               None
NaN  4.000030  (0.999980, 0.000010, 0.000010)      inf
                        None      0.0
45     ((I, J, F), (X, Y))      -1         0
                                               None
NaN       NaN                           None      0.0
                        None      0.0
46     ((I, J, F), (X, Z))      -1         0
                                               None
3.00000       NaN                           None      0.0
(0.166667, 0.000000, 0.833333)      1.0
47     ((I, J, F), (Y, Z))      -1         0
                                               None
NaN  4.499985  (0.000010, 0.833328, 0.166662)      inf
                        None      0.0
```

```
48  ((I, J, F), (X, Y, Z))      -1        0
                                                  None
NaN        NaN                              None       0.0
                          None       0.0
```

- Meaning of the columns
  - Index column: Start from 0.
  - `support` : Actions with positive possibility (done by support numeration).
  - `NE_type` : The type of the Nash Equilibrium. 3 Possible values :
    - `-1`: The support does not admit a Nash Equilibrium.
    - `PNE` : The support admits an pure Nash Equilibrium.
    - `MNE` : The support admits mixed Nash Equilibria.
  - `NE_count` : The number of Nash Equilibria supported by the `support`. 3 Possible values :
    - `0` : The support does not admit a Nash Equilibrium.
    - `1` : The support admits one Nash Equilibrium.
    - `inf` : The support admits infinitely many Nash Equilibria.
  - `NE_value` : Value of the Nash Equilibria.
    - If `NE_count==0`, it shows `None`.
    - If `NE_count==1`, it is the value of the MNE.
    - If `NE_count==inf`, it shows an example of the MNE based on the support.
  - `w1` : Expected payoff of **player 1**. `NaN` if **player 2** fails to have a strategy `y` to form a Nash Equilibrium.

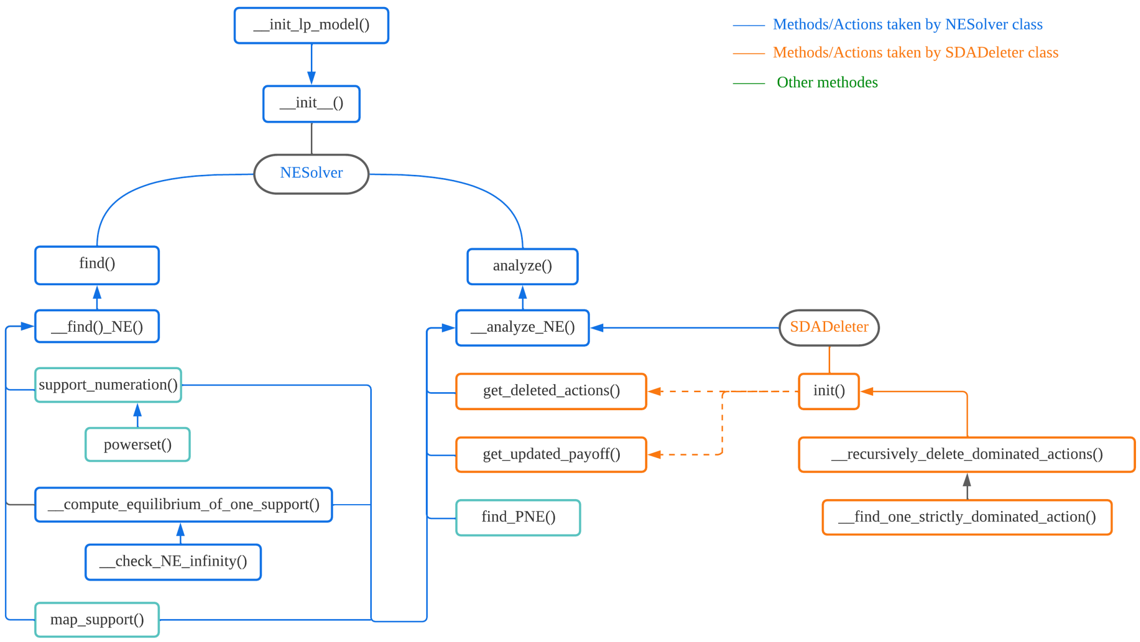
    `w2` : Expected payoff of **player 2**. `NaN` if **player 1** fails to have a strategy `x` to form a Nash Equilibrium.
  - `x_value` : A stochastic vector representing (an example of) player1's strategy or `None` if player1 fails to have one.

    `y_value` : A stochastic vector representing (an example of) player2's strategy or `None` if player2 fails to have one.
  - `x_count`: The number of player1's possible strategies : `0.0`, `1.0` or `inf`

`y_count`: The number of player2's possible strategies : `0.0`, `1.0` or `inf`

4.3 Please see other examples in Test3.py

5. Code Structure



- Explanations of each method is within the code comment.

6. Time Complexity

Assumption:

- Player**1** is the **row** player having payoff matrix **A** and **n** actions.

  Her strategy is represented as a stochastic vector **x** and her expected payoff is **w1**.

- Player**2** is the **column** player having payoff matrix **B** and **m** actions.

Her strategy is represented as a stochastic vector **y** and her expected payoff is **w2**.

6.1 Delete Strictly Dominated Actions (SDA)

Implemented by `__find_one_strictly_dominated_action()` and `__recursively_delete_dominated_actions()`.

Main Idea:

- An action is detected as a SDA if in the payoff matrix its column (or row) is strictly smaller than another column (or row) element-wisely.
- `__recursively_delete_dominated_actions()` will check dominated actions for row (with payoff matrix A) and column (with payoff matrix B) players repeatedly.

  ```
  (checkA, checkB, checkA… each is called one turn)
  ```

- `__find_one_strictly_dominated_action()` will be called to check whether one strictly dominated action exists in one turn.

  It may be called more than once in one turn if there're multiple SDA in one turn.

Principles (priority from high to low):

- Each player will be checked for at least once
- Stop checking when either player don't have a dominated action in her turn
- Will change to check another player if no more dominated actions can be found in her turn

6.1.1 Worst case time complexity

Assume player1 has n actions and player2 has m actions. The payoff matrices A and B are of shape n * m. In the worst case, the first turn do not find any strictly dominated actions. Then starting from the second turn, one strictly dominated action is found and deleted at the last comparison in each turn until each player has only one action left. (When check A, we may delete one row. When check B, we may delete one column.)

Turn Index	**Payoff matrix**	**Whether SDA exists**	**Original shape**	**Outcome shape**
1	A	F	n * m	n * m
2	B	T	n * m	n * (m-1)
3	A	T	n * (m-1)	(n-1) * (m-1)
...	...			
n+m	A or B	T	1 * 2 or 2 * 1	1 * 1

- To find one SDA in one turn, given payoff matrices of shape i * j
    - Player1 with payoff matrix A: O(j * i * (i-1))
        - Compare the elements between two row: O(j)
        - The number of comparisons: i * (i-1)   (row order matters)
    - Player2 with payoff matrix B: O(i * j * (j-1)) , with similar reasons
- From the aspect of player1, the shape of payoff matrix we need to check:
    - n * m
    - n * (m-1)
    - (n-1) * (m-2)
    - (n-2) * (m-3)
    - ...

  From the aspect of player2, the shape of payoff matrix we need to check:
    - n * m
    - (n-1) * (m-1)
    - (n-2) * (m-2)
    - ...
- **Assume n = m, the overall time complexity is:**

$\sum_{k=1}^{n-1}[k(k+1)k+kk(k-1)]+2nn(n-1)$
$=\sum_{k=1}^{n-1}2k^3+2n^2(n-1)$, (according to
$Sn=1^3+2^3+...+n^3=[n(n+1)/2]^2$)
$=O(n^4)$

- **Since we only consider small games here, let n = 3, the complexity is 54.**

6.2 Find pure Nash Equilibria (PNE)

Implemented by `find_PNE`.

Main idea:

- Assume row player1 has payoff matrix A, column player2 has payoff matrix B
- If a strategy profile is detected as PNE if and only if
  - it's payoff of player1 is the max value in the corresponding column of A and
  - the payoff of player2 is the max value in the corresponding row of B.
- Store the location of column max values and row max values into two sets, and find the intersection of the two sets which shows the location of PNE.

6.2.1 Worst case time complexity

Assume player1 has n actions, player 2 has m actions.

- Find all column max for player1: O(mn)
  - Find the max value of one column: O(n)
  - Number of columns: m

  Find all row max for player2: O(mn)
  - Find the max value of one row: O(m)
  - Number of rows: n

- Assume there are i column max values and j row max values, we use python set `intersection()` to find PNE. The time complexity of this step is O(min(i,j)). (by hash table, Ref: link)
- Since the upper bound of i or j is mn, the overall time complexity is $O(mn)$ when all strategies have the same payoff for one player.

6.3 Compute Nash Equilibria (NE) based on one support

Implemented by `__init_lp_model()`, `__compute_equilibrium_of_one_support()` and `__check_NE_infinity()`.

6.3.1 Main idea

- We use two LP models to find the NE for each support. One is `lpm_1`, which handles x and w2. One is `lpm_2`, which handles y and w1.
- `__init_lp_model()` builds the two LP with basic variables and constraint that will be used for each support. Thus it will be called only once. `__compute_equilibrium_of_one_support()` updates the constraints for each support and solve the two LP. A NE exists only when the two LP both have a feasible solution.
- If an LP model has a feasible solution. We use `__check_NE_infinity()` to find out whether it has infinitely many solutions.

6.3.2 Steps

(Steps for player1 and player2 is symmetric except A * y and B.T * x. The explanation below is flattened for easy understanding.)

Suppose `I` is the set of action index in player1's support, `J` is the set of action index in player2's support.

- **Initialize**
    - lpm_1

x = [x_1, .., x_n].T	add n variables	[1-1]
w2	add 1 variable	[1-2]
B*	add 1 variable	[1-3]
x_1,...,x_n >= 0	add n constraint	[1-4]
x_1+x_2+..x_n=1	add 1 constraint	[1-5]
B* = B.T * y	add n constraint	[1-6]

- lpm_2

y = [y_1, .., y_m].T	add m variables	[2-1]
w1	add 1 variable	[2-2]
A*	add 1 variable	[2-3]
y_1,...,y_m >= 0	add m constraint	[2-4]
y_1+y_2+..y_m=1	add 1 constraint	[2-5]
A* = A * y	add m constraint	[2-6]

- **Update constraints** for each support
  - lpm_1

x_i = 0 if i not in `I`; x_i > 0 if action i in `I`	add n constraints	[1-7]
B_*i <= w2 if i not in* `I`*; B*_i = w2 if action i in `I`	add n constraints	[1-8]

  - lpm_2

y_j = 0 if action j not in `J`; y_j > 0 if action j in `J`	add m constraints	[2-7]
A_j <= w1 if action j not in `J`; A_j = w1 if action j in `J`	add m constraints	[2-8]

- **Solve** LP models
  - lpm_1
    - #variables: n+1

      [1-1], [1-2]
    - #equality or inequality constraints: 2n+1

      [1-5], [1-7], [1-8]
    - (no B* and B* = B.T * x since they functions as a bridge between w2 and x).
  - lpm_2
    - #variables: m+1

      [2-1], [2-2]
    - #equality or inequality constraints: 2m+1

      [2-5], [2-7], [2-8]
    - (no A* and A* = A * y since they functions as a bridge between w1 and y).
- **If** an LP has a feasible solution, we then **extract** the value of corresponding payoff and strategy vector, and **check** whether the LP have infinitely many solutions:
  - lpm_1:
    - trim payoff matrices B so that it only contains rows and columns representing actions that are in the support `I`, `J`
    - The value of x is determined by equations in [1-5], [1-8]
    - Reformate these equations in to matrix computation M*z=b with z = [x_i1, x_i2, ..., w2].T with i1, i2... in `I`.

Now M is of shape `(|J|+1, |I|+1)`, b is of shape `(|I|+1, 1)`.

- If `rank(M) == rank(M|b) < |I|+1`, player1 has infinitely many strategies that may form NE under this support.

- lpm_2:
  - trim payoff matrices A so that it only contains rows and columns representing actions that are in the support `I`, `J`
  - The value of x is determined by equations in [2-5], [2-8]
  - Reformate these equations in to matrix computation M*z=b with z = [y_j1, y_j2, ..., w1].T with j1, j2... in `J`.

    Now M is of shape `(|I|+1, |J|+1)`, b is of shape `(|J|+1, 1)`.
  - If `rank(M) == rank(M|b) < |J|+1`, player2 has infinitely many strategies that may form NE under this support.
- Determine whether the support admits NE, update relevant information.

6.3.3 Time Complexity

Assume that adding one variable or constraint, extracting one variable takes a constant time.

Assume the time complexity of gurobi solving the two LP models with player1 has n actions and player2 has m actions is T_gurobi(n, m).

Assume the time complexity of computing the rank of matrix with shape (a, b) or (b, a) is T_rank(a,b).

- Initialize: $O(mn)$ (for matrix computation in [1-6], [2-6])
- Update constraints for each support: $O(n+m)$
- Solve LP models: $T\_gurobi(n, m)$

If an LP has a feasible solution:

- Extract outcome: $O(n)$ or $O(m)$, depends on which LP model to extract

- Check whether there are infinitely many NE:
  - Trim payoff matrix: $O(|I||J|)$
  - Check `lpm_1`: $O(|I||J|) + T\_rank(|J|+1, |I|+2)$
  - Check `lpm_2`: $O(|I||J|) + T\_rank(|I|+1, |J|+2)$

In summary, the time complexity of computing NE based on one support is highly depends on whether the LP models have solution and the size of support.

- Lower bound without initialization:

  The two LP models do not have feasible solution:
  $O(n+m) + T\_gurobi(n, m)$
- Upper bound without initialization:

  The support admit NE:
  $O(n+m) + T\_gurobi(n, m) + O(|I||J|) + T\_rank(|I|, |J|)$

6.4 Support numeration

Implemented by `support_numeration()` and `powerset()`.

Main idea:

- Given the list of index of the actions, we use `powerset()` to find all the subset of action indices of each player.
- Then `support_numeration()` is used to find all the combinations between the index subsets of the two players.

6.4.1 Time complexity

- Find all index subsets for players: $O(2^n) + O(2^m)$
- Find all combination between two players' index subsets:
  $O((2^n - 1) * (2^m - 1)) = O(2^{m+n})$
- Overall time complexity: $O(2^{m+n})$

6.5 Conclusion

6.5.1 Find()

- If method `find()` is used to compute all the Nash Equilibria of a game
  - it will first do support numeration,
  - and then solve NE based on each support.
- Therefore the overall time complexity should be the sum of the time complexity of finding NE based on each support.
  - As in 6.4.1, we have overall $(2^n - 1) * (2^m - 1)$ supports.
  - As in 6.3.3, the time complexity of computing NE of one support is highly depend on whether the two LP are feasible and the size of each support.

6.5.2 Analyze()

- If method `Analyze()` is used to compute all the Nash Equilibria of a game,
  - it will first delete all the strictly dominated actions (6.1),
  - then find pure Nash Equilibria without SDA (6.2),
  - conduct support numeration without SDA (6.4),
    - will delete supports of pure strategy profile
    - will delete supports that containing one pure strategy if no PNE is found
  - and solve NE based on each support (6.3)
- The time complexity highly varies between different games.