Q3. [15 Marks] (a) Write the formal definitions of Big-O, Big-Omega and Theta.

(i) f(n) = O(g(n)) if and only if (fill in the remaining statement)

If $\forall n_0 \in N$, $\exists c \in R$ s.t. $f(n) \leq c\,g(n)$ $\forall n \geq n_0$, then $f(n) = O(g(n))$

(ii) f(n) = big-omega(g(n)) if and only if (fill in the remaining statement)

$f(n) = \Omega(g(n))$ if and only if $g(n) = O(f(n))$

So, $\forall n_0 \in N$, $\exists c \in R$ s.t. $c\,f(n) \geq g(n)$ $\forall n \geq n_0$, then $f(n) = \Omega(g(n))$

(iii) f(n) = theta(g(n)) if and only if (fill in the remaining statement)

If $\forall n_0 \in N$, $\exists c_1, c_2 \in R$, s.t. $c_1 g(n) \leq f(n) \leq c_2 g(n)$ $\forall n \geq n_0$

then $f(n) = \Theta(g(n))$

(b) If f(n) > 0 and g(n) > 0 for all n, then formally prove that max{f(n), g(n)} = O(f(n) + g(n)).

$$\max\{f(n), g(n)\} = \frac{f(n)+g(n)}{2} + \left|\frac{f(n)-g(n)}{2}\right|$$

$$\leq \frac{f(n)+g(n)}{2} + \frac{|f(n)| + |g(n)|}{2}$$

$$\leq f(n) + g(n)$$

$\rightarrow$ ~~$|x+y| \leq |x|+$~~ $|x-y| \leq |x| + |y|$ since f, g are true $\leq x + y$.

$\Rightarrow$ For $c = 1$ let $h(n) = f(n) + g(n)$

$\max\{f(n), g(n)\} \leq h(n)$ $\forall n \geq n_0$

(c) If f(n) > 0 and g(n) > 0 for all n, then is the following statement true: max{f(n), g(n)} = big-Omega(f(n)+g(n)). Formally prove if yes, show a counter-example if not.

$$\max\{f(n), g(n)\} = \frac{f(n)+g(n)}{2} + \left|\frac{f(n)-g(n)}{2}\right|$$

$$\max\{f(n), g(n)\} \geq \frac{f(n)+g(n)}{2}.$$

So if we put $c = \frac{1}{2}$,

$h(n) = f(n) + g(n)$ $(c = \frac{1}{2})$ s.t.

$\forall n_0 \in N$ $\exists c$ $\forall n \geq n_0$.

$\max\{f(n), g(n)\} \geq c\,h(n)$

So, $\max\{f(n), g(n)\} = \Omega(f(n) + g(n))$