Fase 1 - Actividad 2

Estudiar la Estabilidad del Sistema

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$$\begin{cases} p'(t) = \alpha_1 p(t) - \alpha_2 p(t) d(t) \\ d'(d) = -\beta_1 d(t) + \beta_2 p(t) d(t) \end{cases}$$

donde $p(t)$ son las presas y $d(t)$ los depredadores

$$f(p,d) = \alpha_1 p(t) - \alpha p(t) d(t)$$

$$g(p,d) = -\beta_1 d(t) + \beta_2 p(t) d(t)$$

Puntos de equilibrio $P_1 = (p,d) = (0,0)$ y $P_2 = (p,d) = \left(\frac{\beta_1}{\beta_2}, \frac{\alpha_1}{\alpha_2}\right)$

Matriz Jacobiana

$$J(p,d) = \begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial d} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial d} \end{pmatrix}$$

$$\frac{\partial f}{\partial p} = \alpha_1 - \alpha_2 d(t) \qquad \frac{\partial f}{\partial d} = -\alpha_2 p(t)$$

$$\frac{\partial g}{\partial d} = \beta_2 d(t) \qquad \frac{\partial g}{\partial d} = -\beta_1 + \beta_2 p(t)$$

$$J(p,d) = \begin{pmatrix} \alpha_1 - \alpha_2 d(t) & -\alpha_2 p(t) \\ \beta_2 d(t) & -\beta_1 + \beta_2 p(t) \end{pmatrix}$$

Evaluada en P_1

$$J(0,0) = \begin{pmatrix} \alpha_1 - \alpha_2(0) & -\alpha_2(0) \\ \beta_2(0) & -\beta_1 + \beta_2(0) \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}$$

Evaluada en P_2

$$J\left(\frac{\beta_1}{\beta_2}, \frac{\alpha_1}{\alpha_2}\right) = \begin{pmatrix} \alpha_1 - \alpha_2\left(\frac{\alpha_1}{\alpha_2}\right) & -\alpha\left(\frac{\beta_1}{\beta_2}\right) \\ \beta_2\left(\frac{\alpha_1}{\alpha_2}\right) & -\beta_1 + \beta_2\left(\frac{\beta_1}{\beta_2}\right) \end{pmatrix} = \begin{pmatrix} 0 & -\alpha_2\frac{\beta_1}{\beta_2} \\ \beta_2\frac{\alpha_1}{\alpha_2} & 0 \end{pmatrix}$$

Valores propios $J(0,0)$

$$A = \begin{bmatrix} \alpha_1 & 0 \\ 0 & -\beta_1 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} \alpha_1 - \lambda & 0 \\ 0 & -\beta_1 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (\alpha_1 - \lambda)(-\beta_1 - \lambda) - (0)(0) = (\alpha_1 - \lambda)(-\beta_1 - \lambda)$$

$$\boxed{\begin{matrix} \lambda_1 = \alpha_1 \\ \lambda_2 = -\beta_1 \end{matrix}} \qquad \lambda_2 < 0 < \lambda_1 \quad \therefore \quad (x_0, y_0) \text{ es un punto silla}$$

Vectores propios $J(0,0)$

$\lambda_1 = \alpha_1$

$$(A - \lambda_1 I)\vec{v} = 0 \Rightarrow (A - (\alpha_1)I)\vec{v} = 0 \Rightarrow \begin{bmatrix} \alpha_1 - \alpha_1 & 0 \\ 0 & -\beta_1 - \alpha_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 0 + v_2 0 = 0$$
$$v_1 0 + (-\beta_1 - \alpha_1)v_2 = 0$$

$$\boxed{(v_1, 0)}$$

$\lambda_2 = -\beta_1$

$$(A - \lambda_2 I)\vec{v} = 0 \Rightarrow (A - (-\beta_1)I)\vec{v} = 0 \Rightarrow \begin{bmatrix} \alpha_1 + \beta_1 & 0 \\ 0 & -\beta_1 + \beta_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1(\alpha_1 + \beta_1) + v_2 0 = 0$$
$$v_1 0 + v_2 0 = 0$$

$$\boxed{(0, v_2)}$$

Valores propios $J\left(\frac{\beta_1}{\beta_2}, \frac{\alpha_1}{\alpha_2}\right)$

$$A = \begin{bmatrix} 0 & -\alpha_2 \frac{\beta_1}{\beta_2} \\ \beta_2 \frac{\alpha_1}{\alpha_2} & 0 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} -\lambda & -\alpha_2 \frac{\beta_1}{\beta_2} \\ \beta_2 \frac{\alpha_1}{\alpha_2} & -\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (-\lambda)(-\lambda) - \left(-\alpha_2 \frac{\beta_1}{\beta_2}\right)\left(\beta_2 \frac{\alpha_1}{\alpha_2}\right) = \lambda^2 + \beta_1\alpha_1 \Rightarrow \lambda^2 = -\beta_1\alpha_1 \Rightarrow \lambda = \pm\sqrt{-\beta_1\alpha_1} \Rightarrow \lambda = \pm i\sqrt{\beta_1\alpha_1}$$

$$\boxed{\begin{array}{l} \lambda_1 = i\sqrt{\beta_1\alpha_1} \\ \lambda_2 = -i\sqrt{\beta_1\alpha_1} \end{array}}$$

λ_1, λ_2 no son real, $Re(\lambda_1)=0$ y $Re(\lambda_2)=0$

λ_1 y λ_2 son imaginarios puros

$\therefore$ el punto crítico es un centro y es estable, pero no asintóticamente estable

Vectores propios $J\left(\frac{\beta_1}{\beta_2}, \frac{\alpha_1}{\alpha_2}\right)$

$\lambda_1 = i\sqrt{\beta_1\alpha_1}$

$$(A - (\lambda_1)I)\vec{v} = 0 \Rightarrow (A - (i\sqrt{\beta_1\alpha_1})I)\vec{v} = 0 \Rightarrow \begin{bmatrix} -i\sqrt{\beta_1\alpha_1} & -\alpha\frac{\beta_1}{\beta_2} \\ \beta_2\frac{\alpha_1}{\alpha_2} & -i\sqrt{\beta_1\alpha_1} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-V_1(i\sqrt{\beta_1\alpha_1}) - V_2\left(\alpha_2\frac{\beta_1}{\beta_2}\right) = 0$$

$$V_1\left(\beta_2\frac{\alpha_1}{\alpha_2}\right) - V_2(i\sqrt{\beta_1\alpha_1}) = 0$$

$$\boxed{\left(\frac{i\alpha_2\sqrt{\beta_1}}{\beta_2\sqrt{\alpha_1}}V_2, V_2\right)}$$

$$V_1 = \frac{-V_2\frac{\alpha_2\beta_1}{\beta_2}}{i\sqrt{\beta_1\alpha_1}} = \frac{-V_2\alpha_2\beta_1}{i\sqrt{\beta_1\alpha_1}\,\beta_2} = \frac{i\alpha_2\sqrt{\beta_1}}{\beta_2\sqrt{\alpha_1}}V_2$$

$\lambda_2 = -i\sqrt{\beta_1\alpha_1}$

$$(A - \lambda_2 I)\vec{v} = 0 \Rightarrow (A - (-i\sqrt{\beta_1\alpha_1})I)\vec{v} = 0 \Rightarrow \begin{bmatrix} i\sqrt{\beta_1\alpha_1} & -\alpha_2\frac{\beta_1}{\beta_2} \\ \beta_2\frac{\alpha_1}{\alpha_2} & i\sqrt{\beta_1\alpha_1} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V_1(i\sqrt{\beta_1\alpha_1}) + V_2\left(-\alpha_2\frac{\beta_1}{\beta_2}\right) = 0$$

$$V_1\left(\beta_2\frac{\alpha_1}{\alpha_2}\right) + V_2(i\sqrt{\beta_1\alpha_1}) = 0$$

$$\boxed{\left(\frac{-i\alpha_2\sqrt{\beta_1}}{\beta_2\sqrt{\alpha_1}}V_2, V_2\right)}$$

$$V_1 = \frac{V_2\alpha_2\frac{\beta_1}{\beta_2}}{i\sqrt{\beta_1\alpha_1}} = \frac{V_2\alpha_2\beta i}{\beta_2\sqrt{\beta_1\alpha_1}\,i} = \frac{-i\alpha_2\sqrt{\beta_1}}{\beta_2\sqrt{\alpha_1}}V_2$$