MATHEMATICAL ASSOCIATION
supporting mathematics in education

Rotating the Fifteen Puzzle
Author(s): A. L. Davies
Source: *The Mathematical Gazette*, Vol. 54, No. 389 (Oct., 1970), pp. 237-240
Published by: Mathematical Association
Stable URL: http://www.jstor.org/stable/3613773
Accessed: 01-11-2015 06:33 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*.

http://www.jstor.org

This content downloaded from 128.111.121.42 on Sun, 01 Nov 2015 06:33:06 UTC
All use subject to JSTOR Terms and Conditions

ROTATING THE FIFTEEN PUZZLE

BY A. L. DAVIES

The Fifteen puzzle is well known. In its original form it consists of a shallow square box that can just hold sixteen small squares. Squares, numbered 1 to 15, are placed in any order in the box, leaving the sixteenth position empty (e.g. Fig. 1). The problem is to slide the squares, without lifting them out, until the numbers are in their natural order (Fig. 2).

2	1	3	5
4	6	7	8
12	11	10	9
15	13	14	

FIG. 1

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

FIG. 2

For the problem to have a solution, the initial order must be an even permutation of the natural order; or, what is equivalent, the natural order must be obtainable from the initial order by an even number of transpositions. For example, the permutation in figure 2 can be obtained from that in Fig. 1 by interchanging 2 and 1, 5 and 4, 12 and 9, 11 and 10, 15 and 13, and then 15 and 14. Six transpositions are required, an even number, so the position illustrated in Fig. 1 is one in which the puzzle can be solved. If, however, the last row in Fig. 1 read 13, 15, 14, in that order, a total of five transpositions would be needed and the puzzle would not be soluble. A proof of this is indicated in *Mathematical Recreations and Essays* by Rouse Ball.

My interest in the puzzle was aroused by Professor Ledermann who, in a fascinating talk to the Leicester Branch of the M.A., not only established the necessity but also the sufficiency of this condition, a point which generally appears to have been overlooked. His proof applies to any square or rectangular puzzle of the same type, provided it contains at least two rows and two columns. One example of a commercially produced puzzle he showed us had the numbers printed diagonally on the squares so that there are two possible final positions (Figs. 3 and 4), depending on which way up the board is held. In Fig. 4 the board has been rotated through a

This content downloaded from 128.111.121.42 on Sun, 01 Nov 2015 06:33:06 UTC
All use subject to JSTOR Terms and Conditions

negative quarter turn. Any initial position can be reduced to one or other of these final positions. This does not happen in general with puzzles of different sizes and Professor Ledermann suggested it would be interesting to find out for which puzzles it holds. Rouse Ball (op. cit.) states that for a puzzle of m rows and n columns it is true when m and n are both even and for certain other cases as well. This cannot be entirely correct for it is clearly false for $m = n = 2$. To show that the property holds for the standard size puzzle we note that the position in Fig. 4 can be obtained from that in Fig. 5 by rotating the board through a negative quarter turn and then moving the squares in the bottom row to the left. The proof is completed by showing that the position in Fig. 5 is an odd permutation of Fig. 3.

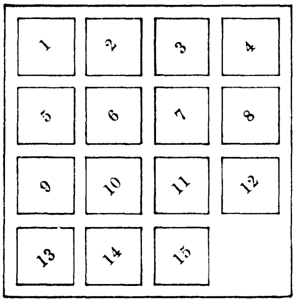

FIG. 3

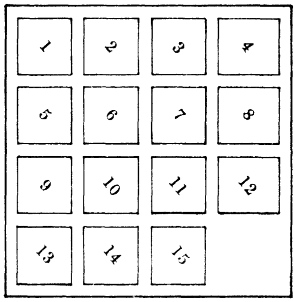

FIG. 4

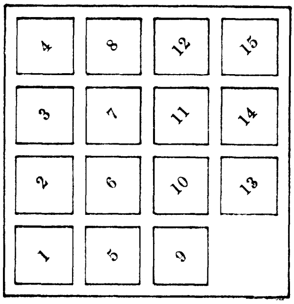

FIG. 5

For the general puzzle we consider situations similar to those in Figs. 3 and 5 to determine for what values of m and n the permutation is odd and so gives rise to a solution one way or the other for every initial position. We start with the elements in their natural order, numbering the empty position as well, thus:

$$\begin{array}{cccc} 1 & 2 & \dots & n \\ n+1 & \dots & \dots & 2n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & mn \end{array}$$

This content downloaded from 128.111.121.42 on Sun, 01 Nov 2015 06:33:06 UTC
All use subject to JSTOR Terms and Conditions

We move the numbers from the board, without changing their order, to a temporary store of mn positions. These positions are considered as following in order the last position on the board. We then place them back one at a time, starting with 1, until we arrive at the following array:

$$\begin{array}{ccc} m & \dots & nm \\ \dots & \dots & \dots \\ 2 & \dots & \dots \\ 1 & \dots & \dots \end{array}$$

At each step of this procedure we record the number of transpositions we make in the complete set of mn numbers. The situation when the $(k-1)$th column has been completed is:

$$\begin{array}{cccl} m & \dots & (k-1)m & \\ \dots & \dots & \dots & \\ 2 & \dots & \dots & \\ 1 & \dots & \dots & (k-1)m+1, \quad (k-1)m+2, \quad \dots, \quad nm \end{array}$$

Moving the next element, $(k-1)m+1$, from the store to the board does not change the permutation. However, in moving $(k-1)m+2$ to the board we put it before the k elements already placed in the last row and so perform k transpositions. The next element is placed before the $2k$ elements in the last two rows, requiring $2k$ transpositions, and so on. The last element to be placed in this column is km and $(m-1)k$ transpositions are required. For the whole column then we have

$$k \sum_{r=0}^{m-1} r \quad \text{transpositions.}$$

Summing over all the columns we obtain

$$\sum_{k=1}^{n} k \,.\, \sum_{r=0}^{m-1} r \quad \text{transpositions.}$$

To complete the similarity with Fig. 5, we must move mn (the space) down the last column to its position in the bottom right hand corner, which requires $m-1$ transpositions.

The total number of transpositions is then

$$\tfrac{1}{2}(m-1)m \,.\, \tfrac{1}{2}n(n+1) + m - 1.$$

We are interested only in whether this number is odd or even; that is, in its value modulo 2. Let us represent this by $f(m, n)$. To find the value of $f(m, n)$ for all values of m and n, we note that $\frac{1}{2}n(n+1)$ is an integer, which will be odd or even depending only on the value of n, modulo 4; similarly for m. So we can tabulate the

This content downloaded from 128.111.121.42 on Sun, 01 Nov 2015 06:33:06 UTC
All use subject to JSTOR Terms and Conditions

function thus:

	f	$n \pmod 4$: 0	1	2	3
	0	1	1	1	1
m	1	0	0	0	0
(mod 4)	2	1	0	0	1
	3	0	1	1	0

If the numbers on the squares are marked by dots, or by some other method that will allow the board to be orientated in any way, the above result can easily be extended. The function $f(m, n)$ applies to a puzzle starting with n rows and m columns for a quarter turn in the opposite direction. To keep m for the rows and n for the columns, we define a new function $g(m, n)$ whose table will be the transpose of that for $f(m, n)$. The function for the half turn, $h(m, n)$ say, is then obtained as the sum of f and g.

	g	$n \pmod 4$: 0	1	2	3
	0	1	0	1	0
m	1	1	0	0	1
(mod 4)	2	1	0	0	1
	3	1	0	1	0

	h	$n \pmod 4$: 0	1	2	3
	0	0	1	0	1
m	1	1	0	0	1
(mod 4)	2	0	0	0	0
	3	1	1	0	0

A simpler method, by means of transformation geometry, can be used for a square puzzle of side n. A quarter turn is equivalent to successive reflections in two lines at 45°. We reflect in a diagonal and mediator of the square. Reflection in a diagonal leaves the n elements on the diagonal unmoved and interchanges the remaining elements in pairs, giving $\frac{1}{2}(n^2 - n)$ transpositions. Reflection in a mediator, if n is even, interchanges all the elements in pairs, giving $\frac{1}{2}n^2$ transpositions; if n is odd, n elements on the mediator remain unchanged, giving $\frac{1}{2}(n^2 - n)$ transpositions. Again we require a further $n - 1$ transpositions to move the space back to the bottom right hand corner.

The total number of transpositions is $\begin{cases} n^2 - 1, & \text{if } n \text{ is odd.} \\ n^2 - 1 + \frac{1}{2}n, & \text{if } n \text{ is even.} \end{cases}$

So the total number of transpositions is odd only if $\frac{1}{2}n$ is even, or the sides of the square must be doubly even, which is so in the case of the Fifteen puzzle.

For the general puzzle, it is possible to find $h(m, n)$ by considering reflections in the two mediators. Is it possible to find $f(m, n)$ without summing an A.P.?

Ratcliffe College,
Leicester LE7 8SG

A. L. Davies

This content downloaded from 128.111.121.42 on Sun, 01 Nov 2015 06:33:06 UTC
All use subject to JSTOR Terms and Conditions