BLG368E Operations Research

HOMEWORK 1

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1. (50 pts.) A paper-recycling plant processes box board, tissue paper, newsprint, and book paper into pulp that can be used to produce three grades of recycled paper (grades 1, 2, and 3). The prices per ton and the pulp contents of the four inputs are shown in the following Table. Two methods, de-inking and asphalt dispersion, can be used to process the four inputs into pulp. It costs \$20 to de-ink a ton of any input. The process of de-inking keeps 90% of the input's pulp to produce de-inked pulp. It costs \$15 to apply asphalt dispersion to a ton of material. The asphalt dispersion process keeps 80% of the input's pulp. At most, 3,000 tons of input can be run through each process (the asphalt dispersion process and the de-inking process). Grade 1 paper can only be produced with newsprint or book paper pulp; grade 2 paper, only with book paper, tissue paper, or box board pulp; and grade 3 paper, only with newsprint, tissue paper, or box board pulp. To meet its current demands, the company needs 500 tons of pulp for grade 1 paper, 500 tons of pulp for grade 2 paper, and 600 tons of pulp for grade 3 paper.

Input	Cost (\$)	Pulp Content (%)
Box board	5	15
Tissue paper	6	20
Newsprint	8	30
Book paper	10	40

a. Formulate an LP to minimize the cost of meeting the demands for pulp in general form (by using ∀ and ∑ notation).

Decision variables:

x_{ijk}: the amount of input i to produce grade j paper with process k (ton)

($i = 1$(box board), 2(tissue paper), 3(newsprint), 4(book paper); $j = 1,2,3$; $k = 1$(de-inking), 2(asphalt dispersion))

Parameters:

C_k: cost of applying process k to a ton of material (\$)

B_i: cost of input i per ton (\$)

P_i: pulp content ratio for input i per ton

D_j: requirement for grade j paper to meet demand (ton)

R_k: keeping ratio for process k ($R_1 = 0.9, R_2 = 0.8$)

Q: capacity of process k (ton) ($Q = 3000$)

$Y_{ij} = 1$ if input i is used to produce grade j paper, 0 otherwise.

LP model:

$$\min z = \sum_i \sum_j \sum_k C_k Y_{ij} x_{ijk} + \sum_i \sum_j \sum_k B_i Y_{ij} x_{ijk} \tag{1}$$

$$\text{s.t.} \quad \sum_i \sum_k R_k P_i Y_{ij} x_{ijk} \geq D_j \quad \forall j \tag{2}$$

$$\sum_i \sum_j Y_{ij} x_{ijk} \leq Q \quad \forall k \tag{3}$$

$$x_{ijk} \geq 0 \quad \forall i, \forall j, \forall k \tag{4}$$

Objective function (1) minimizes total cost. Constraints (2) satisfy total requirement for grade j paper to meet demand. Constraints (3) ensure that total amount of input run through the process k cannot exceed the capacity (3000 tons).

b. Find the optimal solution with GAMS.

GAMS code:

```
sets
i      inputs        /I1*I4/
j      paper grade    /G1*G3/
k       process        /1,2/;

Parameters
C(k)    cost of applying process k to a ton of material ($)
/
1     20
2     15/

B(i)    cost of input i per ton ($)
/
I1      5
I2      6
I3      8
I4      10/

P(i)    pulp content ratio for input i per ton
/
I1      0.15
I2      0.20
I3      0.30
I4      0.40/

D(j)    requirement for grade j paper to meet demand (ton)
/
G1      500
G2      500
G3      600/

R(k)    removing ratio for process k
/
1     0.9
2     0.8/;

Scalars
Q       capacity of process k (ton)  /3000/;

Table
```

```
Y(i,j)   1 if input i is used to produce grade j paper - 0 otherwise
       G1     G2     G3
I1      0      1      1
I2      0      1      1
I3      1      0      1
I4      1      1      0;

positive variables
x(i,j,k) the amount of input i to produce grade j paper with process k (ton);

variables
z       total cost;

Equations
objective       total cost
demand(j)       total requirement for grade j paper to meet demand
capacity(k)     capacity constraint for process k;

objective..     z =e= sum((i,j,k),C(k)*Y(i,j)*x(i,j,k))+ sum((i,j,k),B(i)*Y(i,j)*x(i,j,k));
demand(j)..     sum((i,k),R(k)*P(i)*Y(i,j)*x(i,j,k)) =g= D(j);
capacity(k)..   sum((i,j),x(i,j,k)) =l= Q;

model  recycling  /all/;
solve recycling using LP minimizing z;
display x.L;
```

<u>Optimal solution:</u>

1388.9 tons of book paper (input 4) are used to produce grade 1 paper by de-inking.

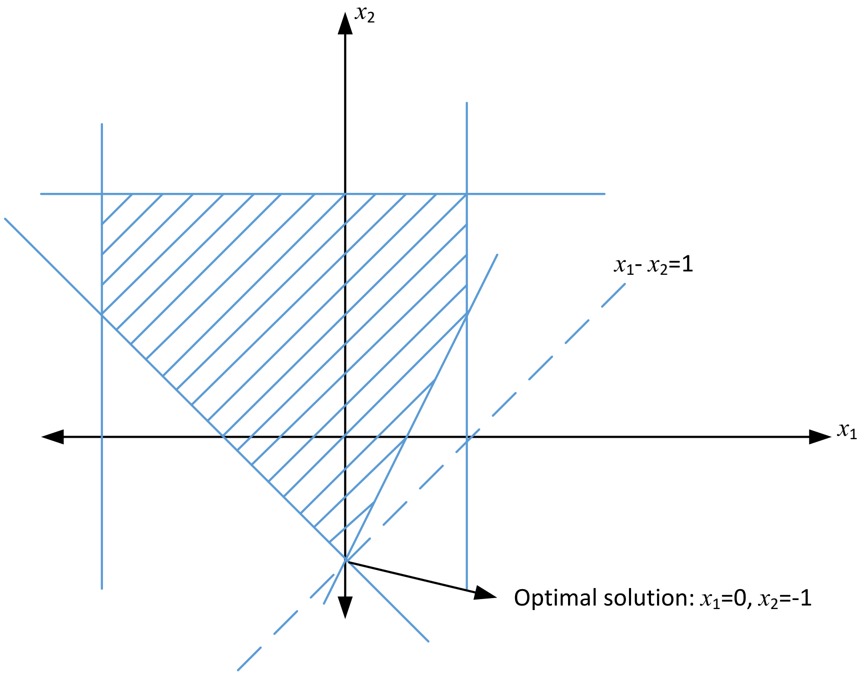
944.4 tons and 500 tons of book paper (input 4) is used to produce grade 2 paper by de-inking and asphalt dispersion, respectively.

2500 tons of newprint (input 3) are used to produce grade 3 paper by asphalt dispersion.

Total cost will be 140,000$.

2. (10 pts.) Use the graphical method to find the optimal solution to the following LP.

$$\begin{aligned} \text{max} \quad & z = x_1 - x_2 \\ \text{s.t.} \quad & -x_1 - x_2 \le 1 \\ & 2x_1 - x_2 \le 1 \\ & -2 \le x_1 \le 1 \\ & x_2 \le 2 \end{aligned}$$



Optimal solution is $x_1 = 0, x_2 = -1, z = 1$.

3. (15 pts.) Find all basic solutions of the following system.

$$\begin{aligned} -x_1 + 2x_2 + x_3 + x_4 - 2x_5 &= 4 \\ x_1 - 2x_2 \quad\quad + 2x_4 - x_5 &= 3 \end{aligned}$$

Basic variables	**Basic solution**
$\{x_1, x_2\}$	Since the coefficient vectors of x_1 and x_2 are not linearly independent, no basic solution exists with x_1 and x_2 as basic variables
$\{x_1, x_3\}$	$x_2, x_4, x_5 = 0, x_1 = 3, x_3 = 7$
$\{x_1, x_4\}$	$x_2, x_3, x_5 = 0, x_1 = -5/3, x_4 = 7/3$
$\{x_1, x_5\}$	$x_2, x_3, x_4 = 0, x_1 = 2/3, x_5 = -7/3$
$\{x_2, x_3\}$	$x_1, x_4, x_5 = 0, x_2 = -3/2, x_3 = 7$
$\{x_2, x_4\}$	$x_1, x_3, x_5 = 0, x_2 = 5/6, x_4 = 7/3$
$\{x_2, x_5\}$	$x_1, x_3, x_4 = 0, x_2 = -1/3 , x_5 = -7/3$
$\{x_3, x_4\}$	$x_1, x_2, x_5 = 0, x_3 = 5/2, x_4 = -3$
$\{x_3, x_5\}$	$x_1, x_2, x_4 = 0, x_3 = -2, x_5 = -3$
$\{x_4, x_5\}$	$x_1, x_2, x_3 = 0, x_4 = 2/3, x_5 = -5/3$

4. (10 pts.) Use the simplex algorithm to find the optimal solution to the following LP. Determine the leaving and entering variables at each iteration.

$$\begin{aligned} \max \quad & z = -x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + 4x_2 \le 12 \\ & 2x_1 - \ \ x_2 \le 12 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Standard form:

$$\begin{aligned} \max \quad & z = -x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + 4x_2 + s_1 = 12 \\ & 2x_1 - x_2 + s_2 = 12 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

Initial tableau:

z	x_1	x_2	s_1	s_2	RHS	Ratio test
1	1	-2	0	0	0	
0	3	4	1	0	12	12/4=3
0	2	-1	0	1	12	-

x_2 enters to the solution, s_1 leaves the solution.

z	x_1	x_2	s_1	s_2	RHS
1	2.5	0	0.5	0	6
0	0.75	1	0.25	0	3
0	2.75	0	0.25	1	15

Optimal solution is obtained. Optimal solution is $x_1 = 0, x_2 = 3, z = 6$.

5. (15 pts.) Use the 2-phase simplex algorithm to find the optimal solution to the following LP.

$$\begin{aligned} \max \quad & z = 5x_1 - 2x_2 + x_3 \\ \text{s.t.} \quad & 2x_1 + 4x_2 + x_3 \leq 6 \\ & 2x_1 + x_2 + 3x_3 \geq 2 \\ & x_1, x_2 \geq 0 \\ & x_3 \text{ urs.} \end{aligned}$$

Standard form:

$$\begin{aligned} \max \quad & z = 5x_1 - 2x_2 + x_3^+ - x_3^- \\ \text{s.t.} \quad & 2x_1 + 4x_2 + x_3^+ - x_3^- + s_1 = 6 \\ & 2x_1 + x_2 + 3x_3^+ - 3x_3^- - e_2 = 2 \\ & x_1, x_2, x_3^+, x_3^-, s_1, s_2 \geq 0 \end{aligned}$$

Phase 1:

$$\begin{aligned} \min \quad & w = a_2 \\ \text{s.t.} \quad & 2x_1 + 4x_2 + x_3^+ - x_3^- + s_1 = 6 \\ & 2x_1 + x_2 + 3x_3^+ - 3x_3^- - e_2 + a_2 = 2 \\ & x_1, x_2, x_3^+, x_3^-, s_1, s_2, a_2 \geq 0 \end{aligned}$$

w	x_1	x_2	x_3^+	x_3^-	s_1	e_2	a_2	RHS
1	0	0	0	0	0	0	-1	0
0	2	4	1	-1	1	0	0	6
0	2	1	3	-3	0	-1	1	2

Initial tableau:

w	x_1	x_2	x_3^+	x_3^-	s_1	e_2	a_2	RHS	Ratio test
1	2	1	3	-3	0	-1	0	2	
0	2	4	1	-1	1	0	0	6	6/1=6
0	2	1	3	-3	0	-1	1	2	2/3=0.66

x_3^+ enters the basis, a_2 leaves the basis.

Optimal tableau:

w	x_1	x_2	x_3^+	x_3^-	s_1	e_2	a_2	RHS
1	0	0	0	0	0	0	-1	0
0	4/3	11/3	0	0	1	1/3	-1/3	16/3
0	2/3	1/3	1	-1	0	-1/3	1/3	2/3

Since $w = 0$ and a_2 is not in the basis at optimal tableau, we continue to Phase 2 by eliminating the column of a_2.

Phase 2:

z	x_1	x_2	x_3^+	x_3^-	s_1	e_2	RHS
1	-5	2	-1	1	0	0	0
0	4/3	11/3	0	0	1	1/3	16/3
0	2/3	1/3	1	-1	0	-1/3	2/3

z	x_1	x_2	x_3^+	x_3^-	s_1	e_2	RHS	Ratio test
1	-13/3	7/3	0	0	0	-1/3	2/3	
0	4/3	11/3	0	0	1	1/3	16/3	(16/3)/(4/3)=4
0	2/3	1/3	1	-1	0	-1/3	2/3	(2/3)/(2/3)=1

x_1 enters the basis, x_3^+ leaves the basis.

z	x_1	x_2	x_3^+	x_3^-	s_1	e_2	RHS	Ratio test
1	0	9/2	13/2	-13/2	0	-5/2	5	
0	0	3	-2	2	1	1	4	4/2=2
0	1	1/2	3/2	-3/2	0	-1/2	1	-

x_3^- enters the basis, s_1 leaves the basis.

z	x_1	x_2	x_3^+	x_3^-	s_1	e_2	RHS
1	0	57/4	0	0	13/4	3/4	18
0	0	3/2	-1	1	1/2	1/2	2
0	1	11/4	0	0	3/4	1/4	4

Optimal solution is obtained. Optimal solution is $x_1 = 4, x_2 = 0, x_3 = -2, z = 18$.