XII. Day of the Week of January 1st

Which day of the week appears more often as the first day of a year, Saturday or Sunday?

Most people would answer: neither. It's intuitive that in the long run the frequency of Saturdays and Sundays that fall on January 1st should be the same. But that's not the case.

Before we go any further, let's remind ourselves of the definition of a leap year. A particular year is a leap year if its number is divisible by 4, but not by 100, unless it's divisible by 400. For example, years 1892 and 1896 were leap years, 1900 was not, 1904, 1908, etc. were leap years, and 2000 is.

Thus, within any period of 400 years we have exactly 97 leap years (i.e., all of the years that are divisible by 4 — and there are 100 of them — except three years whose numbers are divisible by 100 but not by 400). Within any period of 400 years we have exactly

$$97 \cdot 366 + 303 \cdot 365 = 146,097 \text{ days.}$$

The number 146,097 is divisible by 7. This means that any period of 400 years consists of an integer number of whole weeks. It also means that 400 years constitute a cycle. Since 1 January 2001 is a Monday, then 1 January 2401 is without a doubt a Monday as well.

Since 400 is not divisible by 7, different days of the week may have different frequencies. It's impossible to have the same number of Mondays, Tuesdays, Wednesdays, etc. within each 400 years, so it's sufficient to find out whether Saturday or Sunday appears more often as January 1st within an *arbitrary* period of 400 years.

Since we have the freedom of selecting *any* 400-year period, let's do some counting for the period of 2001 – 2400. Within this period we have four long, regular intervals where a leap year occurs precisely every four years. The only exceptions are 2100, 2200, and 2300, which are *not* leap years. Note also that any 28-year period that's within one of these regular intervals (i.e., every period of 28 years that doesn't contain the years 2100, 2200, or 2300) has four instances of each day as the first day of the year. For example, within the period 1 January 2034 – 31 December 2061, there are precisely four Mondays, four Tuesdays, four Wednesdays, etc. as days for January 1st. We needn't be concerned with these periods. During these 28-year intervals there are 7 leap years, and

$$7 \cdot 366 + 21 \cdot 365 = 10,227,$$

which is divisible by 7.

Therefore, let's consider the following set of periods, which amount to 400 consecutive years:

(a) 2001 – 2028	(e) 2101 – 2128	(i) 2201 – 2228	(m) 2301 – 2328
(b) 2029 – 2056	(f) 2129 – 2156	(j) 2229 – 2256	(n) 2329 – 2356
(c) 2057 – 2084	(g) 2157 – 2184	(k) 2257 – 2284	(o) 2357 – 2384
(d) 2085 – 2100	(h) 2185 – 2200	(l) 2285 – 2300	(p) 2385 – 2400

There are four repetitions of each day of the week (as January 1st) in each of the periods: (a), (b), (c), (e), (f), (g), (i), (j), (k), (m), (n), and (o), so the only real counting is needed for the remaining periods: (d), (h), (l), and (p). We have to check (e.g., using a universal calendar) that

1 January 2085 is a Monday,
1 January 2185 is a Saturday,
1 January 2285 is a Thursday, and
1 January 2385 is a Tuesday.

Then it's easy to find out the exact number of days of the week in each of these four periods:[4]

	20—	21—	22—	23—
—85	Mon	Sat	Thu	Tue
—86	Tue	Sun	Fri	Wed
—87	Wed	Mon	Sat	Thu
—88	Thu	Tue	Sun	Fri
—89	Sat	Thu	Tue	Sun
—90	Sun	Fri	Wed	Mon
—91	Mon	Sat	Thu	Tue
—92	Tue	Sun	Fri	Wed
—93	Thu	Tue	Sun	Fri
—94	Fri	Wed	Mon	Sat
—95	Sat	Thu	Tue	Sun
—96	Sun	Fri	Wed	Mon
—97	Tue	Sun	Fri	Wed
—98	Wed	Mon	Sat	Thu
—99	Thu	Tue	Sun	Fri
—00	Fri	Wed	Mon	Sat

Thus, the count, for periods (d), (h), (l), and (p), is

[4]The last row of the table gives the day of the week for the first day of years 2100, 2200, 2300, and 2400, respectively.

Monday:	8
Tuesday:	10
Wednesday:	9
Thursday:	9
Friday:	10
Saturday:	8
Sunday:	10

and the total count for 400 years is

Monday:	56
Tuesday:	58
Wednesday:	57
Thursday:	57
Friday:	58
Saturday:	56
Sunday:	58

Thus, the probability that an arbitrary January 1st is a Sunday is 0.145 (i.e., 58/400), whereas the same probability for a Saturday is only 0.14 (i.e., 56/400).

How can you make some money from that knowledge? Now there's a real-world problem!