$$E(\hat{\theta}_3) = E\left(\frac{\sum_{i=1}^{n} \varpi_i X_i}{\sum_{i=1}^{n} \varpi_i}\right) = \frac{1}{\sum_{i=1}^{n} \varpi_i} E\left(\sum_{i=1}^{n} \varpi_i X_i\right) = \mu$$

因此，估計量 $\overline{X}$ 及 $\sum_{i=1}^{n} \varpi_i X_i / \sum_{i=1}^{n} \varpi_i$ 皆為母體參數 μ 之不偏估計量。

$\Sigma(x_i-\bar{x})^2 = \Sigma(x_i^2 - 2x_i\bar{x} + \bar{x}^2)$

$= \Sigma x_i^2 - \Sigma 2x_i\bar{x} + \bar{x}^2$

$= \Sigma x_i^2 - 2\Sigma n\bar{x}\bar{x} + \Sigma\bar{x}^2 \Rightarrow \Sigma x_i^2 - n\bar{x}^2$

$E(\hat{\theta}_1) = E\left(\frac{\Sigma(x_i-\bar{x})^2}{n}\right) = \frac{1}{n}E(\Sigma x_i^2 - n\bar{x}^2)$

$= \frac{n-1}{n}\sigma^2 \neq \sigma^2$

非不偏估計量

例 6.4

假設統計量 $\hat{\theta}_1 = \sum_{i=1}^{n}(X_i - \overline{X})^2/n$，$\hat{\theta}_2 = \sum_{i=1}^{n}(X_i - \overline{X})^2/(n-1)$ 分別為母體變異數 σ^2 之估計量，請問何者滿足不偏性？

解 依 $E(X_i) = \mu$，$V(X_i) = \sigma^2 = E\left(X_i^2\right) - \mu^2$，

① $E(\Sigma x_i^2) = n\sigma^2 + n\mu^2$

$V(x) = E(x^2) - E^2(x)$

$E(x_i^2) = V(x_i) + E^2(x_i)$

$\Sigma E(x_i^2) = \Sigma[V(x_i) + E^2(x_i)] \Rightarrow \Sigma(E x_i^2)$

$= n\sigma^2 + n\mu^2$

② $E(n\bar{x}^2) = nE(\bar{x}^2)$

$V(x) = E(x^2) - E^2(x)$

$E(\bar{x}^2) = V(\bar{x}) + E^2(\bar{x})$

$E(\bar{x}^2) = \frac{\sigma^2}{n} + \mu^2$

$\Rightarrow nE(\bar{x}^2) = \sigma^2 + n\mu^2$