Quantum Algorithms, Spring 2022: Lecture 6 Scribe

Rutvij Menavlikar, Abhyudit Mohla

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1 Recap

1.1 Universality of Quantum Circuits

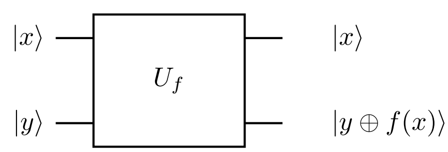
- Single qubit and $CNOT$ gates together can be used to implement an arbitrary two-level unitary operation on the state space of n qubits.[**?**]
- The set $\{CNOT, H, R_{\frac{\pi}{4}}\}$ is universal for quantum computing, i.e. any other quantum circuit can be well approximated using quantum circuits of only these gates.
  And *Solovay-Kitaev thoerem* states that any t-gate quantum circuit can be ϵ approximated using $\mathcal{O}\left(t.\text{polylog}\left(\frac{1}{\epsilon}\right)\right)$ gates from $\{CNOT, H, R_{\frac{\pi}{4}}\}$.

1.2 Quantum Parallelism

To estimate a function f s.t.

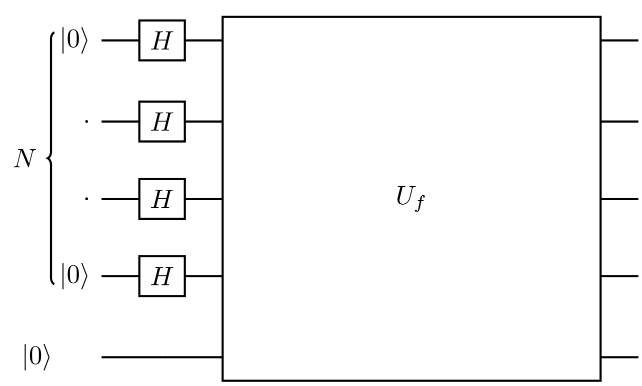
$$f : \{0,1\}^n \to \{0,1\}$$

we use a quantum gate U_f in the following way.



$$\begin{aligned} \text{So, if } f(x) = 0,\ & |x\rangle|y\rangle \xrightarrow{U_f} |x\rangle|y\rangle \\ \text{if } f(x) = 1,\ & |x\rangle|y\rangle \xrightarrow{U_f} |x\rangle|\bar{y}\rangle \end{aligned} \tag{1}$$

Now, we can parallelise this circuit, even for n qubits by adding Hadamard's gates on input qubits before applying the U_f gate in the following way.



Thus, by applying U_f only once, we are able to obtain a quantum state that contains all possible 2^n values of $f(x)$ in superposition.

$$|0\rangle^{\otimes n} |0\rangle \xrightarrow{H^{\otimes n} \otimes I} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |0\rangle \xrightarrow{U_f} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle \tag{2}$$

Note that,

- Quantum parallelism is not enough to demonstrate the power of Quantum Computing
- Quantum parallelism needs to be combined with interference, entanglement to do something better than classical computing.

1.3 Deutsch Algorithm

1.3.1 Problem

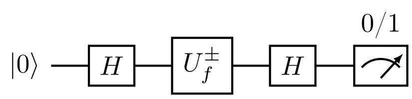
We are given a binary function $f : \{0, 1\} \to \{0, 1\}$ such that either $f(0) = f(1)$ or $f(0) \neq f(1)$. We have determine which using minimum number of queries.

1.3.2 Classical Approach

We query for value of $f(0)$ and $f(1)$ to determine which kind of function is f.
Hence, classical approach requires *2* queries.

1.3.3 Quantum Algorithm

We set up the following circuit



The calculations are as follows

$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \xrightarrow{U_f^{\pm}} \frac{1}{\sqrt{2}} \Big((-1)^{f(0)} |0\rangle + (-1)^{f(1)} |1\rangle \Big) \xrightarrow{H}$$
$$\frac{1}{2} \Big(\Big((-1)^{f(0)} + (-1)^{f(1)} \Big) |0\rangle + \Big((-1)^{f(0)} - (-1)^{f(1)} \Big) |1\rangle \Big) \tag{3}$$

Thus, when $f(0) = f(1)$ the measured state is $|0\rangle$, and when $f(0) \neq f(1)$ the measured state is $|1\rangle$.
And hence, with quantum parallelism we can solve the problem with only 1 query to U_f.

2 Mach-Zehnder Interferometer

In physics, the Mach–Zehnder interferometer is a device used to determine the relative phase shift variations between two collimated beams derived by splitting light from a single source.[**?**] Essentially, Mach-Zehnder Interferometer physically captures Deutsch's problem.

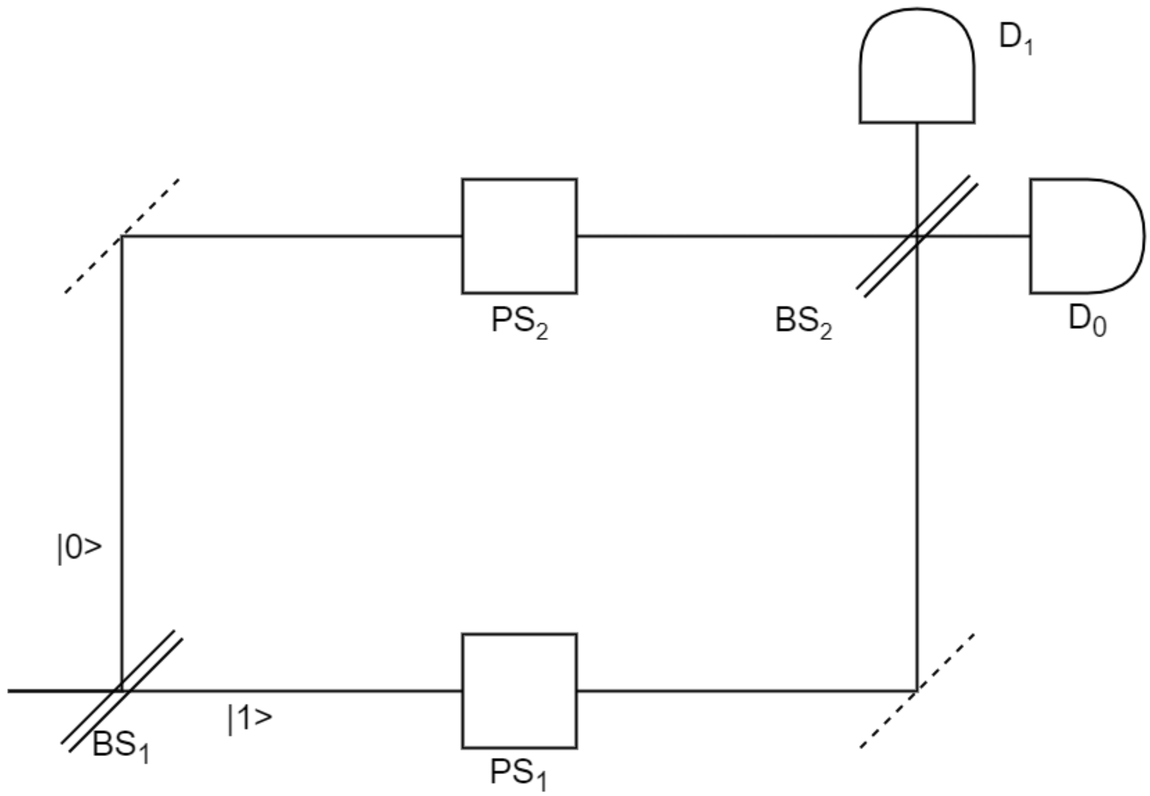


Figure 1: Mach-Zehnder Interferometer set up

The set up includes the following components

- BS_1 and BS_2 are beam splitters. They split a photon into a superposition of *lower path* and *upper path.*
- PS_1 and PS_2 are phase shifters. PS_1 shifts the phase of the beam incident on it by ϕ_1 and PS_2 shifts it by ϕ_0.
- D_0 and D_1 are detectors. D_0 detects a photon in *lower path* and D_1 detects a photon in *upper path.*

These components are arranged as shown in fig 1.

Here we assume that *lower path* is represented by $|0\rangle$ and *upper path* is represented by $|1\rangle$. Thus, $|0\rangle$ undergoes phase shift of ϕ_0 and $|1\rangle$ undergoes phase shift of ϕ_1. This can be represented as

$$|0\rangle \xrightarrow{PS_2} e^{i\phi_0}|0\rangle, |1\rangle \xrightarrow{PS_1} e^{i\phi_1}|1\rangle$$

Also, it is given that $|\phi_0 - \phi_1| = 0$ or π and we have to determine which.

Thus, the calculations are as follows:

$$|0\rangle \xrightarrow{BS_1} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{PS} \frac{1}{\sqrt{2}}\left(e^{i\phi_0}|0\rangle + e^{i\phi_1}|1\rangle\right) \xrightarrow{BS_2} \frac{1}{2}\left(\left(e^{i\phi_0} + e^{i\phi_1}\right)|0\rangle + \left(e^{i\phi_0} - e^{i\phi_1}\right)|1\rangle\right) \tag{4}$$

But we can see that

$$\frac{1}{2}\left(\left(e^{i\phi_0} + e^{i\phi_1}\right)|0\rangle + \left(e^{i\phi_0} - e^{i\phi_1}\right)|1\rangle\right) = \frac{e^{i\phi_0}}{2}\left(\left(1 + e^{i(\phi_1-\phi_0)}\right)|0\rangle + \left(1 - 1 + e^{i(\phi_1-\phi_0)}\right)|1\rangle\right) \tag{5}$$

Thus,

- When $|\phi_0 - \phi_1| = 0$, $|0\rangle$ is observed, i.e. D_0 detects a photon.
- When $|\phi_0 - \phi_1| = \pi$, $|1\rangle$ is observed, i.e. D_1 detects a photon.

Note that: This is a quantum phenomena. Classically, each of the detector would detect a photon 50% of the times.

3 Deutsch-Jozsa Algorithm

This algorithm is a generalisation of Deutsch problem on n qubits.

3.1 Problem

We are given a black box U_f for some Boolean function $f : \{0,1\}^n \to \{0,1\}$ where it is given that f is either *constant* or *balanced*, i.e

$$\begin{aligned} \text{Either, } f(x) = \;& \textit{constant: } \forall x \in \{0,1\}^n, f(x) = e \text{ s.t. } e \in \{0,1\} \\ f(x) = \;& \textit{balanced: } \begin{cases} 0, & \text{for } 2^{n-1} \text{ values of } x \\ 1, & \text{for rest of the } 2^{n-1} \text{ values of } x \end{cases} \end{aligned}$$

We have to determine which using minimum number of queries.

3.2 Classical Approach

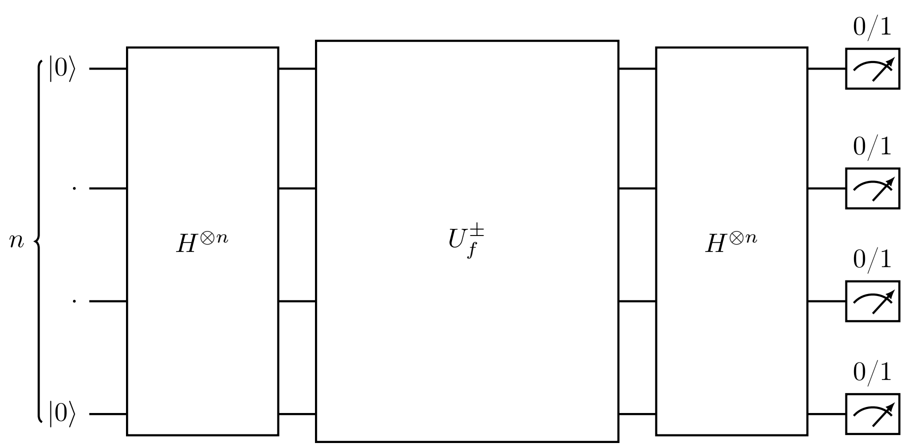
Serially, or randomly we input values of $x \in \{0,1\}^n$ and check their outputs.

- If f is *balanced*:
  If we observe 0 and 1 in output, then f is *balanced.* And by Pigeonhole principle, we would require at-most $2^{n-1} + 1$ queries to reach this conclusion.
- If f is *constant*:
  If we observe the same output after $2^{n-1} + 1$ queries, we can conclude that f is *constant.*

Hence, number of queries required are $2^{n-1} + 1$.

3.3 Quantum Algorithm

We set up the following circuit



This leads to the following changes,

$$|0\rangle^{\otimes n} \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \xrightarrow{U_f^{\pm}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} |x\rangle \xrightarrow{H^{\otimes n}} \frac{1}{2^n} \sum_{x\in\{0,1\}^n} \sum_{z\in\{0,1\}^n} (-1)^{f(x)+x.z} |z\rangle \tag{6}$$

Let $|\psi_f\rangle = \frac{1}{2^n} \sum_{x\in\{0,1\}^n} \sum_{z\in\{0,1\}^n} (-1)^{f(x)+x.z} |z\rangle$
Then, we look at the amplitude of $|0\rangle^{\otimes n}$ in $|\psi_f\rangle$,

$$\langle 00..0|\psi_f\rangle = \frac{1}{2^n} \sum_{x\in\{0,1\}^n} \sum_{z\in\{0,1\}^n} (-1)^{f(x)+x.z} \langle 00..0|z\rangle \tag{7}$$

$$= \frac{1}{2^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} \qquad \langle 00..0|z\rangle = \begin{cases} 1, & z = |0\rangle^{\otimes n} \\ 0, & \text{otherwise} \end{cases}, x.0 = 0 \tag{8}$$

Thus,

- If f is *balanced*:
  $\langle 00..0|\psi_f\rangle = 0$
- If f is *constant*:
  $\langle 00..0|\psi_f\rangle = 1$ or $\langle 00..0|\psi_f\rangle = -1$

Thus, if the result of measurement is the state $|0\rangle^{\otimes n}$, then f is *constant*, and for any other result of measurement f is *balanced*.
Hence, only 1 query is required to classify the function. And hence, there is an exponential optimization in query complexity.

3.4 Randomized Deutsch-Jozsa Algorithm

Now, we require our algorithm to classify f with a probability $\geq 1 - \epsilon$ $(\epsilon > 0)$

3.4.1 Classical Approach

We choose d values of $x \in \{0,1\}^n$ to query.

$$\text{Let values of } x\text{: } S = \{x_1, x_2, .., x_d\} \qquad x_i \in \{0,1\}^n$$

Thus, by making the d queries, we get $f(S) = \{f(x_1), f(x_2), .., f(x_d)\}$ Then the following scenarios are possible,

- *Case 1*: $\forall x_i \in S, f(x_i) = 0$ or $\forall x_i \in S, f(x_i) = 1$
  In this case, if $d < 2^{n-1} + 1$, then we still cannot be sure is f is *balanced* or *constant*. And hence, there is an uncertainty in classification. But the probability of observing this case is $\frac{1}{2^d} + \frac{1}{2^d} = \frac{1}{2^{d-1}}$
- *Case 2*: $\exists x_i, x_j \in S$, s.t. $f(x_i) \neq f(x_j)$
  In this case, we can definitely conclude that f is *balanced*.

Thus, we can choose d in a way that we reduce the probability of observing the case where we are not certain about the outcome. That is,

$$\frac{1}{2^{d-1}} < 1 - (1 - \epsilon) \Rightarrow d > \log_2 \left(\frac{2}{\epsilon}\right) \tag{9}$$

Hence, the query complexity for the classical approach to the randomised Deutsch-Jozsa Problem is $\mathcal{O} \log\left(\frac{1}{\epsilon}\right)$

4 Bernstein-Vazirani Algorithm

A related problem was given by Ethan Bernstein and Umesh Vazirani.

4.1 Problem

We are given a black box U_f for some Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ where it is given that $\forall x \in \{0,1\}^n f(x) = s.x(\mod 2)$ for some unknown string $s \in \{0,1\}^n$. i.e.,

$$\forall x \in \{0,1\}^n f(x) = s_1x_1 \oplus s_2x_2 \oplus ... \oplus s_nx_n$$

We have to determine s using minimum number of queries.

4.2 Classical Approach

We pass a set of inputs $I = \{x_1, x_2, .., x_n\}$ such that

$$\forall x_i \in I, x_i(j) = \begin{cases} 1, & j = i \\ 0, & \text{otherwise} \end{cases}$$

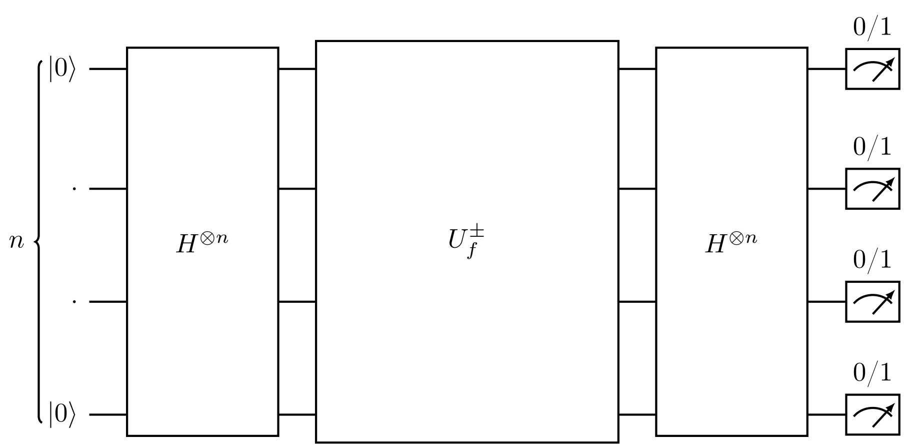
Thus, the output of each input x_i looks like

$$\begin{aligned} f(x_i) &= s_1x_i(1) \oplus s_2x_i(2) \oplus .. \oplus s_nx_i(n) && (10) \\ &= 0 \oplus 0 \oplus .. \oplus 0 \oplus s_i \oplus 0 \oplus .. \oplus 0 && (11) \\ &= s_i && (12) \end{aligned}$$

Hence, n queries are required to determine s.

4.3 Quantum Algorithm

We set up the circuit in the same way as we did for Deutsch-Jozsa Algorithm.



The calculations are as follows,

$$|0\rangle^{\otimes n} \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \xrightarrow{U_f^{\pm}} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \tag{13}$$

and

$$\begin{aligned} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle &= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x.s(\mod 2)} |x\rangle && (14) \\ &= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x.s} |x\rangle && (15) \end{aligned}$$

But note that $|s\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{x.s} |x\rangle$ and $H^{\otimes n} \times H^{\otimes n} = \mathbb{I}_n \Rightarrow (H^{\otimes n})^{-1} = H^{\otimes n}$
Thus,

$$\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} |x\rangle \xrightarrow{H^{\otimes n}} |s\rangle \tag{16}$$

Hence, the measured output is the string s.
And hence, we are able to determine s in 1 query, which gives us a linear speed up in query complexity.