STA2002: Probability and Statistics II
Confidence Interval I

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Introduction

In this lecture we will introduce the technique of **confidence interval (CI)**, which is an important way to construct interval estimators for parameters.

We will discuss confidence intervals for means:

- The observations are normally distributed or not
- The variance of the observations is known or unknown

Suggested reading: Chapter 7.1 of the textbook.

Confidence Interval

Definition (Confidence Interval)

Let $X_1, X_2, \ldots, X_n$ be a sample on a random variable X, where X has PDF $f(x;\theta)$, $\theta \in \Omega$. Let $L = L(X_1, \ldots, X_n)$ and $U = U(X_1, \ldots, X_n)$ be two statistics. Given a level $\alpha \in (0, 1)$, we say (L, U) is a $(1-\alpha)100\%$ confidence interval for θ if

$$1 - \alpha = P_\theta[\theta \in (L, U)].$$

That is, the probability that the interval includes θ is $1-\alpha$, which is called the **confidence coefficient** of the interval.

CI for mean μ

We consider an i.i.d. random sample $X_1, \ldots, X_n$ of size n for estimating the mean $\mu = \mathbb{E}(X_1)$ under the following four cases:

1. $X_i \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$, σ^2 known
2. X_i i.i.d. with mean μ, variance $\sigma^2 = \mathrm{Var}(X_1)$, σ^2 known
3. $X_i \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$, σ^2 unknown
4. X_i i.i.d. with mean μ, variance σ^2, σ^2 unknown

Case 1: Normal, σ^2 Known

Motivation

Suppose $X_i \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$ and σ^2 is known. Please show that the probability that μ belongs to the random interval

$$[\bar{X} - \frac{\sigma \cdot z_{\alpha/2}}{\sqrt{n}}, \bar{X} + \frac{\sigma \cdot z_{\alpha/2}}{\sqrt{n}}]$$

is $1 - \alpha$, where $\Pr(Z > z_{\alpha/2}) = \alpha/2$ for $Z \sim N(0, 1)$.

Based on the property of normal random variables, we have

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \Rightarrow Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

Then,

$$\Pr\left(-z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) = 1 - \alpha$$
$$\Rightarrow \Pr\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

Case 1: Normal, σ^2 Known

- Once the sample is observed, we can compute the sample mean $\bar{x}$. The computed interval

$$\bar{x} \pm \frac{\sigma \cdot z_{\alpha/2}}{\sqrt{n}} \quad \text{or} \quad \left[\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

  is called a $100(1-\alpha)\%$ (two-sided) **confidence interval (CI)** of μ.

- Properties:
  - CI is centered at $\bar{x}$, a point estimate of μ
  - Width: $2z_{\alpha/2}\sigma/\sqrt{n}$
  - Larger $n \Rightarrow$ shorter CI
  - Larger $\alpha \Rightarrow$ smaller $z_{\alpha/2} \Rightarrow$ shorter CI

Example

Let X be the length of lifetime of a light bulb, and assume that $X \sim N(\mu, 1296)$. We have a random sample of $n = 27$ with $\bar{x} = 1478$. A 95% confidence interval for μ is

$$\begin{aligned}
&\left[\bar{x} - z_{0.025}\left(\frac{\sigma}{\sqrt{n}}\right),\ \bar{x} + z_{0.025}\left(\frac{\sigma}{\sqrt{n}}\right)\right] \\
&= \left[1478 - 1.96\left(\frac{36}{\sqrt{27}}\right),\ 1478 + 1.96\left(\frac{36}{\sqrt{27}}\right)\right] \\
&= [1464.42,\ 1491.58].
\end{aligned}$$

Note that $z_{0.025}$ can be computed by qnorm(0.975) in R.

Case 2: Non-Normal, σ^2 Known

- $X_1, \ldots, X_n$ are i.i.d. with mean $\mu = \mathbb{E}(X_1)$ and variance $\sigma^2 = \mathrm{Var}(X_1)$, and σ^2 is known.
- Note that we release the normal distribution assumption
- By Central Limit Theorem, for large n:

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \overset{\text{approx}}{\sim} N(0, 1).$$

Thus,

$$\Pr\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \approx 1 - \alpha.$$

The **approximate** (two-sided) $100(1-\alpha)\%$ CI is:

$$\bar{x} \pm z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

Case 2: Non-Normal, σ^2 Known

Example

Let X be the amount of orange juice consumed by an American per day. We know that $\sigma = 96$. To estimate μ, we have a sample of $n = 576$ and $\bar{x} = 133$. An approximate 90% confidence interval for μ is:

$$133 \pm 1.645 \left(\frac{96}{\sqrt{576}} \right) = [126.42, 139.58]$$

Case 3: Normal, σ^2 Unknown

- Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$ and σ^2 is unknown.
- As σ is unknown, we cannot use the CI in Case 1 since the interval
$$\bar{x} \pm z_{\alpha/2} \left(\frac{\sigma}{\sqrt{n}} \right)$$
depends on σ.
- Instead, recall the sample variance
$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$
and
$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1).$$

Case 3: Normal, σ^2 Unknown

- Consider

$$\Pr\left(-t_{\alpha/2}(n-1) \le \frac{\bar{X}-\mu}{S/\sqrt{n}} \le t_{\alpha/2}(n-1)\right) = 1-\alpha$$

$$\Pr\left(\bar{X} - \frac{S \cdot t_{\alpha/2}(n-1)}{\sqrt{n}} \le \mu \le \bar{X} + \frac{S \cdot t_{\alpha/2}(n-1)}{\sqrt{n}}\right) = 1-\alpha.$$

- Once we have the realized sample, we can compute $\bar{x}$ and s, and

$$\left[\bar{x} - \frac{s \cdot t_{\alpha/2}(n-1)}{\sqrt{n}}, \bar{x} + \frac{s \cdot t_{\alpha/2}(n-1)}{\sqrt{n}}\right]$$

is called a $100(1-\alpha)\%$ confidence interval (CI) of μ.

- Here, s represents the realized value of S.

Case 3: Normal, σ^2 Unknown

Example

Let $X \sim N(\mu, \sigma^2)$, and both μ and σ^2 are unknown. We compute that $n = 20$, $\bar{x} = 507.50$, and $s = 89.75$. Since $t_{0.05}(19) = 1.729$, a 90% confidence interval for μ is

$$507.50 \pm 1.729 \left(\frac{89.75}{\sqrt{20}} \right) = [472.80, 542.20].$$

Note that $t_{0.05}(19)$ can be computed by qt(0.975,19) in R.

Case 4: Non-Normal, σ^2 Unknown

- $X_1, \ldots, X_n$ are i.i.d. with mean $\mu = E(X_1)$ and variance $\sigma^2 = \text{Var}(X_1)$, and σ^2 is unknown.
- If n is large enough (say $n \geq 100$), or if each X_i is approximately normal, then

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \overset{\text{approx}}{\sim} t(n-1).$$

- As a result,

$$\left[\bar{x} - t_{\alpha/2}(n-1)\left(\frac{s}{\sqrt{n}}\right),\ \bar{x} + t_{\alpha/2}(n-1)\left(\frac{s}{\sqrt{n}}\right)\right]$$

is called an **approximate** (two-sided) $100(1-\alpha)\%$ confidence interval (CI) of μ.

Case 4: Non-Normal, σ^2 Unknown

Note that,

- If n is large enough, then $t_{\alpha/2}(n-1) \approx z_{\alpha/2}$, and so we can also use the interval

$$\left[\bar{x} - z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right),\ \bar{x} + z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right)\right]$$

  as an approximate (two-sided) $100(1-\alpha)\%$ CI for μ.
- If n is small or when each X_i does not look like a normal distribution (e.g., when X_i are very skewed), this method has to be used with caution.

One-sided Confidence Intervals

- Up until now we have only been discussing **two-sided** confidence intervals, since the CI are often of the form $\bar{x} \pm a$ for some constant a.
- Suppose that $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$ and σ^2 is known. Consider

For $X_i \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$, σ^2 known:

$$\Pr\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_\alpha\right) = 1 - \alpha$$

$$\Rightarrow P\left(\mu \geq \bar{X} - z_\alpha \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

One-sided Confidence Intervals

- Thus,

$$\left[\bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}}, \infty\right)$$

  is a one-sided $100(1-\alpha)\%$ CI for μ.

- Similarly,

$$\left(-\infty, \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}\right]$$

  is another one-sided CI for μ.

- When we only care about the a lower or upper bound on μ, say the lower bound of bulb lifetimes, one-sided confidence interval is of the interest.

Two-sided confidence interval for μ

Distribution of X_i	**Distribution**	**Two-sided $1-\alpha$ CI**
$N(\mu, \sigma^2)$, σ^2 known	$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$	$\bar{x} \pm \frac{\sigma \cdot z_{\alpha/2}}{\sqrt{n}}$
Any distribution with large n, σ^2 known	$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \overset{\text{approx}}{\sim} N(0,1)$	$\bar{x} \pm \frac{\sigma \cdot z_{\alpha/2}}{\sqrt{n}}$
$N(\mu, \sigma^2)$, σ^2 unknown	$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$	$\bar{x} \pm \frac{s \cdot t_{\alpha/2}(n-1)}{\sqrt{n}}$
Any distribution with large n, σ^2 unknown	$\frac{\bar{X}-\mu}{S/\sqrt{n}} \overset{\text{approx}}{\sim} t(n-1)$	$\bar{x} \pm \frac{s \cdot t_{\alpha/2}(n-1)}{\sqrt{n}}$