STA2002: Probability and Statistics II

Hypothesis Testing III

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Introduction

- In this lecture we will continue our journey in hypothesis testing and discuss hypothesis testing for proportions.
- Recall that, we have already learned the following tests:
  - Pooled t-test
  - Welch's t-test
  - Paired t-test
  - F-test
- Suggested reading: Chapter 8.3 of the textbook.

Hypothesis testing for proportions

- Let p be the proportion of population who has a certain characteristic.
- To estimate p, we draw n i.i.d. random samples

$$X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p),$$

where each X_i is 1 if the individual has the characteristic, and is 0 if the individual does not have that characteristic.

Hypothesis testing for proportions

- Denote $Y = \sum_{i=1}^{n} X_i \sim \text{Binomial}(n, p)$, then

$$\mathbb{E}(Y) = np$$

  and

$$\text{Var}(Y) = np(1-p).$$

- Moreover, if n is large enough, then by Central Limit Theorem, we have

$$\hat{p} = \frac{Y}{n} = \frac{\sum_{i=1}^{n} X_i}{n} \overset{\text{approx}}{\sim} N\left(p, \frac{p(1-p)}{n}\right)$$

Hypothesis testing for proportions

- Given the null hypothesis test $H_0 : p = p_0$, we construct the following test statistic,

$$T := \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}.$$

- Under H_0, $T \sim N(0, 1)$ approximately.
- Once we observe the data $x_1, \ldots, x_n$, we can compute

$$t = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}},$$

which is the realized value of T.

(One-sided test) $H_0 : p = p_0$, $H_1 : p > p_0$

- p-value is
$$\Pr(T > t; H_0) = \Pr(Z > t),$$
where $Z \sim N(0, 1)$.
- Reject H_0 at significance level α when
$$\begin{aligned} \text{p-value} &= \Pr(Z > t) \leq \alpha \\ \Leftrightarrow & t \geq z_\alpha \\ \Leftrightarrow & \hat{p} \geq p_0 + z_\alpha \sqrt{p_0(1 - p_0)/n} \end{aligned}$$

(One-sided test) $H_0 : p = p_0$, $H_1 : p < p_0$

- p-value is

$$\Pr(T < t; H_0) = \Pr(Z < t).$$

- Reject H_0 at significance level α when

$$\begin{aligned} &\text{p-value} = \Pr(Z < t) \le \alpha \\ \Leftrightarrow &t \le -z_\alpha \\ \Leftrightarrow &\hat{p} \le p_0 - z_\alpha \sqrt{p_0(1-p_0)/n} \end{aligned}$$

(Two-sided test) $H_0 : p = p_0$, $H_1 : p \neq p_0$

- p-value is
$$\Pr(|T| > |t|; H_0) = 2\Pr(Z > |t|),$$
- Reject H_0 at significance level α when

$$\begin{aligned} & \text{p-value} = 2\Pr(Z > |t|) \leq \alpha \\ \Leftrightarrow & |t| \geq z_{\alpha/2} \\ \Leftrightarrow & \hat{p} \geq p_0 + z_{\alpha/2}\sqrt{p_0(1-p_0)/n} \text{ or } \hat{p} \leq p_0 - z_{\alpha/2}\sqrt{p_0(1-p_0)/n} \end{aligned}$$

Example

Example

It was claimed that many commercially manufactured dice are not fair because the "spots" are really indentations, so that, for example, the 6-side is lighter than the 1-side. Let p equal the probability of rolling a 6 with one of these dice. To test $H_0 : p = 1/6$ against the alternative hypothesis $H_1 : p \neq 1/6$ with the significance level $\alpha = 5\%$, several such dice will be rolled to yield a total of $n = 8000$ observations, and the number 6 appears 1389 times.

- Let Y equal the number of times that 6 resulted in the 8000 trials. The test statistics is

$$Z = \frac{Y/n - 1/6}{\sqrt{1/6 \cdot 5/6/n}} \overset{\text{approx}}{\sim} N(0,1)$$

Example

Solution:

- Compute
$$t = \frac{1389/8000 - 1/6}{\sqrt{(1/6)(5/6)/8000}} = 1.67$$
$$z_{\alpha/2} = z_{0.025} = 1.96$$
- Critical region approach: $|t| = 1.67 < 1.96$, so we fail to reject H_0 at 5% significance level.
- p-value approach: $2\Pr(Z > 1.67) = 0.0949 > 0.05$, and so we fail to reject H_0 at 5% significance level.
- However, if we set $\alpha = 0.1$ with $z_{0.05} = 1.64$, then we would reject H_0 at 10% significance level.

Hypothesis testing for equality of two proportions

- Suppose that we have two populations, namely population 1 and population 2.
- Let p_1 and p_2 be the proportion of population 1 and population 2, respectively, who has a certain characteristic.
- Our goal is to carry out testing on $H_0 : p_1 = p_2$ against possible alternative hypothesis H_1.

Hypothesis testing for equality of two proportions

- Denote the samples to be $X_{1,1}, \ldots, X_{1,n} \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p_1)$ and $X_{2,1}, \ldots, X_{2,m} \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p_2)$, and assume that they are independent.
- Denote the sums to be $Y_1 = \sum_{i=1}^{n} X_{1,i} \sim \text{Binomial}(n, p_1)$ and $Y_2 = \sum_{i=1}^{m} X_{2,i} \sim \text{Binomial}(m, p_2)$.
- MLEs of p_1 and p_2 are,

$$\hat{p}_1 = \frac{Y_1}{n}, \quad \hat{p}_2 = \frac{Y_2}{m}.$$

- If n and m are large enough, then by Central Limit Theorem, we have

$$\hat{p}_1 - \hat{p}_2 \overset{\text{approx}}{\sim} N\left(p_1 - p_2, \frac{p_1(1-p_1)}{n} + \frac{p_2(1-p_2)}{m}\right)$$

- We will only derive the two-sided test. The one-sided test is very similar.
- Under H_0, $p_1 = p_2$, and we have

$$\hat{p}_1 - \hat{p}_2 \overset{\text{approx}}{\sim} N\left(0, p_1(1-p_1)\left(\frac{1}{n} + \frac{1}{m}\right)\right)$$

- However, we only know $p_1 = p_2$ but the value of p_1 is unknown.
- Question: What is the distribution of $Y_1 + Y_2$ under H_0?

(Two-sided test) $H_0 : p_1 = p_2$, $H_1 : p_1 \neq p_2$

- Approximate p_1 by
$$\hat{p} = \frac{Y_1 + Y_2}{n + m}.$$
- Construct the test statistic,
$$T := \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1 - \hat{p})(1/n + 1/m)}}.$$
- Under H_0, $T \overset{\text{approx}}{\sim} N(0, 1)$.
- Once we observe the data y_1 and y_2, we can compute
$$t = \frac{y_1/n - y_2/m}{\sqrt{\hat{p}(1 - \hat{p})(1/n + 1/m)}}$$

(Two-sided test) $H_0 : p_1 = p_2$, $H_1 : p_1 \neq p_2$

- p-value is

$$\Pr(|T| > |t|; H_0) = 2\Pr(Z > |t|),$$

and we reject H_0 at significance level α when

$$\begin{aligned} & \text{p-value} = 2\Pr(Z > |t|) \leq \alpha \\ \Leftrightarrow & |t| \geq z_{\alpha/2} \\ \Leftrightarrow & |\hat{p}_1 - \hat{p}_2| \geq z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})(1/n + 1/m)} \end{aligned}$$

Example

Example

In a sample of $n = 200$, we have $y_1 = 22$ low birth weight babies in Country 1. In a sample of $m = 100$ we find $y_2 = 16$ low birth weight babies in Country 2. Please test

$$H_0 : p_1 = p_2 \quad \text{vs.} \quad H_1 : p_1 \neq p_2.$$

at a significance level 5%.

Solution:

- We compute

$$t = \frac{\frac{22}{200} - \frac{16}{100}}{\sqrt{\frac{38}{300}\left(1 - \frac{38}{300}\right)\left(\frac{1}{200} + \frac{1}{100}\right)}} = -1.23.$$

- Critical region approach: Since $|t| = 1.23 < z_{0.025} = 1.96$
- p-value approach: $2\,\Pr(Z > 1.23) = 0.219 > 0.05$
- Fail to reject H_0 at 5% significance level.

Table 1: Summary of Hypothesis Tests for Proportions

H_0	H_1	p-value	Critical Region
One-sample test $(T \overset{\text{approx}}{\sim} N(0,1),\ t = \frac{\hat{p}-p_0}{\sqrt{p_0(1-p_0)/n}})$			
$p = p_0$	$p > p_0$	$\Pr(T > t)$	$t \geq z_\alpha$
$p = p_0$	$p < p_0$	$\Pr(T < t)$	$t \leq -z_\alpha$
$p = p_0$	$p \neq p_0$	$2\Pr(T > \lvert t \rvert)$	$\lvert t \rvert \geq z_{\alpha/2}$
Two-sample test $(T \overset{\text{approx}}{\sim} N(0,1),\ t = \frac{\hat{p}_1-\hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})(1/n+1/m)}})$			
$p_1 = p_2$	$p_1 > p_2$	$\Pr(T > t)$	$t \geq z_\alpha$
$p_1 = p_2$	$p_1 < p_2$	$\Pr(T < t)$	$t \leq -z_\alpha$
$p_1 = p_2$	$p_1 \neq p_2$	$2\Pr(T > \lvert t \rvert)$	$\lvert t \rvert \geq z_{\alpha/2}$