DETERMINANT FOR NON-SQUARE MATRICES

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DETERMINANT FOR NON-SQUARE MATRICES

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Abstract

In this paper, the authors generalized the concept of determinant form, square matrix to non square matrix. We also discuss the properties for non square determinant. Using this we investigate the inverse of non square matrix.

1. Introduction

The term of determinant was introduced by Gauss in 1801 while discussing quadratic forms . He used the term because of the determinant determines the properties of quadratic forms. We know that the area of the triangle with vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]. \tag{1.1}$$

Similarly the condition for a second degree equation in x and y to represent a pair of straight line is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0. \tag{1.2}$$

To minimize the difficulty in remembering these type of expressions, Mathematicians developed the idea of representing the expression in determinant form.

Key Words and Phrases : *Matrix, Determinant of matrix.*

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The above expression (1.1) and (1.2) can be represented in the form

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Again if we determine x, y, z from the three equations

$$a_1x + b_1y + c_1z; a_2x + b_2y + c_2z; a_3x + b_3y + c_3z,$$

we obtain $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0$.

This can be written as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Thus a determinant is a particular type of expression written in a special concise form. Note that the quantities are arranged in the form of a square between two vertical lines. This arrangement is called a determinant.

1.1 Determinants for Square Matrix

Definition 1.1 [2] : To every square matrix A of order n with entries as real or complex numbers, we can associate a number called determinant of matrix A and is denoted by $|A|$ or $det(A)$.

Thus determinant formed by the elements of A and is said to be determinant of matrix A.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then its $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

Definition 1.2 [2] : Let $|A| = |(a_{ij})|$ be a determinant of square matrix of order n. The minor of an arbitrary element a_{ij} is the determinant obtained by deleting the i-th row and j-th column in which the element a_{ij} stands. The minor of a_{ij} is denoted by M_{ij}.

Definition 1.3 [2] : The cofactor is a signed minor. The cofactor of a_{ij} is denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j}M_{ij}$.

Definition 1.4 [2] : Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then its determinant is defined as $|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$.

Definition 1.5 [2] : A square matrix A is said to be singular if $|A| = 0$, otherwise is said to be non-singular matrix.

1.2 Properties of Determinants for Square Matrix

The following properties are true for determinant of square matrices of any order.

Theorem 1.6 [1] : The value of a determinant not changed when we interchanged rows into columns and columns into rows. that is $|A| = |A^T|$ for any square matrix A.

Theorem 1.7 [1] [2] : If any two rows (columns) of a determinant are interchanged, then the determinant changes in sign but its numerical value is unaltered.

Theorem 1.8 [1] : If any two rows (columns) of a determinant are identical, then the value of the determinant is zero.

Theorem 1.9 [1] : If every element in a row (or column) of a determinant is multiplied by a constant k, then the value of the determinant is multiplied by k.

Theorem 1.10 [1] : If every element in any row (column) can be expressed as the the sum of two quantities then the given determinant can be expressed as the sum of two determinants of the same order with the elements of the remaining rows (columns) of the both being the same.

Theorem 1.11 [1] : A determinant is unaltered when to each element of any row (column) is added to those of several other rows (columns) multiplied respectively by constant factors.

In the history of Matrices, yet now mathematicians are interested in finding the value of determinant for square matrix only, Actually the definition of determinant and its properties are discussed only for square matrices. There is no definition of determinant for non square matrices. To break this we introduced a new concept called non square determinant that is determinant for non square matrices and we investigate the inverse of the non square matrices.

2. Non Square Determinant

Definition 2.1 : Let A be a non square matrix of order $m \times n$. If $n > m$ then the matrix A is called horizontal matrix, otherwise A is called vertical matrix.

Definition 2.2 : To every non square matrix A of order $m \times n$ with entries as real or complex numbers, we can associate a number called determinant of matrix A and is denoted by $|A|$ or $det(A)$.

If $A = [a_{11} \ \ a_{12} \ \ a_{13} \cdots a_{1n}]$, then its,

$$|A| = a_{11} - a_{12} + a_{13} - \cdots + (-1)^{1+n} a_{1n} = \sum_{i=1}^{n} (-1)^{1+i} a_{1i}. \tag{2.1}$$

If $A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \cdots \\ a_{m1} \end{bmatrix}$, then its $|A| = a_{11} - a_{21} + a_{31} - \cdots + (-1)^{m+1} a_{m1} = \sum\limits_{i=1}^{m} (-1)^{i+1} a_{i1}$. If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \end{bmatrix}$, then its $|A| = \sum\limits_{i=1}^{n-1} \sum\limits_{j=i+1}^{n} (-1)^{p_{ij}} \begin{vmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{vmatrix}$ and if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{bmatrix}$, then its $|A| = \sum\limits_{i=1}^{n-1} \sum\limits_{j=i+1}^{n} (-1)^{p_{ij}} \begin{vmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{vmatrix}$ where

$$p_{ij} = \begin{cases} j - \frac{i}{2} + \frac{(-1)^i}{4} + \frac{3}{4} & \text{if } n \text{ is even;} \\ j - \frac{i}{2} - \frac{(-1)^i}{4} + \frac{9}{4} & \text{if } n \text{ is odd.} \end{cases}.$$

Definition 2.3 : Let $|A| = |(a_{ij})|$ be a non square determinant of order $m \times n$. The minor of an arbitrary element a_{ij} is the determinant obtained by deleting the i-th row and j-th column in which the element a_{ij} stands. The minor of a_{ij} is denoted by M_{ij}.

Definition 2.4 : The cofactor of non square matrix is a signed minor. The cofactor of a_{ij} is denoted by A_{ij} and is defined as

$$A_{ij} = (-1)^{i+j} M_{ij}. \tag{2.2}$$

Definition 2.5 : Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \end{bmatrix}$ then its determinant is defined as

$$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - \cdots + (-1)^{1+n} M_{1n} = \sum_{i=1}^{n} (-1)^{1+i} a_{1i} M_{1i}. \tag{2.3}$$

Also if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \end{bmatrix}$ then its determinant is defined as

$$|A| = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} - \cdots + (-1)^{m+1}M_{m1} = \sum_{i=1}^{m}(-1)^{i+1}a_{i1}M_{i1}. \quad (2.4)$$

Definition 2.6 : A non square matrix A is said to be singular if $|A| = 0$, otherwise is said to be non-singular matrix.

The following properties are true for non square determinants of any order. But we are going to prove the properties for the non square determinant of order 3×4 and 4×3 only.

Theorem 2.7 : The value of the non square determinant unchanged when we interchanged rows into columns and columns into rows. that is $|A| = |A^T|$ for any non square matrix A.

Proof : Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$.

By the definition of $|A|$, we have

$$\begin{aligned}
|A| &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - a_{14}M_{14} \\
&= a_{11}\left\{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}\right\} \\
&\quad -a_{12}\left\{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}\right\} \\
&\quad +a_{13}\left\{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix}\right\} \\
&\quad -a_{14}\left\{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right\} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{11}a_{22}a_{34} + a_{11}a_{24}a_{32} + a_{11}a_{23}a_{34} \\
&\quad -a_{11}a_{24}a_{33} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{12}a_{21}a_{34} - a_{12}a_{24}a_{31} \\
&\quad -a_{12}a_{23}a_{34} + a_{12}a_{24}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{13}a_{21}a_{34} \\
&\quad +a_{13}a_{24}a_{31} + a_{13}a_{22}a_{34} - a_{13}a_{24}a_{32} - a_{14}a_{21}a_{32} + a_{14}a_{22}a_{31} \\
&\quad +a_{14}a_{21}a_{33} - a_{14}a_{23}a_{31} - a_{14}a_{22}a_{33} + a_{14}a_{23}a_{32}. \qquad (2.5)
\end{aligned}$$

Let us interchange the rows and columns of A. We have,

$$\begin{aligned}
|A^T| &= \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{vmatrix} \\
&= a_{11}M_{11} - a_{12}M_{21} + a_{13}M_{31} - a_{14}M_{41} \\
&= a_{11}\left\{ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \right\} \\
&\quad -a_{12}\left\{ \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \right\} \\
&\quad +a_{13}\left\{ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix} \right\} \\
&\quad -a_{14}\left\{ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \right\} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{11}a_{22}a_{34} + a_{11}a_{24}a_{32} + a_{11}a_{23}a_{34} \\
&\quad -a_{11}a_{24}a_{33} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{12}a_{21}a_{34} - a_{12}a_{24}a_{31} \\
&\quad -a_{12}a_{23}a_{34} + a_{12}a_{24}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{13}a_{21}a_{34} \\
&\quad +a_{13}a_{24}a_{31} + a_{13}a_{22}a_{34} - a_{13}a_{24}a_{32} - a_{14}a_{21}a_{32} + a_{14}a_{22}a_{31} \\
&\quad +a_{14}a_{21}a_{33} - a_{14}a_{23}a_{31} - a_{14}a_{22}a_{33} + a_{14}a_{23}a_{32}. \qquad (2.6)
\end{aligned}$$

From equation (2.5) and (2.6) we obtain the proof of the theorem. □

Theorem 2.8 : If any two rows of horizontal matrix determinant are interchanged, then the horizontal matrix determinant changes in sign but its numerical value is unaltered.

Proof : Consider the horizontal matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$.

By the definition of $|A|$, we have

$$\begin{aligned}
|A| &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - a_{14}M_{14} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{11}a_{22}a_{34} + a_{11}a_{24}a_{32} + a_{11}a_{23}a_{34} \\
&\quad -a_{11}a_{24}a_{33} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{12}a_{21}a_{34} - a_{12}a_{24}a_{31} \\
&\quad -a_{12}a_{23}a_{34} + a_{12}a_{24}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{13}a_{21}a_{34} \\
&\quad +a_{13}a_{24}a_{31} + a_{13}a_{22}a_{34} - a_{13}a_{24}a_{32} - a_{14}a_{21}a_{32} + a_{14}a_{22}a_{31} \\
&\quad +a_{14}a_{21}a_{33} - a_{14}a_{23}a_{31} - a_{14}a_{22}a_{33} + a_{14}a_{23}a_{32}. \qquad (2.7)
\end{aligned}$$

Now, let B be the non square determinant obtained from A by interchanging the first and second rows. Then

$$\begin{aligned} |B| &= a_{21}M_{11} - a_{22}M_{12} + a_{23}M_{13} - a_{24}M_{14} \\ &= -[a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{11}a_{22}a_{34} + a_{11}a_{24}a_{32} + a_{11}a_{23}a_{34} \\ &\quad -a_{11}a_{24}a_{33} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{12}a_{21}a_{34} - a_{12}a_{24}a_{31} \\ &\quad -a_{12}a_{23}a_{34} + a_{12}a_{24}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{13}a_{21}a_{34} \\ &\quad +a_{13}a_{24}a_{31} + a_{13}a_{22}a_{34} - a_{13}a_{24}a_{32} - a_{14}a_{21}a_{32} + a_{14}a_{22}a_{31} \\ &\quad +a_{14}a_{21}a_{33} - a_{14}a_{23}a_{31} - a_{14}a_{22}a_{33} + a_{14}a_{23}a_{32}] \\ &= -|A|. \end{aligned} \tag{2.8}$$

Hence the proof is complete. □

Theorem 2.9 : If any two rows of horizontal matrix determinant are identical, then the value of the horizontal matrix determinant is zero.

Proof : Let $|A|$ be the determinant value of the horizontal matrix A. Assume that the first two rows are identical. By Theorem 2.8 interchange the first two rows of A, we obtain $-|A|$. Since first two rows are identical even after interchange we get the same $|A|$. That is $|A| = -|A|$. Hence we obtain $|A| = 0$. This completes the proof of the theorem. □

Theorem 2.10 : If every element in a row of horizontal matrix determinant is multiplied by a constant k, then the value of the horizontal matrix determinant is multiplied by k.

Proof : Consider the horizontal matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$. Let us multiply the first row of A by a constant k. Thus we get a new matrix,

$$B = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} & ka_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

Then $|B| = ka_{11}M_{11} - ka_{12}M_{12} + ka_{13}M_{13} - ka_{14}M_{14} = k|A|. Hence the proof of the theorem is complete.$ □

Theorem 2.11 : If every element in any row of an horizontal matrix can be expressed as the the sum of two quantities then the given horizontal matrix determinant can be expressed as the sum of two horizontal matrix determinants of the same order with the elements of the remaining rows of the both being the same.

Proof : Consider the horizontal matrix

$$A = \begin{bmatrix} \alpha + a_{11} & \beta + a_{12} & \gamma + a_{13} & \delta + a - 14 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

Then we have,

$$\begin{aligned} |A| &= (\alpha + a_{11})M_{11} - (\beta + a_{12})M_{12} + (\gamma + a_{13})M_{13} - (\delta + a_{14})M_{14} \\ &= (\alpha M_{11} - \beta M_{12} + \gamma M_{13} - \delta M_{14}) + (a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - a_{14}M_{14}) \\ &= \begin{vmatrix} \alpha & \beta & \gamma & \delta \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix}. \end{aligned}$$

Hence the proof. □

Theorem 2.12 : A horizontal matrix determinant is unaltered when to each element of any row is added to those of several other rows multiplied respectively by constant factors.

Proof : Consider the horizontal matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$.

Let B be a determinant obtained when to the elements of the first row of A are added to those of second row and third row multiplied by l and m. Then

$$B = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

where $s_i = a_{1i} + la_{2i} + ma_{3i}$ for all $i = 1, 2, 3, 4$. Using the Theorem 2.11, we have

$$\begin{aligned} |B| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix} + \begin{vmatrix} la_{11} & la_{12} & la_{13} & la_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix} \\ &\quad + \begin{vmatrix} ma_{31} & ma_{32} & ma_{33} & ma_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix}. \end{aligned}$$

Again using the Theorem 2.10 and Theorem 2.9, we have

$$|B| = |A| + l(0) + m(0) = |A|.$$

Hence the proof is complete. □

The following Theorems 2.13, 2.14, 2.15, 2.16 and 2.17 are the immediate consequence of Theorems 2.8, 2.9, 2.10, 2.11, to 2.12 for vertical matrix.

Theorem 2.13 : If any two columns of vertical matrix determinant are interchanged, then the vertical matrix determinant changes in sign but its numerical value is unaltered.

Theorem 2.14 : If any two columns of vertical matrix determinant are identical, then the value of the vertical matrix determinant is zero.

Theorem 2.15 : If every element in a column of vertical matrix determinant is multiplied by a constant k, then the value of the vertical matrix determinant is multiplied by k.

Theorem 2.16 : If every element in any column of an vertical matrix can be expressed as the the sum of two quantities then the given vertical matrix determinant can be expressed as the sum of two vertical matrix determinants of the same order with the elements of the remaining columns of the both being the same.

Theorem 2.17 : A vertical matrix determinant is unaltered when to each element of any column is added to those of several other column multiplied respectively by constant factors.

3. Inverse for Non Square Matrix

In this section, the others discuss about inverse for the non square matrix of any order. We know that, the fundamental ideas for existence of inverse of square matrix ((ie) it must be non singular).

Definition 3.1 : Let $A = (a_{ij})$ be a non square matrix. The transpose of the cofactor matrix of A is called the adjoint matrix of A. It is denoted by $adj(A)$.

Now for the non square matrix, we introduce the new concept "Left inverse" and "Right inverse" using the following definitions.

Definition 3.2 : A non singular non square matrix A having Left inverse if there exists a matrix A_L^{-1} such that $A_L^{-1}A = I$, where I denote the identity matrix.

Definition 3.3 : A non singular non square matrix A having Right inverse if there exists a matrix A_R^{-1} such that $AA_R^{-1} = I$, where I denote the identity matrix.

The following Lemma is the immediate consequence of the above definitions and the definition of inverse for square matrix.

Lemma 3.4 : For any non singular square matrix A, the left inverse and right inverse exist and it is equal to inverse of A, that is

$$A_L^{-1} = A_R^{-1} = A^{-1} = \frac{1}{|A|} adj(A). \tag{3.1}$$

The below Theorems 3.5 and Theorem 3.7 tells that the existence of right inverse and left inverse, for some non-square matrix.

Theorem 3.5 : Every non singular horizontal matrix A having a right inverse A_R^{-1}, such that

$$A_R^{-1} = \frac{1}{|A|} \ adj(A). \tag{3.2}$$

Proof : We are going to prove this theorem for the horizontal matrix of order 3×4. Consider the horizontal matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$. Then the determinant value of the matrix A is

$$\begin{aligned} |A| \ = \ & a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{11}a_{22}a_{34} + a_{11}a_{24}a_{32} + a_{11}a_{23}a_{34} \\ & -a_{11}a_{24}a_{33} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{12}a_{21}a_{34} - a_{12}a_{24}a_{31} \\ & -a_{12}a_{23}a_{34} + a_{12}a_{24}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{13}a_{21}a_{34} \\ & +a_{13}a_{24}a_{31} + a_{13}a_{22}a_{34} - a_{13}a_{24}a_{32} - a_{14}a_{21}a_{32} + a_{14}a_{22}a_{31} \\ & +a_{14}a_{21}a_{33} - a_{14}a_{23}a_{31} - a_{14}a_{22}a_{33} + a_{14}a_{23}a_{32}. \end{aligned} \tag{3.3}$$

Now, the minor of a_{11} is

$$M_{11} = a_{22}a_{33} - a_{23}a_{32} - a_{22}a_{34} + a_{24}a_{32} + a_{23}a_{34} - a_{24}a_{33}.$$

The minor of a_{12} is

$$M_{12} = a_{21}a_{33} - a_{23}a_{31} - a_{21}a_{34} + a_{24}a_{31} + a_{23}a_{34} - a_{24}a_{33}.$$

The minor of a_{13} is

$$M_{13} = a_{21}a_{32} - a_{22}a_{31} - a_{21}a_{34} + a_{24}a_{31} + a_{22}a_{34} - a_{24}a_{32}.$$

: The minor of a_{14} is

$$M_{14} = a_{21}a_{32} - a_{22}a_{31} - a_{21}a_{33} + a_{23}a_{31} + a_{22}a_{33} - a_{23}a_{32}.$$

The minor of a_{21} is

$$M_{21} = a_{12}a_{33} - a_{13}a_{32} - a_{12}a_{34} + a_{14}a_{32} + a_{13}a_{34} - a_{14}a_{33}.$$

The minor of a_{22} is

$$M_{22} = a_{11}a_{33} - a_{13}a_{31} - a_{11}a_{34} + a_{14}a_{31} + a_{13}a_{34} - a_{14}a_{33}.$$

The minor of a_{23} is

$$M_{23} = a_{11}a_{32} - a_{12}a_{31} - a_{11}a_{34} + a_{14}a_{31} + a_{12}a_{34} - a_{14}a_{32}.$$

The minor of a_{24} is

$$M_{24} = a_{11}a_{32} - a_{12}a_{31} - a_{11}a_{33} + a_{13}a_{31} + a_{12}a_{33} - a_{13}a_{22}.$$

The minor of a_{31} is

$$M_{31} = a_{12}a_{23} - a_{13}a_{22} - a_{12}a_{24} + a_{14}a_{22} + a_{13}a_{24} - a_{14}a_{23}.$$

The minor of a_{32} is

$$M_{32} = a_{11}a_{23} - a_{21}a_{13} - a_{11}a_{33} + a_{21}a_{14} + a_{13}a_{24} - a_{14}a_{23}.$$

The minor of a_{33} is

$$M_{33} = a_{11}a_{22} - a_{12}a_{21} - a_{11}a_{24} + a_{14}a_{21} + a_{12}a_{24} - a_{14}a_{22}.$$

The minor of a_{34} is

$$M_{34} = a_{11}a_{22} - a_{12}a_{21} - a_{11}a_{24} + a_{13}a_{21} + a_{12}a_{24} - a_{13}a_{22}.$$

Hence we have the matrix,

$$A_R^{-1} = \frac{1}{|A|}\, adj(A) = \frac{1}{|A|}\begin{bmatrix} M_{11} & -M_{12} & M_{13} & -M_{14} \\ -M_{21} & M_{22} & -M_{23} & M_{24} \\ M_{31} & -M_{32} & M_{33} & -M_{34} \end{bmatrix}^T.$$

Clearly, the matrix A_R^{-1} is the right inverse of A because,

$$\begin{aligned} AA_R^{-1} &= \frac{1}{|A|}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\ &\quad \times \begin{bmatrix} M_{11} & -M_{12} & M_{13} & -M_{14} \\ -M_{21} & M_{22} & -M_{23} & M_{24} \\ M_{31} & -M_{32} & M_{33} & -M_{34} \end{bmatrix}^T \\ &= I. \end{aligned}$$

$\square$

Example 3.6 : The matrix $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 2 \\ 3 & 4 & 1 & 2 \end{bmatrix}$ as a right inverse

$$A_R^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{5}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

It is easy to verify that $AA_R^{-1} = I$.

The next theorem is a consequence of Theorem 3.7, this shows that the existence of left inverse for non singular vertical matrix.

Theorem 3.7 : Every non singular vertical matrix A having a left inverse A_L^{-1}, such that

$$A_L^{-1} = \frac{1}{|A|}\, adj(A). \tag{3.4}$$

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