3. Show that the language $L = \{w : n_a(w) = n_b(w)\}$ is not regular.
Is L^* regular?

A language can be proved whether it is regular or not by using the Pumping Lemma Theorem.

If a language R is regular, then any string $w = xyz$ of R must satisfy

1. For each $i \geq 0$, $xy^iz \in R$
2. $|y| > 0$
3. $|xy| \leq P$

we must Prove L is not regular using contradiction.

Assume L is regular

Consider $w = a^P b^P \Rightarrow aaaaaaabbbbbbb$

Pumping length = 7

Case 1: The y is in the 'a' part

$\underbrace{aa}_{x}\underbrace{aaaa}_{y}\underbrace{abbbbbbb}_{z}$ $\qquad |xy| = 6 \leq 7$

Case 2: The y is in the 'b' part

$\underbrace{aaaaaaabb}_{x}\underbrace{bbbb}_{y}\underbrace{b}_{z}$ $\qquad |xy| = 13 \nleq 7$

Case 3: The y is in both 'a' and 'b' part

$\underbrace{aaaaa}_{x}\underbrace{aabb}_{y}\underbrace{bbbbb}_{z}$ $\qquad |xy| = 9 \nleq 7$

Testing case 1: $xy^iz \Rightarrow xy^2z$

aa aaaa aaaaabbbbbbb

Here, # of a's = 11 ≠ # of b's = 7 , ! ! =

Testing case 2: $xy^iz \Rightarrow xy^2z$

aaaaaaabb bbbbbbbbb

Here, # of a's = 7 ≠ # of b's = 11

Testing case 3: $xy^iz \Rightarrow xy^2z$

aaaaa aabb aabb bbbbb

Here, $w = a^p b^p$ so this string $w \notin L$

Since w does not satisfy all 3 pumping conditions, the Language L is not regular.

We know that $L^* = L^0 \cup L^1 \cup L^2 \cup \ldots$

Thus, strings in L^* contains equal numbers of a's and equal number of b's.

That is $L^* = L$

$\therefore$ L^* is also not regular.

4.3.3 Simpler Way

Let us assume L is regular.

Consider $w = a^m b^m$

If a language R is regular, then any string $w = xyz$ of R must satisfy

1. For each $i \geq 0$, $xy^iz \in R$
2. $|y| \geq 1$
3. $|xy| \leq m$

We can split w as such: $w_1 = xyz$ $\quad m \geq |xy| \geq 1$

$|y| \geq 1$

$i = 0, 1, 2, \dots \quad xy^iz \in L \Rightarrow y = a^k \quad 1 \leq k \leq m$

However,

$$w_2 = xy^2z = xyyz = \underbrace{a^j}_{x}\underbrace{a^k}_{y}\underbrace{a^k}_{y}\underbrace{a^q b^m}_{z}$$

z could be made of $\underbrace{a^j}_{j}\underbrace{a^k}_{y}\underbrace{a^q b^m}_{z}$

$j + k + q = m$

$$w_2 = a^{m+k} b^m \notin L$$

$\therefore$ Language L is not regular.

Alternatively:

Let $i = 0$

$$w_0 = xy^0z = xz = a^j a^q b^m = a^{m-k} b^m \notin L$$

$\therefore$ Language L is not regular.

4. Prove that the following languages are not regular.

a. $L = \{a^n b^l a^k : k \geq n+1\}$

Given m, pick $w = a^m b^m a^{2m}$. The string y must then be a^k and the pumped strings will be

$$w_i = a^{m+(i-1)k} b^m a^{2m}.$$

If we take $i \geq 2$, then $m + (i-1)k > m$, and then w_i is not in L.

d. $L = \{a^n b^l : n \leq l\}$

Assume L is regular

Consider $w = a^m b^m \Rightarrow aaabbb$

Let $m = 3$

$\underbrace{a}_{x}\underbrace{aa}_{y}\underbrace{bbb}_{z}$

$|y| \geq 1$

$|xy| \leq m$

$xyz \in L$

xy^2z

aaaaabbb

As we can see that the string 'aaaaabbb' doesn't follow the language constraint that $a^n b^l : n \leq l$

$\therefore$ L is not regular

4.3.4d: Simpler Way

$L = \{a^n b^\ell : n \le \ell\}$

If a Language R is regular, then any string $w = xyz$ of R must satisfy

1. For each $i \ge 0$, $xy^iz \in R$
2. $|y| \ge 1$
3. $|xy| \le m$

Assume L is regular.

Let m be the pumping length

Let $w = a^m b^m \in L$ $\quad |w| = 2m > m$ comes from $a^m b^m$

We can split w as such: $w_1 = xyz$ $\quad m \ge |xy| \ge 1$

$m \ge |y| \ge 1$

$|y| = k$

$k = 1, 2, \ldots m$

$xy^iz \in L \quad i = 0, 1, 2, \ldots$

Let $i = 0$ can not be used

$w_0 = xy = a^{m-k} b^m \in L$ since $y = a^k$

Let $i = 2$ $\quad w_2 = xy^2z = a^{m+k} b^m \notin L$

This contradicts the P.L. conditions

$\therefore$ L is not regular

15. Consider the languages below. For each, make a conjecture whether or not it is regular. Then prove your conjecture.

15a. $L = L = \{a^n b^l a^k : n + l + k > 5\}$.

The language is regular. This is most easily seen by splitting the problem into cases such as $l = 0$, $k = 0$, $n > 5$, for which one can easily construct regular expressions.

15b. $L = \{a^n b^l a^k : n > 5, l > 3, k \leq l\}$.

This language is not regular. If we choose $w = aaaaaab^m a^m$, our opponent has several choices. If y consists of only a's, we use $i = 0$ to violate the condition $n > 5$. If the opponent chooses y as consisting of b's, we can then violate the condition $k \leq l$.

15f. $L = \{a^n b^l : n \geq 100, l \leq 100\}$.

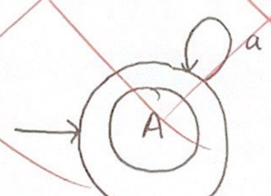
The language is regular.

A language is said to be regular iff a finite state machine recognizes it.

If the language was $a^n b^n$ where $n \geq 100$ or $n \leq 100$ the language would not be considered regular as n is the same for both nodes. This would require the FSM to store or count strings.

Let $n = 100$, $l = 0$

$aaaa \ldots aaa^{100}$



15f. $L = \{ a^n b^\ell : n \geq 100, \ \ell \leq 100 \}$.

The language is regular.

A language is regular iff a FA recognizes it or the language can be represented by a regular expression.

The following RE represents the language shown above:

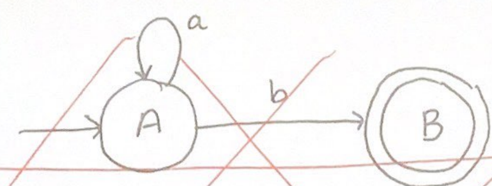
$$a^{100} a^* (b^0 + b^1 + b^2 + \dots + b^{100})$$

Since the Language can be represented by a RE, the Language is regular.

$\therefore$ my conjecture was correct.

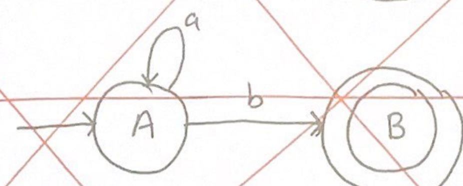
Let $n = 100, \ell = 1$

$a^{100} b$



Let $n = 100, \ell = 100$

$a^{100} b^{100}$



As the above example languages are accepted by a FSM, we conclude that our conjecture is true.

$\therefore$ This language is regular

15g) $L = \{ a^n b^\ell : |n - \ell| = 2 \}$

The language is not regular.

Assume L is regular

Since L is regular there exists a pumping length $m > 0$

Let $w = a^m b^{m+2}$, then $w \in L$ and $|w| > m$

Then from the pumping lemma there exists $xyz \in \Sigma^*$ such that $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$

So $y = a^k$ for some $1 \leq k \leq m$.

Taking $i = 0$, then $xy^iz = a^{m-k} b^{m+2} \notin L$ because

$|m - k - (m+2)| = |-k-2| = k + 2 \neq 2$.

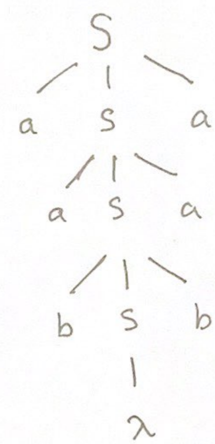
This is a contradiction of the pumping lemma.

$\therefore$ L is not regular.

2. Draw the derivation tree corresponding to the derivation in Example 5.1.

$$S \rightarrow aSa,$$
$$S \rightarrow bSb,$$
$$S \rightarrow \lambda$$

$$S \Rightarrow aSa \Rightarrow aaSaa \Rightarrow aabSbaa \Rightarrow aabbaa$$



3. Grammar $S \to abB,$

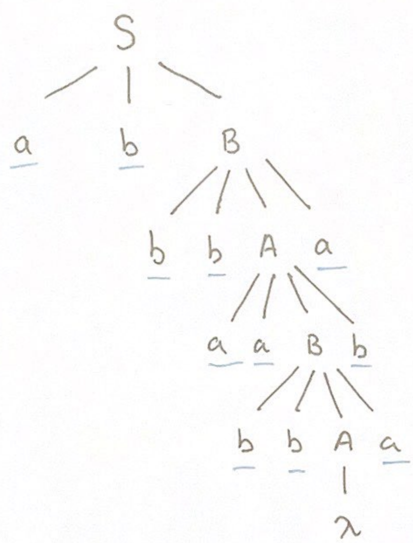
$A \to aaBb,$

$B \to bbAa,$

$A \to \lambda$

$w = abbbaabbaba$

<u>Leftmost Derivation Tree:</u>



<u>Leftmost Derivation:</u>

$S \Rightarrow abB \Rightarrow abbbAa \Rightarrow abbbaaBba \Rightarrow abbbaabbAaba$

$\Rightarrow abbbaabbaba$

7a. $L = \{a^n b^m : n \leq m+3\}$

First, solve the case $n = m+3$. Then add more b's. This can be done by

$$S \to aaaA,$$
$$A \to aAb \mid B,$$
$$B \to Bb \mid \lambda$$

But this is incomplete since it creates at least three a's. To take care of the cases $n = 0, 1, 2$, we add

$$S \to \lambda \mid aA \mid aaA$$

Therefore:

$$S \to \lambda \mid aA \mid aaA,$$
$$A \to aAb \mid B,$$
$$B \to Bb \mid \lambda$$

Alternative Solution

$$S \to aSb \mid A \mid B$$
$$A \to a \mid aa \mid aaa \mid \lambda$$
$$B \to bB \mid b$$

7d. $L = \{a^n b^m : 2n \leq m \leq 3n\}$

$$S \to aSbb \mid aSbbb \mid \lambda$$

These productions nondeterministically produce either bb or bbb for each generated a.

7f. $L = \{ w \in \{a, b\}^* : n_a(v) \geq n_b(v)$, where v is any prefix of $w \}$

$S \to aSb \mid SS \mid S_1$

$S_1 \to aS_1 \mid \lambda$

8a. $L = \{ a^n b^m c^k : n = m \text{ or } m \leq k \}$

For the first case $n = m$ and k is arbitrary. This can be achieved by

$S_1 \to AC$,

$A \to aAb \mid \lambda$,

$C \to Cc \mid \lambda$

In the second case, n is arbitrary and $m \leq k$. Here we use

$S_2 \to BD$,

$B \to aB \mid \lambda$,

$D \to bDc \mid E$,

$E \to Ec \mid \lambda$

Finally, we start productions with $S \to S_1 \mid S_2$.

$S \to S_1 \mid S_2$

$S_1 \to AC$,
$A \to aAb \mid \lambda$,
$C \to Cc \mid \lambda$

~~$S_1 \to AC$,~~
~~$S_2 \to BD$,~~
~~$A \to aAb \mid \lambda$,~~
~~$B \to aB \mid \lambda$,~~
~~$C \to Cc \mid \lambda$,~~
~~$D \to bDc \mid E$,~~
~~$E \to Ec \mid \lambda$~~

$S_2 \to BD$,
$B \to aB \mid \lambda$
$D \to bDc \mid E$
$E \to Ec \mid \lambda$

8b. $L = \{a^n b^m c^k : n = m \text{ or } m \neq k\}$

$S \to S_1 \mid S_2$

$S_1 \to AB$

$A \to aAb \mid \lambda$

$B \to cB \mid \lambda$

$S_2 \to CD$

$C \to aC \mid \lambda$

$D \to bDc \mid E \mid F$

$E \to bE \mid b$

$F \to cF \mid c$

8d. $L = \{a^n b^m c^k : n + 2m = k\}$

$\underbrace{aaa\ldots aa}_{n}\underbrace{bb\ldots bb}_{m}cc\ldots c$

$n + 2m$

Every a add one c

Every b add 2 c's

$S \to aSc \mid B$

$B \to bBcc \mid \lambda$

8h. $L = \{a^n b^m c^k : k \geq 3\}$

$S \to AB$

$A \to aAb \mid \lambda$

$B \to cB \mid ccc$

Another Solution Presented in Discussion:

$S \to ACccc$

$A \to aAb \mid \lambda$

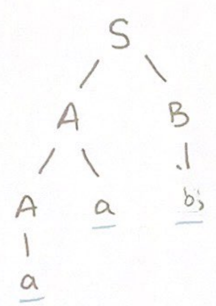
$C \to cC \mid \lambda$

6. $S \to AB \mid aaB$,

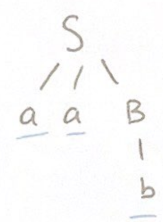
$A \to a \mid Aa$,

$B \to b$

$w = aab$



$w = aab$



String $w = aab$ shows that the above grammar is ambiguous.

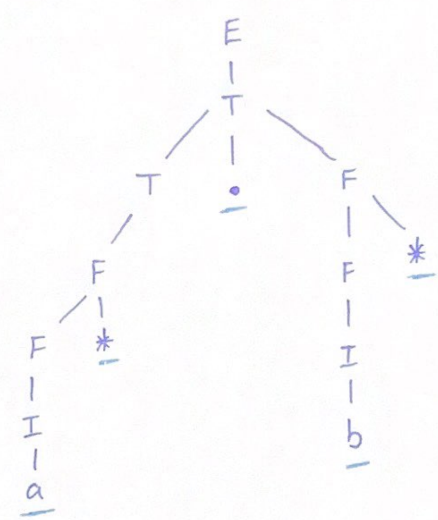
Ambiguous Grammar: A CFG is ambiguous if there exists more than one derivation tree or parse tree.

10. Give an unambiguous grammar that generates the set of all regular expressions on $\Sigma = \{a, b\}$.

set of all regular expressions: $r_1 + r_2$, $r_1 \cdot r_2$, r_1^*, (r_1)

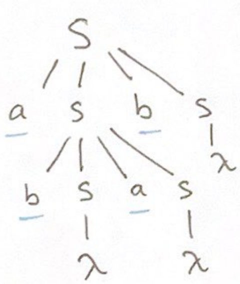
$$E \longrightarrow E + T \mid T$$
$$T \longrightarrow T \cdot F \mid F$$
$$F \longrightarrow F^* \mid I$$
$$I \longrightarrow a \mid b \mid \lambda \mid (E) \mid \phi$$

$a^* \cdot b^*$

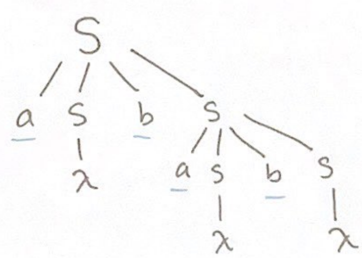


13. Show that the following grammar is ambiguous.

$$S \rightarrow aSbS \mid bSaS \mid \lambda$$



abab



abab

The same string "abab" can be produced more than 1 way, therefore the given grammar is ambiguous.