The Derivative in Calculus

The derivative is one of the fundamental concepts in calculus, along with the integral. It represents the **rate of change** of a function.

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1. Definition and Interpretation

- **Geometrically:** The derivative of a function $f(x)$ at a point $x = a$ is the **slope of the tangent line** to the graph of $f(x)$ at the point $(a, f(a))$.
- **Physically/Conceptually:** The derivative measures the **instantaneous rate of change** of the function's output (y or $f(x)$) with respect to its input (x).
  - If $f(t)$ is the **position** of an object, then $f'(t)$ (the derivative) is its **instantaneous velocity**.

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2. Formal Definition (Limit Definition)

The derivative of $f(x)$, denoted $f'(x)$ or $\frac{dy}{dx}$, is formally defined using a limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is the limit of the **average rate of change** (the slope of a secant line) as the interval (h) approaches zero, turning it into the **instantaneous rate of change** (the slope of the tangent line).

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3. Key Differentiation Rules

To avoid using the limit definition constantly, we use rules:

Rule	Function	Derivative ($f'(x)$ or $\frac{dy}{dx}$)
Constant Rule	$f(x) = c$	0
Power Rule	$f(x) = x^n$	nx^{n-1}
Sum/Difference Rule	$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
Product Rule	$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
Chain Rule	$f(g(x))$	$f'(g(x)) \cdot g'(x)$

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4. Examples with Breakdown

4.1. Power Rule and Constant Multiple Rule

Example: Find the derivative of the function $f(x) = 5x^3$.

Step	Breakdown	Rule Used
Original Function	$f(x) = 5x^3$	
Apply Power Rule	$\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$	Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$
Apply Constant Multiple Rule	$f'(x) = 5 \cdot (3x^2)$	Constant Multiple Rule: $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$
Simplify	$f'(x) = **15x^2**$	

4.2. Sum/Difference Rule

Example: Find the derivative of the polynomial $g(x) = 2x^4 - 6x + 7$.

Step	Breakdown	Rule Used
Original Function	$g(x) = 2x^4 - 6x + 7$	
Differentiate term by term	$g'(x) = \frac{d}{dx}(2x^4) - \frac{d}{dx}(6x) + \frac{d}{dx}(7)$	Sum/Difference Rule
Apply Power/Constant Rules	$g'(x) = (2 \cdot 4x^{4-1}) - (6 \cdot 1x^{1-1}) + 0$	Power Rule & Constant Rule
Final Result	$g'(x) = **8x^3 - 6**$	

4.3. Product Rule

Example: Find the derivative of $y = (x^2 + 1)(3x - 5)$.

Let $f(x) = x^2 + 1$ and $g(x) = 3x - 5$.

Step	Breakdown	Rule Used
Find Components	$f'(x) = 2x$; $g'(x) = 3$	
Apply Product Rule	$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$	Product Rule
Substitute	$\frac{dy}{dx} = (2x)(3x - 5) + (x^2 + 1)(3)$	

Step	Breakdown	Rule Used
Simplify	$\frac{dy}{dx} = (6x^2 - 10x) + (3x^2 + 3)$	
Final Result	$\frac{dy}{dx} = **9x^2 - 10x + 3**$	

4.4. Chain Rule (Composite Functions)

Example: Find the derivative of $h(x) = (2x+1)^4$.

This is $f(g(x))$, where $f(\text{stuff}) = (\text{stuff})^4$ and $g(x) = 2x+1$.

Step	Breakdown	Rule Used
Identify Outer/Inner Derivatives	$f'(u) = 4u^3$; $g'(x) = 2$	Power Rule
Apply Chain Rule	$h'(x) = f'(g(x)) \cdot g'(x)$	Chain Rule
Substitute	$h'(x) = 4(2x+1)^3 \cdot 2$	
Final Result	$h'(x) = **8(2x+1)^3**$	