Quantum Reinforcement Learning - A Literature Review

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***Abstract*—This document is a review of the concepts required to gain an understanding of Quantum Reinforcement Learning.**

***Index Terms*—Quantum Computing, qubits, Linear Algebra, Quantum Machine Learning, Quantum Reinforcement Learning**

I. Introduction

Write about Quantum Reinforcement Learning

II. Linear Algebra

Linear Algebra consists of *vector spaces*, the one we deal with the most is $\mathbb{C}^n$, the space of all n-tuples of complex numbers. These tuples are called vectors, and can also be represented in a column, called a column vector. Operations like addition and scalar multiplication work element-wise, where the scalar is mostly a complex number itself.

Onto the quantum mechanical notation for linear algebra, where we denote a vector by $|w\rangle$, where w is a label to the vector, and $|\cdot\rangle$ is the *ket* notation, indicating that the object is a vector.

Concepts like Linear Independence, Bases, Linear Operators, Matrices are to be read from Quantum Computing and Quantum Information by Nielsen and Chuang.

A. Pauli Matrices

These are 2×2 matrices that are very useful and prominent in the study of Quantum Computing and Quantum Information. They are :

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

B. Inner Products

An *inner product* is a function that takes in two input vectors $|v\rangle$ and $|w\rangle$ from a vector space and produces a complex number as the output.

A function from $V \times V \to \mathbb{C}$ is an inner product iff it satisfies the given conditions :

- It is linear in it's second argument
- When the order of the two vectors is reversed, the new output should be the complex conjugate of the old output.
- When the two input vectors are the same, the output must be ≥ 0 with equality iff the input vectors are 0 (the 0 vector)

A common example of an inner product in $\mathbb{C}^n$ is $\langle v|w\rangle$

A vector space equipped with an inner product is called an *inner product space*, which, in finite dimensional spaces (fortunately, quantum computation involves only those) is exactly the same as a *Hilbert space*

C. Outer Products

The *outer product* representation is a useful way to represent linear operators, and uses the inner product. If $|v\rangle, |w\rangle$ are two vectors part of inner product spaces V, W respectively, then we can define a linear operator from $V \to W$ by $|w\rangle\langle v|$, which takes in a vector from V and outputs a vector from W. Its action is described by :

$$(|w\rangle\langle v|)(|v'\rangle) \equiv |w\rangle \langle v|v'\rangle = \langle v|v'\rangle |w\rangle$$

The *completeness relation* for orthonomal vectors is as follows : Let $|i\rangle\langle i|$ be any orthonormal basis for a vector space V, it follows that

$$\sum_i |i\rangle\langle i| = I$$

This can also be used to represent any operator in the outer product representation (easy enough to get, given orthonormal bases for the vector spaces of the domain and codomain of the operator). Another application of the completeness relation is to prove the *Cauchy-Schwartz inequality*

D. Eigenvectors and Eigenvalues

A *diagonal representation* (or an orthonormal decomposition) for an operator A on a vector space V is a representation $A = \sum_i \lambda_i |i\rangle\langle i|$, where the vectors $|i\rangle$ form an orthonormal set of eigenvectors for A (a little different from the normal definition, but this is Quantum Mech, dealing with orthonormal states more). An oprator is said to be diagonalisable if it has a diagonal representation.

I'll assume you know about Adjoints and Projectors, from QCQI, where projectors satisfy the equation $P^2 = P$

I'll also assume you know about the definition and properties of Tensor Products from QCQI, including operators on tensor-product spaces, inner products on them as well, and the *Kronecker Product* representation.

E. Spectral Decomposition

Any normal operator M on a vector space V is diagonal with respect to some orthonormal basis of V and conversely, any diagonalisable operator is normal.

Note that the definition of diagonalisable might be a bit different to what you're used to, so refer QCQI.

F. Operator Functions

Given a function f from $\mathbb{C} \to \mathbb{C}$, it is possible to define a corresponding matrix function on normal matrices. Let $A = \sum_a a |a\rangle\langle a|$, where A is a normal operator, and $f(A)$ is defined as $\sum_a f(a) |a\rangle\langle a|$

The *trace* of a matrix is defined to be the sum of its diagonal elements. It is cyclic wrt permutation of the order of the product of matrices, and is linear. It is also invariant under the unitary similarity transformation $U \to UAU^\dagger$, using the cyclic nature of the trace and the fact that $UU^\dagger = I$. Therefore the trace of an operator can be defined as the trace of any matrix representation of A, since this invariance ensures that it's well defined.

A very interesting result is that the trace of $A |\psi\rangle\langle\psi|$ is the same as $\langle\psi|A|\psi\rangle$, which could also be thought of as arising from the cyclic property.

G. Commutators and Anti-Commutators

The commutator and anticommutator are defined respectively as :

$$[A, B] = AB - BA \text{ and } \{A, B\} = AB + BA$$

Suppose A and B are Hermition operators, then their commutator is 0 iff there exists an orthonormal basis such that both A and B are simultaneously diagonalisable with respect to that basis. A and B are called *simultaneously diagonalisable*

The concepts of Polar and Singular value decomposition can be read from QCQI as well.

III. Postulates of Quantum Mechanics

A. State Space

Postulate 1: Associated to any isolated Physical system is a complex vector space with inner product (inner product space or Hilbert space) known as the *state space* of the system. The system is completely described by the *state vector*, which is a unit vector in the system's state space.

Note that Quantum Mechanics does not tell us what exactly the state space or the state vector of a given physical system is.

The simplest quantum mechanical system is the qubit, a two dimensional state space. Suppose $|0\rangle, |1\rangle$ form an orthonormal basis for that state space, then a state vector in that space can be written as

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

with an additional normalisation condition of $\langle\psi|\psi\rangle = 1 \equiv a^2 + b^2 = 1$.

B. Evolution

Postulate 2: The evolution of a *closed* (there are different rules for systems with external noise) quantum system is described by a *unitary transformation*. If the system is at a state $|\psi\rangle$ at time t_1 and at a state $|\psi'\rangle$ at time t_2, then

$$|\psi'\rangle = U|\psi\rangle$$

where U is a unitary operator that depends only on the times t_1 and t_2.

Again, Quantum mechanics does not tell us which unitary operators U describe the system, it merely tells us that the evolution can be written in such a way.

For examples of possible unitary operators (important in QC), look at the X and Z gates, the X gate called the quantum NOT gate (or the *bit-flip* matrix), since it takes $|0\rangle$ to $|1\rangle$ and vice-versa and the Z gate takes $|0\rangle$ to $|0\rangle$ and $|1\rangle$ to $-|1\rangle$, hence the name *phase-flip* matrix (not used for this very often though).

I assume that the reader would be familiar with the Schrodinger Equation, which deals with the evolution of quantum systems in continuous time.

C. Quantum Measurement

Postulate 3: Quantum measurements are described by a collection M_m of measurement operators. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = \langle\psi|M^\dagger M|\psi\rangle$$

and the state of the system after measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M^\dagger M|\psi\rangle}}$$

The measurement operators also satisfy the *completeness equation*, similar to outer products, and its physical significance is that the probabilities sum up to one.

The *computational basis* is a basis with basis vectors $|0\rangle$ and $|1\rangle$ and with measurement operators $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. You might know that on measuring a qubit $|\psi\rangle = a|0\rangle + b|1\rangle$, the probability of obtaining measurement outcome 0 is $|a|^2$, which can be derived from the above definitions. The probability of measuring 1 would be $|b|^2 = 1 - |a|^2$. The outcomes in both cases would be $|0\rangle$ and $|1\rangle$ upto a phase factor, which can be ignored.

There is a theorem that says that there is no quantum measurement capable of distinguishing two non-orthogonal states, whose proof is in the book QCQI.

D. Projective Measurements

This is an important special case of the general measurement postulate.

Projective measurements: A projective measurement is described by an *observable*, M, a Hermitian operator on the state space of the system being observed. The observable has a spectral decomposition

$$M = \sum_m m P_m$$

where P_m is the projector onto the eigenspace of M with eigenvalue m and the possible outcomes of the measurement also correspond to the eigenvalues, m of the observable. Here,

$$p(m) = \langle\psi|P_m|\psi\rangle$$

Given that outcome m occurred, the state of the quantum system immediately after the measurement

$$\frac{P_m \left|\psi\right\rangle}{\sqrt{p(m)}}$$

. Also, the average value of the measurement turns out to be

$$\mathbf{E}(M) = \langle\psi|M|\psi\rangle$$

I'll let the book QCQI give a much better explanation of the Heisenberg Uncertainity Principle than I ever can :)

E. POVMs

POVM formalism is a mathematical tool used when one is not concerned about the post-measurement state of the system, merely the probabilities of the respective measurement outcomes, for example, the final measurement of an experiment. This is a simple consequence of the earlier postulates for measurement, but it deserves a special mention here.

There are some cool things about POVMs that can be read from QCQI.

F. Phase

Very commonly used term and has several different meanings depending on the context.

The state $e^{i\theta}\left|\psi\right\rangle$ is equal to the state $\left|\psi\right\rangle$ up to a *global phase factor* $e^{i\theta}$. This does not change the statistics of any measurement, since given any measurement operator, the global phase gets cancelled during the calculation for the measurement.

Relative phase has a different meaning, of a particular amplitude of a basis state having the same magnitudes, yet not the exact same. For an example, consider

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

which both differ in the sign of the amplitudes of the basis state $|1\rangle$. More generally, those amplitudes differ by a relative phase if there is a real θ such that $a = \exp i\theta b$, and two states are said to differ by a relative phase in some basis if each of the amplitudes in that basis is related by such a phase factor.

Postulate 4: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle\otimes|\psi_2\rangle\otimes\cdots\otimes|\psi_n\rangle$

There is a subscript notation to denote states and operators on different systems, when it isn't clear from context. An example, in a system containing 5 qubits, X_3 is the NOT gate acting on the third qubit.

This leads to something very interesting, called *entanglement*. A composite state that cannot be written as a product of states of its component systems is called an *entangled* state. An example is the most studied entangled state,

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

. You can verify that you cannot split it into a tensor product of two single qubit states.

Entanglement is the basis of a procees called *Superdense Coding*, which uses one qubit to send two classical bits of information.

G. Density Operator

The *Density operator* provides an alternate formulation of Quantum Mechanics, not on the basis of state vectors. Although it is mathematically equivalent to the state vector formalism, it is very convenient when thinking about some scenarios.

Suppose a quantum system is in one of a number of states $|\psi_i\rangle$, where i is an index, with respective probabilities p_i. We call $\{p_i, |\psi_i\rangle\}$ and ensemble of pure states. The density operator or density matrix for the system is defined by the equation

$$\rho \equiv \sum_i p_i \left|\psi_i\right\rangle\langle\psi_i|$$

Evolution of the density operator can be represented as $U\rho U^\dagger$, where U is a unitary operator

Measurements can also be described here. If we have some measurement operators M_m that describe a measurement. It turns out that the probability of obtaining result m is the trace of $M_m^\dagger M_m \rho$.

The desnity operator has some properties as well, which you can refer to QCQI for.

IV. Quantum Circuits

Quantum Circuits provide a language to describe quantum algorithms : assemblies of discrete sets of components which describe computational procedures.

A. Single Qubit operations

We start with common operations performed on a single qubit. The Pauli matrices have already been covered, but some other gates which are very important are :

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \quad S = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} \quad T = \begin{bmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{bmatrix}$$

These are called the *Hadamard gate*, H, the *phase gate*, S, and the $\pi/8$ gate, T.

Exponentiating the Pauli matrices give rise to three useful classes of unitary matrices, called the rotation operators around the x, y and z axes respectively.

$$R_x(\theta) \equiv e^{-i\theta X/2} = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$
$$R_y(\theta) \equiv e^{-i\theta Y/2} = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$
$$R_z(\theta) \equiv e^{-i\theta Z/2} = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

Some algebraic connections between these matrices : $H = (X+Z)/\sqrt{2}$ and $S = T^2$

A rotation about a general axis $\hat{n} = (n_x, n_y, n_z)$ can be calculated by :

$$R_{\hat{n}}(\theta) = e^{-i\theta\hat{n}\cdot\vec{\sigma}/2} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(n_xX + n_yY + n_zZ)$$

Every unitary operator can be decomposed into a sequence of $Z-Y$ decompositions as follows :

$$U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$$

This can be extended to any two arbitrary non-parallel directions as well.

There is an important corollary, which is as follows. Suppose U is a unitary gate on a single qubit. Then there exist unitary operators A, B and C on a single qubit such that $ABC = I$ and $U = e^{i\alpha}AXBXC$, where α is some overall phase factor.

Some other common circuit identites are :

$$HXH = Z \quad HYH = -Y \quad HZH = X$$

B. Controlled Operations

This is the first introduction of an implicit logical condition : "If A is true, then do B". The most common controlled operation is called the *controlled-NOT* operator.

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Define abbreviations and acronyms the first time they are used in the text, even after they have been defined in the abstract. Abbreviations such as IEEE, SI, MKS, CGS, ac, dc, and rms do not have to be defined. Do not use abbreviations in the title or heads unless they are unavoidable.

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An excellent style manual for science writers is [7].

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TABLE I
TABLE TYPE STYLES

Table Head	**Table Column Head**		
	Table column subhead	***Subhead***	***Subhead***
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[a]Sample of a Table footnote.

Fig. 1. Example of a figure caption.

Figure Labels: Use 8 point Times New Roman for Figure labels. Use words rather than symbols or abbreviations when writing Figure axis labels to avoid confusing the reader. As an example, write the quantity "Magnetization", or "Magnetization, M", not just "M". If including units in the label, present them within parentheses. Do not label axes only with units. In the example, write "Magnetization (A/m)" or "Magnetization {A[m(1)]}", not just "A/m". Do not label axes with a ratio of quantities and units. For example, write "Temperature (K)", not "Temperature/K".

ACKNOWLEDGMENT

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