# Alternatives to Black-76 Model for Options Valuation of Futures Contracts (Lectures' Notes)

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## Outline of the Lectures

1. Introduction: Problem, Literature, Basics (Aug 13)

2. Forwards, Futures, Options (Aug 17-18)

3. Black-Scholes and Black-76 Models: Positive Prices (Aug 19)

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**Outline of the Lecture II**

4. Alternative Models: Negative Prices (Aug 20)

5. Some Results for Alternative Models (Aug 21)

6. Preparing Your Final Presentation (Aug 24-26) on Aug 27 (Thursday)

7. Appendices A (Martingales), B (Stochastic Calculus) and C (Simulation) (Aug 13)

8. References

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## Introduction: Problem from Scott Dalton (Ovintiv)

The industry problem "Practical Option Valuation with Negative Underlying Prices" was presented by Scott Dalton, Director of Risk Management, Ovintiv.
Excerpt from Scott Dalton's presentation:
"In March 2020, the WTI futures contract settled below zero for the first time in the contract's history. Many market participants apply the Black 76 model or a variation of this model to calculate the value of the options on this futures contract. However, Black 76 requires positive underlying market prices. The goal of this project is to identify alternative models which can accept negative underlying pricing, and assess the suitability of the alternatives."

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**Introduction: Outline**

In these lectures' notes I would like to introduce forwards, futures and options, and to review some results on Black-Scholes-73 and Black-76 models for positive prices, and also on alternatives models for negative prices for option valuation of futures contracts.

I will focus on the first model introduced by Louis Bachelier in 1900, and on other ones, including Ornstein-Uhlenbeck model (1930) and Vasicek model (1977). For these models I will present some results related to the problem and project.

In the end of this lecture notes there are three Appendices, A-C, which contain brief introductions to martingales, stochastic calculus (including Itô formula) and simulation.
I'll start with the References (they are also in the end of the lectures) to pay students' attention to main literatures.

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## Introduction: Topics' Breakdown and Main References

Stochastic Processes: [9, 15]
Stochastic Calculus: [11, 15]
Martingales: [11, 15]
Simulation: [10, 11, 13, 14]
Forward, Futures, Options: [2, 5, 8, 16, 20]
Energy Markets: [2, 16, 19, 20]
Market Price of Risk: [19]
Mathematical Finance: [5, 11, 15]
Black-Scholes Formula: [3]
Black-76 Formula: [4]
Ho-Lee Model: [6]
Hull-White Model: [7]
OU Model: [17]
Vasicek Model: [18]

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**Stochastic Models for Assets in Finance and Energy Markets**

Before we are going to move to forward, futures and options, I'd like to present some stochastic models for assets in finance and energy markets with positive and negative prices.

For one of these models with positive prices we'll present European option price formulas, namely, for GBM, and for other ones we'll present European option price formulas for futures, namely, for Bachelier and OU models.

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**Stochastic Models for Assets with Positive Prices**

- $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$-Geometric Brownian Motion (GBM)

- $dS(t) = (a - bS(t))dt + \sigma S(t)dW(t)$-Continuous-time GARCH Model

- $dS(t) = (a - bS(t))dt + \sigma\sqrt{S(t)}dW(t)$-Cox-Ingersoll-Ross Model ($a \geq \sigma^2/2$)

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**Stochastic Models for Assets with Negative Prices**

- $S(t) = S_0 + \sigma W(t)$-*Bachelier* model ([1], 1900), ($\sigma > 0$)

- $dS(t) = -aS(t)dt + \sigma dW(t)$-*Ornstein-Uhlenbeck (OU)* model ([17], 1930), ($a > 0$)

- $dS(t) = a(b - S(t))dt + \sigma dW(t)$-*Vasicek* model ([18], 1977), ($a, b > 0$)

- $dS(t) = \theta(t)dt + \sigma dW(t)$-*Ho-Lee* model ([6], 1986), ($\theta(t)$-detrministic)

- $dS(t) = (a(t) - b(t))S(t)dt + \sigma(t)dW(t)$-*Hull-White* model ([7], 1987), ($a(t), b(t), \sigma(t)$-deterministic)

- $dS(t) = a(t)S(t)dt + \sigma(t)dI(t)$-*BKM-B* model ([2], 2007) ($I(t)$ process with independent increments)

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## Forward, Futures, Options

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**Forwards, Futures, Options**

Securities such as stocks, which are traded independently of other assets, are called primary securities.

Derivative securities (e.g., forwards, futures, options) are legal contracts conferring certain financial rights or obligations upon the holder, contingent on the prices of other securities, referred to as the underlying securities.

Derivative securities are also referred to as contingent claims because their value is contingent on the underlying securities.

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**Forward Contracts or Forwards**

A forward contractis an agreement to buy or sell a risky asset as a specific future time $T$, known as the delivery date, for a price $K$ fixed at the present moment, called the forward price.

No money is paid at the time when a forward contract is exchanged.

If you buy, you take a long forward position, if you sell, then you take a short forward position.

We will use the notation $F(t, T)$ for a forward price at time $t$ with delivery date $T$.

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## Crude Oil Prices and Forward Curve

We use Department of Energy, USA, source (Courtesy: A. Yanagisawa, IEEJ, July 2009) for crude oil prices, 1/03 – 1/09. See Figure 1 below.

Light sweet crude oil listed at New York Mercantile Exchange, or NYMEX, front month, so-called WTI price (closing price).

The forward curve, a curve of futures price over contract month, is presented on Figure 2 below.

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Figure 1: Crude oil price


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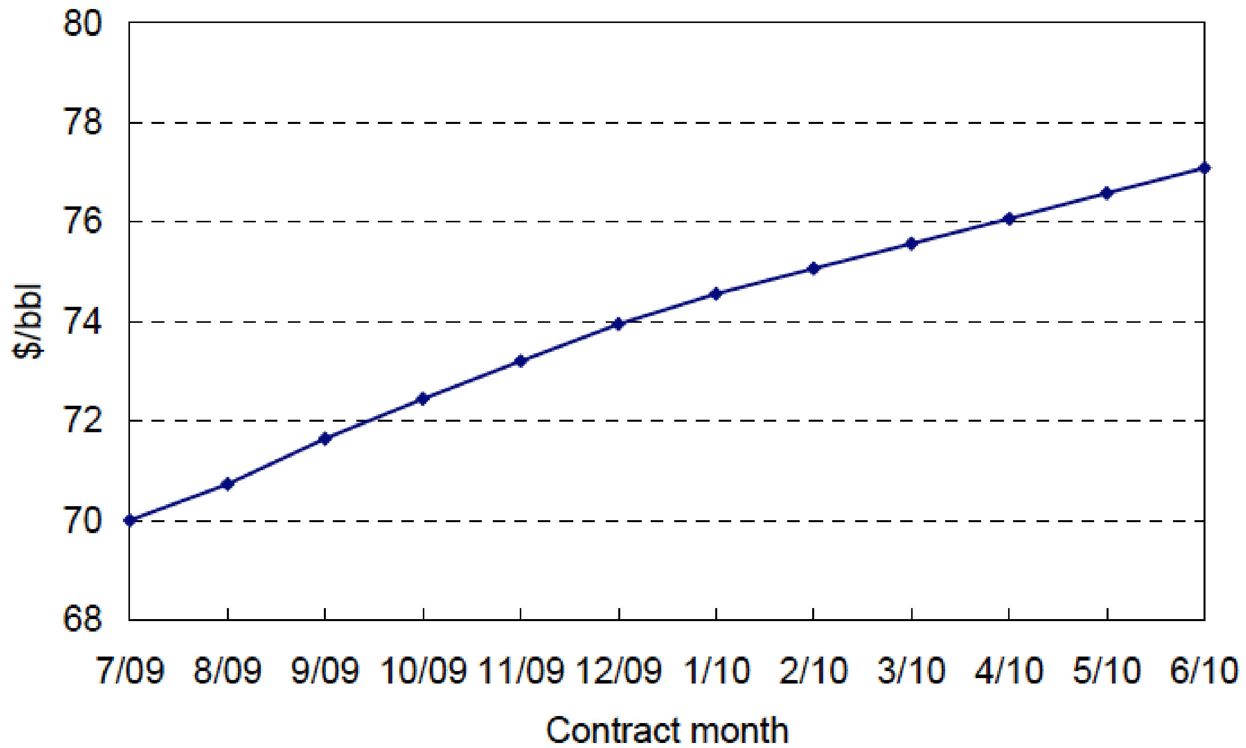
Figure 2: Forward curve (on 9th June 2009)

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**Forwards: Tenor, Contango and Backwardation**

The forward curves, however, form different shapes depending on the period (see Figure 3 below): contango or backwardation.

Tenor refers to the length of time remaining before a financial contract expires.

When forward prices are decreasing with tenor, the curve is said to be backwardated or in backwardation.

When forward prices are increasing with tenor, the curve is said to be in contango.

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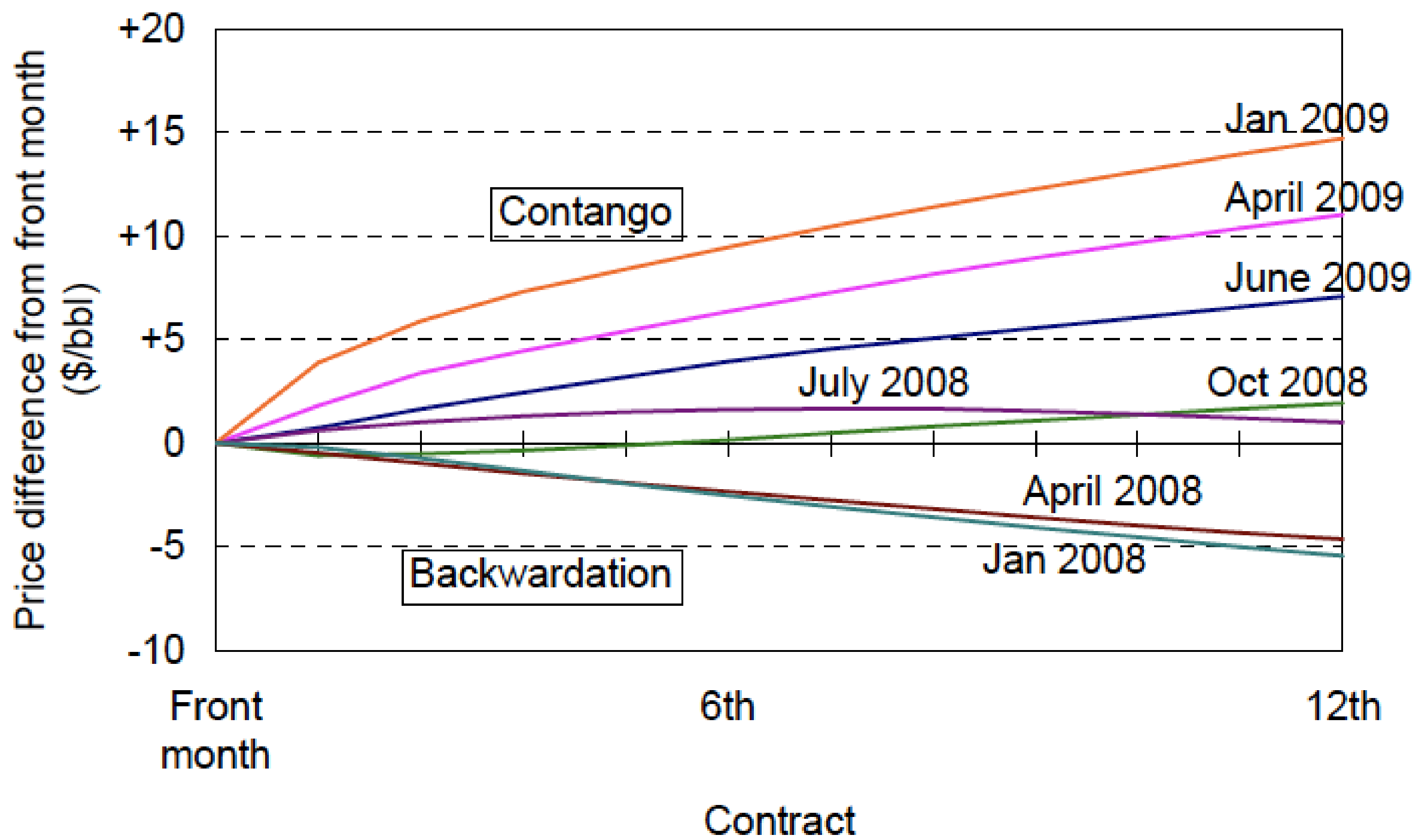
Figure 3: Change of forward curves

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**Forward Curve for Crude Oil and Natural Gas (NG)**

Nest two Figures show the WTI and NG forward curves at a variety of dates (Courtesy: Swingle (2014)).

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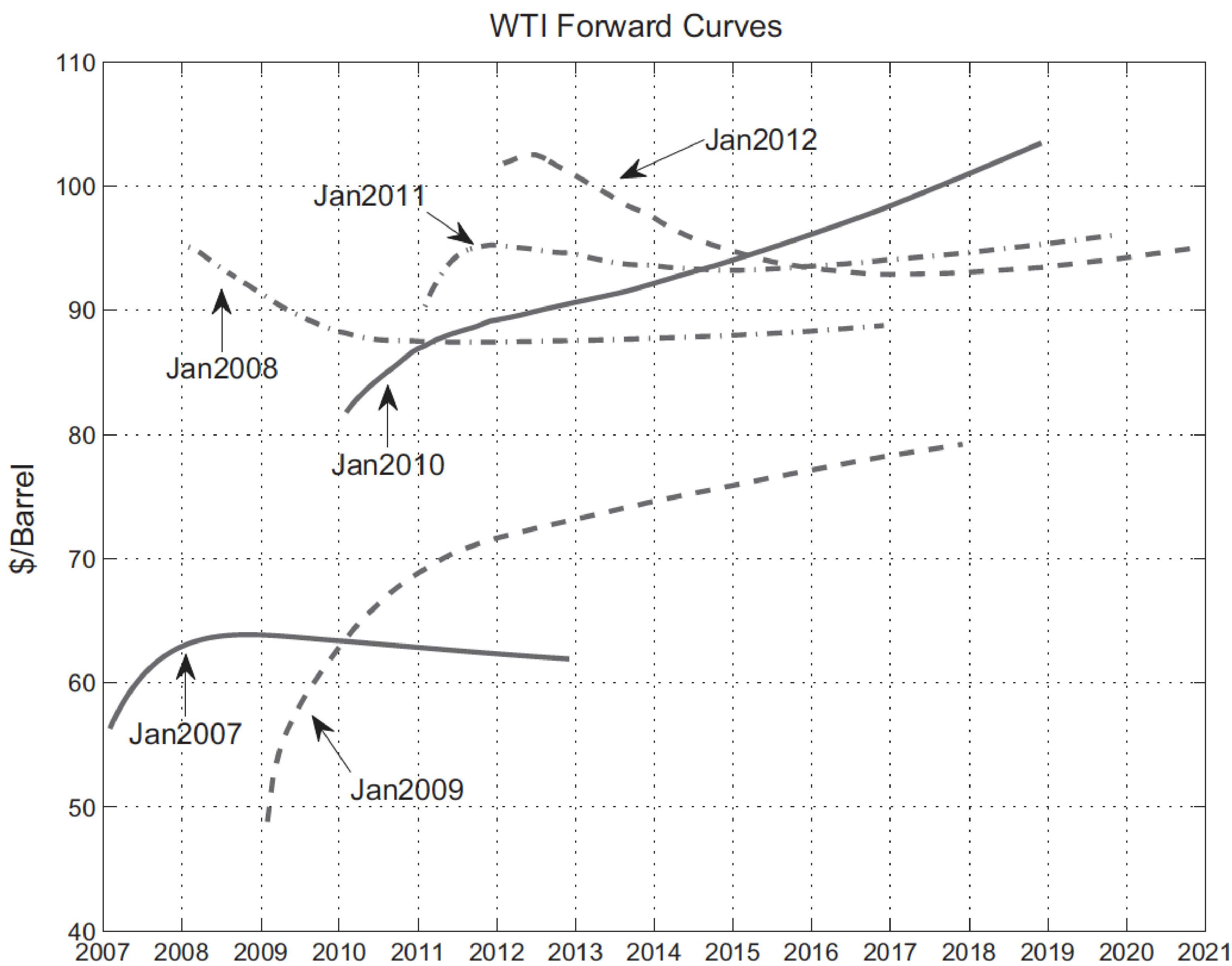


Figure 2.3. WTI historical forward curves.

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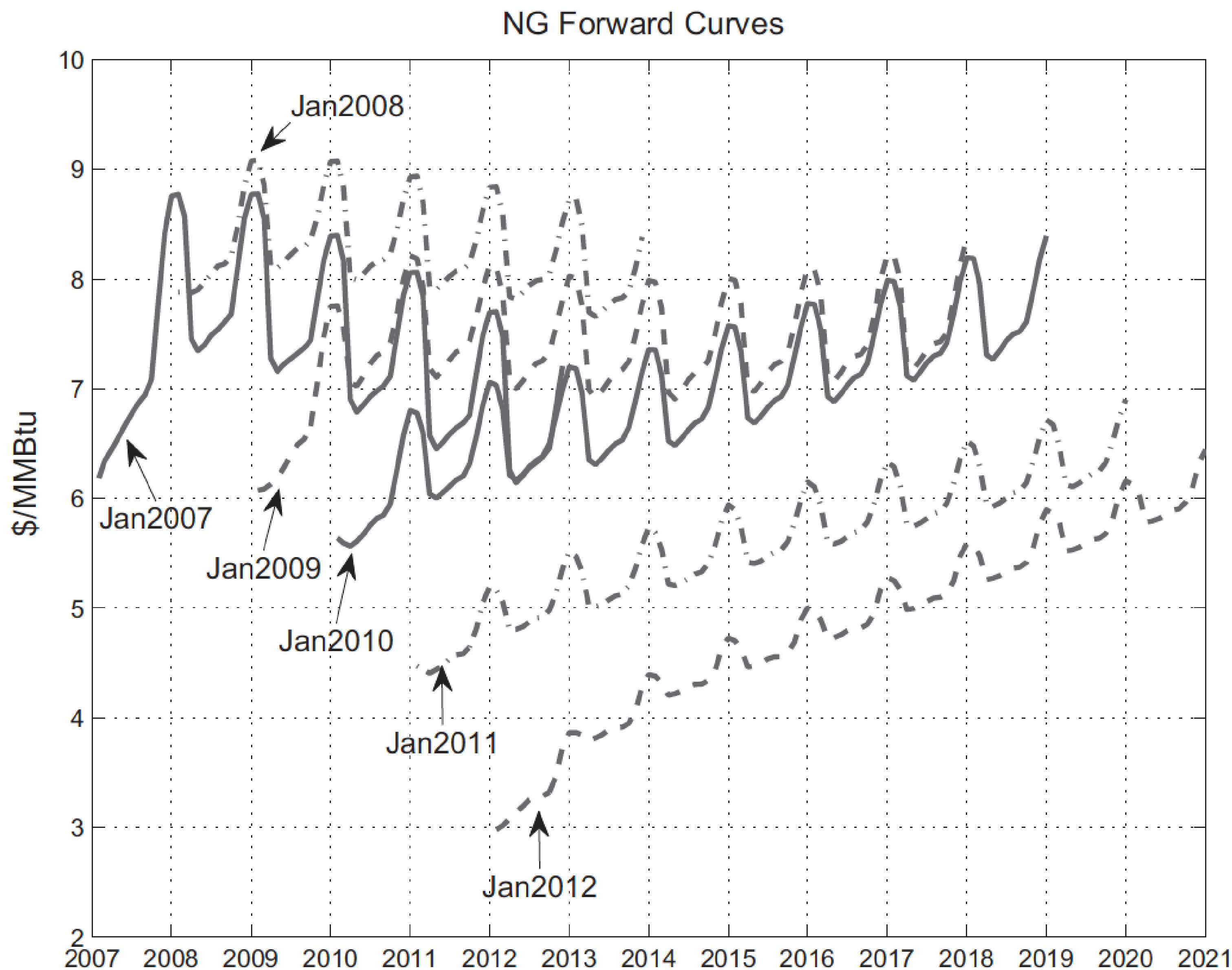


Figure 2.4. NG historical forward curves.

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**Futures Contracts or Futures**

One of the two parties to a forward contract will be losing money. There is also always a risk of default by the party suffering loss.

Futures contracts are designed to eliminate the risk. A futures contracts involves an underlying asset, like a forward contract, and a specified time of delivery $T$. We use the notation $f(t,T)$ for a futures price at time $t$ with delivery date $T$. It costs nothing to initiate a futures position, similar to a forwards.

The difference lies in the cash flow during the lifetime of the futures contract. A long forward contract involves just a single payment $S(T) - F(0,T)$ at delivery.

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## Futures Contracts or Futures II

In the case of futures, the market dictates the so-called futures prices $f(n, T)$ (they are random variables) for each step $n = 0, 1, 2, ...$ such that $n\tau \leq T$, where $\tau$ is one step, say, a day.

A futures contract also involves a random cash flow, known as marking to market, meaning that the holder of a long futures position will receive the amount $f(n, T) - f(n-1, T)$ if positive, or will have to pay it if negative. The opposite payments apply for a short futures position.

Two conditions are imposed: 1) $f(T, T) = S(T)$, and 2) at each step $n = 0, 1, 2, ...$, $n\tau \leq T$, the value of a futures position is zero (i.e., it cost nothing to close, open or alter a futures position).

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## Options

There are many types of options: European, American, exotic, Asian, collar, Boston, etc. Also, we have call and put options.

A European call option is a contract that gives the holder the right, but not obligation, to buy an asset, called the underlying, for a price $K$ fixed in advance, known as the exercise price or strike price, at a specific future time $T_e$, called the exercise or expiry time.

A European put option gives the right, but not obligation, to sell underlying asset for the strike price $K$ at the expiry time $T_e$.

An American option can be exercise at any time up to and including expiry.

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**Options: In the Money, At the Money, Out of the Money**

A call option with strike price $K$ at time $t$ is

- in the money if $S(t) > K$;
- at the money if $S(t) = K$;
- out of the money if $S(t) < K$.

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**Options: Pay-off Functions**

The value of the European call option (i.e., pay-off function) at maturity or expiry $T_e$ is

$$(S(T_e) - K)_+ = \max(S(T_e) - K, 0)).$$

The value of the European put option at maturity or expiry $T_e$ is

$$(K - S(T_e))_+ = \max(K - S(T_e), 0).$$

There are two main problems in option pricing theory, among others:
1) Pricing the option: how should we price at time $t = 0$ an asset worth, e.g., $(S(T_e) - K)_+$ for European call at time $T$?
2) Hedging the option: how should the writer of option, who earns the premium initially, generate an amount $(S(T_e) - K)_+$ for European call at time $T$?

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**Options: Risk-neutral Valuation**

In this lecture, we will focus on the first question.

To answer this question we introduce two key concepts:

i) discounting for interest and

2) the no arbitrage principle

(sometimes referred to as no free lunch opportunity).

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## Options: Discounting Interest

*Discounting interest*: If we have some money in a risk-free savings account (bank deposit) and this investment grows accordingly to a continuously compounded interest rate $r > 0$, then its value increases by a factor $e^{rt}$ over a time length $t$. We will use $r$ to denote the annual rate (so that time is measured in years).

The simplest example is a risk-free bank account with the amount of money $B(t)$ at time $t > 0$. If the initial deposit is $B(0)$, then at time $t > 0$ it will be $B(t) = e^{rt}B(0)$. Hence, $B(t)$ satisfies the equation $dB(t) = rB(t)dt$, $B(0) > 0$, $r > 0$, $t \geq 0$.
Suppose that we have an amount $C(0)$ at time time zero, then it is worth $C(t) = e^{rt}C(0)$ at time $t$ or $C(T) = e^{rT}C(0)$ at expiry $T$. It means that to have $C(T)$ amount of money on saving account at time $T$ we have to have $C(0) = e^{-rT}C(T)$ amount of money at time $t = 0$.

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**Options: No Arbitrage Principle**

*No Arbitrage Principle:* This principle means that there is never an opportunity to make a risk-free profit that gives a greater return than that provided by the interest from the bank deposit. Arguments based on the No Arbitrage principle are the main tools of financial mathematics.

The key role for criteria of No Arbitrage plays the problem of change of measures which is crucial in mathematical finance. We call this measure "martingale measure" or "risk-neutral measure", and denote it by $Q$, to make it different from the initial or physical probability measure $P$.

(See Appendix A for an info about martingales).

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## Options: No Arbitrage Principle II

The technic of change of measure is based on the construction of a new probability measure $Q$ equivalent to the given measure $P$ and such that a process $\tilde{S}(t)$, built on the initial process $S(t)$ satisfies some 'fairness' condition.

In the case of mathematical finance, this process $\tilde{S}(t) = e^{-rt}S(t)$ is a martingale with respect to the new measure $Q$. As long as the asset value $S(t)$ at time $t$ is random or unknown, we have to calculate the expected value of this asset. It means that expectation should be taken with respect to this martingale measure $Q$.

We denote this expectation by $E_Q$, compare with $E_P$-expectation with respect to the initial measure $P$. Thus, martingale property of $\tilde{S}(t) = e^{-rt}S(t)$ means that $E_Q[e^{-rt}S(t)] = S(0)$.

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## Options: Simple Explanation of Risk-neutral Valuation

The simplest explanation of using the measure $Q$ instead of $P$ is the following: the value of $E_Q S(t) = S(0)e^{rt}$ at time $t$ is similar to the value of the riskless asset $B(t)$ at time $t$, namely, $B(t) = B(0)e^{rt}$, where $S(0)$ and $B(0)$ are initial values of stock (risky asset) and bond (riskless asset), respectively. However, it is not the case for the value of $E_P S(t)$, meaning, $E_P S(t) \neq S(0)e^{rt}$.

The risk-neutral valuation means that we act on the stock market as if we put our risky asset $S(t)$ in a bank with interest rate $r > 0$. Note, that $B(t)$ is a deterministic function and $S(t)$ is a stochastic function of time.

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**Options: Option Pricing Formula-European Call**

Returning to our payoff functions, it means that we have to calculate this expected value with respect to the risk-neutral measure $Q$ :

$$E_Q[\max\{S(T_e) - K, 0\}].$$

Therefore, if the initial (at time $t = 0$) fair price of European call option is $C(0, T_e)$, then the value $E_Q[\max\{S(T_e) - K, 0\}]$ at time $T_e$ is equal to $e^{rT_e} C(0, T_e)$ or:

$$C(0, T_e) = e^{-rT_e} E_Q[\max\{S(T_e) - K, 0\}].$$

This formula gives the answer to our previous question: the fair price of the option at time $t = 0$ is defined by the last formula.

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## Options: Option Pricing Formula-European Put

The situation for the European put option is similar, taking into account the payoff function $P(0,T_e) = \max\{K - S(T_e), 0\}$ in this case, thus we have to calculate

$$E_Q[\max(\{K - S(T_e), 0\}].$$

Therefore, the fair price $P(0)$ of the European put option at time $t = 0$ is

$$P(0,T_e) = e^{-rT_e} E_Q[\max\{K - S(T_e), 0\}].$$

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**Options: Call-Put Parity**

We note, that fair prices of European call and put options satisfy the following so-called 'call-put parity':

$$C(0, T_e) + Ke^{-rT_e} = P(0, T_e) + S(0), \qquad (CPP)$$

where $S(0)$ is the initial (at time $t = 0$) asset price.

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## Black-Scholes (1973) and Black (1976) Formulas: Positive Prices

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## Reminder: Black-Scholes Model and Formula

The Black-Scholes model (1973) for a stock price $S(t)$ follows geometric Brownian motion (GBM):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

where $\mu \in R$ is a drift coefficient, $\sigma > 0$ is a volatility, $W(t)$ is a Wiener process or Brownian motion.
Using Itô formula we can get (see Appendix B on stochastic calculus and Itô formula):

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}.$$

It means that $S(t) > 0$ is positive as long as $S(0) > 0$.

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## Reminder: Black-Scholes Model and Formula-Risk-neutral Set Up

In risk-neutral world, i.e, with risk-neutral measure $Q$, the SDE (see Appendix B for SDE) for the stock price $S(t)$ has the following form:

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t),$$

where $W^Q(t)$ is $Q$-Wiener process (we note, that $W(t)$ is $P$-Wiener process).

The solution to this SDE is the following one (using Itô formula:

$$S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma W^Q(t)}.$$

From here it follows that discounted stock price $\tilde{S}(t) := e^{-rt}S(t)$ is a martingale! (Check it!-see Appendix A for martingales).

Thus, $d\tilde{S}(t) = \sigma\tilde{S}(t)dW^Q(t)$, where $W^Q(t) := W(t) + (\mu - r)t/\sigma$.

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## Reminder: Black-Scholes Model and Formula for European Call Option

Let $C(t, T_e)$ be the value for the European call option, then we have well-known Black-Scholes formula:

$$C(t, T_e) = e^{-r(T_e - t)}[S(t)N(d_1) - KN(d_2)], \qquad (BSCall)$$

where $r > 0$ is an interest rate, $K$ is the strike price, $N(x)$ is the cumulative distribution function of a standard normal r.v.,

$$N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy,$$

and

$$d_{1,2} := \frac{\ln(S(0)/K) + (r \pm \frac{1}{2}\sigma^2(T_e - t))}{\sigma\sqrt{T_e - t}},$$

$T_e$ is the expiry/maturity of the option.

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**Reminder: Black-Scholes Model and Formula for European Put Option**

Let $P(t,T_e)$ be the value for the European put option, then we have from call-put parity, formula (CPP):

$$P(t,T_e) = e^{-r(T_e-t)}[KN(-d_2) - S(t)N(-d_1)], \qquad (BSPut)$$

where $r > 0$ is an interest rate, $K$ is the strike price, $N(x)$ is the cumulative distribution function of a standard normal r.v.,

$$N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy,$$

and

$$d_{1,2} := \frac{\ln(S(0)/K) + (r \pm \frac{1}{2}\sigma^2(T_e - t))}{\sigma\sqrt{T_e - t}},$$

$T_e$ is the expiry/maturity of the option.

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**Table of Normal Distributions $N(x)$**

The values $N(d_1)$ and $N(d_2)$ in (BSCall) and (BSPut) formulas can be found in the Table of Normal Distributions for $N(x)$ (see next slide).

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## Table for $N(x)$ When $x \leq 0$

This table shows values of $N(x)$ for $x \leq 0$. The table should be used with interpolation. For example,

$$\begin{aligned} N(-0.1234) &= N(-0.12) - 0.34[N(-0.12) - N(-0.13)] \\ &= 0.4522 - 0.34 \times (0.4522 - 0.4483) \\ &= 0.4509 \end{aligned}$$

| $x$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| −0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |
| −0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| −0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| −0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| −0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| −0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| −0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| −0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| −0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| −0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| −1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| −1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| −1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| −1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| −1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| −1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| −1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| −1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| −1.8 | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| −1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| −2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| −2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| −2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| −2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| −2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| −2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| −2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| −2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| −2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| −2.9 | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| −3.0 | 0.0014 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| −3.1 | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| −3.2 | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| −3.3 | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| −3.4 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| −3.5 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 |
| −3.6 | 0.0002 | 0.0002 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| −3.7 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| −3.8 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| −3.9 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| −4.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |



## Table for $N(x)$ When $x \geq 0$

This table shows values of $N(x)$ for $x \geq 0$. The table should be used with interpolation. For example,

$$\begin{aligned} N(0.6278) &= N(0.62) + 0.78[N(0.63) - N(0.62)] \\ &= 0.7324 + 0.78 \times (0.7357 - 0.7324) \\ &= 0.7350 \end{aligned}$$

| $x$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9986 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.3 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.4 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |
| 3.5 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 |
| 3.6 | 0.9998 | 0.9998 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.7 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.8 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.9 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 4.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |



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**Black-Scholes Formula: Example for European Call and Put Options**

*Example (Hull (1997)).* Let $S(0) = \$42, K = \$40, r = 0.1(10\%), \sigma = 0.2(20\%), T_e - t = 0.5(half \quad a \quad year)$. Find the values of European call and put options.
*Solution.*

$$\begin{aligned} d_1 &= \frac{\ln(1.05)+0.12\times 0.5}{0.2\times\sqrt{0.5}} = 0.7693 \\ d_2 &= \frac{\ln(1.05)+0.08\times 0.5}{0.2\times\sqrt{0.5}} = 0.6278. \end{aligned}$$

Then $Ke^{-r(T_e-t)} = 40e^{-0.05} = 38.049$. Applying formula (BSCall), taking into account the values for $d_1$ and $d_2$ from the Table of Normal Distribution, we have the value of European call option:

$$C(0,T_e) = 42N(0.7693) - 38.049N(0.6278) = 4.76,$$

and the value of European put option (see formula (CPP), Call-Put Parity):

$$P(0,T_e) = 4.76 + 38.049 - 42 = 0.81.$$

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## Black-76 Model: Positive Prices

Black-76 model and formula are a modification of the standard Black-Scholes option pricing framework to the situation in which the underlying price process is that of a generic forward or future contracts.
As with Black-Scholes, returns are normally distributed, and the underlying price process is a geometric Brownian motion (GBM) with constant volatility $\sigma$ :

$$\frac{dF(t,T)}{F(t,T} = \sigma dW(t),$$

where $F(t,T)$ is a forward price at time $t$ with delivery $T$, $W(t)$ is a Wiener process or Brownian motion. (We suppose here that $\mu = 0$. General form is $dF = \mu F dt + \sigma F dW(t)$.)
Using Itô formula (see Appendix B for Itô formula) we can get:

$$F(t,T) = F(0,T)e^{-(1/2)\sigma^2 t + \sigma W(t)},$$

which confirms that $F(t,T) > 0$ is positive as long as $F(0,T) > 0$. It's also a martingale (see Appendix A for martingales).

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## Black-76 Formula: European Call Option Price for a Commodity

Let $C(t,T_e)$ be the value for the European call option written on a forward $F$, where $T > T_e$, then:

$$C(t,T_e) = e^{-r(T_e - t)}[F(t,T)N(d_1) - KN(d_2)], \qquad (BlCall)$$

where $r > 0$ is an interest rate, $K$ is the strike price, $N(x)$ is the cumulative distribution function of a standard normal r.v., and

$$d_{1,2} := \frac{\ln(F/K) \pm \frac{1}{2}\sigma^2(T_e - t)}{\sigma\sqrt{T_e - t}},$$

$T_e$ is the expiry/maturity of the option.
The value of European put option may be found using call-put parity, formula (CPP).

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## Black-76 Formula: European Put Option Price for a Commodity

Let $P(t,T_e)$ be the value for the European call option written on a forward $F$, where $T>T_e$, then:

$$P(t,T_e)=e^{-r(T_e-t)}[KN(-d_2)-F(t,T)N(-d_1)], \qquad (BlPut)$$

where $r>0$ is an interest rate, $K$ is the strike price, $N(x)$ is the cumulative distribution function of a standard normal r.v., and

$$d_{1,2}:=\frac{\ln(F/K)\pm\frac{1}{2}\sigma^2(T_e-t)}{\sigma\sqrt{T_e-t}},$$

$T_e$ is the expiry/maturity of the option.
The value of European put option may be also found using call-put parity, formula (CPP).

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**Black-76 Formula: Example**

*Example (Hull (1997)).* Consider a European put futures option on crude oil. Let the time to maturity is four months, the current futures price is \$20, strike price is \$20, the risk-free interest rate is 9% per annum, and the volatility of the futures price is 25% per annum. Thus, $F(t,T) = 20, K = 20, r = 0.09, T_e - t = 0.3333, \sigma = 0.25$. We note that $\ln(F/K) = 0$, therefore

$$\begin{aligned} d_1 &= \frac{\sigma\sqrt{T_e - t}}{2} = 0.07216 \\ d_2 &= -\frac{\sigma\sqrt{T_e - t}}{2} = -0.07216, \end{aligned}$$

and $N(-d_1) = 0.4712, \quad N(-d_2) = 0.5288$. Thus, applying the European put option price formula (BlPut), we have:

$$P(t, T_e) = e^{-0.09 \times 0.3333}(20 \times 0.5288 - 20 \times 0.4712) = 1.12,$$

or \$1.12.

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## Some Alternative Models: Negative Prices

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**Some Alternative Models**

I introduce some alternative models for negative prices.

They are:
- Bachelier (1900),
- Ornstein-Uhlenbeck (1930),
- Vasicek (1977),
- Ho-Lee (1986) ,
- Hull-White (1987)
- Benth, Kallsen and Meyer-Brandis (BKM-B) (2007)

models.

These models are usually called arithmetic models.

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**Some Alternative Models: Negative Prices**

- $S(t) = S_0 + \sigma W(t)$-*Bachelier* model ([1], 1900), $(\sigma > 0)$

- $dS(t) = -aS(t)dt + \sigma dW(t)$-*Ornstein-Uhlenbeck (OU)* model ([17], 1930), $(a > 0)$

- $dS(t) = a(b - S(t))dt + \sigma dW(t)$-*Vasicek* model ([18], 1977), $(a, b > 0)$

- $dS(t) = \theta(t)dt + \sigma dW(t)$-*Ho-Lee* model ([6], 1986), ($\theta(t)$-detrministic)

- $dS(t) = (a(t) - b(t)S(t))dt + \sigma(t)dW(t)$-*Hull-White* model ([7], 1987), ($a(t), b(t), \sigma(t)$-deterministic)

- $dS(t) = a(t)S(t)dt + \sigma(t)dI(t)$-*BKM-B* model ([2], 2007) ($I(t)$ process with independent increments)

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**Some Results for Alternative Models**

The next slides will contains solutions for some alternative models, and option pricing formulas for them.

I will focus on European call option prices for Bachelier and Ornstein-Uhlenbeck models.

Similar solutions may be found for other models considered above.

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**European Call Option Price for Bachelier Model:** $r = 0$

We consider Bachelier model: $S(t) = S_0 + \sigma W(t)$.

As long as $W(t)$ has normal distribution with mean 0 and variance $t$, then $S(t)$ can take negative values.

Let $r = 0$, $T_e$ is the maturity, $\max(S(T_e) - K, 0)$ is the pay-off function, and $C(S_0, T_e)$ is the European call option price at time $t = 0$. Then, regarding that $S(T_e)$ is normally distributed with mean $S_0$ and variance $\sigma^2 T_e$, we have:

$$\begin{aligned} C(S_0, T_e) &= E[(S(T_e) - K)_+] \\ &= (S_0 - K)N(\tfrac{S_0 - K}{\sigma\sqrt{T_e}}) + \sigma\sqrt{T_e}N'(\tfrac{S_0 - K}{\sigma\sqrt{T_e}}), \end{aligned} \qquad (Ba_1)$$

where $N'(x)$ is the density of normal distribution function $N(x)$.

?Derive formula $(Ba_1)$?

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**European Call Option Price on Forward for Bachelier Model:** $r > 0$

If we consider forward price $F(t,T)$ at time $t$ with maturity $T$, and interest rate $r > 0$, then the European call option price on this forward is:

$$\begin{aligned} C(F(t,T),T_e) &= e^{-r(T_e-t)}E[(F(t,T)-K)_+] \\ &= e^{-r(T_e-t)}[(F-K)N(\tfrac{F-K}{\sigma\sqrt{T_e-t}})+\sigma\sqrt{T_e}N'(\tfrac{F-K}{\sigma\sqrt{T_e-t}})]. \end{aligned} \qquad (Ba_2)$$

We note that $T > T_e$.

?Derive formula $(Ba_2)$?

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## Bachelier Model: Estimation for Volatility

The volatility $\sigma \equiv \sigma^{Ba}$ in the Bachelier model is determined by the price $C(S_0, T_e)$ of an at the money option (i.e., $S_0 = K$) with maturity $T_e$ by the relation:

$$\sigma^{Ba} = C(S_0, T_e)\sqrt{\frac{2\pi}{T_e}}. \qquad (BaVol)$$

It follows directly from formula $(Ba_1)$ after substitution $S_0 = K$.

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**Comparison Bachelier and Black-Scholes Volatilities**

If $\sigma^{Ba}$ is Bachelier implied volatility (see (BaVol)) and $\sigma^{BS}$ is Black-Scholes implied volatility (see (BS)), then

$$0 \le \sigma^{BS} - \frac{\sigma^{Ba}}{S_0} \le \frac{T_e}{12}(\sigma^{BS})^3.$$

The implied volatility of an option contract is that value of the volatility of the underlying instrument which, when input in an option pricing model, will return a theoretical value equal to the current market price of the option. It is not the same as historical volatility, also known as realized volatility or statistical volatility.

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## Solution for OU process

For OU process

$$dS(t) = -aS(t)dt + \sigma dW(t), \qquad (OU)$$

we can solve w.r.t. $S(t)$ (hint: multiply both side by $e^{at}$ and use differential of a product):

$$S(t) = e^{-at}S_0 + \sigma e^{-at}\int_0^t e^{as}dW(s), \qquad (OUSol)$$

where the last integral is Itô integral (see Appendix B for Itô integral). We note that

$$S(t) \equiv Normal(e^{-at}S_0, \sigma^2 e^{-2at}\int_0^t e^{2as}ds).$$

Thus, $S(t)$ can take negative values.

?Find solution (OUSol) for (OU)?

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## Solution for OU process: Risk-neutral Setting

Let $\lambda \in R$ be a market price of risk, $W^Q(t) := W(t) + \lambda t$.

Then in risk-neutral world the OU process has the following form:

$$dS(t) = -a^* S(t)dt + \sigma dW^Q(t),$$

where $a^* := a + \lambda\sigma$.

The solution of this equation is:

$$S(t) = e^{-a^* t} S_0 + \sigma e^{-a^* t} \int_0^t e^{a^* s} dW^Q(s).$$

We note that $W^Q(t)$ is a standard Wiener process under risk-neutral measure $Q$. We note that

$$S(t) \equiv Normal(e^{-a^* t} S_0, \sigma^2 e^{-2a^* t} \int_0^t e^{2a^* s} ds).$$

Thus, $S(t)$ can take negative values.

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**European Call Option Price for OU process:** $r = 0$

Let $C(S_0, T_e)$ be a European call option price for OU process, then:

$$
\begin{aligned}
C(S_0, T_e) &= E_Q(S(T_e) - K)_+ \\
&= (e^{a^* T_e} S_0 - K) N\left(\frac{e^{-a^* T_e} S_0 - K}{\sigma\sqrt{(1-e^{-2a^* T_e})/2a^*}}\right) \qquad (OUCall_1) \\
&+ \sigma\sqrt{(1-e^{-2a^* T_e})/2a^*}\, N'\left(\frac{e^{-a^* T_e} S_0 - K}{\sigma\sqrt{(1-e^{-2a^* T_e})/2a^*}}\right)
\end{aligned}
$$

We note, if $a^* = 0$, then this formula $(OUCall_1)$ coincides with formula $(Ba_1)$.

?Derive formula $(OUCall_1)$?

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**European Call Option Price on Forwards/Futures for OU process:** $r > 0$

Let $C(S_0, T_e)$ be a European call option price for OU process, then:

$$
\begin{aligned}
C(F,T_e) &= e^{-r(T_e-t)}E_Q(F(t,T)-K)_+ \\
&= e^{-r(T_e-t)}[(e^{a^*T_e}F-K)N(\tfrac{e^{-a^*T_e}F-K}{\sigma\sqrt{(1-e^{-2a^*T_e})/2a^*}}) \qquad (OUCall_2) \\
&+ \sigma\sqrt{(1-e^{-2a^*T_e})/2a^*}N'(\tfrac{e^{-a^*T_e}F-K}{\sigma\sqrt{(1-e^{-2a^*T_e})/2a^*}})]
\end{aligned}
$$

We note, if $a^* = 0$, then this formula ($OUCall_2$) coincides with formula ($Ba_2$).

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## Solution to Vasicek Process

For Vasicek process

$$dS(t) = a(b - S(t))dt + \sigma dW(t), \qquad (Vas_1)$$

$(a, b > 0)$, after setting $X(t) = S(t) - b$, we can see that $X(t)$ satisfies an OU process $dX(t) = -aX(t)dt + \sigma dW(t)$. Thus, applying previous result for OU process, we get the solution for Vasicek process:

$$S(t) = S(0)e^{-at} + b(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as} dW(s). \qquad (VasSol_1)$$

Thus,

$$S(t) \approx Normal(S(0)e^{-at} + b(1 - e^{-at}), \sigma^2(1 - e^{-2at})/2a)),$$

which means that $S(t)$ can be negative with positive probability.
European call option price for Vasicek process can be obtained using previous slide's result.
?Derive solution $(VasSol_1)$ for $(Vas_1)$ equation?

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## Solution to Vasicek Process: Risk-neutral Setting

For Vasicek process

$$dS(t) = a(b^* - S(t))dt + \sigma dW^Q(t), \qquad (Vas_2)$$

$(a > 0)$, after setting $X(t) = S(t) - b^*$, we can see that $X(t)$ satisfies an OU process $dX(t) = -aX(t)dt + \sigma dW^Q(t)$. Thus, applying previous result for OU process, we get the solution for Vasicek process:

$$S(t) = S(0)e^{-at} + b^*(1 - e^{-at}) + \sigma e^{-at}\int_0^t e^{as}dW^Q(s). \qquad (VasSol_2)$$

Thus,

$$S(t) \approx Normal(S(0)e^{-at} + b^*(1 - e^{-at}), \sigma^2(1 - e^{-2at})/2a)),$$

which means that $S(t)$ can be negative with positive probability.
European call option price for Vasicek process can be obtained using previous slide's result. Here: $b^* = b - \lambda\sigma/a$, $W^Q(t) := W(t) + \lambda t$, and $\lambda \in R$ is a market price of risk.
?Derive solution $(VasSol_2)$ for $(Vas_2)$ equation?

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**European Call Option Price for Vasicek process:** $r = 0$

Let $C(S_0, T_e)$ be a European call option price for Vasicek process in risk-neutral world, then:

$$
\begin{aligned}
C(S_0, T_e) &= E_Q(S(T_e) - K)_+ \\
&= (e^{aT_e}(S_0 - b^*) - K)N\left(\frac{e^{-aT_e}(S_0 - b^*) - K}{\sigma\sqrt{(1 - e^{-2aT_e})/2a}}\right) \qquad (VasCall_1) \\
&+ \sigma\sqrt{(1 - e^{-2aT_e})/2a}\,N'\left(\frac{e^{-aT_e}(S_0 - b^*) - K}{\sigma\sqrt{(1 - e^{-2aT_e})/2a}}\right)
\end{aligned}
$$

?Derive formula $(VasCall_1)$?

We note, if $b^* = 0$, then this formula $(VasCall_1)$ coincides with formula $(OUCall_1)$.

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**European Call Option Price on Forwards/Futures for Vasicek process:** $r > 0$

Let $C(S_0, T_e)$ be a European call option price for Vasicek process in risk-neutral setting, then:

$$\begin{aligned}
C(F, T_e) &= e^{-r(T_e - t)} E_Q(F(t,T) - K)_+ \\
&= e^{-r(T_e - t)}[(e^{aT_e}(F - b^*) - K)N(\tfrac{e^{-aT_e}(F-b^*)-K}{\sigma\sqrt{(1-e^{-2aT_e})/2a}}) \qquad (VasCall_2) \\
&+ \sigma\sqrt{(1 - e^{-2aT_e})/2a}\,N'(\tfrac{e^{-aT_e}(F-b^*)-K}{\sigma\sqrt{(1-e^{-2aT_e})/2a}})].
\end{aligned}$$

We note, if $b^* = 0$, then this formula $(VasCall_2)$ coincides with formula $(OUCall_2)$.

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# Appendices

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**Appendix A, Appendix B and Appendix C**

**Appendix A: A Brief Intro to Martingales**

**Appendix B: A Brief Intro to Stochastic Calculus and Itô Formula**

**Appendix C: A Brief Intro to Simulation**

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**Appendix A:**

**A Brief Intro to Martingales**
**See [9,11,15]**

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## Appendix A: A Brief Intro to Martingales

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete filtered probability space with $\sigma$-algebra $\mathcal{F}$, filtration $\mathcal{F}_t$ and physical probability measure $P$. Filtration $\mathcal{F}_t$ can be thought as an information available for us until moment $t$.

Stochastic process $M(t)$ is called martingale if

1) $E_P(M(t))^2 < +\infty$

and

2) $E_P[M(t)|\mathcal{F}_s] = M(s), \quad 0 \le s \le t.$

Here: $E_P$ is the expectation w.r.t. $P$, and $E_P[\cdot|\cdot]$ is a conditional expectation.

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**Appendix A: A Brief Intro to Martingales: Examples**

We consider two the most significant examples of martingales: Wiener process and Poisson process.

The former process has continuous paths, and the latter has jump paths.

These two processes are the basics models in math finance and risk theory, and many models for stock and commodity prices, as well as risk process, can be modelled using Wiener and Poisson processes and their combinations.

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**Appendix A: A Brief Intro to Martingales-Wiener Process**

Wiener Process (or Brownian motion) $W(t)$: stochastic process with independent and stationary and normally distributed increments: $E_P[W(t)] = 0$ and $Var[W(t)] = t$. It is a process with continuous paths.

The Wiener process is a martingale (both conditions 1) and 2) follows from the above definition):

$$E_P[W(t)|\mathcal{F}_s] = W(s).$$

? Prove that process $M(t) := W(t)$ is a martingale ?

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## Appendix A: A Brief Intro to Martingales-Poisson Process

Poisson Process $N(t)$ : stochastic process with independent, stationary and Poisson distribution:

$$P(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, 3...,$$

where $\lambda > 0$ is an intensity of jumps, $k! := 1 \times 2 \times 3 \times ... \times k, 0! = 1$.

Poisson process is pure jump process, i.e., its paths are not continuous.

Poisson process itself is not a martingale, but centralized Poisson process, i.e., $M(t) := N(t) - \lambda t$ is a martingale, which follows from $E_P[N(t)] = \lambda t$ and $Var[N(t)] = \lambda t$.

? Prove that process $M(t) := N(t) - \lambda t$ is a martingale ?

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## Appendix A: A Brief Intro to Martingales-Exponential Martingales

We note that the following exponential stochastic processes are martingales (so-called exponential martingales):

$$i) \quad M(t) = e^{\sigma W(t) - \frac{\sigma^2}{2} t}$$

where $\sigma > 0$ and $W(t)$ is a Wiener process;

$$ii) \quad M(t) := e^{N(t) - \lambda t(e-1)},$$

where $\lambda > 0$, $N(t)$ is a Poisson process, and $e$ is a Euler constant $e \approx 2,718...$

These two exponential martingales are very important in mathematical finance for risk-neutral valuation of stochastic models with Wiener and Poisson components.

? Prove that i) and ii) above are martingales ?

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**Appendix B:**

**A Brief Intro to Stochastic Calculus**
**See [11, 15]**

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## Appendix B: A Brief Intro to Stochastic Calculus and Itô Formula-Itô Integral

Let us consider an $\mathcal{F}_t$ Brownian motion $W(t)$ and an $\mathcal{F}_t$-adapted process $H_t, 0 \leq t \leq T$.

If $\int_0^T E_P[H_s^2]ds < +\infty$, then we define Itô stochastic integral as

$$I_t(H) := \int_0^t H_s dW(s).$$

It satisfies the following conditions:

1) $I_t(H)$ is a martingale;

2) $E_P[I_t(H)] = 0$;

3) $E_P[I_t(H)]^2 = \int_0^t E_P[H_s^2)]ds$, $0 \leq t \leq T$.

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**Appendix B: A Brief Intro to Stochastic Calculus and Itô Formula-Itô Calculus vs. Classical Calculus**

If we differentiate the function $t \to W^2(t)$ in classical calculus case, then it should be $dW^2(t) = 2W(t)dW(t)$, i.e., $W^2(t) = 2\int_0^t W(s)dW(t)$.

However, in stochastic calculus we know that $E_P[W^2(t)] = t$, but $E_P[\int_0^t W(s)dW(s)] = 0$, which is a contradiction.

It looks like we are missing some additional term, namely, $t$.

The Itô formula allows us to differentiate such function as $t \to f(W(t))$, if $f$ is twice differentiable function.

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## Appendix B: A Brief Intro to Stochastic Calculus and Itô Formula-Itô Process

We call $X_t$ an $R$-valued Itô process if it can be written as

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW(s), \qquad (*)$$

where $X_0$ is $\mathcal{F}_0$-measurable, $K_t$ and $H_t$ are $\mathcal{F}_t$-adapted processes, such that $\int_0^T |K_s| ds < +\infty$, and $\int_0^T H_s^2 ds < +\infty$ $P$ almost sure.

Equivalent differential form of (*) is:

$$dX_t = K_t dt + H_t dW(t).$$

We note, that $X_t$ is a martingale if and only if $\int_0^t K_s ds = 0$.

We also note that $W(t)$ is an Itô process, because we can take $K_t = 0$ and $H_t = 1$ in the previous definition of Itô process.

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**Appendix B: A Brief Intro to Stochastic Calculus and Itô Formula-Ito Formula**

Let $X_t$ be an Itô process defined in (*), and $f$ be a twice differentiable function. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d < X, X >_s, \qquad (**)$$

where, by definition,

$$< X, X >_t := \int_0^t H_s^2 ds$$

and

$$\int_0^t f'(X_s)dX_s = \int_0^t f'(X_s)K_s ds + \int_0^t f'(X_s)H_s dW(s).$$

Formula (**) is called the Itô formula for the Itô process (*).

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## Appendix B: A Brief Intro to Stochastic Calculus and Itô Formula-Example 1 (Wiener Process)

*Example 1* (*Itô Formula for Wiener Process*). If $f(x)=x^2$ and $X_t=W(t)$, then we identify $K_s=0$ and $H_s=1$, and from Itô formula (**) it follows that

$$W^2(t)=2\int_0^t W(s)dW(s)+\frac{1}{2}\int_0^t 2ds,$$

i.e.,

$$W^2(t)-t=2\int_0^t W(s)dW(s),$$

and we have got that additional term $t$!

We also note that $W^2(t)-t$ is a martingale, because $2\int_0^t W(s)dW(s)$ is a martingale as an Itô integral.

?Derive Itô formula for $W^3(t)$ and for $W^n(t), n\geq 4$, in general, and find the conditions for $W^3(t)$ and $W^n(t)$ to be martingales?

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## Appendix B: A Brief Intro to Stochastic Calculus and Itô Formula-Example 2 (GBM)

Let us find the solution to the GBM $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$ applying Itô formula (**).

We can rewrite GBM in integral form

$$S(t) = S(0) + \int_0^t \mu S(u)du + \int_0^t \sigma S(u)dW(u).$$

Thus, GBM is an Itô process with $K_u = \mu S(u)$ and $H_u = \sigma S(u)$.

We now apply Itô formula (**) to $f(x) = \ln(x)$ :

$$\ln(S(t)) = \ln(S(0)) + \int_0^t \frac{dS(u)}{S(u)} + \frac{1}{2}\int_0^t (\frac{-1}{S^2(u)})\sigma^2 S^2(u)du.$$

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**Appendix B: A Brief Intro to Stochastic Calculus and Itô Formula-Example 2 (GBM)-Continued**

Thus,

$$\ln(S(t)) = \ln(S(0)) + \int_0^t (\mu - \sigma^2/2)ds + \int_0^t \sigma dW(s)$$

or,

$$\ln(S(t)) = \ln(S(0)) + (\mu - \sigma^2/2)t + \sigma W(t).$$

From here we get the solution:

$$S(t) = S(0)e^{(\mu-\sigma^2/2)t+\sigma W(t)},$$

which we used to derive Black-Scholes formula.

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**Appendix B: A Brief Intro to Stochastic Calculus and Itô Formula-SDE**

Itô process $X_t$ in (*) can be formally written in differential form

$$dX_t = K_t dt + H_t dW(t),$$

or, if we consider even more general equations of the type,

$$dX_t = a(t, X_t)dt + \sigma(t, X_t)dW(t),$$

or, in integral form,

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t \sigma(s, X_s)dW(s).$$

We call these types of equations stochastic differential equations (SDE) or stochastic integral equations (SIE), respectively.

We note that in the GBM case for $S(t)$, $a(t, X_t) = \mu X(t)$ and $\sigma(t, X_t) = \sigma X(t)$.

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**Appendix C:**

**A Brief Intro to Simulation**
**See [10, 11, 13, 14]**

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**Appendix C: A Brief Intro to Simulation**

This Appendix C contains a brief introductions to simulation:

- the Monte-Carlo method,
- simulation of r.v.s,
- stochastic processes,
- SDEs, and solution to the BS model.

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**Appendix C: A Brief Intro to Simulation-Monte-Carlo Method**

The Monte-Carlo method is used to calculate/compute an expectation applying the strong law of large numbers (LLN).

Let $X$ be a random variable (r.v.), and $X_1, X_2, ..., X_n, ...$ be a sequence of independent trials of $X$, following the distribution of $X$, then, applying the LLN,

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{1 \le k \le n} f(X_k) = E[f(X)], \qquad (M - C)$$

where the function $f(x)$ is such that $E[|f(X)|] < +\infty$.

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**Appendix C: A Brief Intro to Simulation-Monte-Carlo Method II**

To implement this method, we proceed as follows:

1) build $U_k, k \geq 1$ a sequence of independent, uniformly distributed r.vs. on $[0, 1]$;

2) look for a function $(u_1, ..., u_p) \to F(u_1, ..., u_p)$ such that the r.v. $F(U_1, ..., U_p)$ follows the distribution of $X$;

3) the sequence of r.vs. $X_n := F(U_{(n-1)p+1}, ..., U_{np})$ is a sequence of i.r.vs. following the distribution of $X$.

For example, we can apply formula (M-C) above to the functions $f(x) = x$ and $f(x) = x^2$ to estimate the first and second-order moments of $X$. Also, *rand* or *random* functions in the *stdlib C* library provide pseudo-random numbers on $[0, 1]$.

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**Appendix C: A Brief Intro to Simulation-**
**Simulation of Random Variables: Uniform R.Vs.**

A uniform distribution on $[0, 1]$ ca be obtained by a sequence of integers $(x_n), n \geq 1$, between 0 and $m - 1$ as follows:

$$x_0 = initial \quad value \in \{0, 1, ..., m - 1\}$$

$$x_{n+1} = ax_n + b \quad (modulo \quad m).$$

The choice for $a, b, m$ may be the following (Sedgewick (1987)): $a = 31415821, \quad b = 1, \quad m = 10^8$.

See Knuth (1981) for random number generators and http://random.mat.sbg.ac.at/links for Monte-Carlo simulations.

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**Appendix C: A Brief Intro to Simulation-**
**Simulation of Random Variables: Gaussian R.Vs.**

A Gaussian r.v. $G$ can be simulated if we take $(U_1, U_2)$ two independent uniformly distributed r.vs. on $[0,1]$, then

$$\sqrt{-2\ln(U_1)}\cos(2\pi U_2)$$

follows a standard Gaussian distribution (i.e., zero-mean and with variance 1).

To simulate a Gaussian r.v. with mean $m$ and variance $\sigma^2$, we can set $X = m + \sigma G$, where $G$ is a standard Gaussian r.v.

? Simulate a Gaussian r.v. with mean $m = 2$ and volatility $\sigma = 3$ ?

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## Appendix C: A Brief Intro to Simulation-Brownian Motion

First Method: A Brownian motion $W(t)$ can be simulated by using $G_i, i \geq 0$, independent standard normal r.vs..
We set

$$S_0 = 0$$

$$S_{n+1} - S_n = G_n.$$

Then the distribution of $(\sqrt{\Delta t}S_0, ..., \sqrt{\Delta t}S_n)$ is identical to the distribution of

$$(W(0), W(\Delta t), W(2\Delta t)..., W(n\Delta t)).$$

The Brownian motion $W(t)$ then can be approximated by

$$X_t^n = \sqrt{\Delta t}S_{[t/\Delta t]},$$

where $[x]$ is the floor function, i.e., the largest integer less than or equal to $x$.

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## Appendix C: A Brief Intro to Simulation-Brownian Motion II

Second Method: A Brownian motion $W(t)$ can be simulated as follows. Let $X_i, i \geq 1$, be a sequence of i.i.d.r.vs. with distribution $P(X_i = 1) = 1/2$ and $P(X_i = -1) = 1/2$. We have $E[X_i] = 0$ and $E[X_i^2] = 1$. Set $S_n = X_1 + ... + X_n$.

Then we approximate the Brownian motion by the process $X_t^n$ such that

$$X_t^n = \frac{1}{n} S_{[nt]},$$

where $[x]$ is the floor function.

? Simulate a Brownian motion $W(t)$ ?

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## Appendix C: A Brief Intro to Simulation-SDEs

A SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW(t), \quad X_0 = 0,$$

can be simulated using the following approach (so-called 'Euler approximation').
We construct a discrete-time process $S_n, n \geq 0$, approximating the solution of the SDE at times $n\Delta t$, setting

$$S_{n+1} - S_n = \{b(S_n)\Delta t + \sigma(S_n)(W((n+1)\Delta t) - W(n\Delta t))\}, \quad S_0 = x.$$

If $X_t^n = S_{[t/\Delta t]}$, then the process $X_t^n$ approximates $X_t$ in the sense that for any $T > 0$ $E(\sup_{t\leq T}|X_t^n - X_t|^2|) \leq C_T\Delta T$, where constant $C_T$ depends only on $T$.

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**Appendix C: A Brief Intro to Simulation-Solution to the Black-Scholes Model**

To simulate the solution of the GBM

$$dS_t = S_t(rdt + \sigma dW(t)), \quad S_0 = s,$$

we use the Euler approximation (see above simulation of SDEs):

$$S_{n+1} = S_n(1 + r\Delta t + \sigma G_n\sqrt{\Delta t}), \quad S_0 = s,$$

and simulate $S_t$ by $S_t^n = S_{[t/\Delta t]}$.
Another method is to use the solution to the GBM and then set

$$S_n = s \times \exp{(r - \sigma^2/2)n\Delta t + \sigma\sqrt{\Delta t}\sum_{i=1}^{n} G_i}.$$

Then approximate $S_t$ by $S_t^n = S_{[t/\Delta t]}$. (See Pardoux and Talay (1985) for more details).

? Simulate the solution to the GBM! ?

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## Plan for Students' Presentation on August 27, 2020

- Introduction to the Problem from Scott Dalton (Ovintiv)
- Review of the Stochastic Model for Positive Prices
- Forwards, Futures and Options
- Black-Scholes (1973) and Black (1976) Formulas
- Review of Stochastic Models for Negative Prices
- Option Pricing on Forwards/Futures
- Review of the Data Presented by Scott Dalton (WTI NYMEX Crude Oil and NYMEX Natural Gas Futures)
- Choice of the Stochastic Model for Negative Prices based on the Data
- Options Valuation of Futures Contracts: Main Results
- References

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