Funding Value Adjustment for General Financial Instruments: Theory and Practice

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In the first part of this paper [AB13] we developed the theoretical framework for pricing financial instruments under multiple sources of funding, leading to a non-linear pricing PDE and to Funding Value Adjustment (FVA).

In this second part we develop the numerical framework for computing FVA for general financial instruments including callable features. Our main technical result is an efficient approximation of the FVA in a fully universal way. Usage of non-linear effective discounting rates permits an exact handling of all solvable special cases (the collateral as linear function of the value) by a single formula. Furthermore, the formula delivers a very accurate approximation for general instruments (barriers, Bermudans, etc.). The proof is based on our second technical result: *exact* calculation of prices of automatically exercisable instruments (e.g. European-style or Barrier) having different stochastic discount rates before and after exercise.

We also address the implementation workflow of the FVA calculation, allowing parallel deal-by-deal computation, and we provide a concrete example of FVA calculation for a Bermudan Swaption with partial collateralization, proving that the quality of the numerical approximation is excellent.

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1. Introduction

Pricing financial instruments including credit and funding risks amounts to solving non-linear PDEs, as discussed e.g. in [Pit10] and [BK12], or to computing recursive expectations, as discussed e.g. in [FST09], [PPB12], [PB13], [CGGN12]. In particular, [Pit10] was the first author to extend the classical Black-Scholes-Merton logic to take into account that the derivative, the hedging instruments, the funding strategy, and the possible collateral may have different growth rates. In general, the inclusion of multiple sources of funding, such as unsecured money market funding, collateral and repo, leads to a non-linear PDE with multiple discounting rates. Solving this non-linear PDE is a typically hard numerical problem. This is probably the reason why, despite the huge amount of theoretical research on this subject, to our knowledge the literature lacks of numerical results.

In the first part of this paper [AB13] we have developed the theoretical framework for pricing financial instruments including multiple sources of funding, leading to Funding Value Adjustment (FVA) based on a generalised replication procedure with respect to [Pit10].

In this second part we develop the numerical framework for computing FVA, and we apply it to general financial instruments, with callable and, possibly, path-dependent features. First, we start with a concise review of the classical and modern replication approaches, following [AB13]. In particular, we bring the reader's attention to the inclusion into the replicating self-financing portfolio of risky assets which are related to collateralized and uncollateralized zero coupon bonds. However, even for stochastic collateral and funding rates, the result, written in probabilistic terms, coincides with [Pit10]. Then, we generalize the funding parts of portfolio, assuming that the portfolio value is distributed between general, possibly non-linear, functions of portfolio, funded with general stochastic rates.

After considering the problems that a direct implementation of this framework would face, we propose a *universal* efficient approximation of the FVA. Usage of non-linear effective discounting rates permits an *exact* handling of all solvable special cases (the collateral as linear function of the value) by a single formula. Furthermore, the formula delivers a very accurate approximation for general instruments (barriers, Bermudans, etc.) provided that we use the *right* values of the instruments. These are the *future prices*, which take into account all eventual exercises prior to the observation dates. We explain the subtle difference between future prices and continuation values and, moreover, we calculate the *exact* price of an automatically exercisable instrument (e.g. European-style or Barrier) having different stochastic discount rates before and after exercises[1]. This result underlies our approximation. We also address the implementation workflow of the FVA adjustment, permitting parallel deal-by-deal computation.

Finally, we provide a concrete example of a Bermudan swaption with partial collateralization. For both underlying swap and Bermudan option we calculate the price exactly, by solving numerically the underlying non-linear PDE, and compare it with the FVA

[1] One real-life example of such a deal is a European option to enter into a cleared swap.

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approximation. We ascertain that the approximation quality is excellent.

As mentioned above, certain authors consider the funding and credit issues at the same time, either introducing default processes into the expectations [PPB11], or considering an explicit portfolio replication by self and counterparty risky bonds to hedge the corresponding default risks [BK12]. However, others doubt in the possibility of hedging against its own risk in replicating the DVA ([Cas12]). [BK12] speak also about inevitable hedge errors. Given the absence of consensus among practitioners, we exclude the default risk from the funding replication. Instead we consider CVA and DVA as separate adjustments, but leave the possibility to add the default indicators into the FVA average. Note that this approximation in the FVA is of the second or third orders in funding spreads, and it can be, at least in the present context, ignored.

The paper is organized as follows. In Section 2 we recall the foundations of the classical single-rate theory. Then, in Section 3 we give a concise review of the modern replication approaches (generalizing it in Section 3.6). In Section 4, after considering the problems with a direct implementation of this framework, we propose our main FVA formula. Finally, in Section 5, we provide a concrete example of a Bermudan swaption with partial collateralization, followed by a conclusions in Section 6.

2. Classical Single-Curve Theory

We start with the classical single-curve theory. Our portfolio consists of multiple instruments described by cashflows and exercises. Suppose that we have a classical model with certain *tradable assets* $\mathcal{A}$ (bonds, swaps etc.). We postulate a Markovian *vector* diffusion on the assets

$$d\mathcal{A}(t) = \cdots + \sigma(t, \mathcal{A}) \cdot dW, \tag{1}$$

for a general, asset dependent, volatility matrix $\sigma(t, \mathcal{A})$ and vector independent standard Brownian motion $dW(t) = \{dW_1(t), \cdots, dW_F(t)\}$ with $\langle dW_i, dW_j \rangle = \delta_{ij}\, dt$. Here and below the dot "$\cdot$" means either matrix-vector product or inner vector product.

Finally, we introduce a *short rate* as some function of the assets: $r(t) = r(t, \mathcal{A}(t))$. The instrument cashflows depend uniquely on the assets $\mathcal{A}(t)$. Optimal exercises are determined by comparing exercise and continuation values.

An arbitrage-free argument shows that the drift of the tradable assets $\mathcal{A}(t)$ is $r(t)$ in an appropriate measure. The proof is based on a self-financing strategy. Indeed, construct a self-financing portfolio replicating the derivative $V(t)$ and containing $\theta(t)$ units of tradable assets $\mathcal{A}(t)$ and $\psi(t)$ units of money account $B(t)$ growing with the rate $r(t)$,

$$dB(t) = B(t)\, r(t)\, dt.$$

The value of the portfolio at time t is then given by

$$V(t) = V(t, \mathcal{A}(t)) = \theta(t) \cdot \mathcal{A}(t) + \psi(t)\, B(t).$$

The self-financing property implies

$$dV(t) = \theta(t) \cdot d\mathcal{A}(t) + \psi(t)\, dB(t).$$

Applying Ito's lemma and equating the diffusion and drift terms, we get, assuming that the number of factors F is greater or equal than the number of assets,

$$\theta(t) = \partial_{\mathcal{A}} V \quad \text{and} \quad \left(\partial_t + \frac{1}{2}\sigma^2(t,\mathcal{A})\partial_{AA}\right) V(t)dt = \psi(t)\, dB(t),$$

which leads to

$$\left(\partial_t + r(t,\mathcal{A})\mathcal{A}\,\partial_{\mathcal{A}} + \frac{1}{2}\sigma^2(t,\mathcal{A})\partial_{AA}\right) V(t) = r(t,\mathcal{A})\, V(t). \tag{2}$$

The solution to this PDE is equivalent (by the Feynman-Kac lemma, see e.g [KS97], [Duf01], [Bjo09], [AP10]) to

$$V(t,\mathcal{A}) = \mathbb{E}\left[e^{-\int_t^T r(s)\,ds} P \,\middle|\, \mathcal{A}(t) = \mathcal{A}\right], \tag{3}$$

where the stochastic variable P (or payoff) is an arbitrary function of the assets at time T, $P = P(\mathcal{A}(T))$, the evolution of the assets being

$$d\mathcal{A} = r(t)\,\mathcal{A}(t)\,dt + \sigma(t,\mathcal{A})\cdot dW.$$

Thus, an arbitrage-free argument allows us to find the drift of the assets. The corresponding measure $\mathbb{E}[dW] = 0$ is called the *risk-neutral* measure. Having determined the evolution of the asset (with fixed drift), we can specify the short-rate process from the initially postulated function $r(t) = r(t,\mathcal{A}(t))$.

Introduce the vector x of Markovian *states* such that $r(t) = r(t,x(t))$ and $\mathcal{A}(t) = \mathcal{A}(t,x(t))$. The number of states is chosen so as to describe the assets' evolution in a "closed" Markovian way. For example, for a two-factor Hull-White model we prepare two states. Then the solution (3) can be presented as a function of x:

$$V'(t,x) = \mathbb{E}\left[e^{-\int_t^T r(s)\,ds} P \,\middle|\, x(t) = x\right] = V(t,\mathcal{A}(x)), \tag{4}$$

or simply

$$V(t) = \mathbb{E}\left[e^{-\int_t^T r(s)\,ds} P \,\middle|\, \mathcal{F}_t\right]. \tag{5}$$

This gives a standard setup for short-rate models, where we *postulate* the short-rate evolution in the risk-neutral measure, $r(t)$, related with the *savings account* numéraire

$$N(t) = e^{\int_0^t r(s)\,ds}.$$

Then the payoff at T is seen from time t as

$$V(t) = N(t)\,\mathbb{E}\left[\frac{P}{N(T)} \,\middle|\, \mathcal{F}_t\right]. \tag{6}$$

This classical model calibrated to a market can calculate conditional expectations, zero-coupon bonds, Libors, rates, etc. to deliver the arbitrage-free derivative price.

3. Modern Multiple-Curve Theory

In this section we partially follow [Pit10] to generalize the arbitrage-free theory with a single rate to an economy containing multiple interest rate curves corresponding to different possibilities to borrow and lend money. This theoretical part is summarized here in a concise manner and discussed in much more detail in the first part of this paper [AB13].

3.1. Pricing PDE by Replication

The financial crisis brought the splitting of a single risk-free rate into several rates corresponding to different guarantees for the lender. These rates are:

- r_C for fully collateralized borrowing (almost the risk-free rate),
- r_R for equity secured borrowing ("repo"),
- r_F for unsecured borrowing.

In general, we have

$$r_C,\, r_R < r_F. \tag{7}$$

We now introduce our tradable[2] assets:

- Collateralized assets $A_C(t)$ (for example, fully collateralized zero-coupon bonds),
- Uncollateralized assets $A_F(t)$ (for example, uncollateralized zero-coupon bonds),
- Equity secured assets $A_R(t)$. We denote the corresponding equity by $S(t)$, which is a component of the vector $A_R(t) = \left\{A_R^{(1)}(t),\, A_R^{(2)}(t), \cdots\right\}$; for example, $S(t) = A_R^{(1)}(t)$.

To lighten the formulas, we will often write A_α where the index α can take the letter values $\{C,\, F,\, R\}$. Let us stress again that all A_α are vectors.

To construct the model we should assume diffusion coefficients of our tradable assets

$$dA_\alpha = \cdots + \sigma_\alpha(t, \mathcal{A}) \cdot dW \quad \text{for} \quad \alpha \in \{C,\, F,\, R\}$$

for general asset dependent volatility matrix $\sigma_\alpha(t, \mathcal{A})$ and vector independent Brownian motion $dW(t)$. We also use the block vector notation $\mathcal{A}(t) = \{A_C(t),\, A_F(t),\, A_R(t)\}$ using calligraphic style.

We suppose that our portfolio contains cashflows depending on the tradable assets $\mathcal{A}(t)$ with possible exercises.

The classical portfolio replication arguments should be modified for an economy with multiple rates. Indeed, consider a portfolio replicating our derivative in the self-financing manner that contains hedge instruments and money accounts. Start first with the hedge instruments

[2] Even if the assets are not directly tradable, they can be composed from the market observables.

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- $\theta_C(t)$ units of collateralized assets $A_C(t)$,
- $\theta_F(t)$ units of uncollateralized assets $A_F(t)$,
- $\theta_R(t)$ units of equity secured assets $A_R(t)$.

We will also use the block vector notation $\vartheta(t) = \{\theta_C(t), \theta_F(t), \theta_R(t)\}$.

The *multiple* money accounts are

- $\psi_C(t)$ units of money account $B_C(t)$ growing at the *risk-free* rate $r_C(t)$,
$$dB_C(t) = B_C(t)\, r_C(t)\, dt,$$
- $\psi_F(t)$ units of money account $B_F(t)$ growing at the *funding* rate $r_F(t)$,
$$dB_F(t) = B_F(t)\, r_F(t)\, dt,$$
- $\psi_R(t)$ units of money account $B_R(t)$ growing at the *repo* rate $r_R(t)$,
$$dB_R(t) = B_R(t)\, r_R(t)\, dt.$$

The value of the portfolio at time t is then given by

$$V(t) = \vartheta(t) \cdot \mathcal{A}(t) + \psi_C(t)\, B_C(t) + \psi_F(t)\, B_F(t) + \psi_R(t)\, B_R(t), \tag{8}$$

where we have denoted

$$\vartheta(t) \cdot \mathcal{A}(t) = \theta_C(t) \cdot A_C(t) + \theta_F(t) \cdot A_F(t) + \theta_R(t) \cdot A_R(t).$$

The distribution of the money between the accounts is done in the following way. Denote the portfolio collateral with $C(t)$. Let us isolate the collateralized part $\psi_C(t)\, B_C(t) + \theta_C(t) \cdot A_C(t)$ of the replicating portfolio. The replication will be perfect if the collateral of the derivative $C(t)$ is equal to the collateral of the equivalent portfolio leading to

$$\psi_C(t)\, B_C(t) = C(t) - \theta_C(t) \cdot A_C(t). \tag{9}$$

The repo account corresponds to the price of the equity secured assets,

$$\psi_R(t)\, B_R(t) = -\theta_R(t) \cdot A_R(t). \tag{10}$$

Indeed, imagine we purchased $\theta_S(t)$ units of the stock $S(t)$, which can secure money from the repo account and the repo assets $A_R^{(i)}(t)$ for $i \geq 2$:

$$\psi_R(t)\, B_R(t) + \sum_{i \geq 2} \theta_R^{(i)}(t) \cdot A_R^{(i)}(t) = -\theta_S(t)\, S(t).$$

This leads to (10) by our convention $S(t) = A_R^{(1)}(t)$ and $\theta_S(t) = \theta_R^{(1)}(t)$.

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The last part of the account corresponds to the unsecured borrowing rate r_F (see (8)):

$$\psi_F(t)\, B_F(t) = V(t) - C(t) - \theta_F(t) \cdot A_F(t). \tag{11}$$

The self-financing property[3]

$$dV(t) = \vartheta(t) \cdot d\mathcal{A}(t) + \psi_C(t)\, dB_C(t) + \psi_F(t)\, dB_F(t) + \psi_R(t)\, dB_R(t). \tag{12}$$

permits to equate both diffusion and drifts terms. The diffusion ones give $\vartheta(t) = \partial_{\mathcal{A}} V(t)$ while the drifts equating results in

$$dV(t) - \vartheta(t) \cdot dA(t) = \psi_C(t)\, dB_C(t) + \psi_F(t)\, dB_F(t) + \psi_R(t)\, dB_R(t),$$

or

$$\left(\partial_t + \frac{1}{2}\sigma^2(t, \mathcal{A})\partial_{AA}\right) V(t) = \sum_{\alpha \in \{C,F,R\}} r_\alpha(t)\, \psi_\alpha(t)\, B_\alpha(t), \tag{13}$$

with the second order evolution part

$$\sigma^2(t, \mathcal{A})\partial_{\mathcal{A}\mathcal{A}} \equiv \sum_{\alpha,\beta \in \{C,F,R\}} \sigma_\alpha(t, \mathcal{A}) \cdot \sigma_\beta(t, \mathcal{A})\, \partial_{A_\alpha}\, \partial_{A_\beta}.$$

Substituting the accounts (9-11) in the the r.h.s. of (13) permits to obtain the main PDE

$$\mathcal{L}_{\mathcal{A}}(t)\, V(t) = r_C\, C(t) + r_F\, (V(t) - C(t)), \tag{14}$$

where the diffusion operator $\mathcal{L}_A$ is given by

$$\mathcal{L}_{\mathcal{A}}(t) = \partial_t + \sum_{\alpha \in \{C,F,R\}} r_\alpha(t)\, A_\alpha(t) \cdot \partial_{\alpha_C} + \frac{1}{2}\sigma^2(t, \mathcal{A})\partial_{\mathcal{A}\mathcal{A}}. \tag{15}$$

This nonlinear pricing PDE (14) delivers the multi-curve security price. We need, of course, the portfolio collateral as a function of its value as well as cashflows as functions of our rates. If the portfolio contains callable instruments, we should establish their exercises. This procedure requires the global optimization of the portfolio value because the notion of *individual* instruments is lost. We address this issue in the following sections.

3.2. Multi-Curve Price as a Conditional Expectation

In this section we show how the PDE solution can be represented in terms of *conditional expectations*. As in the previous section we will use a calligraphic $\mathcal{A}$ to denote the vector of all assets, $\mathcal{A} = \{A_C, A_F, A_R\}$.

[3]Note that, although $\theta_R(t) \cdot A_R(t) = -\psi_R(t)\, B_R(t)$, the increments do not necessarily cancel: $\theta_R(t) \cdot dA_R(t) \neq -\psi_R(t)\, dB_R(t)$.

First of all, let us notice that the diffusion operator $\mathcal{L}_{\mathcal{A}}(t)$ will cancel an arbitrary conditional expectation, $\mathcal{L}_{\mathcal{A}}(t)\, g(t, \mathcal{A}) = 0$, for

$$g(t, \mathcal{A}) = \mathbb{E}\left[G(\mathcal{A}(T))|\, \mathcal{A}(t) = \mathcal{A}\right], \tag{16}$$

provided that the tradable assets evolution is

$$dA_\alpha = r_\alpha(t)\, A_\alpha(t)\, dt + \sigma_\alpha(t, \mathcal{A}) \cdot dW \quad \text{for} \quad \alpha \in \{C,\, F,\, R\}$$

This fixes us the measure which we will call *multi-risk-neutral* measure. Naturally it is equipped with the corresponding average operator $\mathbb{E}[dW] = 0$.

It is easy to check that the price can be represented as

$$V(t) = \mathbb{E}\left[e^{-\int_t^T r_F(s)\, ds} V(T) \,\middle|\, \mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T e^{-\int_t^\tau r_F(s)\, ds} \left(r_F(\tau) - r_C(\tau)\right) C(\tau)\, d\tau \,\middle|\, \mathcal{F}_t\right]. \tag{17}$$

(One can also obtain this from the Feynman-Kac lemma.) Here the filtration $\mathcal{F}_t$ means the sigma-algebra generated by the assets $\mathcal{A}(t)$. Equivalently, we can change evolution variables to a set of Markovian states underlying our rates, together with the equity $S(t)$. Denote x_C a vector of Markovian states underlying the collateralized rate r_C. For example, we have $r_C = x_{C1} + x_{C2}$ in the two-factor case. We use the same notations for the funding and repo states, x_F and x_R, respectively. Then our new state space consists of

$$\{x_C, x_F, x_R, S\}, \tag{18}$$

with the corresponding diffusion operator $\mathcal{L}_x(t)$. The portfolio value being a function of these states, $V(t) = V(t, x_C, x_F, x_R, S)$, satisfies the PDE

$$\mathcal{L}_x(t)\, V(t) = r_C\, C(t) + r_F\, (V(t) - C(t)). \tag{19}$$

with the solution given by (17). Thus, the theory can be constructed in the following way:

- Postulate a measure, the evolution of our states $\{x_C, x_F, x_R\}$, the rates $\{r_C, r_F, r_R\}$ as functions of states and diffusion of the equity $dS = \cdots + \sigma_S(t, \mathcal{A}) \cdot dW$.
- Arbitrage-free arguments will give the equity drift,

  $$dS = r_R(t)\, S(t)\, dt + \sigma_S(t, \mathcal{A}) \cdot dW,$$

  and determine the portfolio price satisfying (19) with solution (17).

3.3. Special Cases of Collateral

There are, in particular, two special cases of collateral. The solution for an *uncollateralized* deal ($C = 0$) is given by

$$V(t) = \mathbb{E}\left[e^{-\int_t^T r_F(s)\, ds} V(T) \,\middle|\, \mathcal{F}_t\right]. \tag{20}$$

The solution for a *fully-collateralized* situation ($C = V$) reduces to

$$V(t) = \mathbb{E}\left[e^{-\int_t^T r_C(s)\, ds} V(T) \,\middle|\, \mathcal{F}_t\right]. \tag{21}$$

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3.4. Measure Change

It is important to notice that the numeraire is dissociated from our multi-risk-neutral measure. Indeed, a fully-collateralized security $V_C(t)$ is discounted with the collateralized savings account

$$V_C(t) = \mathbb{E}\left[e^{-\int_t^T r_C(s)\,ds} V_C(T) \;\middle|\; \mathcal{F}_t\right],$$

while the uncollateralized security is discounted with the pure funding account

$$V_F(t) = \mathbb{E}\left[e^{-\int_t^T r_F(s)\,ds} V_F(T) \;\middle|\; \mathcal{F}_t\right].$$

Note that both conditional expectations are done in the *same* measure. Of course, one can change our pricing measure using, for example, the Radon-Nikodym derivative

$$M(t) = P_C(t,T)/B_C(t), \tag{22}$$

where $P_C(t,T)$ is a collateralized zero bond. The collateralized deal propagation is then

$$\begin{aligned} V_C(t) &= \mathbb{E}\left[e^{-\int_t^T r_C(s)\,ds} V_C(T) \;\middle|\; \mathcal{F}_t\right] = M(t)\mathbb{E}_{C_T}\left[e^{-\int_t^T r_C(s)\,ds} V_C(T)\,\frac{1}{M(T)} \;\middle|\; \mathcal{F}_t\right] \\ &= P_C(t,T)\mathbb{E}_{C_T}\left[\frac{V_C(T)}{P_C(T,T)} \;\middle|\; \mathcal{F}_t\right] = P_C(t,T)\mathbb{E}_{C_T}\left[V_C(T) \mid \mathcal{F}_t\right]. \end{aligned}$$

For the pure funding security, such a measure change gives

$$V_F(t) = \mathbb{E}\left[e^{-\int_t^T r_F(s)\,ds} V_F(T) \;\middle|\; \mathcal{F}_t\right] = M(t)\mathbb{E}_{C_T}\left[e^{-\int_t^T r_F(s)\,ds} V_F(T)\,\frac{1}{M(T)} \;\middle|\; \mathcal{F}_t\right]$$

and will not lead to a standard numeraire-deflated form.

3.5. Simplest Nontrivial Example

In this example we consider the simplest evolution of the underlyings.

We postulate the evolution of the rates in the multi-risk-neutral measure ($\mathbb{E}[\cdots]$) following a one-factor HW model:

- collateralized rate:
$$dr_C = (\kappa_C - a_C\, r_C)\,dt + \sigma_C \cdot dW,$$
- repo rate:
$$dr_R = (\kappa_R - a_R\, r_R)\,dt + \sigma_R \cdot dW,$$
- funding rate:
$$dr_F = (\kappa_F - a_F\, r_F)\,dt + \sigma_F \cdot dW.$$

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We consider correlations between the rates through their volatility vectors. All the parameters are considered to be time dependent; we removed time arguments for brevity. The parameters κ are chosen to fix the discount factor; i.e.,

$$\mathbb{E}\left[e^{-\int_0^T dt\, r_\alpha(t)}\right] = DF_\alpha(T), \tag{23}$$

where the index α takes values in $\{C,\, R,\, F\}$.

The model asset will follow an SDE with a log-normal diffusion term

$$dS = S(r_S(t)\, dt + \sigma_S(t) \cdot dW). \tag{24}$$

Its drift coincides with the repo rate as we have seen in the previous sections.

The model evolution variables consist of the Markovian rates and the stock,

$$\{r_C, r_F, r_R, S\}.$$

Their diffusion operator reads

$$\mathcal{L}_x(t) = \sum_{\alpha \in \{C,\, R,\, F\}} (\kappa_\alpha - a_\alpha\, r_\alpha)\partial_{r_\alpha} + r_R\, S\, \partial_S + \frac{1}{2}\Sigma^2 D^2$$

where we have denoted

$$\Sigma^2\, D^2 \equiv \sum_{\alpha,\beta \in \{C,\, R,\, F\}} \sigma_\alpha \cdot \sigma_\beta\, \partial_{r_\alpha}\, \partial_{r_\beta} + S \sum_{\alpha \in \{C,\, R,\, F\}} \sigma_\alpha \cdot \sigma_S\, \partial_{r_\alpha}\, \partial_S + S^2\, \sigma_S \cdot \sigma_S\, \partial_S\, \partial_S.$$

The portfolio value, a function of these states $V(t) = V(t, r_C, r_F, r_R, S)$, satisfies the following PDE

$$\mathcal{L}_x(t)\, V(t) = r_C\, C(t) + r_F\, (V(t) - C(t)). \tag{25}$$

It has the solution (17) where the filtration $\mathcal{F}_t$ is the sigma-algebra generated by the states at time t. This finishes the example construction.

For completeness, we finish this section with the tradable assets. Apart from the stock (24) we have the following zero-bonds as tradable assets:

- fully-collateralized zero-coupon bonds $A_C(t) = P_C(t, T)$ with some maturity T,
- uncollateralized zero-coupon bonds $A_F(t) = P_F(t, T)$ with some maturity T,
- equity-secured zero-coupon bonds $A_R(t) = P_R(t, T)$ with some maturity T.

We define them as usual:

$$P_\alpha(t, T) = P_\alpha(t, T;\, r) = \mathbb{E}\left[e^{-\int_t^T ds\, r_\alpha(s)} \mid r_\alpha = r\right]. \tag{26}$$

This leads to the SDE

$$dP_\alpha(t, T) = P_\alpha(t, T)\, (r_\alpha\, dt - \sigma_\alpha(t, T) \cdot dW), \tag{27}$$

where the volatility vector of the bond is

$$\sigma_\alpha(t, T) = \sigma_\alpha(t) \int_t^T d\tau e^{-\int_t^\tau a_\alpha(s)\, ds}. \tag{28}$$

We see that the zero bonds have appropriate drifts corresponding to their rates.

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3.6. Generalization of the pricing PDE

Equation (19) can be generalized to multiple funding parts or "accounts". Indeed, without loss of generality we can consider the rates to be dependent on the portfolio value and the collateral. For example, the collateralized rate can be "different" for positive and negative collateral value:

$$r_C = r_{C^+}\,\mathbb{I}_{C\geq 0} + r_{C^-}\,\mathbb{I}_{C<0}. \tag{29}$$

Similarly, for the funding part,

$$r_F = r_{F^+}\,\mathbb{I}_{V\geq C} + r_{F^-}\,\mathbb{I}_{V<C}. \tag{30}$$

This is equivalent to the existence of multiple "accounts"

$$\mathcal{L}_x(t)\,V(t) = r_{C^+}\,(C(t))^+ + r_{C^-}\,(C(t))^- + r_{F^+}\,(V(t)-C(t))^+ + r_{F^-}\,(V(t)-C(t))^-, \tag{31}$$

where $x^+ = \max(x,0)$ and $x^- = \min(x,0)$.

The financial meaning of the posivite/negative split corresponds to different underlying rates:

- Positive part of collateral C^+ growing with rate r_{C^+}.
- Negative part of collateral C^- growing with rate r_{C^-}.
- Positive part of funding amount $(V-C)^+$ growing with rate r_{F^+}.
- Negative part of funding amount $(V-C)^-$ growing with rate r_{F^-}.

Generalizing further, we can come up with arbitrary funding parts $\Phi_i(V)$ and corresponding rates r_i giving rise to the PDE

$$\mathcal{L}_x(t)\,V(t) = \sum_i \Phi_i(V(t))\,r_i(t) \tag{32}$$

with the obvious condition

$$\sum_i \Phi_i(V) = V.$$

Equation (32) is our main evolution equation presented in a compact and general form. For example, the parts corresponding to (31) are

$$\Phi_1(V) = C^+, \quad \Phi_2(V) = C^-, \quad \Phi_3(V) = (V-C)^+, \quad \Phi_4(V) = (V-C)^-.$$

If we choose a discounting rate $r(t)$[4] and transform the PDE (32) into

$$\mathcal{L}_x(t)\,V(t) = r(t)\,V(t) + \sum_i \Phi_i(V(t))\,(r_i(t) - r(t)), \tag{33}$$

we obtain the price

$$V(t) = \mathbb{E}\left[e^{-\int_t^T r(s)\,ds} V(T)\,\middle|\,\mathcal{F}_t\right] - \sum_i \mathbb{E}\left[\int_t^T e^{-\int_t^\tau r(s)\,ds}\,\Phi_i(V(\tau))\,(r_i(\tau) - r(\tau))\,d\tau\,\middle|\,\mathcal{F}_t\right]. \tag{34}$$

[4]A deterministic function of the Markovian states in the model.

4. Funding Valuation Adjustment

4.1. Portfolio and the single-rate model

In previous sections we considered PDEs describing the portfolio evolution *between* eventual payment and exercise dates. Here we will detail properties of our general instruments. We suppose that each instrument (k-th one, for clarity) has payments $B_i^{(k)}$ on dates $t_i^{(k)}$, possibly depending on model rates r_i. Assume that we can exercise into another instrument ℓ_k on dates $T_j^{(k)}$. Note that the exercise indicator can be "chooser" (e.g. Bermudan) or "automatic" (e.g. Barrier). For clarity we consider that the instrument ℓ_k itself is not callable, i.e. has no exercises but only payments $B_i^{(\ell_k)}$ on dates $t_i^{(\ell_k)}$.

Suppose that we want to evaluate the portfolio with a *single-rate* (classic) model, with rate $r(t)$ being a deterministic function of our multiple funding rates $\{r_i(t)\}$. We will use this model as *proxy* for the the multi-rate portfolio evaluation (32).

We denote the individual instrument prices for the single-rate model by $v^{(k)}(t)$, which satisfy

$$\mathcal{L}_x(t)\, v^{(k)}(t) = r(t)\, v^{(k)}(t)$$

between the payment and exercise dates. One can aggregate the cashflows and exercises into the PDE r.h.s. using delta-functions:

$$\begin{aligned}\mathcal{L}_x(t)\, v^{(k)}(t) = r(t)\, v^{(k)}(t) \quad &- \quad \sum_i \delta\left(t - t_i^{(k)}\right)\, B_i^{(k)} \\ &- \quad \sum_j \delta\left(t - T_j^{(k)}\right)\, \left(v^{(\ell_k)}(t_+) - v^{(k)}(t_+)\right)\, I_j^{(k)},\end{aligned} \tag{35}$$

where the first sum in the r.h.s. corresponds to payments $B_i^{(k)}$ on dates $t_i^{(k)}$ and the second to exercises, with $I_j^{(k)}$ an exercise indicator, i.e. we exercise into the (non-callable) instrument $v^{(\ell_k)}(t)$ on date $T_j^{(k)}$ if $I_j^{(k)} = 1$. The pricing equation for the ℓ_k-th instrument looks as follows

$$\mathcal{L}_x(t)\, v^{(\ell_k)}(t) = r(t)\, v^{(\ell_k)}(t) - \sum_i \delta\left(t - t_i^{(\ell_k)}\right)\, B_i^{(\ell_k)}. \tag{36}$$

Our portfolio's value at zero is given by aggregating the values of the individual callable instruments,

$$v(0) = \sum_k v^{(k)}(0). \tag{37}$$

As the last step of the preparation for the FVA we will clarify the difference between continuation and future values.

The price $v^{(k)}(t)$ has the meaning of a *continuation value* (CV), i.e. the future price of the instrument observed at t provided that it was not exercised before. Introduce the *future-continuation value* (FCV) $v_t^{(k)}(u)$, which is the future price at time u *provided that the instrument was not exercised prior to time* t. Note that in the PDEs, we use the pure continuation values, $v^{(k)}(t) = v_t^{(k)}(t)$.

One can represent the FCV via

$$v_t^{(k)}(u) = \left(1 - \mathcal{E}_t^{(k)}(u)\right) v^{(k)}(u) + \mathcal{E}_t^{(k)}(u)\, v^{(\ell_k)}(u) \tag{38}$$

where $\mathcal{E}_t^{(k)}(u)$ is an indicator of exercise at u provided that the instrument was not exercised at time t.

We can also write a "PDE" for the FCV $v_t^{(k)}(u)$. Indeed, starting from $v_t^{(k)}(u)\,|_{t=u} = v^{(k)}(u)$, we proceed backwards by

$$\partial_t\, v_t^{(k)}(u) = -\sum_j \delta\left(t - T_j^{(k)}\right)\left(v_t^{(\ell_k)}(u) - v_{t_+}^{(k)}(u)\right) I_j^{(k)}. \tag{39}$$

Alternatively, one can rewrite it in terms of the global exercise indicator

$$\partial_t\, \mathcal{E}_t^{(k)}(u) = -\sum_j \delta\left(t - T_j^{(k)}\right)\left(1 - \mathcal{E}_{t_+}^{(k)}(u)\right) I_j^{(k)}, \tag{40}$$

with the terminal condition $\mathcal{E}_u^{(k)}(u) = 0$.

The *future price* of the portfolio at t coincides with the FCV conditioned to the origin:

$$v_0(t) = \sum_k v_0^{(k)}(t).$$

4.2. Modern theory vs. classical theory

In this section we will address principal new features of the modern theory with respect to the classic single-rate one.

First of all, we notice that the modern multi-curve PDE (32) is written for the whole portfolio. For example, in the special case of two funding parts (19), the r.h.s. depends on the collateral, eventually netted on the portfolio level. Thus, the notion of individual instruments is lost: we do not know how a single instrument was funded, see [Pit10]. Thus, individual instrument exercises should be calculated by a whole-portfolio optimization. Given the high complexity of such global optimization, the approximation is inevitable.

Of course, if the collateral agreement covers an individual instrument, then its price is perfectly defined.

We can rewrite the PDE (32) between payment/exercise dates as

$$\mathcal{L}(t)\, V(t) = r_\Phi(t, V(t))\, V(t), \tag{41}$$

where we have defined the *effective non-linear rate* as

$$r_\Phi(t, V) = \frac{\sum_i \Phi_i(V)\, r_i(t)}{V}. \tag{42}$$

We see that the effective discounting rate depends explicitly on the portfolio/instrument price and its collateralization agreement. For example, if the portfolio is fully collateralized then the discounting rate coincides with the collateral rate r_C.

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All this non-trivial rate dependence is absent in the classical theory: all the instruments are well defined and have a unique discounting rate r, with prices given by the PDE (35).

Sometimes, the collateral agreement can change during the life of the instrument; for example, this is the case of an option (non-collateralized) to enter into a cleared (collateralized) swap. Note that such option price can be calculated exactly provided that the exercise is automatic, i.e. depends on the model states (e.g. European, Barrier etc.). This is one of our important results. Indeed, in appendix A we consider an option with a discount rate $R(t)$ to enter into a swap with a rate $\tilde{R}(t)$ (both of rates can be stochastic but instrument independent). Then, the difference between the option price and its single-rate value can be written in terms of the single-rate *future* prices (67).

4.3. Main results

In this section, we address the *funding adjustment*, i.e. the difference between the portfolio's multi-curve value (32) and its single-rate value (37). We will use the single-rate model not only to calculate the base price, but also as our *computational tool* for evaluating the FVA.

The main technical result of this Section is the FVA general formula for the portfolio

$$V(0) - v(0) = -\mathbb{E}\left[\int_0^T du\, F_\Phi(u, v_0(u))\, e^{-\int_0^u ds\, \frac{F_\Phi(s, v_0(s)) - F_\Phi(s,0)}{v_0(s)}}\; e^{-\int_0^u ds\, r(s)}\right], \tag{43}$$

where we denoted the single-rate prices with small letters v and multi-rate prices with capital letters V. As explained in the previous section, $v_0(s)$ is the future value of the portfolio and the function F_Φ is the difference between the multi-curve r.h.s. and the single-curve one

$$F_\Phi(t, v) \equiv \sum_i \Phi_i(v)\, r_i(t) - v\, r(t). \tag{44}$$

Thus, to calculate the FVA one should first obtain the portfolio future values for the single-rate model and then aggregate them taking into account the non-linear r.h.s. of the multi-curve setup.

The formula (43) is *exact* for a portfolio containing non-callable instruments provided that the multi-curve r.h.s. is a linear function of the portfolio price. The other cases are treated approximately but with very high accuracy (see Section 5).

Now we will pass to the derivation of the main formula (43). First of all, the difference between the modern and classical prices has the order of spreads, i.e. somehow averaged difference between the multiple rates r_i and the single-rate r

$$V - v = O(\text{spreads}) = O(r_i - r).$$

First, we transform the pricing PDE into its effective rate form (41) and approximate the rate with the corresponding single-rate model values,

$$r_\Phi(t, V(t)) = r_\Phi(t, v(t)) + O(\text{spread}).$$

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This immediately defines the individual instrument prices and linearizes the r.h.s. However, as we will see below, we are not simply getting back to the single-rate theory, because the discounting rate can still be dependent on instrument *exercises*.

The next step is to approximate exercises of the multi-rate model by the single-rate one. For example, we exercise a Bermudan option if this decision is optimal from the point of view of the corresponding single-rate model.

These two steps result in the PDE for the k-th instrument

$$\begin{aligned} \mathcal{L}_x(t)\, V^{(k)}(t) &= r_\Phi(t, v(t))\, V^{(k)}(t) \\ &- \sum_i \delta\left(t - t_i^{(k)}\right) B_i^{(k)} - \sum_j \delta\left(t - T_j^{(k)}\right) \left(V^{(\ell_k)}(t_+) - V^{(k)}(t_+)\right) I_j^{(k)} \\ &+ O(\text{spread}). \end{aligned} \tag{45}$$

If the exercise is automatic (e.g. given by a barrier condition), its indicator is a certain function of model states. If the exercise is an option (e.g. related to a Bermudan option), the indicator $I_j^{(k)}$ used in the individual instrument definition (45) should be calculated using optimal consideration coming from the single-rate prices $v^{(k)}$. That is,

$$I_j^{(k)} = \mathbb{I}_{v^{(\ell_k)}(t_+) > v^{(k)}(t_+)}$$

for $t = T_j^{(k)}$. In any case, the indicator is a certain function of model states, and its usage in the right-hand side of (45) is well defined.

Let us emphasize that the effective rate $r_\Phi(t, v(t))$ in the PDE (45) is *instrument dependent*. Imagine for simplicity that the portfolio contains only one instrument k, which can be exercised into the ℓ_k-th one. If we stay in instrument k, i.e. haven't exercise prior to t, then the portfolio value $v(t)$ will be the instrument k value:

$$v(t) = v^{(k)}(t).$$

Otherwise, if we already exercised into instrument ℓ_k, then

$$v(t) = v^{(\ell_k)}(t)$$

and the rate $r_\Phi(t, v(t))$ depends now on $v^{(\ell_k)}(t)$. Omitting delta-functions in the right-hand side, we obtain

$$\begin{aligned} \mathcal{L}_x(t)\, V^{(k)}(t) &= r_\Phi(t, v^{(k)}(t))\, V^{(k)}(t), \\ \mathcal{L}_x(t)\, V^{(\ell_k)}(t) &= r_\Phi(t, v^{(\ell_k)}(t))\, V^{(\ell_k)}(t). \end{aligned}$$

We see that the instrument discount rate before exercise coincides with $r_\Phi(t, v^{(k)}(t))$, while the rate after the exercise is $r_\Phi(t, v^{(\ell_k)}(t))$. In both cases, these rates depend on the single-rate prices v and are independent of the multi-rate prices V. As mentioned in the previous section, the situation corresponds to an option where the discount rate changes after exercise. In Appendix A we have calculated the price of such Bermudan option given its single-rate price

$$V^{(k)}(t) = v^{(k)}(t) - \mathbb{E}\left[\int_t^T du\, \left(r_\Phi\left(u, v_t^{(k)}(u)\right) - r(u)\right)\, v_t^{(k)}(u)\, e^{-\int_t^u ds\, r_\Phi\left(s, v_t^{(k)}(s)\right)}\, \middle|\, \mathcal{F}_t\right]. \tag{46}$$

It is important to stress that that the formula uses the *future* value $v_t(u)$ (38) in the effective rate, reflecting the fact that possible exercises change the rate.

We warn against the use of the *continuation* value of the instrument $v^{(k)}(t)$ for the effective rate r_Φ calculation

$$V^{(k)}(t) \neq v^{(k)}(t) - \mathbb{E}\left[\int_t^T du\, \left(r_\Phi\left(u, v^{(k)}(u)\right) - r(u)\right)\, v_t^{(k)}(u)\, e^{-\int_t^u ds\, r_\Phi\left(s, v^{(k)}(s)\right)}\, \middle|\, \mathcal{F}_t\right]. \tag{47}$$

As we will see in the numerical experiments section, such misusage can lead to significant errors.

For a general portfolio, containing multiple callable and non-callable instruments, we approximate

$$V(t) \simeq v(t) - \mathbb{E}\left[\int_t^T du\, (r_\Phi(u, v_t(u)) - r(u))\, e^{-\int_t^u ds\, r_\Phi(s, v_t(s))}\, v_t(u)\, \middle|\, \mathcal{F}_t\right], \tag{48}$$

where $v(t)$ is the portfolio continuation value and $v_t(s)$ is its future value.

Introducing the "perturbation" of the true PDE's right-hand side (32) over the single-rate PDE right-hand side

$$F_\Phi(t, v) \equiv \sum_i \Phi_i(v)\, r_i(t) - v\, r(t) = (r_\Phi(v) - r(t))\, v, \tag{49}$$

we can rewrite the *adjustment* as

$$J(t) = V(t) - v(t) = -\mathbb{E}\left[\int_t^T du\, F_\Phi(u, v_t(u))\, e^{-\int_t^u ds\, \frac{F_\Phi(s, v_t(s))}{v_t(s)}}\; e^{-\int_t^u ds\, r(s)}\, \middle|\, \mathcal{F}_t\right].$$

For rare cases the r.h.s. can be nonzero for $V = 0$ which leads to the degeneration of the effective rate. To regularize our approximation we proceed as follows (see Appendix A)

$$J(t) = V(t) - v(t) = -\mathbb{E}\left[\int_t^T du\, F_\Phi(u, v_t(u))\, e^{-\int_t^u ds\, \frac{F_\Phi(s, v_t(s)) - F_\Phi(s, 0)}{v_t(s)}}\; e^{-\int_t^u ds\, r(s)}\, \middle|\, \mathcal{F}_t\right]. \tag{50}$$

The calculation workflow has the following steps:

1. Build the single-rate pricing model equipped with least-squares Monte Carlo.

2. Simulate the model rates $\{r_i\}$, r and all payment indexes.

3. Calculate future values $v_0^{(k)}(t)$ for all instruments in the portfolio on this model. This can be done independently "instrument-by-instrument" using the Algorithmic Exposure methods ([AIM12]); *this is important for parallel computation.*

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4. Aggregate the instrument future prices into the portfolio future prices: $v_0(t) = \sum_k v_0^{(k)}(t)$.

5. Calculate the FVA from the obtained future values $v_0(t)$ by (50) for $t = 0$:

$$J(0) = V(0) - v(0) = -\mathbb{E}\left[\int_0^T du\, F_\Phi(u, v_0(u))\, e^{-\int_0^u ds\, \frac{F_\Phi(s, v_0(s)) - F_\Phi(s,0)}{v_0(s)}}\; e^{-\int_0^u ds\, r(s)}\right]. \tag{51}$$

We conclude the section with several remarks. As we have seen in the previous section, the multi-rate model evolution is written for the whole portfolio price as far as the funding procedures are defined at the portfolio level. This makes the notion of individual instrument quite uncertain. If our portfolio consists of non-callable deals, we can aggregate the cashflows in one "superswap" and work with it as if it were a single instrument. If we have callable exotics we cannot calculate the exercise boundary in the usual manner because we don't know how the individual instruments were funded. The exercises calculation becomes a whole-portfolio optimization procedure, especially so when the portfolio contains multiple callable deals. This means that there is, numerically, a fundamental uncertainty in the funding adjustment for exotic portfolios.

In the derivation above we have *defined* individual instruments applying proportional funding related with the approximate rate $r_\Phi(t, v(t))$. However, a different choice of funding can lead to different results.

Note also that instrument path-dependence can be also included into the single-rate pricing mechanism. For this it is sufficient to increase the regression space by extra path-dependent states.

For non-callable instruments the adjustments (51) are *exact* if the r.h.s. of (32) is a linear function of V. Otherwise, the adjustment is an approximation. Note that for a non-linear r.h.s., the true price for a portfolio of non-callable instruments, as well as for an individual Bermudan instrument, can be obtained numerically by a small-timestep backwards propagation (see the next section for more details). This numerical procedure is relatively simple with respect to the global optimization underlying the exercise calculation for exotics.

5. Numerical experiments

We present here the results of a few numerical experiments. The setting is as follows. The three needed yield curves - model, collateral and funding - are set up with log-linear interpolation on the discount factors and are defined by the continuous zero rates in Table 1 ($DF(t) = e^{-R(t)\,t}$). Thus the model and the collateral curve coincide.

We use a Hull-White 1-factor model, with constant volatility of 1% and constant mean-reversion of 5%. The spreads between the model short rate and the collateral and funding rates are considered to be *deterministic*. We price using a backward Monte-Carlo, averaging the results for several runs with different seeds, and compute future prices using the algorithmic method from [AIM12].

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Table 1: Yield curves

Curve	Maturity 1Y	Maturity 20Y
model curve	1.50%	2.00%
collateral curve	1.50%	2.00%
funding curve	2.50%	2.50%

Table 2: Bermudan swaption dates

Float leg fixing dates:	1Y	1.5Y	2Y	...	9.5Y
Float leg payment dates:	1.5Y	2Y	2.5Y	...	10Y
Fixed leg start dates:	1Y	2Y	3Y	...	9Y
Fixed leg payment dates:	2Y	3Y	4Y	...	10Y
Exercise dates:	1Y	2Y	3Y	...	9Y

The instrument we consider is a 10Y Bermudan swaption giving annually the right to enter into a $10,000$ notional swap where we receive annually the fixed rate and pay semi-annually the floating rate, as indicated in Table 2.

We calculate two prices: the *single-rate* price, which is the "non-perturbed" classical price, corresponding to the PDE

$$\mathcal{L}(t)\, v(t) = r(t)\, v(t),$$

and the *exact* price, which corresponds to the "true" PDE (41)

$$\mathcal{L}(t)\, V(t) = r_\Phi(V(t))\, V(t).$$

We compute the exact price via a fine-tenor backward propagation taking into account the non-linear r.h.s. and the optimal exercise condition. Indeed, formally, this PDE has solution

$$V(t) = \mathbb{E}\left[e^{-\int_t^T r_\Phi(V(s))\, ds} V(T) \,\middle|\, \mathcal{F}_t \right].$$

Choosing a small spacing Δt we approximate

$$\begin{aligned} V(t) &= \mathbb{E}\left[e^{-\int_t^{t+\Delta t} r_\Phi(V(s))\, ds} V(t+\Delta t) \,\middle|\, \mathcal{F}_t \right] \\ &\simeq \mathbb{E}\left[e^{-\int_t^{t+\Delta t} r(s)\, ds}\; e^{-\Delta t [r_\Phi(V(t+\Delta t)) - r(t+\Delta t)]}\, V(t+\Delta t) \,\middle|\, \mathcal{F}_t \right]. \end{aligned} \tag{52}$$

Thus, the *exact* price calculation can be done using a backward discounted propagation by the single-rate model, provided that we modify the underlying as follows

$$V(t+\Delta t) \longrightarrow e^{-\Delta t [r_\Phi(V(t+\Delta t)) - r(t+\Delta t)]}\, V(t+\Delta t).$$

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For this computation we use a 50 points-per-year time discretization (500 time steps for the whole lifetime of the instrument).

The non-linear effective rate makes the instrument pricing somewhat different from what is usual in the single-rate setting. For example, a payment at T of an index I fixed at a previous time t cannot be simply added to the instrument leg L at time t,

$$L_{new}(t) \neq L(t) + P(t,T)\, I(t)$$

where $P(t,T)$ is a zero bond. Instead, non-linear effects force us to add the payments to legs when they are really paid, i.e.

$$L_{new}(T) = L(T) + I(t)$$

The subsequent backwards propagation (52) should take into account the partial "path-dependence" of the new value of the leg $L_{new}(T)$. Indeed, it will depend on the model states at time T as well as on the model states at time t via the index $I(t)$. This means that we should augment the space of regression variables. In our concrete case, for each observation date t in a period of the floating leg $[T_n, T_{n+1}]$, we add the Libor rate observed at T_n as an extra regression variable at t.

Also, on each exercise date, we enter into the swap if its *exact* continuation value is bigger than the option *exact* value (and not the *single-curve* approximations). Notice that we avoid a global optimization as far as the portfolio contains a single callable instrument.

The collateral, as a function of portfolio value, is defined as

$$C(V) = (V - H)^+$$

where the threshold H has been set at $H = 500$, corresponding to an asymmetric CSA (only our counterparty posts collateral, to cover any exposure beyond the threshold). The uncollateralized part is then

$$V - C = \min(V, H),$$

giving rise to the PDE

$$\mathcal{L}\, V = (V - H)^+\, r_C + \min(V, H)\, r_F = r_\Phi(V)\, V,$$

where the effective non-linear rate $r_\Phi(V)$ is defined implicitly by the last equality, as previously.

We use several values for the strike, ranging from ATM $-$ 2% to ATM + 8%. The results for both the Bermudan swaption and the underlying 10Y swap are summarized in Table 3. The difference between the *exact* and *single-rate* prices represents the "true" FVA and is our benchmark in assessing the FVA approximations below. As is obviously to expect, for higher values of the strike the swap is far in the money and the Bermudan price is practically equal to the swap price (the probability to exercise on the first exercise date approaches 100%). See also Figure 1.

Table 3: Prices

Strike	Swap prices		Bermudan prices	
	Single-rate price	Exact price	Single-rate price	Exact price
0.05%	-1,604.54	-1,554.05	85.21	82.18
1.05%	-802.27	-776.71	210.82	204.14
2.05%	0.00	3.20	469.89	458.04
3.05%	802.27	790.23	941.75	925.68
4.05%	1,604.54	1,585.77	1,625.61	1,606.25
5.05%	2,406.82	2,385.07	2,408.26	2,386.47
6.05%	3,209.09	3,185.99	3,209.10	3,186.01
7.05%	4,011.36	3,987.66	4,011.36	3,987.66
8.05%	4,813.63	4,789.68	4,813.63	4,789.68
9.05%	5,615.90	5,591.84	5,615.90	5,591.84
10.05%	6,418.17	6,394.06	6,418.17	6,394.06

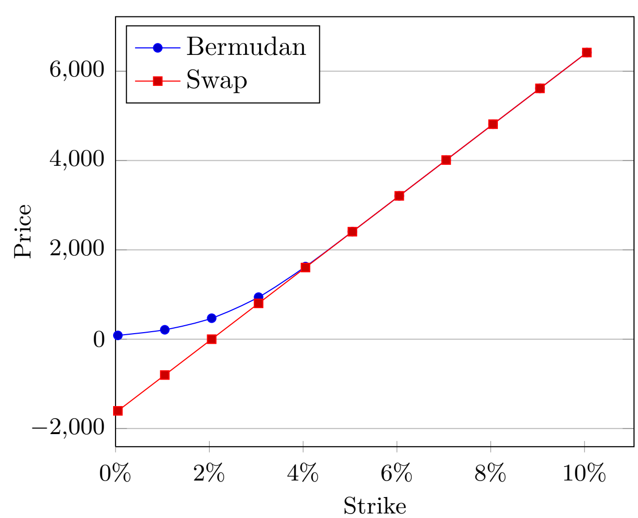

Figure 1: Bermudan swaption and Underlying swap prices

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We further compute two FVA approximations. The first, which we designate as the *approximate* FVA, is given by the adjustment in equation (51), where the effective rate r_Φ is evaluated on the *future* values $v_0(u)$. Then, for comparison, we compute the *naive* FVA, which is given by the adjustment in equation (47), where the effective rate is evaluated on the *continuation* values $v(u)$. The results are summarized in Table 4.[5] We

Table 4: FVAs

Strike	Swap FVA		Bermudan swaption FVA		
	True	Approx.	True	Approx.	Naive
0.05%	50.49	50.49	-3.02	-3.03	-3.07
1.05%	25.56	25.57	-6.67	-6.66	-6.80
2.05%	3.20	3.24	-11.85	-11.77	-12.27
3.05%	-12.04	-11.93	-16.07	-15.93	-17.21
4.05%	-18.77	-18.63	-19.36	-19.21	-21.52
5.05%	-21.75	-21.62	-21.79	-21.65	-24.79
6.05%	-23.10	-22.98	-23.10	-22.98	-26.66
7.05%	-23.70	-23.59	-23.70	-23.59	-27.62
8.05%	-23.95	-23.87	-23.95	-23.87	-28.11
9.05%	-24.06	-23.99	-24.06	-23.99	-28.35
10.05%	-24.11	-24.05	-24.11	-24.05	-28.44

notice the very good agreement of the *approximate* FVA with the benchmark *true* FVA, for both swaption and swap. On the contrary, the *naive* FVA drifts significantly away from the benchmark as the probability to exercise early increases, see Figure 3.

To explain the behavior of the FVA we rewrite the adjustment (51) for our concrete case of the collateral $C(V) = (V - H)^+$ and the model rate r coinciding with the collateralized rate r_C,

$$J(0) = -\mathbb{E}\left[\int_0^T du\, s_F(u) \min(H, v_0(u))\, e^{-\int_0^u dt\, \frac{\min(H, v_0(t))}{v_0(t)}\, s_F(t)}\; e^{-\int_0^u dt\, r_C(t)}\right]. \qquad (53)$$

where $s_F(u) = r_F(u) - r_C(u)$ is the funding spread. The swaption future value $v_0(t)$ is very likely to be positive, thus, the adjustment will be negative due to the term $\min(H, v_0(u))$. For large and positive future values, e.g. for the far in-the-money options, the FVA saturates on the level

$$J_+(0) = -H \int_0^T du\, \mathbb{E}\left[s_F(u)\;\; e^{-\int_0^u dt\, r_C(t)}\right] \qquad (54)$$

If, on the other hand, we have highly negative values, say for out-of-money swap, the FVA is likely to be positive as far as $\min(H, v_0(u)) < 0$.

[5] The *naive* FVA is meaningful only for callable instruments and we thus ignore it for the swap.

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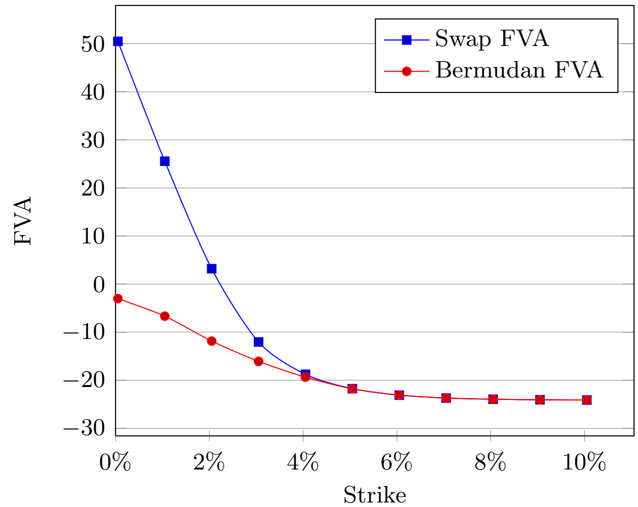

Figure 2: Bermudan swaption and underlying Swap FVAs

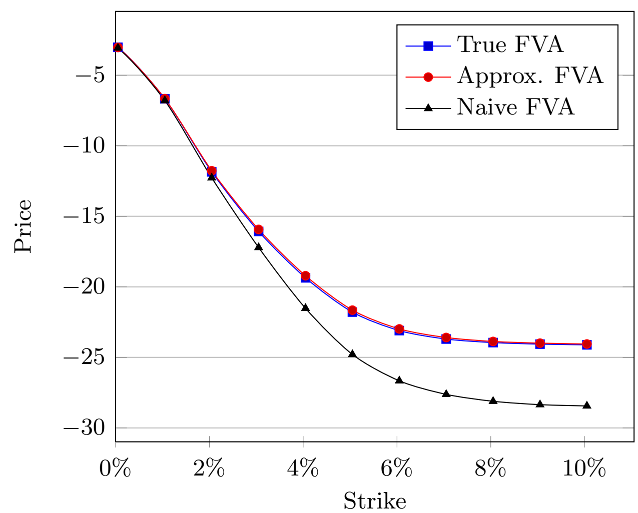

Figure 3: Bermudan swaption FVA approximations

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6. Conclusions

Following the first part of this paper [AB13], we presented a generalized funding theory, where the replication portfolio is split between general, non-linear, functions of the portfolio value, funded with different rates. We also proposed an implementation framework for this general theory, which provides a practical and very accurate approximation for portfolios containing both vanilla and exotic instruments. All along we maintained an understanding of theory tolerances, by keeping an eye on the amplitude of the approximation errors. Finally, we presented the results of a few numerical experiments on a Bermudan swaption, showcasing the high accuracy of the approximation.

A. Different Discount-Rate Adjustment

In this appendix, we address the price of a callable instrument having different stochastic discount rates *before and after* exercises along with different instantaneous cashflows. Such price can be *exactly* calculated given a standard single-discounting pricing results provided that the exercise decisions depend on the model states (automatic), e.g. for Barrier or European options.

Suppose we have a callable instrument with a continuation value v for a classical model with rate r. The rate process $r(t)$ is a certain deterministic function of model states, the Markovian vector process $x(t)$. The instrument payments B_i on dates t_i possibly depend on the model states. We exercise on dates T_j into another instrument with continuation value $\tilde{v}$ provided that the exercise indicator $I_j = 1$. We suppose this instrument is itself non-callable and has payments $\tilde{B}_i$ on dates $\tilde{t}_i$. For simplicity, we suppose that the non-callable instrument does not have payments on the exercise dates. The PDEs of the two instruments read

$$\mathcal{L}_x(t)\, v(t) = r(t)\, v(t) - \sum_i \delta\left(t - t_i\right)\, B_i - D(t)\, \left(\tilde{v}(t) - v(t_+)\right) \tag{55}$$

$$\mathcal{L}_x(t)\, \tilde{v}(t) = r(t)\, \tilde{v}(t) - \sum_i \delta\left(t - \tilde{t}_i\right)\, \tilde{B}_i \tag{56}$$

where we denoted the exercise delta-function part by

$$D(t) = \sum_j \delta\left(t - T_j\right)\, I_j$$

We put the non-callable continuation value $\tilde{v}(t)$ at t in the delta-functional r.h.s. part as far as it is a continuous function of time at $t = T_j$: no payments happen on the exercises.

The evolution operator $\mathcal{L}_x(t)$ is related with model Markovian state processes $x(t)$. The notation $t_+ = t + \varepsilon$ reflects the post-exercise-time logic. We consider the indicator to be automatic, i.e. a certain function of model states, $I_j = I_j\left(x\left(T_j\right)\right)$.

Suppose that now we modify the r.h.s. of (55-56) for general linear functions of values. They include common instantaneous payments $h(t)$ and discounting rates $R(t)$ and $\tilde{R}(t)$

corresponding to the callable and non-callable instrument, respectively. We define the new continuation values denoted with capital letters as

$$\mathcal{L}_x(t)\, V(t) = h(t) + R(t)\, V(t) - \sum_i \delta\left(t - t_i\right)\, B_i - D(t)\, \left(\tilde{V}(t) - V(t_+)\right) \tag{57}$$

$$\mathcal{L}_x(t)\, \tilde{V}(t) = h(t) + \tilde{R}(t)\, \tilde{V}(t) - \sum_i \delta\left(t - \tilde{t}_i\right)\, \tilde{B}_i \tag{58}$$

We note the same cashflows and exercises as in the initial single-rate setup. The discounting rates difference for the callable and non-callable instruments moves the theory out of our usual single-rate setup.

It is easy to see that the adjustment

$$J(t) = V(t) - v(t)$$

satisfies

$$\mathcal{L}_x(t)\, J(t) = R(t)\, J(t) + h(t) + (R(t) - r(t))\, v(t) - D(t)\, \left(\tilde{J}(t) - J(t_+)\right), \tag{59}$$

where $\tilde{J}(t) = \tilde{V}(t) - \tilde{v}(t)$ is the non-callable instrument adjustment. This PDE is obtained as the difference between (57) and (55). The delta-functional part of the r.h.s. of (59) corresponds to the instrument transformation on exercise dates. For example, at $t = T_j$, the callable adjustment $J(T_{j_+})$ jumps into the non-callable one $\tilde{J}(T_j)$ which is continuous at T_j.

Let us quickly prove that the non-callable adjustment can be reduced to the integral

$$\tilde{J}(t) = \tilde{V}(t) - \tilde{v}(t) = -\mathbb{E}\left[\int_t^T du\, \left\{h(u) + (\tilde{R}(u) - r(u))\, \tilde{v}(u)\right\}\, e^{-\int_t^u ds\, \tilde{R}(s)}\, \middle|\, \mathcal{F}_t\right] \tag{60}$$

Indeed, subtracting (56) form (58) we obtain PDE

$$\mathcal{L}_x(t)\, \tilde{J}(t) = \tilde{R}(t)\, \tilde{J}(t) + h(t) + (\tilde{R}(t) - r(t))\, \tilde{v}(t)$$

which is obviously satisfied by (60).

Now we come to the more complicated case of the callable adjustment. We will prove that the adjustment

$$J(t) = -\,\mathbb{E}\left[\int_t^T du\, \left(1 - \mathcal{E}_t(u)\right)\, \left\{h(u) + (R(u) - r(u))\, v(u)\right\}\, e^{-\int_t^u ds\, R_t(s)}\, \middle|\, \mathcal{F}_t\right] \tag{61}$$

$$-\,\mathbb{E}\left[\int_t^T du\, \mathcal{E}_t(u)\, \left\{h(u) + (\tilde{R}(u) - r(u))\, \tilde{v}(u)\right\}\, e^{-\int_t^u ds\, R_t(s)}\, \middle|\, \mathcal{F}_t\right] \tag{62}$$

including the *future rate* in the exponent

$$R_t(s) \equiv \left(1 - \mathcal{E}_t(s)\right)\, R(s) + \mathcal{E}_t(s)\, \tilde{R}(s) \tag{63}$$

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satisfies the PDE (59). Here $\mathcal{E}_t(u)$ is the *global* exercise indicator at u provided that we did not exercise prior to t, satisfying, as mentioned in (40),

$$\partial_t \, \mathcal{E}_t(u) = -D(t) \, \left(1 - \mathcal{E}_{t_+}(u)\right) \tag{64}$$

with the terminal condition $\mathcal{E}_u(u) = 0$. Note that if at exercise time T_j the *local* exercise indicator $I_j = 1$, the global indicator $\mathcal{E}_{T_j}(u) = 1$ for all $u > T_j$.

For times t other than the exercise dates T_j we have the following

$$\mathcal{L}_x(t) \, J(t) = R(t) \, J(t) + h(t) + (R(t) - r(t)) \, v(t) \quad \text{for} \quad t \neq T_j \tag{65}$$

where we have used $\mathcal{E}_t(t) = 0$. Time t derivative in the indicators on the exercise dates should be handled with more care. Indeed, we cannot apply standard derivative rules because $\mathcal{E}_t(u)$ and $\mathcal{E}_t(s)$ can jump at the same time. Thus, we will concentrate on the jump of the adjustment $J(t)$ while t crosses T_j for $I_j = 1$. When time approaches T_j from above, the adjustment is simply $J(T_{j+})$. When time t is exactly the exercise time, $t = T_j$, we have an exercise because $I_j = 1$ which means that $\mathcal{E}_t(u) = 1$ for all $u > t$. Thus,

$$J(T_j) = -\,\mathbb{E}\left[\int_{T_j}^{T} du \, \left\{h(t) + (\tilde{R}(u) - r(u)) \, v(u)\right\} \, e^{-\int_{T_j}^{u} ds \, \tilde{R}(s)} \, \middle| \, \mathcal{F}_{T_j}\right] = \tilde{J}(T_j) \tag{66}$$

which results into a jump of size $I_j(\tilde{J}(T_j) - J(T_{j+}))$ at T_j. We put the indicator I_j in from of the jump because for $I_j = 0$ the $\mathcal{E}_t(u)$ does not change while t crosses T_j. Repeating this procedure for all exercise dates restores the delta-functional term in (59). This finishes the proof.

As mentioned in (38) the callable deal future price is defined as

$$v_t(u) = (1 - \mathcal{E}_t(u)) \, v(u) + \mathcal{E}_t(u) \, \tilde{v}(u)$$

This permits us to rewrite the adjustment in terms of the future price and the future rate (63)

$$J(t) = -\mathbb{E}\left[\int_{t}^{T} du \, \left\{h(t) + (R_t(u) - r(t)) \, v_t(u)\right\} \, e^{-\int_{t}^{u} ds \, R_t(s)} \, \middle| \, \mathcal{F}_t\right] \tag{67}$$

For coinciding callable and non-callable rates, $R = \tilde{R}$, we simply have

$$J(t) = -\mathbb{E}\left[\int_{t}^{T} du \, \left\{h(t) + (R(u) - r(t)) \, v_t(u)\right\} \, e^{-\int_{t}^{u} ds \, R(s)} \, \middle| \, \mathcal{F}_t\right] \tag{68}$$

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