

# Theoretical Foundations of Buffer Stock Saving

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# Drawbacks of Numerical Solutions

## A Black Box

- Can Construct Solution to Model Without Really Understanding It
- Hard Even To Be Sure Your Numerical Solution Is *Right*
- Little Intuition for How Results Might Change With
  - Calibration
  - Structure
- *Very Hard To Teach!*

## I Am A *Big* Fan Of Numerical Methods

- Have Done A Good Deal Of Work With Them Myself
- But As A Result, Have Felt All These Drawbacks Keenly

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Foundations For Microeconomic Household's Problem With

- Uncertain Labor Income
- No Liquidity Constraints
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# Key Result

Restrictions On Parameter Values Such That

- Problem Defines A Contraction Mapping
  - $\Rightarrow \exists$  A Unique Consumption Function  $c(m)$
- There Is A 'Target' Ratio Of Assets to Permanent Income
  - Requires A Key 'Impatience' Condition To Hold
  - Good News
    - Condition Is Weaker (Easier To Satisfy) Than Previous Papers Imposed



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Limit as horizon  $T$  goes to infinity of

$$\max \mathbb{E}_t \left[ \sum_{n=0}^{T-t} \beta^n u(c_{t+n}) \right] \quad (1)$$

$$a_t = m_t - c_t \quad (2)$$

$$b_{t+1} = a_t R$$

$$\mathbf{p}_{t+1} = \mathbf{p}_t \underbrace{\Gamma^{\psi_{t+1}}}_{\equiv \Gamma_{t+1}}$$

$$m_{t+1} = b_{t+1} + \mathbf{p}_{t+1} \xi_{t+1},$$

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/\wp & \text{with probability } \wp \equiv (1 - \wp) \end{cases} \quad (3)$$

- $u(\bullet) = \bullet^{1-\rho}/(1-\rho)$ ;  $\mathbb{E}_t[\psi_{t+n}] = \mathbb{E}_t[\xi_{t+n}] = 1 \ \forall \ n > 0$ ;  $\beta < 1, \rho > 1$

# Surely This Problem Has Been Solved?

No.

- Can't Use Stokey et. al. theorems because CRRA utility
- Lit thru Matkowski and Nowak (2011) Can't Handle Permanent Shocks
- Must Use Boyd's 'Weighted' Contraction Mapping Theorem
- Surprisingly Subtle

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# Benchmark: Perfect Foresight Model

Definitions:

Absolute Patience Factor

$$\mathbf{P} = (R\beta)^{1/\rho}$$

Return Patience Factor

$$\mathbf{P}_R = \mathbf{P}/R$$

Perfect Foresight Growth Patience Factor

$$\mathbf{P}_\Gamma = \mathbf{P}/\Gamma$$

Name	Condition	Implication
(AIC) Absolute Impatience Condition	$\mathbf{P} < 1$	$c \downarrow$ over time
(RIC) Return Impatience Condition	$\mathbf{P}_R < 1$	$c/a \downarrow$ over time
(PF-GIC) Growth Impatience Condition	$\mathbf{P}_\Gamma < 1$	$c/\mathbf{p} \downarrow$ over time

# When Does A Useful Limiting Solution Exist?

Finite Human Wealth (FHW) condition:

$$\Gamma < R \quad (4)$$

Return Impatience Condition:

$$\mathbb{D}_R < R \quad (5)$$



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# What If There Are Liquidity Constraints?

- FHWC is *not* necessary for solution to exist
- Other Key Condition For Useful Solution is  
'Perfect Foresight Finite Value of Autarky Condition (PF-FVAC)':

$$\beta \Gamma^{1-\rho} < 1 \quad (6)$$

- Without RIC, Constraints Are Irrelevant
  - Because Wealth Always Wants To Rise, So Constraint Never Binds

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# Liquidity Constraints and Uncertainty

- Introduce permanent shocks to income
- Finite Value of Autarky Condition Becomes

$$\overline{\overline{\overline{\beta \Gamma^{1-\rho}}}} < 1 \quad (7)$$

$$\beta < \overline{\overline{\Gamma^{\rho-1}}}$$

# Contraction Mapping Requirements

Finite Value of Autarky Condition: Same As In Liq Constr Problem!

$$\overbrace{\beta \Gamma^{1-\rho}}^{\equiv \Gamma} < 1 \quad (8)$$

$$\beta < \underline{\Gamma}^{\rho-1}$$

'Weak Return Impatience Condition' (WRIC)

$$0 \leq \beta^{1/\rho} \underline{\Gamma} < 1 \quad (9)$$

## Requirement For Existence Of A Target

Definitions: 'Uncertainty-Adjusted' Growth:

$$\underline{\Gamma} = \Gamma \psi \quad (10)$$

Adjusted Growth Patience Factor:

$$\mathbf{P}_{\hat{\Gamma}} = \mathbf{P} / \underline{\Gamma} \quad (11)$$

Growth Impatience Condition:

$$\mathbf{P}_{\hat{\Gamma}} < 1 \quad (12)$$

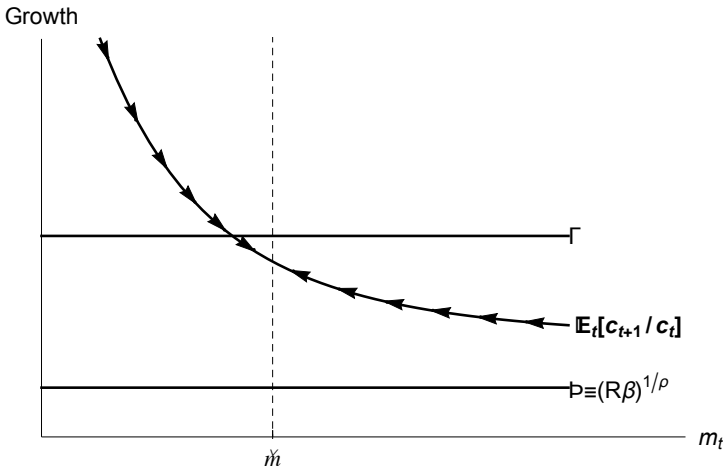
Why? Because it can be shown that

$$\lim_{m_t \rightarrow \infty} \mathbb{E}_t \left[ \frac{m_{t+1}}{m_t} \right] = \mathbf{P}_{\hat{\Gamma}} \quad (13)$$

# Five Propositions

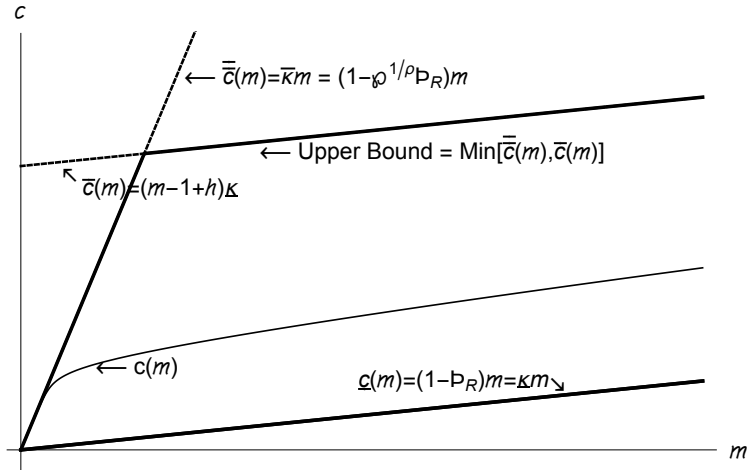
- ①  $\lim_{m_t \rightarrow \infty} \mathbb{E}_t[c_{t+1}/c_t] = \mathbf{P}$
- ②  $\lim_{m_t \rightarrow 0} \mathbb{E}_t[c_{t+1}/c_t] = \infty$
- ③  $\exists$  a unique target value of  $m$ , called  $\check{m}$
- ④  $\mathbb{E}_t[c_{t+1}/c_t | m_t = \check{m}] = \Gamma - \epsilon$
- ⑤  $\left( \frac{d\mathbb{E}_t[c_{t+1}/c_t]}{dm_t} \right) < 0$

# The Target Saving Figure

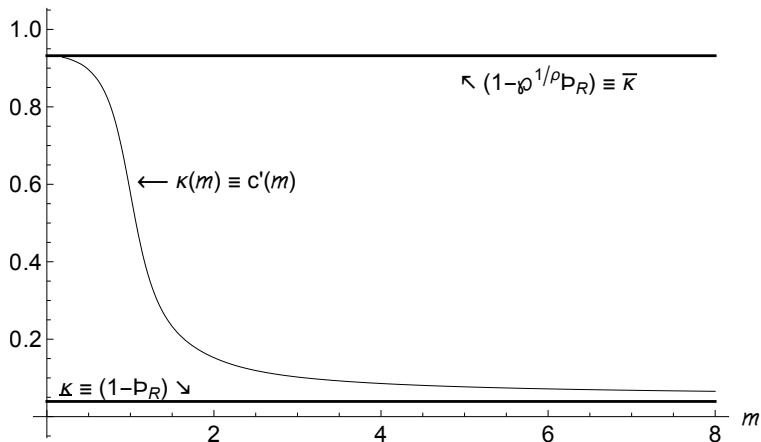




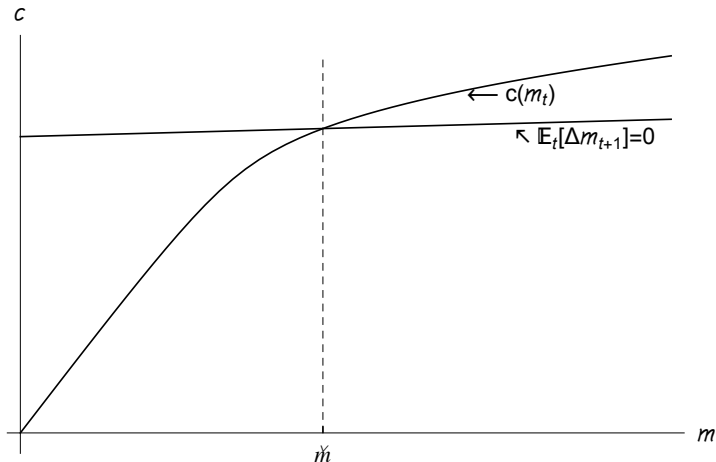
# Bounds On the Consumption Function



# The Marginal Propensity to Consume

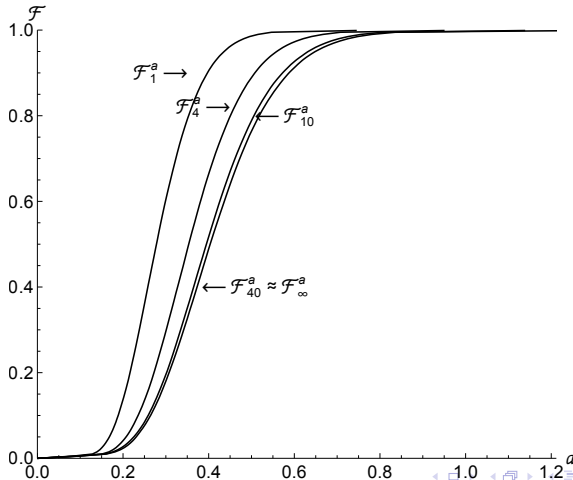


# The Consumption Function and Target Wealth



# Convergence To The Invariant Distribution

Szeidl (2012) Proves Existence of an Invariant Distribution of  $m, c, a$ , etc.



# Balanced Growth Equilibrium

Achieved When Cross Section Distribution Reaches Invariance

$$Y_{t+1}/Y_t = C_{t+1}/C_t = \Gamma \quad (14)$$

Fisherian Separation Fails, Even Without Liquidity Constraints!

Insight:

- Precautionary Saving  $\approx$  Liquidity Constraints
- If  $\tilde{c}(m)$  is solution for constrained consumer,

$$\lim_{\varphi \downarrow 0} c(m; \varphi) = \tilde{c}(m) \quad (15)$$

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# The MPC Out Of Permanent Shocks

<http://econ.jhu.edu/people/ccarroll/papers/MPCPerm.pdf>

Lots of Recent Papers Trying to Measure the MPCP

Paper Proves:

- $MPCP < 1$
- But not a lot less:
  - 0.75 to 0.95 (annual rate) for wide range of parameter values



- Defined Conditions Under Which Widely Used Problem Has Solution
  - Finite Value of Autarky Condition Guarantees Contraction (with WRIC)
  - Growth Impatience Condition Prevents  $m \rightarrow \infty$
- Economy Of Buffer Stock Consumers Exhibits Balanced Growth
  - Even In Absence of General Equilibrium Adj of Interest Rate

- MATKOWSKI, JANUSZ, AND ANDRZEJ S. NOWAK (2011): "On Discounted Dynamic Programming With Unbounded Returns," *Economic Theory*, 46, 455–474.
- SZEIDL, ADAM (2012): "Stable Invariant Distribution in Buffer-Stock Saving and Stochastic Growth Models," *Manuscript, Central European University*.