X - rand var, $x \in X$ - value of r.v

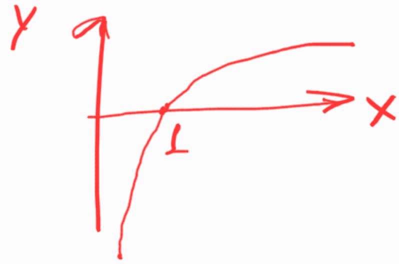
$p(x) = \Pr[X = x]$

x, y - diff val of rand var

Def $H(X)$ of discrete rand var X:

$$H(x) = -\sum_{x} p(x) \log p(x)$$

$y = \log x$



$$\{0 \log 0 = 0\}$$

Average of rand var

$$\sum_{x} p(x) \cdot x = \langle x \rangle$$

$$E_p g(x) = \sum_{x} g(x) p(x)$$

$$H(X) = E_p \log \frac{1}{p(x)}$$

Features of Entropy:

1) $H(X) \geq 0$ for $\forall X$

2) $$\frac{\log_a b}{\log_a c} = \log_c b$$

$$H_b(X) = (\log_b a) H_a(X)$$

Example

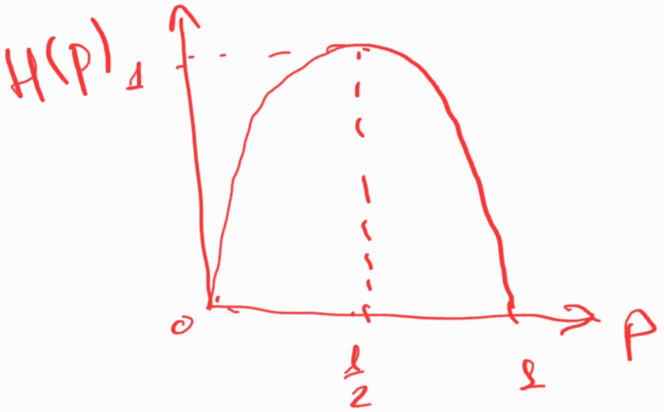
$p \in [0, 1]$

$$X = \begin{cases} 0 & p \\ 1 & 1-p \end{cases}$$

$$H(X) = -p\log p - (1-p)\log(1-p) = H(p)$$

$p = 0, 1 \Rightarrow H = 0$

$p = \frac{1}{2} \Rightarrow H = 1$ bit



Def $H(X, Y) = -\sum_{x,y} p(x,y) \log p(x,y)$

$H(X,Y) = -E_p \log p(X,Y)$

Joint Entropy

Conditional Entropy

$$H(Y|X) = \sum_{x} p(x) H(Y|X=x) =$$

$$= -\underline{\sum_{x} p(x)} \underline{\sum_{y} p(y|x)} \log p(y|x) =$$

$$\{ p(y,x) = p(y|x)\, p(x) \}$$

$$= -\sum p(y,x) \log p(y|x)$$

x,y

$$H(X,Y) = -\sum_{x,y} p(y,x)\log p(y,x) =$$

$$\{p(y,x) = p(y|x)p(x)\} = -\sum_{x,y} p\cdot\log p(y|x)p(x) =$$

$$= -\underbrace{\sum_{x,y} p\log p(y|x)}_{H(y|x)} - \underbrace{\sum_{x,y} p\log p(x)}_{H(x)}$$

$$H(y,x) = H(y|x) + H(x) \text{ - ch. rule}$$

$$p(y,x) = p(y|x)\cdot p(x)$$

$$\log p(y,x) = \log p(y|x) + \log p(x)$$

Example

		X: 1	2	3	4
Y	1	1/8	1/16	1/32	1/32
	2	1/16	1/8	1/32	1/32
	3	1/16	1/16	1/16	1/16
	4	1/4	0	0	0

$$Y = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

$$X = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$$

$$H(y) = 2 \text{ bit}$$

$$H(x) = 1\tfrac{3}{4} \text{ bit}$$

$$H(X|Y) = \sum_{y} p(y) H(X|Y=i) =$$

$$= \frac{1}{4}H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right) + \frac{1}{4}H\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right) +$$

$$+\frac{1}{4}H\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)+\frac{1}{4}H(1,0,0,0)=$$

$$\frac{1}{4}\cdot 1\frac{3}{4}+\frac{1}{4}1\frac{3}{4}+\frac{1}{4}2+\frac{1}{4}0=\frac{11}{8}\text{ bit}$$

$$H(X)=\frac{7}{4}=\frac{14}{8}\text{ bit} \qquad I(X,Y)=\frac{14}{8}-\frac{11}{8}=\frac{3}{8}\text{ bit}$$

Def Relative Entropy

$$D(p\|q)=\sum_{x}p(x)\log\frac{p(x)}{q(x)}$$

$$D(p\|q)\neq D(q\|p)$$

Mutual Information

Def $I(X,Y)=\sum_{x,y}p(x,y)\log\frac{p(x,y)}{p(x)p(y)}=$

$$=D(p(x,y)\,\|\,p(x)p(y))$$

$$I(X,Y)=\sum_{x,y}p\log\frac{p(x|y)}{p(x)}=$$

$$\underbrace{\sum_{x,y}p\log p(x|y)}_{=-H(X|Y)}-\underbrace{\sum_{x,y}p\log p(x)}_{+H(X)}=$$

$$=H(X)-H(X|Y)$$

$$I(X,Y)=H(X)-H(X|Y)$$

$$I(Y,X)=H(Y)-H(Y|X)$$

$$I(Y,X)=H(X)+H(Y)-H(X,Y)$$

$$I(X,X) = H(X) - \underbrace{H(X|X)}_{0} = H(X)$$

$$I(X,Y) = I(Y,X)$$