BEAM DESIGN FORMULAS WITH SHEAR AND MOMENT DIAGRAMS
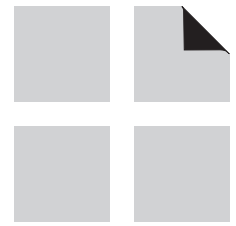
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**NATIONAL DESIGN SPECIFICATION®
FOR WOOD CONSTRUCTION**

WITH COMMENTARY AND
SUPPLEMENT: DESIGN VALUES FOR WOOD CONSTRUCTION

DESIGN AID No. 6

**American
Forest &
Paper
Association**

BEAM FORMULAS WITH SHEAR AND MOMENT DIAGRAMS

The American Wood Council (AWC) is part of the wood products group of the American Forest & Paper Association (AF&PA). AF&PA is the national trade association of the forest, paper, and wood products industry, representing member companies engaged in growing, harvesting, and processing wood and wood fiber, manufacturing pulp, paper, and paperboard products from both virgin and recycled fiber, and producing engineered and traditional wood products. For more information see www.afandpa.org.

Introduction

Figures 1 through 32 provide a series of shear and moment diagrams with accompanying formulas for design of beams under various static loading conditions.

Shear and moment diagrams and formulas are excerpted from the *Western Woods Use Book*, 4th edition, and are provided herein as a courtesy of Western Wood Products Association.
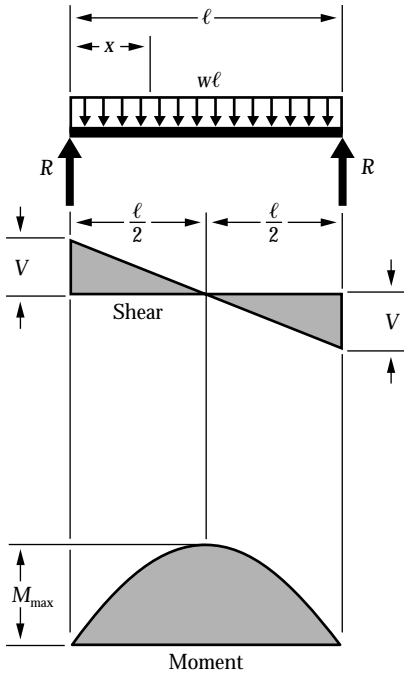
Notations Relative to "Shear and Moment Diagrams"

E = modulus of elasticity, psi
I = moment of inertia, in.^{4}
L = span length of the bending member, ft.
ℓ = span length of the bending member, in.
M = maximum bending moment, in.-lbs.
P = total concentrated load, lbs.
R = reaction load at bearing point, lbs.
V = shear force, lbs.
W = total uniform load, lbs.
w = load per unit length, lbs./in.
Δ = deflection or deformation, in.
x = horizontal distance from reaction to point on beam, in.

List of Figures

Figure 1   Simple Beam – Uniformly Distributed Load

$$R = V \quad \ldots \ldots \ldots \ldots \ldots = \frac{w\ell}{2}$$

$$V_x \quad \ldots \ldots \ldots \ldots \ldots = w\left(\frac{\ell}{2} - x\right)$$

$$M_{max} \text{ (at center)} \quad \ldots \ldots \ldots = \frac{w\ell^2}{8}$$

$$M_x \ldots \ldots \ldots \ldots \ldots \ldots = \frac{wx}{2}(\ell - x)$$

$$\Delta_{max} \text{ (at center)} \quad \ldots \ldots \ldots = \frac{5w\ell^4}{384\ EI}$$

$$\Delta_x \quad \ldots \ldots \ldots \ldots \ldots = \frac{wx}{24\ EI}(\ell^3 - 2\ell x^2 + x^3)$$

Figure 2   Simple Beam – Uniform Load Partially Distributed
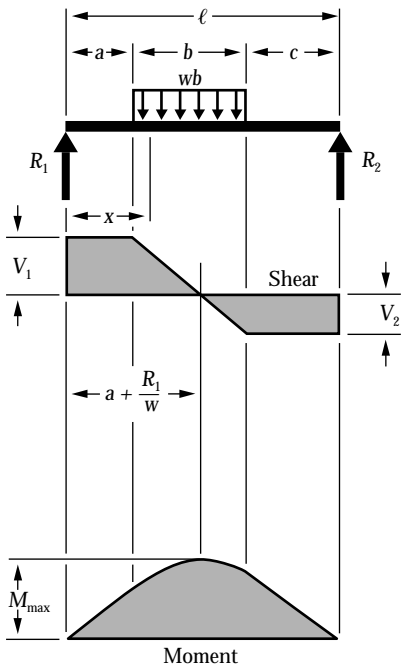
$$R_1 = V_1 \ \text{(max when } a < c) \quad \ldots = \frac{wb}{2\ell}(2c + b)$$

$$R_2 = V_2 \ \text{(max when } a > c) \quad \ldots = \frac{wb}{2\ell}(2a + b)$$

$$V_x \left(\text{when } x > a \text{ and } < (a + b)\right) \ldots = R_1 - w(x - a)$$

$$M_{max} \left(\text{at } x = a + \frac{R_1}{w}\right) \quad \ldots \ldots = R_1\left(a + \frac{R_1}{2w}\right)$$

$$M_x \ \text{(when } x < a) \ \ldots \ldots \ldots = R_1 x$$

$$M_x \left(\text{when } x > a \text{ and } < (a + b)\right) \ldots = R_1 x - \frac{w}{2}(x - a)^2$$

$$M_x \left(\text{when } x > (a + b)\right) \quad \ldots \ldots = R_2(\ell - x)$$

Figure 3 — Simple Beam – Uniform Load Partially Distributed at One End
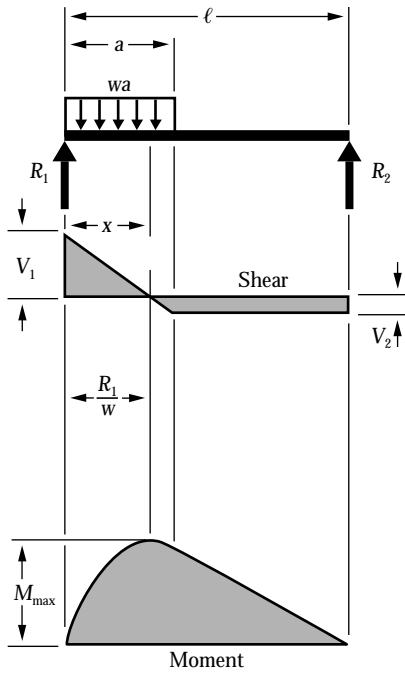
$$R_1 = V_1 \quad \ldots \ldots \ldots \quad = \frac{wa}{2\ell}(2\ell - a)$$

$$R_2 = V_2 \quad \ldots \ldots \ldots \quad = \frac{wa^2}{2\ell}$$

$$V_x \ (\text{when } x < a) \quad \ldots \ldots \quad = R_1 - wx$$

$$M_{max}\left(\text{at } x = \frac{R_1}{w}\right) \quad \ldots \ldots \quad = \frac{R_1^{\,2}}{2w}$$

$$M_x \ (\text{when } x < a) \quad \ldots \ldots \quad = R_1 x - \frac{wx^2}{2}$$

$$M_x \ (\text{when } x > a) \quad \ldots \ldots \quad = R_2(\ell - x)$$

$$\Delta_x \ (\text{when } x < a) \quad \ldots \ldots \quad = \frac{wx}{24\,EI\ell}\left(a^2(2\ell - a)^2 - 2ax^2(2\ell - a) + \ell x^3\right)$$

$$\Delta_x \ (\text{when } x > a) \quad \ldots \ldots \quad = \frac{wa^2(\ell - x)}{24\,EI\ell}(4x\ell - 2x^2 - a^2)$$

Figure 4 — Simple Beam – Uniform Load Partially Distributed at Each End
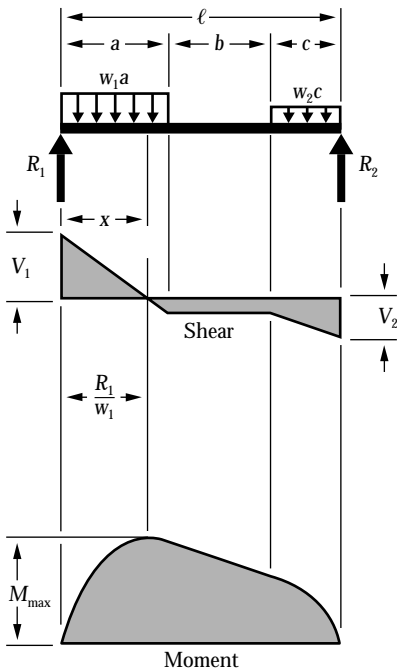
$$R_1 = V_1 \quad \ldots \ldots \ldots \ldots \quad = \frac{w_1 a\,(2\ell - a) + w_2 c^2}{2\ell}$$

$$R_2 = V_2 \quad \ldots \ldots \ldots \ldots \quad = \frac{w_2 c\,(2\ell - c) + w_1 a^2}{2\ell}$$

$$V_x \ (\text{when } x < a) \quad \ldots \ldots \ldots \quad = R_1 - w_1 x$$

$$V_x \left(\text{when } x > a \text{ and } < (a+b)\right) \quad \ldots \quad = R_1 - w_1 a$$

$$V_x \left(\text{when } x > (a+b)\right) \quad \ldots \ldots \quad = R_2 - w_2(\ell - x)$$

$$M_{max}\left(\text{at } x = \frac{R_1}{w_1} \text{ when } R_1 < w_1 a\right) \ldots \quad = \frac{R_1^{\,2}}{2w_1}$$

$$M_{max}\left(\text{at } x = \ell - \frac{R_2}{w_2} \text{ when } R_2 < w_2 c\right) = \frac{R_2^{\,2}}{2w_2}$$

$$M_x \ (\text{when } x < a) \quad \ldots \ldots \ldots \quad = R_1 x - \frac{w_1 x^2}{2}$$

$$M_x \left(\text{when } x > a \text{ and } < (a+b)\right) \ldots \quad = R_1 x - \frac{w_1 a}{2}(2x - a)$$

$$M_x \left(\text{when } x > (a+b)\right) \quad \ldots \ldots \quad = R_2(\ell - x) - \frac{w_2(\ell - x)^2}{2}$$

Figure 5   Simple Beam – Load Increasing Uniformly to One End

Shear

Moment
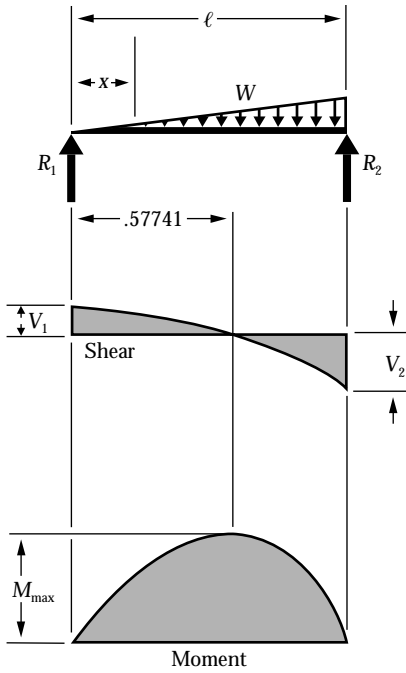
$$R_1 = V_1 \quad\ldots\ldots\ldots\ldots\ldots\quad = \frac{W}{3}$$

$$R_2 = V_2 \quad\ldots\ldots\ldots\ldots\ldots\quad = \frac{2W}{3}$$

$$V_x \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad = \frac{W}{3} - \frac{Wx^2}{\ell^2}$$

$$M_{max}\left(at\ x = \frac{\ell}{\sqrt{3}} = .5774\ell\right)\ldots\ldots\quad = \frac{2W\ell}{9\sqrt{3}} = .1283W\ell$$

$$M_x \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad = \frac{Wx}{3\ell^2}(\ell^2 - x^2)$$

$$\Delta_{max}\left(at\ x = \ell\sqrt{1 - \sqrt{\frac{8}{15}}} = .5193\ell\right)\ .\quad = .01304\frac{W\ell^3}{EI}$$

$$\Delta_x \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad = \frac{Wx}{180EI\ell^2}(3x^4 - 10\ell^2 x^2 + 7\ell^4)$$

Figure 6   Simple Beam – Load Increasing Uniformly to Center

Shear

Moment
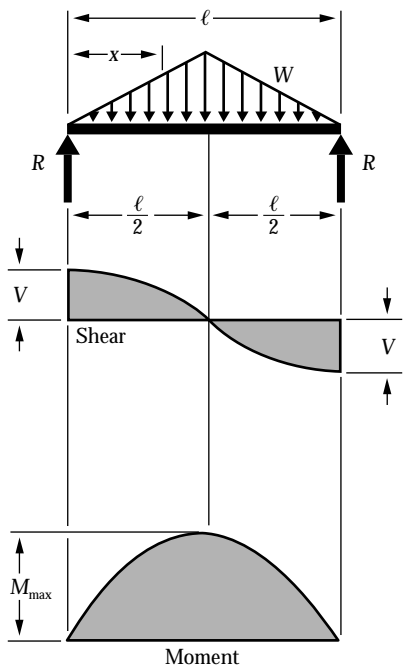
$$R = V \quad\ldots\ldots\ldots\ldots\ldots\quad = \frac{W}{2}$$

$$V_x\left(when\ x < \frac{\ell}{2}\right)\ldots\ldots\ldots\quad = \frac{W}{2\ell^2}(\ell^2 - 4x^2)$$

$$M_{max}\ (at\ center)\ \ldots\ldots\ldots\ldots\quad = \frac{W\ell}{6}$$

$$M_x\left(when\ x < \frac{\ell}{2}\right)\ \ldots\ldots\ldots\quad = Wx\left(\frac{1}{2} - \frac{2x^2}{3\ell^2}\right)$$

$$\Delta_{max}\ (at\ center)\ \ldots\ldots\ldots\ldots\quad = \frac{W\ell^3}{60EI}$$

$$\Delta_x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad = \frac{Wx}{480EI\ell^2}(5\ell^2 - 4x^2)^2$$

Figure 7 Simple Beam – Concentrated Load at Center
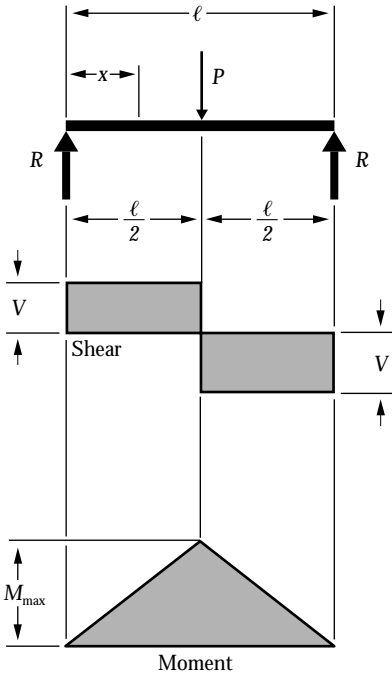
$$R = V \quad \ldots \ldots \ldots \ldots \ldots = \frac{P}{2}$$

$$M_{max} \text{ (at point of load)} \quad \ldots \ldots = \frac{P\ell}{4}$$

$$M_x \left(\text{when } x < \frac{\ell}{2} \right) \quad \ldots \ldots = \frac{Px}{2}$$

$$\Delta_{max} \text{ (at point of load)} \ldots \ldots = \frac{P\ell^3}{48EI}$$

$$\Delta_x \left(\text{when } x < \frac{\ell}{2} \right) \quad \ldots \ldots = \frac{Px}{48EI}(3\ell^2 - 4x^2)$$

Figure 8 Simple Beam – Concentrated Load at Any Point

$$R_1 = V_1 \text{ (max when } a < b) \quad \ldots \ldots = \frac{Pb}{\ell}$$

$$R_2 = V_2 \text{ (max when } a > b) \quad \ldots \ldots = \frac{Pa}{\ell}$$

$$M_{max} \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{Pab}{\ell}$$

$$M_x \text{ (when } x < a) \quad \ldots \ldots \ldots \ldots = \frac{Pbx}{\ell}$$

$$\Delta_{max} \left(\text{at } x = \sqrt{\frac{a(a + 2b)}{3}} \text{ when } a > b \right) \quad = \frac{Pab(a + 2b)\sqrt{3a(a + 2b)}}{27EI\ell}$$

$$\Delta_a \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{Pa^2b^2}{3EI\ell}$$

$$\Delta_x \text{ (when } x < a) \quad \ldots \ldots \ldots \ldots = \frac{Pbx}{6EI\ell}(\ell^2 - b^2 - x^2)$$

$$\Delta_x \text{ (when } x > a) \quad \ldots \ldots \ldots \ldots = \frac{Pa(\ell - x)}{6EI\ell}(2\ell x - x^2 - a^2)$$

Figure 9   Simple Beam – Two Equal Concentrated Loads Symmetrically Placed

$R = V$ . . . . $= P$
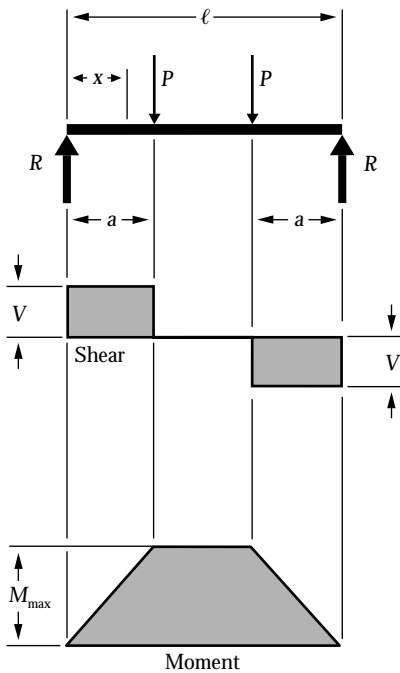
M_{max} (between loads) . . . . $= Pa$

M_x (when $x < a$) . . . . $= Px$

Δ_{max} (at center) . . . . $= \dfrac{Pa}{24EI}(3\ell^2 - 4a^2)$

Δ_x (when $x < a$) . . . . $= \dfrac{Px}{6EI}(3\ell a - 3a^2 - x^2)$

$\Delta_x \left(\text{when } x > a \text{ and} < (\ell - a)\right)$ . . . . $= \dfrac{Pa}{6EI}(3\ell x - 3x^2 - a^2)$

Figure 10   Simple Beam – Two Equal Concentrated Loads Unsymmetrically Placed
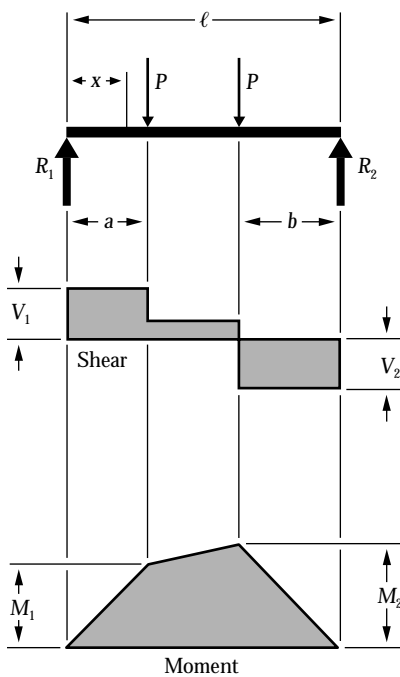
$R_1 = V_1$ (max when $a < b$) . . . . . . $= \dfrac{P}{\ell}(\ell - a + b)$

$R_2 = V_2$ (max when $a > b$) . . . . . . $= \dfrac{P}{\ell}(\ell - b + a)$

$V_x \left(\text{when } x > a \text{ and} < (\ell - b)\right)$ . . . $= \dfrac{P}{\ell}(b - a)$

M_1 (max when $a > b$) . . . . $= R_1a$

M_2 (max when $a < b$) . . . . $= R_2b$

M_x (when $x < a$) . . . . $= R_1x$

$M_x \left(\text{when } x > a \text{ and} < (\ell - b)\right)$ . . . $= R_1x - P(x - a)$

Figure 11   Simple Beam – Two Unequal Concentrated Loads Unsymmetrically Placed
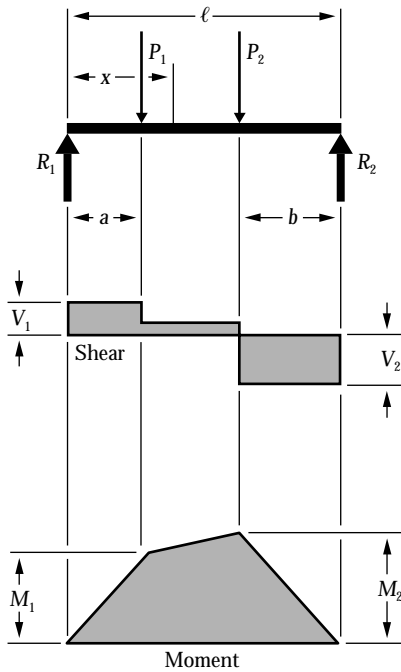
$$R_1 = V_1 \quad \ldots \ldots \ldots \ldots \ldots \quad = \frac{P_1(\ell - a) + P_2 b}{\ell}$$

$$R_2 = V_2 \quad \ldots \ldots \ldots \ldots \ldots \quad = \frac{P_1 a + P_2(\ell - b)}{\ell}$$

$$V_x \left(\text{when } x > a \text{ and } < (\ell - b) \right) \ldots \quad = R_1 - P_1$$

$$M_1 \ (\text{max when } R_1 < P_1) \ . \ . \ . \ . \ . \ . \quad = R_1 a$$

$$M_2 \ (\text{max when } R_2 < P_2) \ \ . \ . \ . \ . \ . \ . \quad = R_2 b$$

$$M_x \ (\text{when } x < a) \ \ . \ . \ . \ . \ . \ . \ . \ . \quad = R_1 x$$

$$M_x \left(\text{when } x > a \text{ and } < (\ell - b) \right) \ldots \quad = R_1 x - P_1(x - a)$$

Figure 12   Cantilever Beam – Uniformly Distributed Load

$$R = V \ \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad = w\ell$$
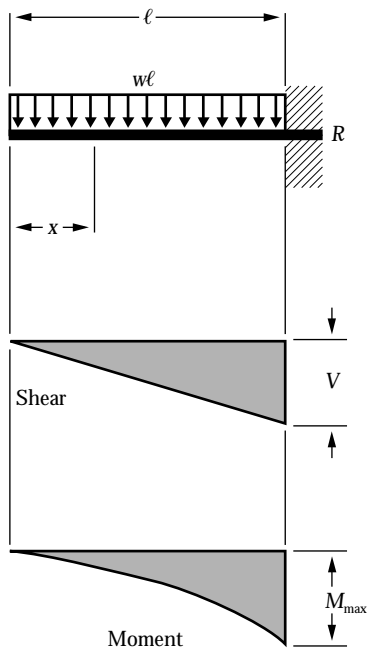
$$V_x \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad = wx$$

$$M_{max} \ (\text{at fixed end}) \ . \ . \ . \ . \ . \ . \ . \ . \quad = \frac{w\ell^2}{2}$$

$$M_x \ \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad = \frac{wx^2}{2}$$

$$\Delta_{max} \ (\text{at free end}) \ \ . \ . \ . \ . \ . \ . \ . \ . \quad = \frac{w\ell^4}{8EI}$$

$$\Delta_x \ \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad = \frac{w}{24EI}(x^4 - 4\ell^3 x + 3\ell^4)$$

Figure 13   Cantilever Beam – Concentrated Load at Free End

$R = V \quad \ldots \ldots \ldots \ldots \ldots = P$
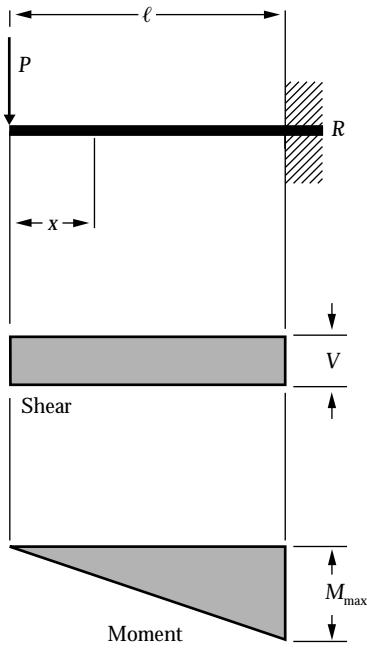
M_{max} (at fixed end) $\ldots \ldots \ldots \ldots = P\ell$

$M_x \quad \ldots \ldots \ldots \ldots \ldots \ldots = Px$

Δ_{max} (at free end) $\quad \ldots \ldots \ldots \ldots = \dfrac{P\ell^3}{3EI}$

$\Delta_x \ldots \ldots \ldots \ldots \ldots \ldots = \dfrac{P}{6EI}(2\ell^3 - 3\ell^2 x + x^3)$

Shear

Moment

Figure 14   Cantilever Beam – Concentrated Load at Any Point

$R = V \quad \ldots \ldots \ldots \ldots \ldots = P$
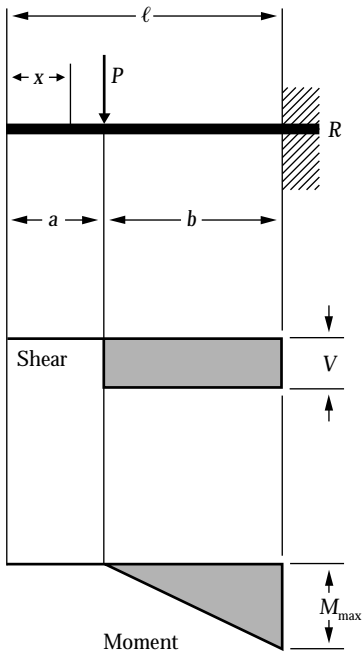
M_{max} (at fixed end) $\ldots \ldots \ldots \ldots = Pb$

M_x (when $x > a$) $\quad \ldots \ldots \ldots \ldots = P(x - a)$

Δ_{max} (at free end) $\quad \ldots \ldots \ldots \ldots = \dfrac{Pb^2}{6EI}(3\ell - b)$

Δ_a (at point of load) $\quad \ldots \ldots \ldots = \dfrac{Pb^3}{3EI}$

Δ_x (when $x < a$) $\ldots \ldots \ldots \ldots = \dfrac{Pb^2}{6EI}(3\ell - 3x - b)$

Δ_x (when $x > a$) $\ldots \ldots \ldots \ldots = \dfrac{P(\ell - x)^2}{6EI}(3b - \ell + x)$

Shear

Moment

Figure 15   Beam Fixed at One End, Supported at Other — Uniformly Distributed Load
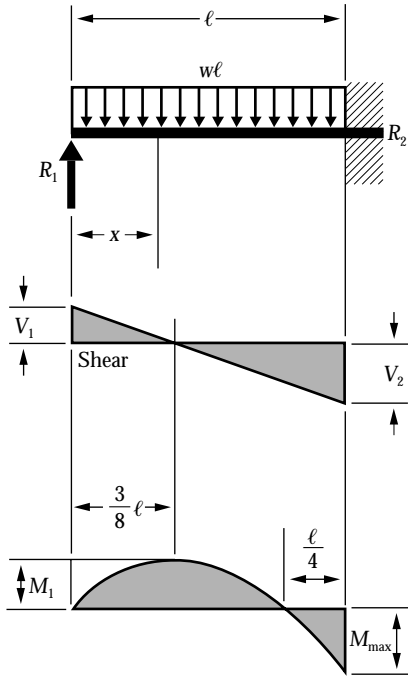
$$R_1 = V_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{3w\ell}{8}$$

$$R_2 = V_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{5w\ell}{8}$$

$$V_x \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = R_1 - wx$$

$$M_{max} \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{w\ell^2}{8}$$

$$M_1 \left(at\ x = \frac{3}{8}\,\ell \right) \ldots \ldots \ldots = \frac{9}{128}\,w\ell^2$$

$$M_x \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = R_1 x - \frac{wx^2}{2}$$

$$\Delta_{max} \left(at\ x = \frac{\ell}{16}(1 + \sqrt{33}) = .4215\,\ell \right) \quad = \frac{w\ell^4}{185EI}$$

$$\Delta_x \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{wx}{48EI}(\ell^3 - 3\ell x^2 + 2x^3)$$

Figure 16   Beam Fixed at One End, Supported at Other — Concentrated Load at Center
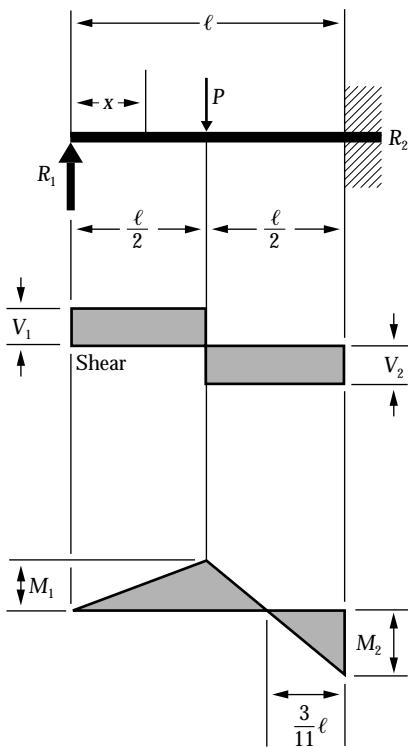
$$R_1 = V_1 \quad \ldots \ldots \ldots \ldots \ldots = \frac{5P}{16}$$

$$R_2 = V_2 \quad \ldots \ldots \ldots \ldots \ldots = \frac{11P}{16}$$

$$M_{max}\ (at\ fixed\ end) \quad \ldots \ldots \ldots = \frac{3P\ell}{16}$$

$$M_1\ (at\ point\ of\ load) \ldots \ldots \ldots = \frac{5P\ell}{32}$$

$$M_x \left(when\ x < \frac{\ell}{2} \right) \quad \ldots \ldots \ldots = \frac{5Px}{16}$$

$$M_x \left(when\ x > \frac{\ell}{2} \right) \quad \ldots \ldots \ldots = P\left(\frac{\ell}{2} - \frac{11x}{16} \right)$$

$$\Delta_{max} \left(at\ x = \ell\sqrt{\frac{1}{5}} = .4472\ell \right) \quad \ldots \ldots = \frac{P\ell^3}{48EI\sqrt{5}} = .009317\frac{P\ell^3}{EI}$$

$$\Delta_x\ (at\ point\ of\ load) \ldots \ldots \ldots = \frac{7P\ell^3}{768EI}$$

$$\Delta_x \left(when\ x < \frac{\ell}{2} \right) \quad \ldots \ldots \ldots = \frac{Px}{96EI}(3\ell^2 - 5x^2)$$

$$\Delta_x \left(when\ x > \frac{\ell}{2} \right) \quad \ldots \ldots \ldots = \frac{P}{96EI}(x - \ell)^2(11x - 2\ell)$$

Figure 17　Beam Fixed at One End, Supported at Other – Concentrated Load at Any Point
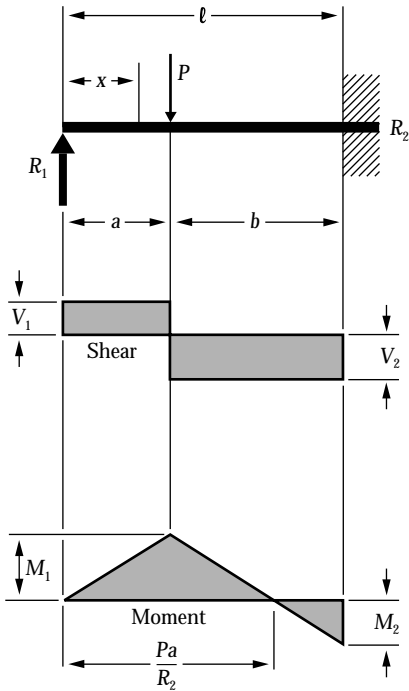
$$R_1 = V_1 \ldots \ldots \ldots \ldots \ldots \ldots = \frac{Pb^2}{2\ell^3}(a + 2\ell)$$

$$R_2 = V_2 \ldots \ldots \ldots \ldots \ldots \ldots = \frac{Pa}{2\ell^3}(3\ell^2 - a^2)$$

$$M_1 \text{ (at point of load)} \ldots \ldots \ldots \ldots = R_1 a$$

$$M_2 \text{ (at fixed end)} \ldots \ldots \ldots \ldots = \frac{Pab}{2\ell^2}(a + \ell)$$

$$M_x \text{ (when } x < a) \ldots \ldots \ldots \ldots = R_1 x$$

$$M_x \text{ (when } x > a) \ldots \ldots \ldots \ldots = R_1 x - P(x - a)$$

$$\Delta_{max}\left(\text{when } a < .414\ell \text{ at } x = \ell\frac{\ell^2 + a^2}{3\ell^2 - a^2}\right) = \frac{Pa}{3EI}\frac{(\ell^2 - a^2)^3}{(3\ell^2 - a^2)^2}$$

$$\Delta_{max}\left(\text{when } a > .414\ell \text{ at } x = \ell\sqrt{\frac{a}{2\ell + a}}\right) = \frac{Pab^2}{6EI}\sqrt{\frac{a}{2\ell + a}}$$

$$\Delta_a \text{ (at point of load)} \ldots \ldots \ldots = \frac{Pa^2 b^3}{12EI\ell^3}(3\ell + a)$$

$$\Delta_x \text{ (when } x < a) \ldots \ldots \ldots = \frac{Pb^2 x}{12EI\ell^3}(3a\ell^2 - 2\ell x^2 - ax^2)$$

$$\Delta_x \text{ (when } x > a) \ldots \ldots \ldots = \frac{Pa}{12EI\ell^3}(\ell - x)^2(3\ell^2 x - a^2 x - 2a^2\ell)$$

Figure 18　Beam Overhanging One Support – Uniformly Distributed Load
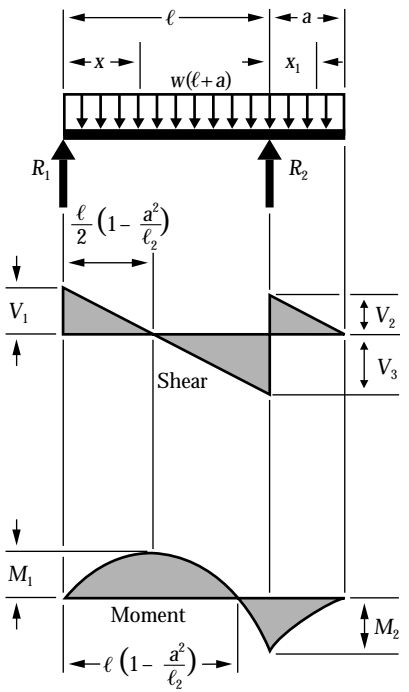
$$R_1 = V_1 \ldots \ldots \ldots \ldots \ldots = \frac{w}{2\ell}(\ell^2 - a^2)$$

$$R_2 = V_2 + V_3 \ldots \ldots \ldots \ldots = \frac{w}{2\ell}(\ell + a)^2$$

$$V_2 \ldots \ldots \ldots \ldots \ldots \ldots = wa$$

$$V_3 \ldots \ldots \ldots \ldots \ldots \ldots = \frac{w}{2\ell}(\ell^2 + a^2)$$

$$V_x \text{ (between supports)} \ldots \ldots = R_1 - wx$$

$$V_{x_1} \text{ (for overhang)} \ldots \ldots \ldots = w(a - x_1)$$

$$M_1\left(\text{at } x = \frac{\ell}{2}\left[1 - \frac{a^2}{\ell^2}\right]\right) \ldots \ldots = \frac{w}{8\ell^2}(\ell + a)^2(\ell - a)^2$$

$$M_2 \text{ (at } R_2) \ldots \ldots \ldots \ldots = \frac{wa^2}{2}$$

$$M_x \text{ (between supports)} \ldots \ldots = \frac{wx}{2\ell}(\ell^2 - a^2 - x\ell)$$

$$M_{x_1} \text{ (for overhang)} \ldots \ldots \ldots = \frac{w}{2}(a - x_1)^2$$

$$\Delta_x \text{ (between supports)} \ldots \ldots = \frac{wx}{24EI\ell}(\ell^4 - 2\ell^2 x^2 + \ell x^3 - 2a^2\ell^2 + 2a^2 x^2)$$

$$\Delta_{x_1} \text{ (for overhang)} \ldots \ldots \ldots = \frac{wx_1}{24EI}(4a^2\ell - \ell^3 + 6a^2 x_1 - 4ax_1^2 + x_1^3)$$

Figure 19   Beam Overhanging One Support – Uniformly Distributed Load on Overhang

$$R_1 = V_1 \quad\ldots\ldots\ldots\ldots \quad = \frac{wa^2}{2\ell}$$
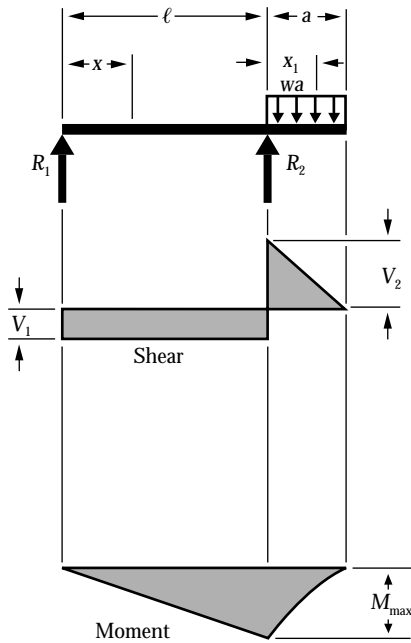
$$R_2 = V_1 + V_2 \quad\ldots\ldots\ldots \quad = \frac{wa}{2\ell}(2\ell + a)$$

$$V_2 \quad\ldots\ldots\ldots\ldots\ldots \quad = wa$$

$$V_{x_1} \text{ (for overhang)} \quad\ldots\ldots \quad = w(a - x_1)$$

$$M_{max} \text{ (at } R_2) \quad\ldots\ldots\ldots \quad = \frac{wa^2}{2}$$

$$M_x \text{ (between supports)} \quad\ldots\ldots \quad = \frac{wa^2 x}{2\ell}$$

$$M_{x_1} \text{ (for overhang)} \quad\ldots\ldots\ldots \quad = \frac{w}{2}(a - x_1)^2$$

$$\Delta_{max}\left(\text{between supports at } x = \frac{\ell}{\sqrt{3}}\right) = \frac{wa^2\ell^2}{18\sqrt{3}EI} = .03208\frac{wa^2\ell^2}{EI}$$

$$\Delta_{max} \text{ (for overhang at } x_1 = a) \quad\ldots \quad = \frac{wa^3}{24EI}(4\ell + 3a)$$

$$\Delta_x \text{ (between supports)} \quad\ldots\ldots \quad = \frac{wa^2 x}{12EI\ell}(\ell^2 - x^2)$$

$$\Delta_{x_1} \text{ (for overhang)} \quad\ldots\ldots\ldots \quad = \frac{wx_1}{24EI}(4a^2\ell + 6a^2 x_1 - 4ax_1^2 + x_1^3)$$

Figure 20   Beam Overhanging One Support – Concentrated Load at End of Overhang
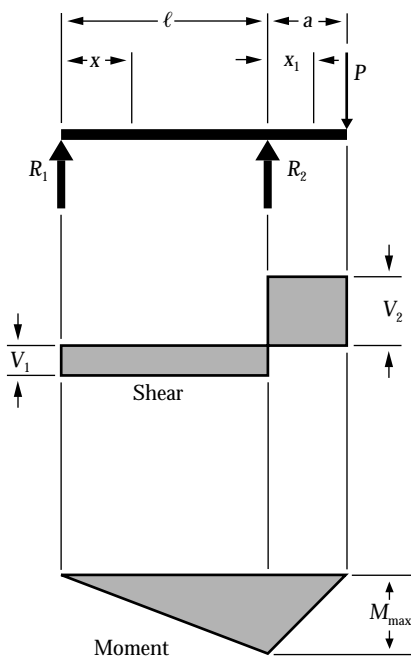
$$R_1 = V_1 \quad\ldots\ldots\ldots\ldots \quad = \frac{Pa}{\ell}$$

$$R_2 = V_1 + V_2 \quad\ldots\ldots\ldots \quad = \frac{P}{\ell}(\ell + a)$$

$$V_2 \quad\ldots\ldots\ldots\ldots\ldots \quad = P$$

$$M_{max} \text{ (at } R_2) \quad\ldots\ldots\ldots \quad = Pa$$

$$M_x \text{ (between supports)} \quad\ldots\ldots \quad = \frac{Pax}{\ell}$$

$$M_{x_1} \text{ (for overhang)} \quad\ldots\ldots\ldots \quad = P(a - x_1)$$

$$\Delta_{max}\left(\text{between supports at } x = \frac{\ell}{\sqrt{3}}\right) = \frac{Pa\ell^2}{9\sqrt{3}EI} = .06415\frac{Pa\ell^2}{EI}$$

$$\Delta_{max} \text{ (for overhang at } x_1 = a) \quad\ldots \quad = \frac{Pa^2}{3EI}(\ell + a)$$

$$\Delta_x \text{ (between supports)} \quad\ldots\ldots \quad = \frac{Pax}{6EI\ell}(\ell^2 - x^2)$$

$$\Delta_{x_1} \text{ (for overhang)} \quad\ldots\ldots\ldots \quad = \frac{Px_1}{6EI}(2a\ell + 3ax_1 - x_1^2)$$

Figure 21   Beam Overhanging One Support – Concentrated Load at Any Point Between Supports

$$R_1 = V_1 \text{ (max when } a < b) \quad \ldots \ldots \ldots = \frac{Pb}{\ell}$$

$$R_2 = V_2 \text{ (max when } a > b) \ldots \ldots \ldots = \frac{Pa}{\ell}$$

$$M_{max} \text{ (at point of load) } \ldots \ldots \ldots \ldots = \frac{Pab}{\ell}$$

$$M_x \text{ (when } x < a) \quad \ldots \ldots \ldots \ldots \ldots = \frac{Pbx}{\ell}$$

$$\Delta_{max} \left(\text{at } x = \sqrt{\frac{a(a + 2b)}{3}} \text{ when } a > b \right) . \quad = \frac{Pab(a + 2b)\sqrt{3a(a + 2b)}}{27EI\ell}$$
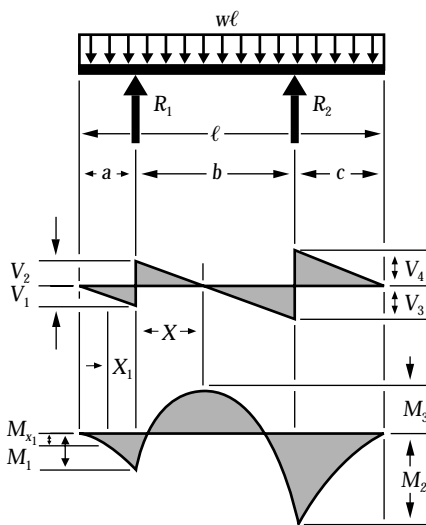
$$\Delta_a \text{ (at point of load) } \ldots \ldots \ldots \ldots = \frac{Pa^2b^2}{3EI\ell}$$

$$\Delta_x \text{ (when } x < a) \quad \ldots \ldots \ldots \ldots = \frac{Pbx}{6EI\ell}(\ell^2 - b^2 - x^2)$$

$$\Delta_x \text{ (when } x > a) \quad \ldots \ldots \ldots \ldots = \frac{Pa(\ell - x)}{6EI\ell}(2\ell x - x^2 - a^2)$$

$$\Delta_{x_1} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{Pabx_1}{6EI\ell}(\ell + a)$$

Figure 22   Beam Overhanging Both Supports – Unequal Overhangs – Uniformly Distributed Load

$$R_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{w\ell \, (\ell - 2c)}{2b}$$

$$R_2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{w\ell \, (\ell - 2a)}{2b}$$

$$V_1 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = wa$$

$$V_2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = R_1 - V_1$$

$$V_3 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = R_2 - V_4$$

$$V_4 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = wc$$

$$V_{x_1} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = V_1 - wx_1$$

$$V_x \text{ (when } x < \ell) . \ldots \ldots \ldots \ldots \ldots = R_1 - w(a + x_1)$$

$$V_m \text{ (when } a < c) \quad \ldots \ldots \ldots \ldots \ldots = R_2 - wc$$

$$M_1 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = -\frac{wa^2}{2}$$

$$M_2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = -\frac{wc^2}{2}$$

$$M_3 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = R_1\left(\frac{R_1}{2w} - a\right)$$

$$M_x \left(\text{max when } x = \frac{R_1}{w} - a \right) \ldots \ldots = R_1 x - \frac{w(a + x)^2}{2}$$

Figure 23   Beam Fixed at Both Ends – Uniformly Distributed Load

$$R = V \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{w\ell}{2}$$

$$V_x \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = w\left(\frac{\ell}{2} - x\right)$$

$$M_{max} \text{ (at ends)} \quad \ldots \ldots \ldots \ldots = \frac{w\ell^2}{12}$$
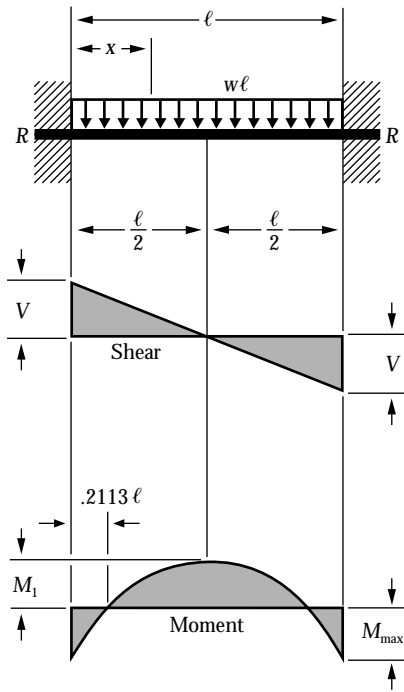
$$M_1 \text{ (at center)} \quad \ldots \ldots \ldots \ldots = \frac{w\ell^2}{24}$$

$$M_x \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{w}{12}(6\ell x - \ell^2 - 6x^2)$$

$$\Delta_{max} \text{ (at center)} \ldots \ldots \ldots \ldots = \frac{w\ell^4}{384EI}$$

$$\Delta_x \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{wx^2}{24EI}(\ell - x)^2$$

Figure 24   Beam Fixed at Both Ends – Concentrated Load at Center
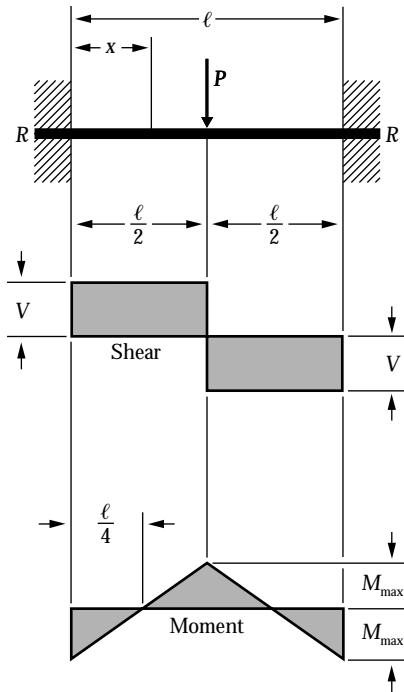
$$R = V \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{P}{2}$$
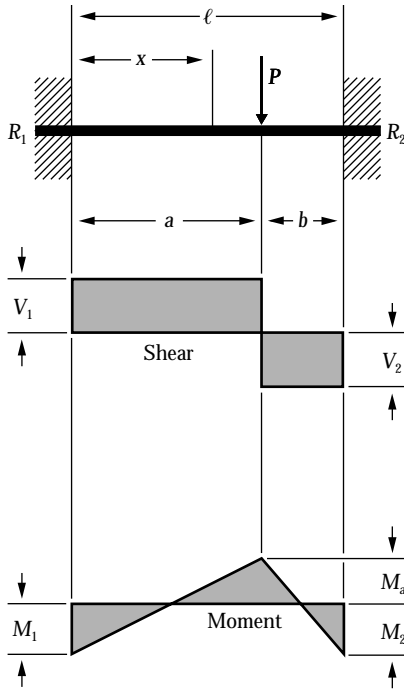
$$M_{max} \text{ (at center and ends)} \quad \ldots \ldots = \frac{P\ell}{8}$$

$$M_x\left(\text{when } x < \frac{\ell}{2}\right) \ldots \ldots \ldots \ldots = \frac{P}{8}(4x - \ell)$$

$$\Delta_{max} \text{ (at center)} \ldots \ldots \ldots \ldots = \frac{P\ell^3}{192EI}$$

$$\Delta_x\left(\text{when } x < \frac{\ell}{2}\right) \ldots \ldots \ldots \ldots = \frac{Px^2}{48EI}(3\ell - 4x)$$

Figure 25　Beam Fixed at Both Ends – Concentrated Load at Any Point

$$R_1 = V_1 \text{ (max when } a < b) \quad \ldots \ldots = \frac{Pb^2}{\ell^3}(3a + b)$$

$$R_2 = V_2 \text{ (max when } a > b) \quad \ldots \ldots = \frac{Pa^2}{\ell^3}(a + 3b)$$

$$M_1 \text{ (max when } a < b) \quad \ldots \ldots \ldots = \frac{Pab^2}{\ell^2}$$

$$M_2 \text{ (max when } a > b) \quad \ldots \ldots \ldots = \frac{Pa^2b}{\ell^2}$$

$$M_a \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{2Pa^2b^2}{\ell^3}$$
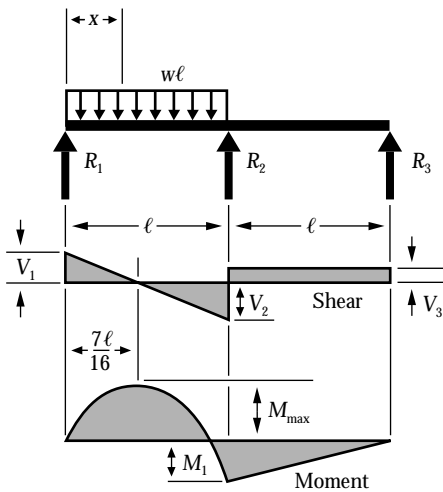
$$M_x \text{ (when } x < a) \quad \ldots \ldots \ldots = R_1x - \frac{Pab^2}{\ell^2}$$

$$\Delta_{max}\left(\text{when } a > b \text{ at } x = \frac{2a\ell}{3a + b}\right) \ldots = \frac{2Pa^3b^2}{3EI(3a + b)^2}$$

$$\Delta_a \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{Pa^3b^3}{3EI\ell^3}$$

$$\Delta_x \text{ (when } x < a) \quad \ldots \ldots \ldots = \frac{Pb^2x^2}{6EI\ell^3}(3a\ell - 3ax - bx)$$

Figure 26　Continuous Beam – Two Equal Spans – Uniform Load on One Span

$$R_1 = V_1 \quad \ldots \ldots \ldots \ldots \ldots = \frac{7}{16}w\ell$$

$$R_2 = V_2 + V_3 \quad \ldots \ldots \ldots \ldots = \frac{5}{8}w\ell$$

$$R_3 = V_3 \quad \ldots \ldots \ldots \ldots \ldots = -\frac{1}{16}w\ell$$
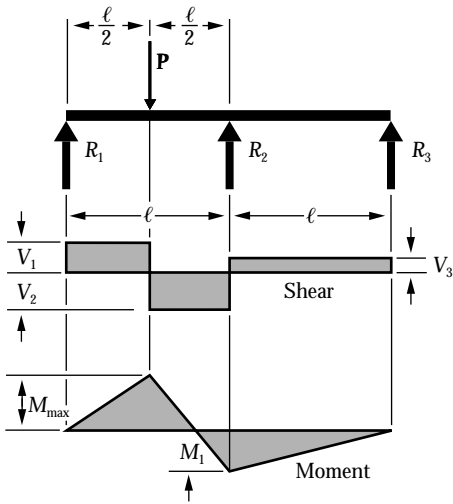
$$V_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{9}{16}w\ell$$

$$M_{max}\left(\text{at } x = \frac{7}{16}\ell\right) \quad \ldots \ldots \ldots = \frac{49}{512}w\ell^2$$

$$M_1 \text{ (at support } R_2) \quad \ldots \ldots \ldots = \frac{1}{16}w\ell^2$$

$$M_x \text{ (when } x < \ell) \quad \ldots \ldots \ldots = \frac{wx}{16}(7\ell - 8x)$$

Figure 27 Continuous Beam – Two Equal Spans – Concentrated Load at Center of One Span

$$R_1 = V_1 \quad \ldots \ldots \ldots \ldots \ldots \quad = \frac{13}{32}P$$

$$R_2 = V_2 + V_3 \ldots \ldots \ldots \ldots \quad = \frac{11}{16}P$$

$$R_3 = V_3 \quad \ldots \ldots \ldots \ldots \ldots \quad = -\frac{3}{32}P$$

$$V_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad = \frac{19}{32}P$$

$$M_{max} \text{ (at point of load)} \ldots \ldots \ldots \quad = \frac{13}{64}P\ell$$

$$M_1 \text{ (at support } R_2) \ldots \ldots \ldots \ldots \quad = \frac{3}{32}P\ell$$

Figure 28 Continuous Beam – Two Equal Spans – Concentrated Load at Any Point
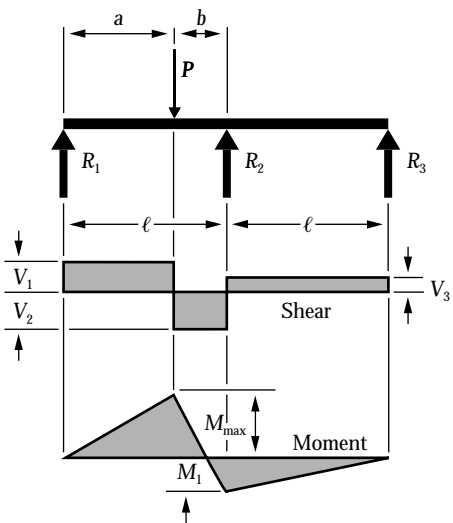
$$R_1 = V_1 \quad \ldots \ldots \ldots \ldots \ldots \quad = \frac{Pb}{4\ell^3}\left(4\ell^2 - a(\ell + a)\right)$$

$$R_2 = V_2 + V_3 \ldots \ldots \ldots \ldots \quad = \frac{Pa}{2\ell^3}\left(2\ell^2 + b(\ell + a)\right)$$

$$R_3 = V_3 \quad \ldots \ldots \ldots \ldots \ldots \quad = -\frac{Pab}{4\ell^3}(\ell + a)$$

$$V_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad = \frac{Pa}{4\ell^3}\left(4\ell^2 + b(\ell + a)\right)$$

$$M_{max} \text{ (at point of load)} \ldots \ldots \ldots \quad = \frac{Pab}{4\ell^3}\left(4\ell^2 - a(\ell + a)\right)$$

$$M_1 \text{ (at support } R_2) \ldots \ldots \ldots \ldots \quad = \frac{Pab}{4\ell^2}(\ell + a)$$

Figure 29   Continuous Beam – Two Equal Spans – Uniformly Distributed Load
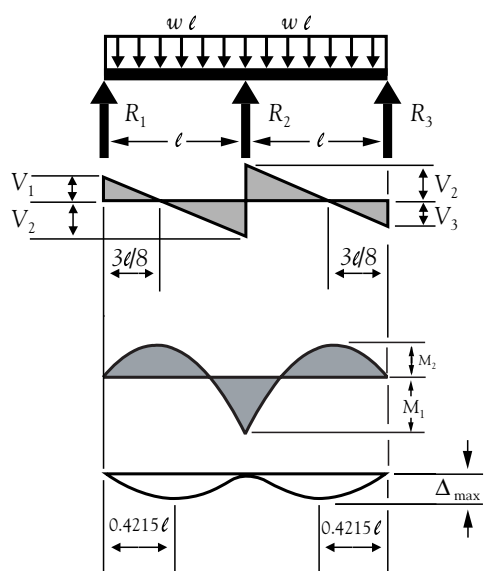
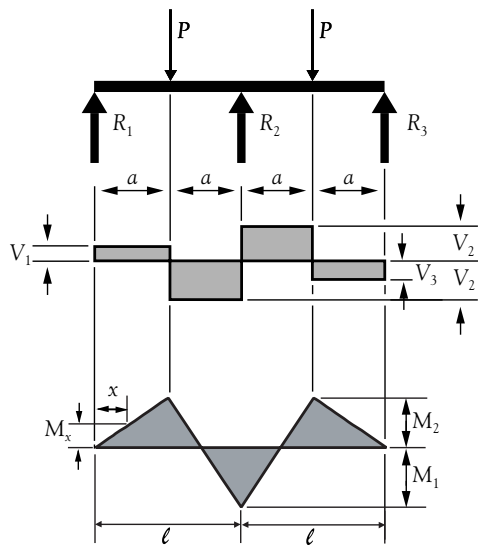$$R_1 = V_1 = R_3 = V_3 \quad \ldots \ldots \ldots \ldots \ldots = \frac{3w\ell}{8}$$

$$R_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{10w\ell}{8}$$

$$V_2 = V_{max} \quad \ldots \ldots \ldots \ldots \ldots = \frac{5w\ell}{8}$$

$$M_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{w\ell^2}{8}$$

$$M_2 \left(at \; \frac{3\ell}{8} \right) \quad \ldots \ldots \ldots \ldots = \frac{9w\ell^2}{128}$$

$$\Delta_{max} \; (at \; 0.4215\,\ell, \; approx. \; from \; R_1 \; and \; R_3) \ldots = \frac{w\ell^4}{185EI}$$

Figure 30   Continuous Beam – Two Equal Spans – Two Equal Concentrated Loads Symmetrically Placed

$$R_1 = V_1 = R_3 = V_3 \quad \ldots \ldots \ldots = \frac{5P}{16}$$

$$R_2 = 2V_2 \quad \ldots \ldots \ldots \ldots \ldots = \frac{11P}{8}$$

$$V_2 = P - R_1 \quad \ldots \ldots \ldots \ldots = \frac{11P}{16}$$

$$V_{max} \quad \ldots \ldots \ldots \ldots \ldots \ldots = V_2$$

$$M_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots = -\frac{3P\ell}{16}$$

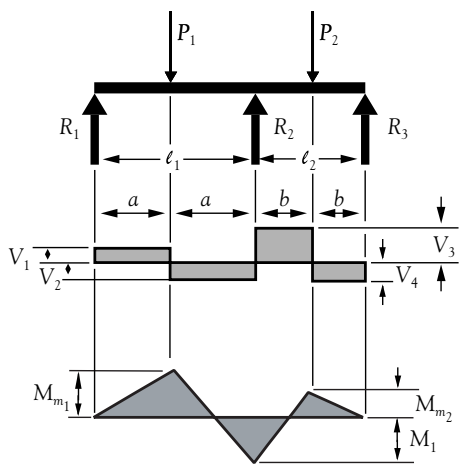$$M_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{5P\ell}{32}$$

$$M_x \; (when \; x < a) \quad \ldots \ldots \ldots = R_1 x$$

Figure 31   Continuous Beam — Two Unequal Spans — Uniformly Distributed Load
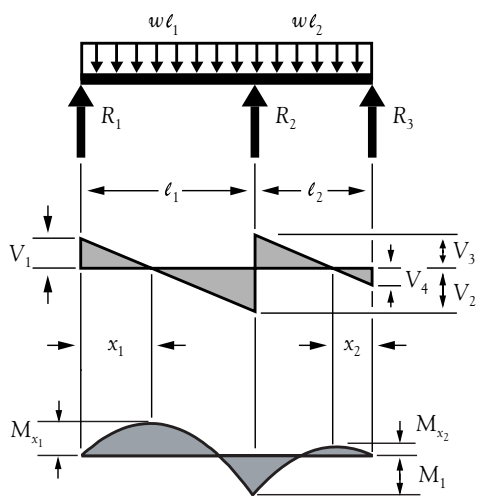
$$R_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{M_1}{\ell_1} + \frac{w\ell_1}{2}$$

$$R_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = w\ell_1 + w\ell_2 - R_1 - R_3$$

$$R_3 = V_4 \quad \ldots \ldots \ldots \ldots \ldots = \frac{M_1}{\ell_2} + \frac{w\ell_2}{2}$$

$$V_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = R_1$$

$$V_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = w\ell_1 - R_1$$

$$V_3 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = w\ell_2 - R_3$$

$$V_4 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = R_3$$

$$M_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = -\frac{w\ell_2{}^3 + w\ell_1{}^3}{8(\ell_1 + \ell_2)}$$

$$M_{x_1}\left(\text{when } x_1 = \frac{R_1}{w}\right) \quad \ldots \ldots \ldots = R_1 x_1 - \frac{w x_1{}^2}{2}$$

$$M_{x_2}\left(\text{when } x_2 = \frac{R_3}{w}\right) \quad \ldots \ldots \ldots = R_3 x_2 - \frac{w x_2{}^2}{2}$$

Figure 32   Continuous Beam — Two Unequal Spans — Concentrated Load on Each Span Symmetrically Placed

$$R_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{M_1}{\ell_1} + \frac{P_1}{2}$$

$$R_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots = P_1 + P_2 - R_1 - R_3$$

$$R_3 \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{M_1}{\ell_2} + \frac{P_2}{2}$$

$$V_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots = R_1$$

$$V_2 \quad \ldots \ldots \ldots \ldots \ldots \ldots = P_1 - R_1$$

$$V_3 \quad \ldots \ldots \ldots \ldots \ldots \ldots = P_2 - R_3$$

$$V_4 \quad \ldots \ldots \ldots \ldots \ldots \ldots = R_3$$

$$M_1 \quad \ldots \ldots \ldots \ldots \ldots \ldots = -\frac{3}{16}\left(\frac{P_1\ell_1{}^2 + P_2\ell_2{}^2}{\ell_1 + \ell_2}\right)$$

$$M_{m_1} \quad \ldots \ldots \ldots \ldots \ldots \ldots = R_1 a$$

$$M_{m_2} \quad \ldots \ldots \ldots \ldots \ldots \ldots = R_3 b$$