Integer Linear Programming Problems

1.1.1.

$\max z = x_1 + 4x_2$
st
$x_1 + x_2 \le 7$
$-x_1 + 3x_2 \le 3$
$x_1 x_2 \ge 0$ and integer

1st $\Rightarrow$

Relaxation
$\max z = x_1 + 4x_2$
st
$x_1 + x_2 \le 7$
$-x_1 + 3x_2 \le 3$
$x_1 x_2 \ge 0$

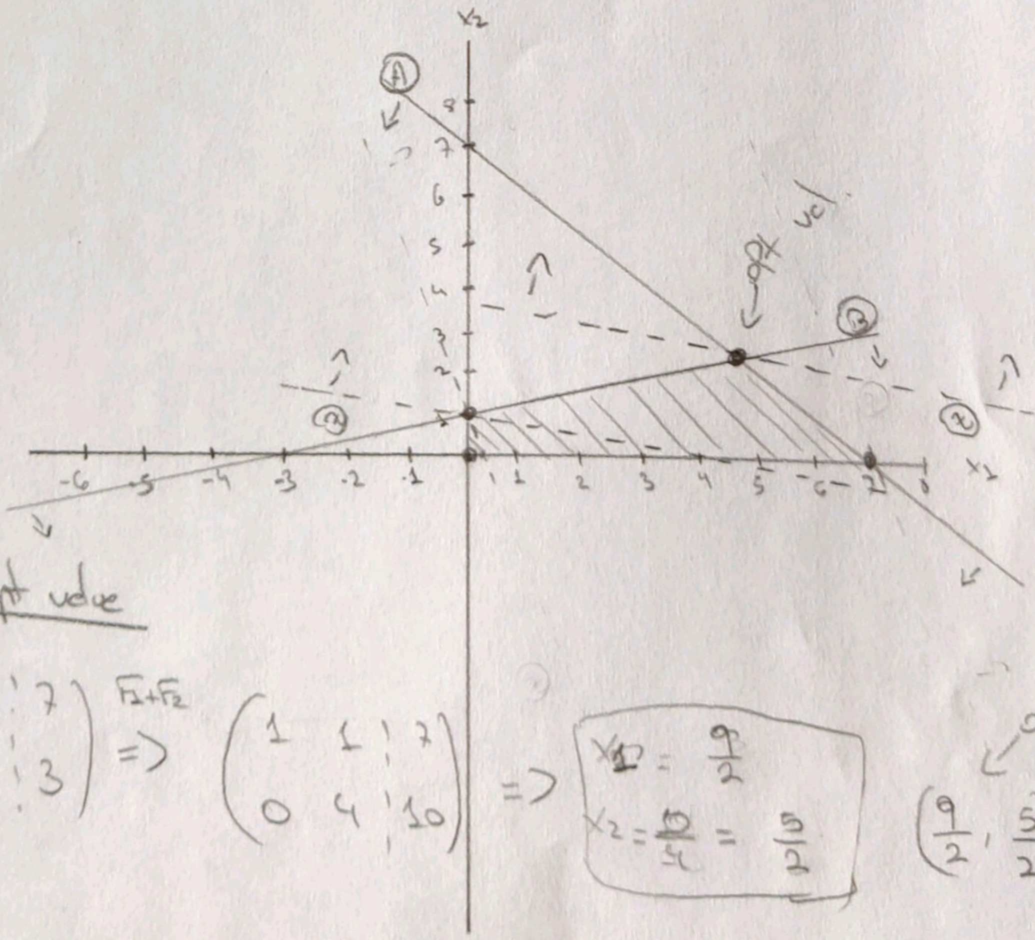
2nd

Solve relaxed problem

(A) $x_1 + x_2 \le 7$ $\begin{cases} x_1 = 0, x_2 = 7 \\ x_2 = 0, x_1 = 7 \end{cases}$

(B) $-x_1 + 3x_2 = 3$ $\begin{cases} x_1 = 0, x_2 = 1 \\ x_2 = 0, x_1 = -3 \end{cases}$

(z) $\cancel{x_1} + 4x_2 = 4$ $\begin{cases} x_1 = 0, x_2 = 1 \\ x_2 = 0, x_1 = 4 \end{cases}$



Calculating opt value

$$\left(\begin{array}{cc|c} 1 & 1 & 7 \\ -1 & 3 & 3 \end{array}\right) \xrightarrow{F_1+F_2} \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 4 & 10 \end{array}\right) \Rightarrow \boxed{\begin{array}{l} x_1 = \frac{9}{2} \\ x_2 = \frac{10}{4} = \frac{5}{2} \end{array}}$$

opt value

$\left(\frac{9}{2}, \frac{5}{2}\right)$

ILP2

② $\max z = x_1 + 4x_2$

st

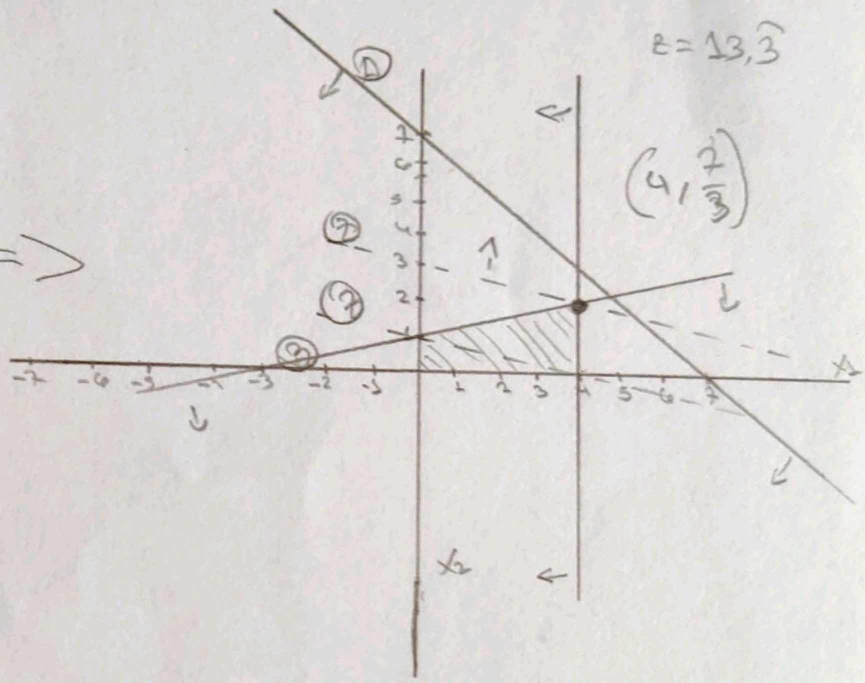
① $x_1 + x_2 \leq 7$

② $-x_1 + 3x_2 \leq 3$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

$\Rightarrow$



ILP3

② $\max z = x_1 + 4x_2$

st

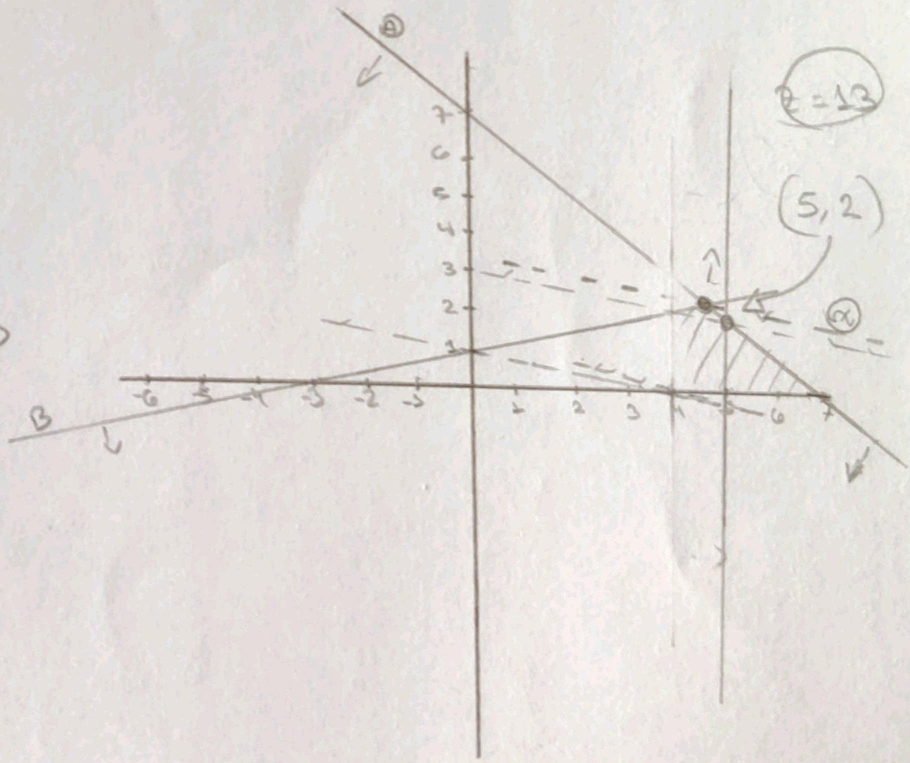
Ⓐ $x_1 + x_2 \leq 7$

Ⓑ $-x_1 + 3x_2 \leq 3$

$x_1 \geq 5$

$x_1, x_2 \geq 0$

$\Rightarrow$



LP4

$\max z = x_1 + 4x_2$

st

(A) $x_1 + x_2 \le 7$

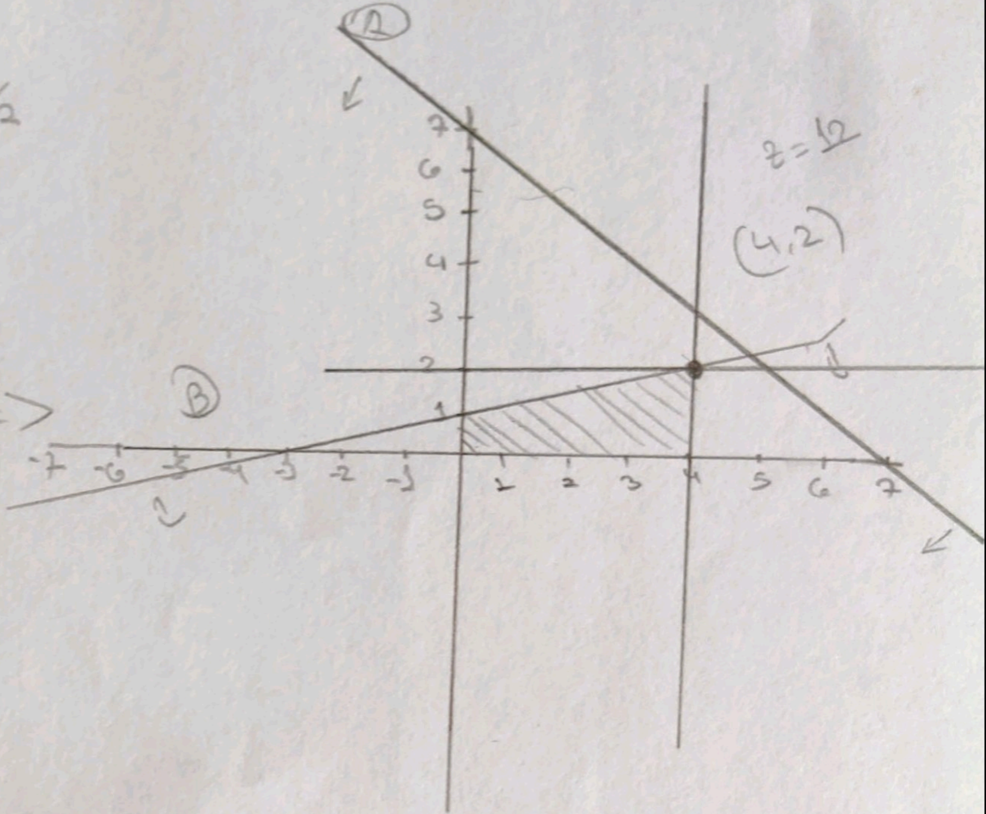
(B) $-x_1 + 3x_2 \le 3$

$x_1 \le 4$

$x_2 \le 2$

$x_1, x_2 \ge 0$

$\Rightarrow$



LP5

$\max = x_1 + 4x_2$

st

(A) $x_1 + x_2 \le 7$

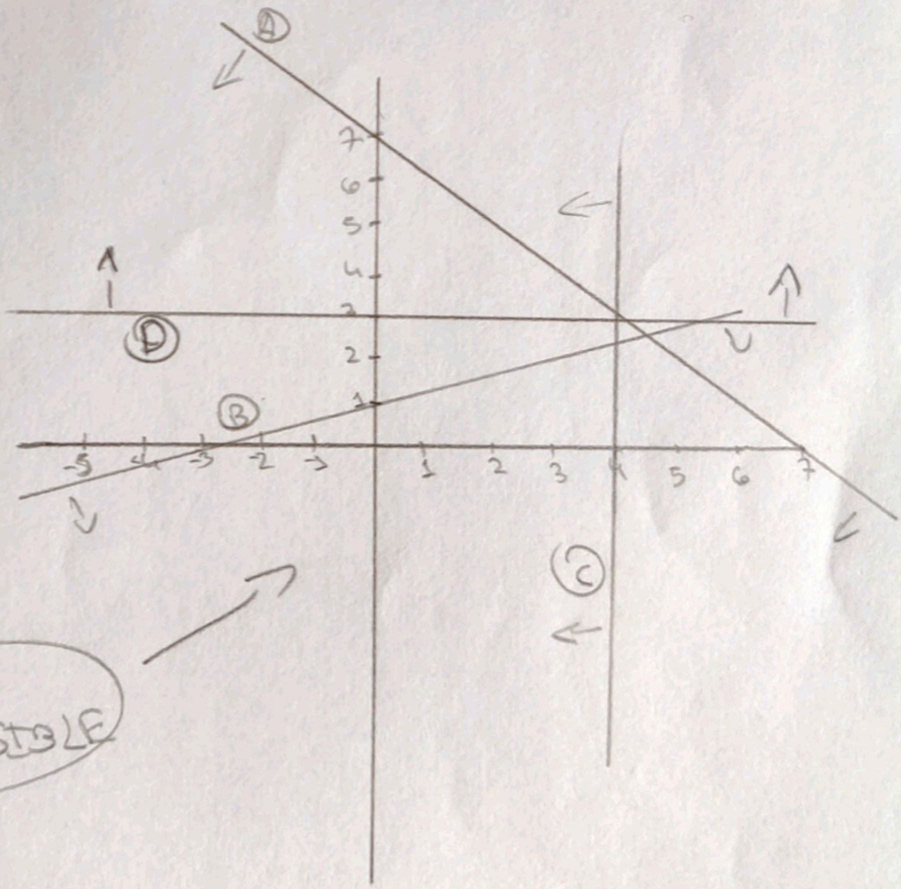
(B) $-x_1 + 3x_2 \le 3$

(C) $x_1 \le 4$

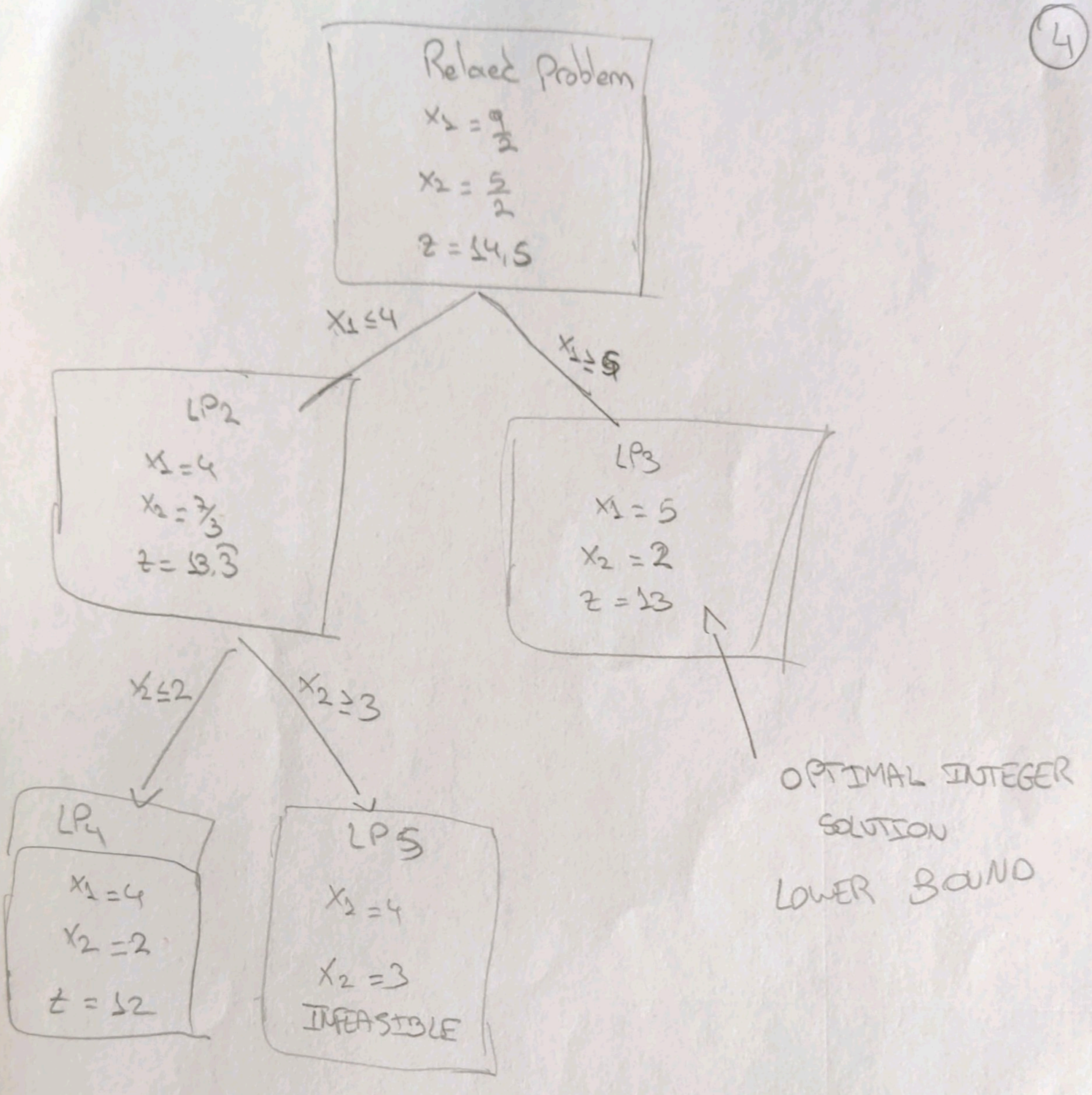
(D) $x_2 \ge 3$

$x_1, x_2 \ge 0$

$\Rightarrow$



INFEASIBLE


Relaxed Problem
$x_1 = \frac{9}{2}$
$x_2 = \frac{5}{2}$
$z = 14.5$
$x_1 \leq 4$
$x_1 \geq 5$
LP2
$x_1 = 4$
$x_2 = 7/3$
$z = 13.\overline{3}$
LP3
$x_1 = 5$
$x_2 = 2$
$z = 13$
$x_2 \leq 2$
$x_2 \geq 3$
LP4
$x_1 = 4$
$x_2 = 2$
$z = 12$
LP5
$x_1 = 4$
$x_2 = 3$
INFEASIBLE
OPTIMAL INTEGER SOLUTION
LOWER BOUND

1.1.2-

max $z = 6x_1 + 8x_2$

st

$2x_1 + 4x_2 \le 36$

$3x_1 - 4x_2 \le 40$

$x_1, x_2 \ge 0$ and integer

Relax problem $\Rightarrow$

max $z = 6x_1 + 8x_2$

st

$2x_1 + 4x_2 \le 36$

$3x_1 - 4x_2 \le 40$

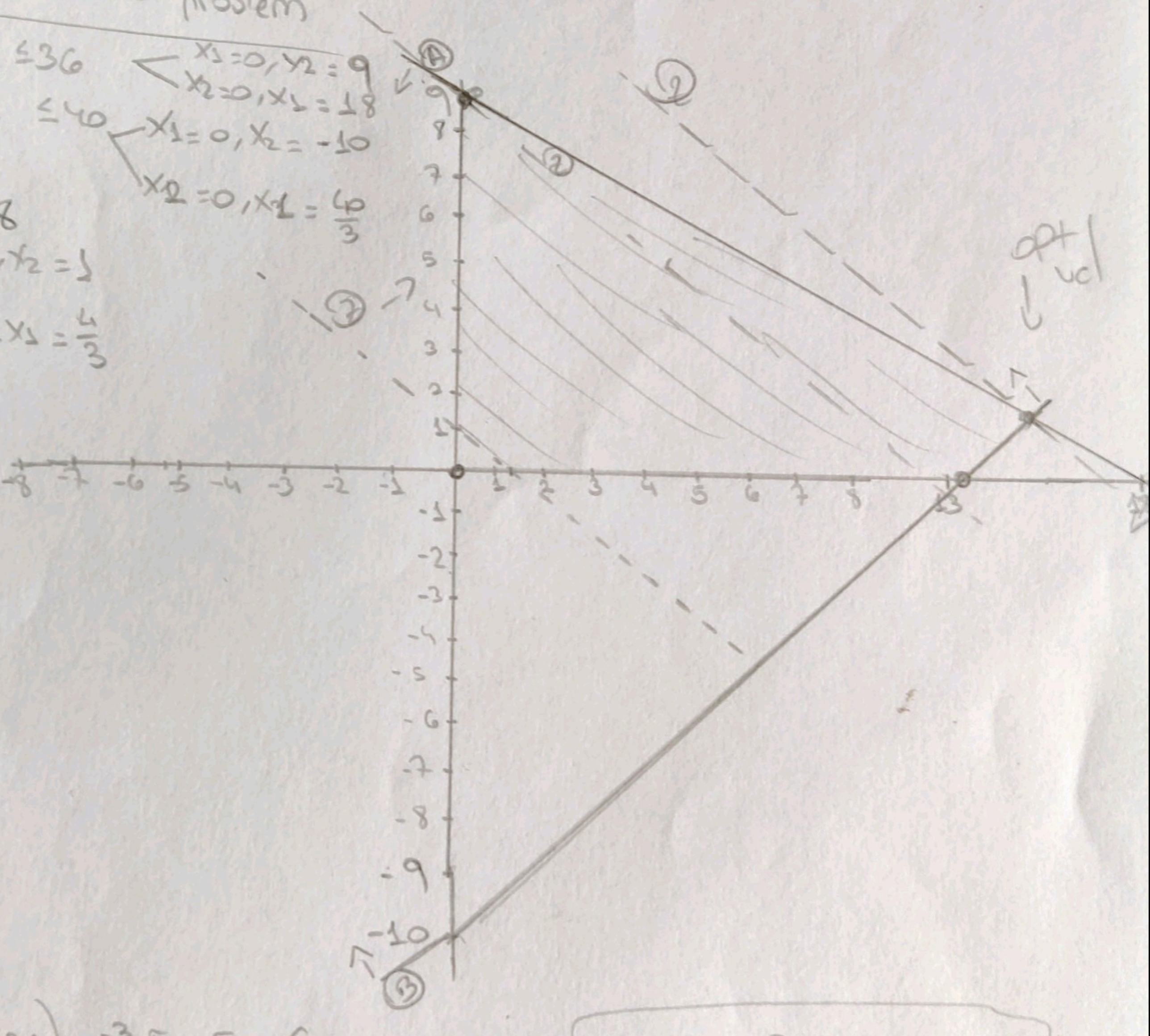
$x_1, x_2 \ge 0$

- Solve relaxed problem

(A) $2x_1 + 4x_2 \le 36$ — $x_1 = 0, x_2 = 9$; $x_2 = 0, x_1 = 18$

(B) $3x_1 - 4x_2 \le 40$ — $x_1 = 0, x_2 = -10$; $x_2 = 0, x_1 = \frac{40}{3}$

(Z) $6x_1 + 8x_2 = 8$ — $x_1 = 0, x_2 = 1$; $x_2 = 0, x_1 = \frac{4}{3}$



$$\begin{pmatrix} 2 & 4 & \vdots & 36 \\ 3 & -4 & \vdots & 40 \end{pmatrix} \xrightarrow{-\frac{3}{2}F_1 + F_2} \left(\begin{array}{cc|c} 2 & 4 & 36 \\ 0 & -10 & -14 \end{array}\right)$$

$$\boxed{\Rightarrow x_1 = \frac{76}{5}, \quad \Rightarrow x_2 = \frac{7}{5} \qquad z = 102{,}4}$$

LP2

(Z) $\max z = 6x_1 + 8x_2$

st

(A) $2x_1 + 4x_2 \leq 36$

(B) $3x_1 - 4x_2 \leq 40$

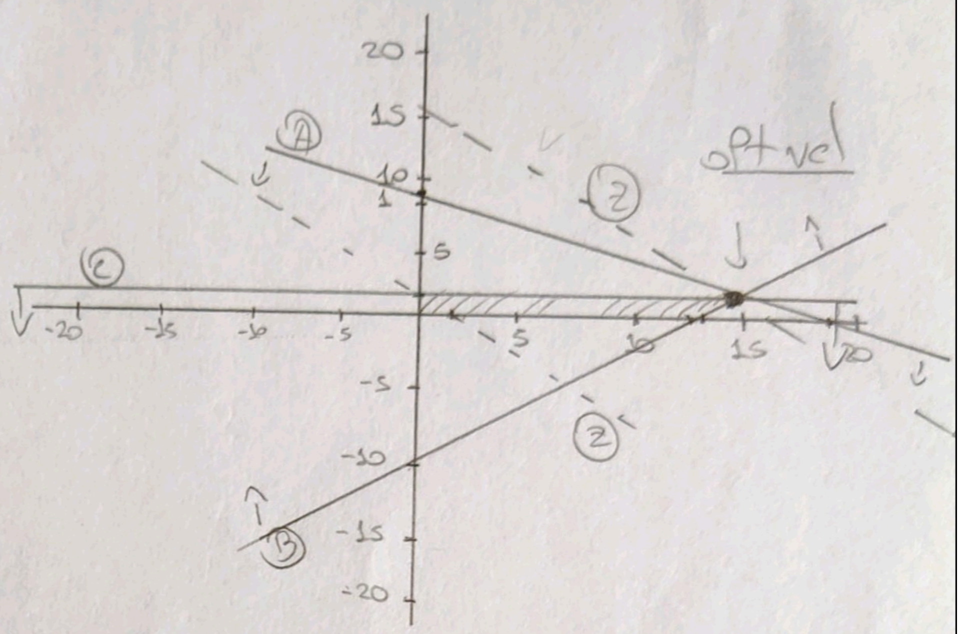
(C) $x_2 \leq 1$

$x_1, x_2 \geq 0$ and integer

$x_1 = \frac{44}{3}$

$x_2 = 1$

$z = 96$



LP3

$\max z = 6x_1 + 8x_2$

st

(A) $2x_1 + 4x_2 \leq 36$

(B) $3x_1 - 4x_2 \leq 40$

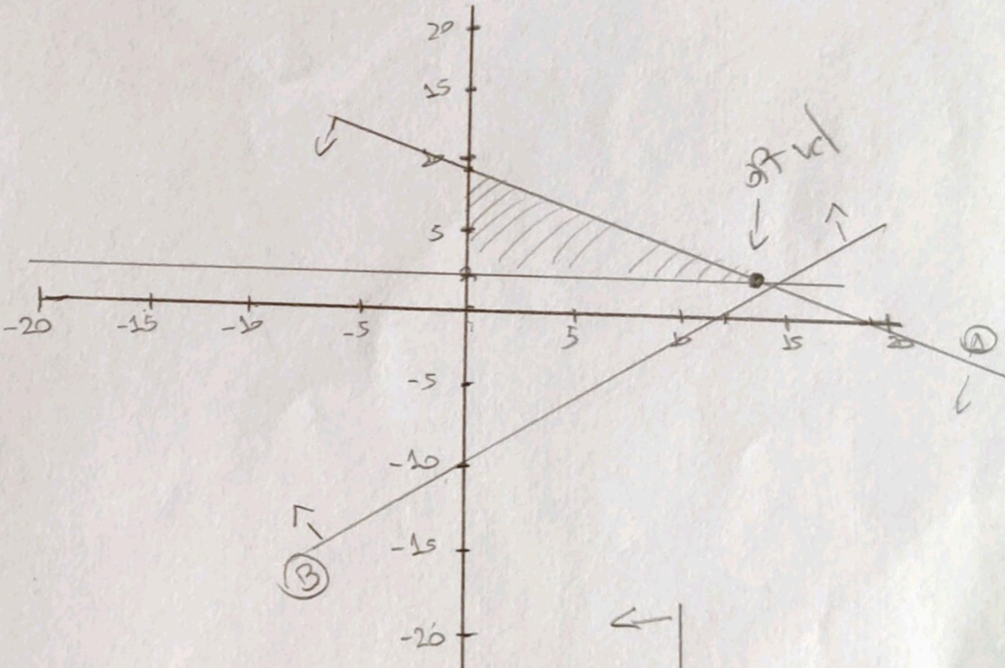
(C) $x_2 \geq 2$

$x_1, x_2 \geq 0$

$x_1 = 14$

$x_2 = 2$

$z = 100$



LP4

$\max z = 6x_1 + 8x_2$

st

(A) $2x_1 + 4x_2 \leq 36$

(B) $3x_1 - 4x_2 \leq 40$

(C) $x_2 \leq 1$

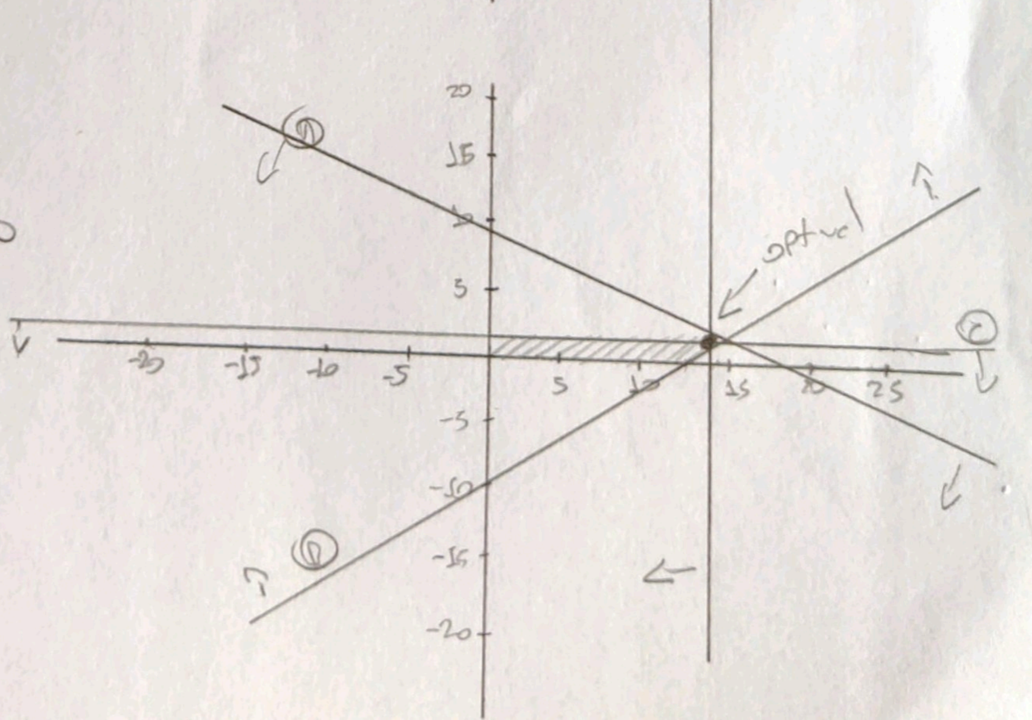
(D) $x_1 \leq 14$

$x_1, x_2 \geq 0$

$x_1 = 14$

$x_2 = 1$

$z = 92$



LP5

max $z = 6x_1 + 8x_2$

st

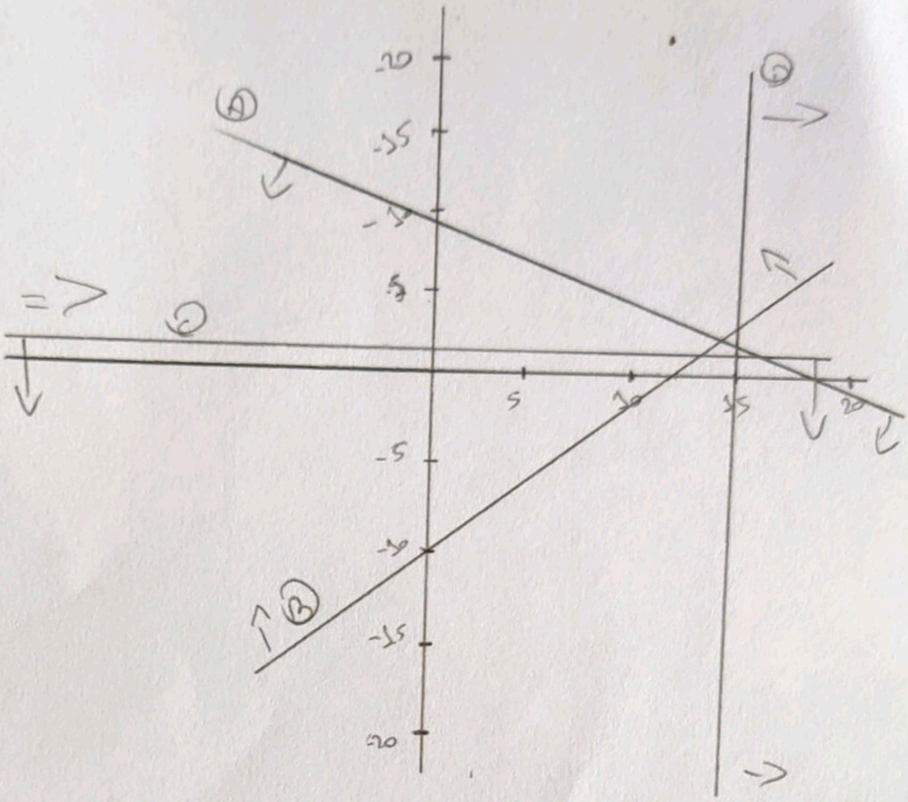
(A) $2x_1 + 4x_2 \le 36$

(B) $2x_1 - 4x_2 \le 40$

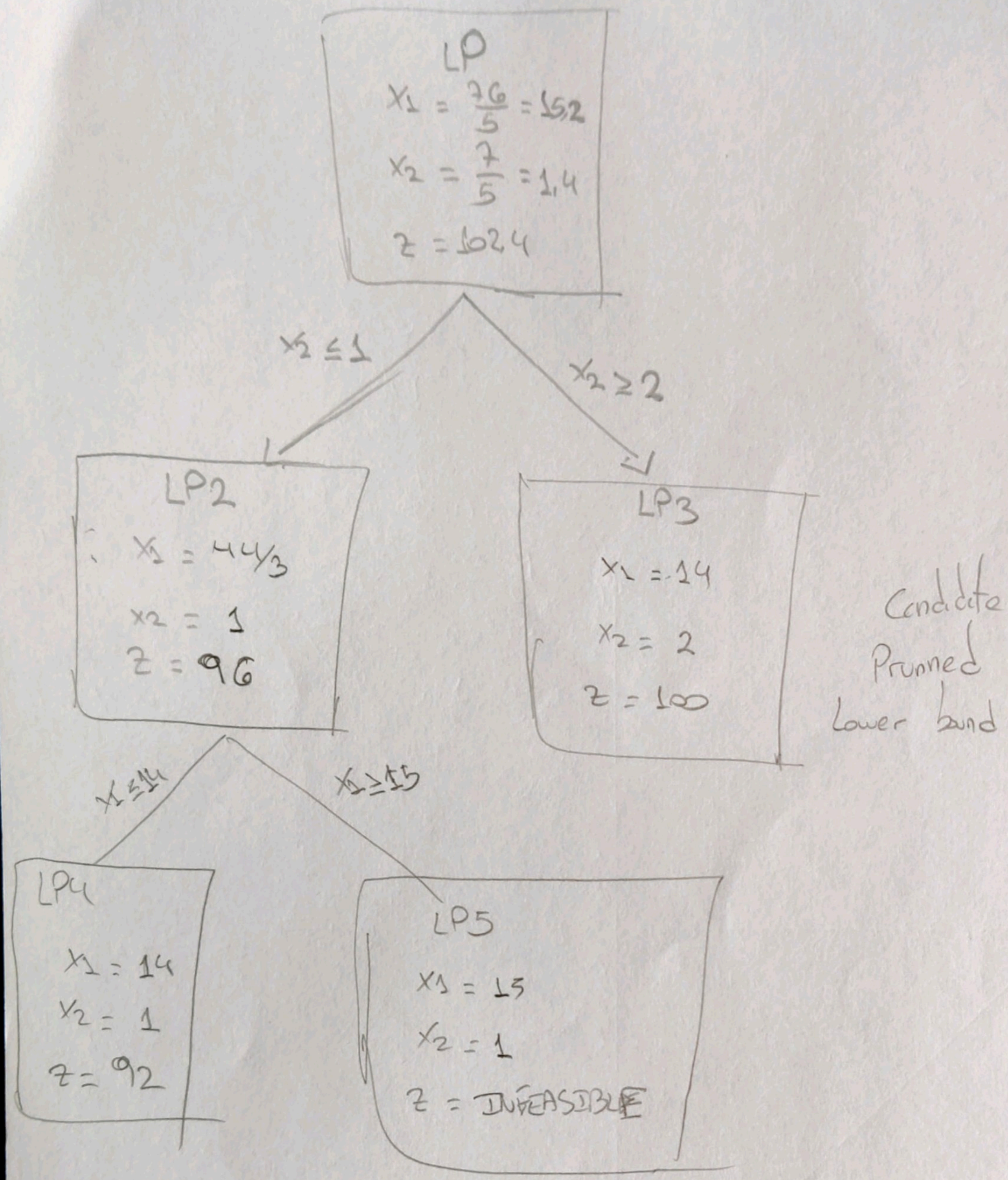
(C) $x_2 \le 1$

(D) $x_1 \ge 15$

$x_1, x_2 \ge 0$



INFEASIBLE


LP
x1 = 76/5 = 15,2
x2 = 7/5 = 1,4
z = 102,4
x2 ≤ 1
x2 ≥ 2
LP2
x1 = 44/3
x2 = 1
z = 96
LP3
x1 = 14
x2 = 2
z = 100
Candidate
Prunned
Lower bound
x1 ≤ 14
x1 ≥ 15
LP4
x1 = 14
x2 = 1
z = 92
LP5
x1 = 15
x2 = 1
z = INFEASIBLE

1.2.1. *Rebred*

$\max z = x_1 + 4x_2$

st

$x_1 + x_2 \le 7$

$-x_1 + 3x_2 \le 3$

$x_1, x_2 \ge 0$

=>

	x_1	x_2	x_3	x_4	z
	0	0	$\frac{7}{4}$	$\frac{3}{4}$	$\frac{29}{2}$
a_1	1	0	$3/4$	$-\frac{1}{4}$	$9/2$
a_2	0	1	$\frac{1}{4}$	$\frac{1}{4}$	$5/2$

$x_1 = 9/2 = 4{,}5$

$x_2 = 5/2 = 2{,}5$

$z = \frac{29}{2} = 14{,}5$

- $x_1 \le 4$ $\boxed{LP2}$

$\max z = x_1 + 4x_2$

st

$x_1 + x_2 \le 7$

$-x_1 + 3x_2 \le 3$

$x_1 \le 4$

$x_1, x_2 \ge 0$

=>

$x_1 + x_2 + x_3 \le 7$

$-x_1 + 3x_2 + x_4 \le 3$

$x_1 + x_5 = 4$

$x_1, x_2 \ge 0$

	x_1	x_2	x_3	x_4	x_5	z
	0	0	$\frac{7}{4}$	$\frac{3}{4}$	0	$29/2$
a_1	1	0	$3/4$	$-1/4$	0	$9/2$
a_2	0	1	$1/4$	$1/4$	0	$5/2$
a_5	1	0	0	0	1	4

$-r_2 + r_3$ =>

0	0	$7/4$	$3/4$	0	$29/2$
1	0	$3/4$	$-1/4$	0	$9/2$
0	1	$1/4$	$1/4$	0	$5/2$
0	0	$-3/4$	$1/4$	1	$-1/2$

Not primal feasible

Dual will be used

	0	0	0	$4/3$	$7/3$	$40/3$
a_1	1	0	0	0	1	4
a_2	0	1	0	$1/3$	$1/3$	$7/3$
a_5	0	0	1	$-1/3$	$-4/3$	$\frac{2}{3}$

$-7/4$

$-3/4$

$-1/4$

$x_1 = 4$

$x_2 = 7/3$

$z = 40/3 = 13.\overline{3}$

• $x_1 \ge 5$ $\boxed{LP3}$

$$\begin{aligned} x_1 + x_2 &\le 7 \\ -x_1 + 3x_2 &\le 3 \\ x_1 &\ge 5 \\ x_1, x_2 &\ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 + x_2 + x_3 &= 7 \\ -x_1 + 3x_2 + x_4 &= 3 \\ -x_1 + x_5 &= -5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

	0	0	7/4	3/4	0	29/2
a_1	1	0	3/4	-1/4	0	9/2
a_2	0	1	1/4	1/4	0	5/2
a_5	-1	0	0	0	1	-5

=>

	0	0	7/4	3/4	0	29/2
a_1	1	0	3/4	-1/4	0	9/2
a_2	0	1	1/4	1/4	0	5/2
a_5	0	0	3/4	-1/4	1	-1/2

Not primal feasible, dual will be used.

	0	0	7/4	3/4	0	29/2	-3/4
a_1	1	0	3/4	-1/4	0	9/2	1/4
a_2	0	1	1/4	1/4	0	5/2	-1/4
a_5	0	0	-3	1	-4	2	

=>

	0	0	4	0	3	13
a_1	1	0	0	0	-1	5
a_2	0	1	1	0	1	2
a_5	0	0	-3	1	-4	2

$\boxed{\begin{aligned} x_1 &= 5 \\ x_2 &= 2 \\ z &= 13 \end{aligned}}$ lower bound candidate

• $LP_2 \to LP4 \quad x_2 \le 2,\ x_1 \le 4$

$\max z = x_1 + 4x_2$

st

$x_1 + x_2 \le 7$

$-x_1 + 3x_2 \le 3$

$x_1 + x_5 = 4$

$x_2 + x_6 = 3$

=>

0	0	0	4/3	7/3	0	40/3
1	0	0	0	1	0	4
0	1	0	1/3	1/3	0	7/3
0	0	1	-1/3	-4/3	0	2/3
0	1	0	0	0	1	2

$-F_3 + F_5$

	0	0	0	4/3	7/3	0	40/3
a_1	1	0	0	0	1	0	4
a_2	0	1	0	1/3	1/3	0	7/3
a_5	0	0	1	-1/3	-4/3	0	2/3
a_6	0	0	0	-1/3	-1/3	1	-1/3

=>

=>

0	0	0	4/3	7/3	0	40/3	-4/3
1	0	0	0	1	0	4	
0	1	0	1/3	1/3	0	7/3	-1/3
0	0	1	-1/3	-4/3	0	2/3	1/3
0	0	0	1	1	-3	1	

=>

	x_1	x_2	x_3	x_4	x_5	x_6	z
	0	0	0	0	1	-4	12
a_1	1	0	0	0	1	0	4
a_2	0	1	0	0	0	1	2
a_5	0	0	1	0	-1	-1	1
a_6	0	0	0	1	1	-3	1

$x_1 = 4$

$x_2 = 2$

$z = 12$

Pruned.

○ LP2 → LP5, $x_1 \le 4$ $x_2 \ge 3$

$$\max z = x_1 + 4x_2$$

st

$$\begin{aligned} x_1 + x_2 &\le 7 \\ -x_1 + 3x_2 &\le 3 \\ x_1 \qquad\quad + x_5 &= 4 \\ -x_2 \qquad\quad + x_6 &= -3 \end{aligned}$$

$\Rightarrow$

$$\begin{array}{cc|cccc|c} 0 & 0 & 0 & 4/3 & 7/3 & 0 & 40/3 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1/3 & 1/3 & 0 & 7/3 \\ 0 & 0 & 1 & -1/3 & -4/3 & 0 & 2/3 \\ 0 & -1 & 0 & 0 & 0 & 1 & -3 \end{array}$$

$$\begin{array}{cc|cccc|c} 0 & 0 & 0 & 4/3 & 7/3 & 0 & 40/3 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1/3 & 1/3 & 0 & 7/3 \\ 0 & 0 & 1 & -1/3 & -4/3 & 0 & 2/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1 & -2/3 \end{array}$$

Not primal nor dual feasible problem, so

INFEASIBLE

2.1.2 -

$\max z = 6x_1 + 8x_2$

st

$2x_1 + 4x_2 \leq 36$

$3x_1 - 4x_2 \leq 40$

$x_1, x_2 \geq 0$

Relaxed =>

	0	0	12/5	2/5	512/5
a_2	0	1	3/20	-1/10	7/5
a_1	1	0	1/5	1/5	76/5

- LP1 $x_1 \leq 15$

	0	0	12/5	2/5	0	512/5	
a_2	0	1	3/20	-1/10	0	7/5	-2/5
a_1	1	0	1/5	1/5	0	76/5	1/10
a_5	0	0	1	1	-5	1	-1/5

	0	0	2	0	2	102
a_2	0	1	4/20	0	-5/10	3/2
a_1	1	0	1/5	0	1	15
a_4	0	0	1	1	-5	1

$x_1 = 15$

$x_2 = 3/2$

$z = 102$

- LP2 $x_1 \geq 16$

	0	0	14/5	2/5	0	512/5
a_2	0	1	3/20	-1/10	0	7/5
a_1	1	0	1/5	1/5	0	76/5
a_5	0	0	1/5	1/5	1	-4/5

Not primal nor dual feasible

INFEASIBLE

- LP3 -> LP1

$x_1 \leq 15$, $x_2 \geq 2$

	0	0	2	0	2	0	102	-2
a_2	0	1	4/20	0	-5/10	0	3/2	5/10
a_1	1	0	14/5	0	1	0	15	-
a_4	0	0	1	1	-5	0	1	5
a_5	0	0	-2/5	0	1	-2	1	

	0	0	14/5	0	0	4	100
a_2	0	1	0	0	0	-1	2
a_1	1	0	3/5	0	0	2	14
a_4	0	0	-1	1	0	-10	6
a_5	0	0	-2/5	0	1	-2	1

$$\boxed{\begin{array}{l} x_1 = 14 \\ x_2 = 2 \\ z = 100 \end{array}}$$

pruned

lower bound

• LP4 $\rightarrow$ LP1 , $x_1 \leq 15$, $x_2 \leq 1$

	0	0	2	0	2	0	102	-2
a_2	0	1	1/5	0	-1/2	0	3/2	-1/3
a_1	1	0	1/5	0	1	0	15	-1/5
a_4	0	0	1	1	-5	0	1	-
a_4	0	0	1	0	-5/2	-5	5/2	

0	0	0	0	7	10	97
0	1	0	0	0	1	1
1	0	0	0	3/2	1	29/2
0	0	0	1	-5/2	-5	-3/2
0	0	1	0	-5/2	-5	5/2

x_1 is desired again and $z_{LP4} < z_{LP3}$ so no need to go further is needed.

best solution integer is LP3.

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$\max z = 2x_1 + x_2 + 3x_3$

st

$x_1 + x_2 + 3x_2 \leq 17$

$3x_1 + 2x_2 + 2x_3 \leq 11$

$x_1, x_2, x_3 \geq 0$

Relaxed $\Rightarrow$

	5/2	2	0	0	3/2	33/2
a_4	-7/2	-2	0	1	-3/2	1/2
a_3	3/2	1	1	0	1/2	11/2

$x_1 = 0$

$x_2 = 0$

$x_3 = 11/2 = 5.5$ $\qquad z = 33/2 = 16.5$

- LP2 $\quad x_3 \leq 5$

	5/2	2	0	0	3/2	0	33/2
a_4	-7/2	-2	0	1	-3/2	0	1/2
a_3	3/2	1	1	0	1/2	0	11/2
a_6	0	0	1	0	0	1	5

$\Rightarrow$

	5/2	2	0	0	3/2	0	33/2
a_4	-7/2	-2	0	1	-3/2	0	1/2
a_3	3/2	1	1	0	1/2	0	11/2
a_6	-3/2	-1	0	0	-1/2	1	-1/2
	-1/6	-2			-3		

	5/2	2	0	0	3/2	0	33/2	-5/2
a_4	-7/2	-2	0	1	-3/2	0	1/2	7/2
a_3	3/2	1	1	0	1/2	0	11/2	-3/2
a_1	1	2/3	0	0	1/3	-2/3	1/3	

	0	1/3	0	0	2/3	5/3	47/3
a_4	0	1/3	0	1	-1/3	-7/3	5/3
a_3	0	-1	1	0	0	1	5
a_1	1	2/3	0	0	1/3	-2/3	1/3

$x_1 = 1/3 = 0,\overline{3}$

$x_2 = 0$

$x_3 = 5$ $\qquad z = 47/3 = 15,6$

- LP3 $x_3 \geq 6$

$$\max z = 2x_1 + x_2 + 3x_3$$

$$x_1 + x_2 + 3x_3 \leq 17$$

$$3x_1 + 2x_2 + 2x_3 \leq 11$$

$$-x_3 + x_6 = -6$$

	5/2	2	0	0	3/2	0	33/2
a_4	-7/2	-2	0	1	-3/2	0	1/2
a_3	3/2	1	1	0	1/2	0	11/2
a_6	0	0	-1	0	0	1	-6

=>

	5/2	2	0	0	3/2	0	33/2
a_4	-7/2	-2	0	1	-3/2	0	1/2
a_3	3/2	1	1	0	1/2	0	11/2
a_6	3/2	1	0	0	1/2	1	-1/2

Not primal nor dual feasible

<u>INFEASIBLE</u>

- LP4 -> LP2 $x_3 \leq 5$, $x_1 \leq 0$

$\max z = 2x_1 + x_2 + 3x_3$

st

$x_1 + x_2 + 3x_3 \leq 17$

$3x_1 + 2x_2 + 2x_3 \leq 11$

$x_1 \leq 0$

$x_3 \leq 5$

	0	1/3	0	0	2/3	5/3	0	47/3
a_4	0	1/3	0	1	-1/3	-7/3	0	5/3
a_3	0	-1	1	0	0	1	0	5
a_1	1	2/3	0	0	1/3	-2/3	0	1/3
a_7	1	0	0	0	0	0	1	0

$x_1 = 0$

$x_2 = 1/2$ $z = 31/2$

$x_3 = 5$

	0	1/3	0	0	2/3	5/3	0	47/3	-1/3
a_4	0	1/3	0	1	-1/3	-7/3	0	5/3	-1/3
a_3	0	-1	1	0	0	1	0	5	+
a_1	1	2/3	0	0	1/3	-2/3	0	1/3	-2/3
a_7	0	-2/3	0	0	-1/3	2/3	1	-1/3	
$a_7/(-2/3)$	0	1	0	0	1/2	-1	-3/2	1/2	

	0	0	0	0	1/2	2	1/2	31/2
a_4	0	0	0	1	-1/2	-2	1/2	3/2
a_3	0	0	1	0	0	1	0	5
a_1	1	0	0	0	0	0	1	0
a_2	0	1	0	0	1/2	-1	-3/2	1/2

• LP5 → LP2 $x_3 \leq 5$, $x_1 \geq 1$

$\max z = 2x_1 + x_2 + 3x_3$

st

$x_1 + x_2 + 3x_3 \leq 17$

$3x_1 + 2x_2 + 2x_3 \leq 11$

$x_1 \geq 1$

$x_3 \leq 5$

	0	1/3	0	0	2/3	5/3	0	47/3
a_4	0	1/3	0	1	-1/3	-7/3	0	5/3
a_3	0	-1	1	0	0	1	0	5
a_1	1	2/3	0	0	1/3	-2/3	0	1/3
a_7	0	2/3	0	0	1/3	-2/3	1	-2/3

	0	1/3	0	0	2/3	5/3	0	47/3	-5/3
a_4	0	1/3	0	1	-1/3	-7/3	0	5/3	7/3
a_3	0	-1	1	0	0	1	0	5	-
a_1	1	2/3	0	0	1/3	-2/3	0	1/3	2/3
a_6	0	-1	0	0	-1/2	1	-3/2	1	

	0	2	0	0	3/2	0	5/2	14
a_4	0	-2	0	1	-3/2	0	-7/2	4
a_3	0	0	1	0	1/2	0	3/2	4
a_1	1	0	0	0	0	0	-1	1
a_6	0	-1	0	0	-1/2	1	-3/2	1

$x_1 = 1$

$x_2 = 0$

$x_3 = 4$ $z = 14$

prunned

lower bound

• LP6 -> LP4 -> LP2 , $x_3 \leq 5$, $x_1 \leq 0$, $x_2 \leq 0$

	0	0	0	0	1/2	2	1/2	0	3/2	-3/2
a_4	0	0	0	1	-1/2	-2	1/2	0	3/2	3/2
a_3	0	0	1	0	0	1	0	0	5	0
a_1	1	0	0	0	0	0	1	0	0	0
a_2	0	1	0	0	1/2	-1	-3/2	0	1/2	-1/2
a_8	0	0	0	0	1	-2	-3	-2	1	

Aquí he hecho todo en la misma tabla.

	0	0	0	0	0	3	2	1	15
a_4	0	0	0	1	0	-3	-1	-1	2
a_3	0	0	1	0	0	1	0	0	5
a_1	1	0	0	0	0	0	1	0	0
a_2	0	1	0	0	0	0	0	1	0
a_5	0	0	0	0	1	-2	-3	-2	1

$x_1 = 0$

$x_2 = 0$ $\quad z = 15$

$x_3 = 5$

primal

lower bound

new

• LP7 → LP4 → LP2 , $x_3 \le 5$, $x_1 \le 0$, $x_2 \ge 1$

	0	0	0	0	1/2	2	1/2	0	3/2	-1/2
a_4	0	0	0	1	-1/2	-2	1/2	0	3/2	-1/2
a_3	0	0	1	0	0	1	0	0	5	0
a_1	1	0	0	0	0	0	1	0	0	0
a_2	0	1	0	0	1/2	-1	-3/2	0	1/2	3/2
a_5	0	0	0	0	-1/3	2/3	1	-2/3	1/3	-3/2

0	0	0	0	2/3	5/3	0	1/3	46/3
0	0	0	1	-1/3	-7/3	0	1/3	4/3
0	0	1	0	0	1	0	0	5
1	0	0	0	1/3	-2/3	0	2/3	-1/3
0	1	0	0	0	0	0	-1	1
0	0	0	0	-1/3	2/3	1	-2/3	1/3

Whatever happened here, it is not the solution and you stop here because there is no more iterations

sol opt is LP6.