MATH3204: Assignment 01

Due: 24-August-2020 @11:59pm

1. Vector Spaces:

(a) Determine whether each of the following is a subspace. In all these cases the underlying field is $\mathbb{R}$. **[2 marks each]**

i. $\{(a,b,c) \in \mathbb{R}^3 \mid a + 2b + 2c = 0\}$

ii. $\{(a,b,c) \in \mathbb{R}^3 \mid a^2 = b\}$

iii. $\{f \in C[0,1] \mid f(1/2) = 1\}$ with the standard function addition and scalar multiplication where $C[0,1]$ denotes the set of all continuous functions on the interval $0 \le x \le 1$.

iv. $\left\{ \boldsymbol{A} \in \mathbb{R}^{2\times 2} \mid \boldsymbol{A} = \begin{bmatrix} a & b \\ 0 & a+b \end{bmatrix},\ a, b \in \mathbb{R} \right\}$ with the standrad matrix addition and scalar multiplication.

(b) Find a basis for the subspace of $\mathbb{R}^5$ defined by **[4 marks]**

$$\mathcal{U} \triangleq \{(a,b,c,d,e) \in \mathbb{R}^5 \mid a = 3b \text{ and } c = 7d\}$$

(c) Show that $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. **[4 marks]**

(d) Prove that any subspace contains the zero vector. **[4 marks]**

2. Linear Maps:

(a) Find the matrix representation of the operator $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\boldsymbol{f}(x,y,z) = (2x - z, x + y)$ with respect to the bases $\mathcal{B}_{\mathbb{R}^3} = \{(1,0,1), (1,-1,0), (0,0,2)\}$ and $\mathcal{B}_{\mathbb{R}^2} = \{(3,4), (4,-3)\}$. **[5 marks]**

(b) Let $\mathcal{P}_1$ denote the space of polynomial of degree at most one with real coefficients. Find the matrix representation of the operator $T : \mathcal{P}_1 \to \mathcal{P}_1$ given by $T(at + b) = (a + 2b)t + (4a + 3b)$ with respect to the standard basis for $\mathcal{P}^1$. **[5 marks]**

(c) Do the same as above with $\{2, t/2\}$ for a basis of the range of T (still use the standard basis for the domain of T). **[5 marks]**

(d) Let $\boldsymbol{f} : \mathcal{V} \to \mathcal{W}$ be an isomorphism, i.e., linear and invertible map. Show that $\boldsymbol{f}^{-1} : \mathcal{W} \to \mathcal{V}$ is also an isomorphism. **[5 marks]**

3. Matrix Inverse:

(a) Show that the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ cannot have an inverse. **[5 marks]**

(b) One of the most important functions in the calculus is the exponential function e^x. For similar reasons, "matrix exponentials" are highly important and useful. Recall that the exponetial function is defined, using the power series, as

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

This is then used to define the notion of exponential of a square matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ as

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{k!} = \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \dots.$$

Note that $e^{\boldsymbol{A}} \in \mathbb{C}^{n \times n}$. Furthermore, just as the series for e^x converges for all values of x, it can be shown that $e^{\boldsymbol{A}}$ also converges for all square matrices $\boldsymbol{A}$. Show that $\left(e^{\boldsymbol{A}}\right)^{-1} = e^{-\boldsymbol{A}}$, i.e., inverse of matrix exponential behaves the same way that you expect it to! **[15 marks]**

4. Spectral Properties:

(a) Suppose $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n$, $n \geq 2$, $\boldsymbol{u} \neq \boldsymbol{0}, \boldsymbol{v} \neq \boldsymbol{0}$ and let $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^*$. **[2 mark each]**

i. Find all the eigenvalues of $\boldsymbol{A}$.

ii. When is $\boldsymbol{A}$ diagonalizable and when is it not? In other words, find conditions on $\boldsymbol{u}$ and $\boldsymbol{v}$ that ensure $\boldsymbol{A}$ is not defective.

iii. When is $\boldsymbol{A}$ unitarily diagonalizable and when is it not? In other words, find conditions on $\boldsymbol{u}$ and $\boldsymbol{v}$ that ensure $\boldsymbol{A}$ is normal.

iv. Find all the singular values of $\boldsymbol{A}$.

v. What is Rank($\boldsymbol{A}$)?

vi. Find the left and right singular vectors corresponding to the largest singular value of $\boldsymbol{A}$.

(b) You are given a matrix $\boldsymbol{A} \in \mathbb{C}^{4\times 4}$ whose eigenvalues are 2 and 3, both with the algebraic multiplicity of 2. **[1 mark each]**

i. What are the possible Jordan matrices for $\boldsymbol{A}$?

ii. For each possibility, indicate the geometric multiplicity of each eigenvalue.

iii. For each possibility, indicate whether $\boldsymbol{A}$ is diagonalizable.

(c) Let $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\intercal$ be the fill SVD of $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ with $m \geq n$ **[1 mark each]**

$$\boldsymbol{U} = \begin{bmatrix}\boldsymbol{u}_1 \mid \boldsymbol{u}_2 \mid \dots \mid \boldsymbol{u}_m\end{bmatrix} \in \mathbb{C}^{m\times m},$$
$$\boldsymbol{V} = \begin{bmatrix}\boldsymbol{v}_1 \mid \boldsymbol{v}_2 \mid \dots \mid \boldsymbol{v}_n\end{bmatrix} \in \mathbb{C}^{n\times n},$$
$$\boldsymbol{\Sigma} = \begin{bmatrix}\boldsymbol{\Sigma}_n \\ \boldsymbol{0}\end{bmatrix} \in \mathbb{R}^{m\times n},$$

where

$$\boldsymbol{\Sigma}_n = \begin{pmatrix} \sigma_1 & & & & & & \\ & \sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & \sigma_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \in \mathbb{R}^{n\times n},$$

and $\sigma_i > 0,\ i = 1, 2, \dots, r$. Using SVD decomposition, show that

i. $\mathrm{Rank}(\boldsymbol{A}) = \mathrm{Rank}(\boldsymbol{A}^*)$

ii. $\mathrm{Rank}(\boldsymbol{A}) + \dim(\mathrm{Null}(\boldsymbol{A})) = n$

iii. $\mathrm{Null}(\boldsymbol{A}) = \mathrm{Span}\{\boldsymbol{v}_{r+1}, \boldsymbol{v}_{r+2}, \dots, \boldsymbol{v}_n\}$

iv. $\boldsymbol{A}^\dagger = \boldsymbol{V}\boldsymbol{\Sigma}^\dagger\boldsymbol{U}^*$ where

$$\boldsymbol{\Sigma}^\dagger = \begin{bmatrix}\boldsymbol{\Sigma}_n^\dagger & \boldsymbol{0}\end{bmatrix} \in \mathbb{R}^{n\times m}, \quad \text{and} \quad \boldsymbol{\Sigma}_n^\dagger = \begin{pmatrix} 1/\sigma_1 & & & & & & \\ & 1/\sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & 1/\sigma_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

v. $\left\| \boldsymbol{A}\boldsymbol{A}^{\dagger} \right\|_2 = 1$ where the matrix norm is the ℓ-2 induced norm, i.e., spectral norm

5. In this question, you will use SVD to compress your image. Take a picture of yourself and load it using a programming language of your choosing. For `Matlab` and `Python`, you can respectively look into functions `imread` and `open` from `Python Imaging Library (PIL)`.

(a) After loading your image, turn it into gray-scale. Write a short script for computing its truncated SVD. You can use the inbuilt function `svd` for both `Matlab` and `Python` (in the `NumPy` library). Start with rank $r = 2$ and go up by powers of 2, to $r = 64$. Show the resulting images. **[5 marks]**

(b) Comment on the performance of the truncated SVD. State how much storage is required as a function of r and matrix dimensions and compare it with the storage required for the original picture. **[5 marks]**

(c) Now do the same but keep the colors, i.e., don't turn your image into gray-scale. Your code for this part should output colored compressions of your original image that (Hint: consider performing SVD on each RGB color band separately and then combine the results). **[10 marks]**

100 marks in total

Note:

- This assignment counts for 15% of the total mark for the course.
- Although not mandatory, if you could type up your work, e.g., `LaTex`, it would be greatly appreciated.
- Show all your work and attach your code and all the plots (if there is a programming question).
- Combine your solutions, all the additional files such as your code and numerical results, along with your coversheet, **all in one single PDF file**.
- Please submit your single PDF file on Blackboard.