MATH3405 Final Project

The Hairy Ball Theorem

Ruyun Qi

November 5, 2020

Abstract

Since the 1880s, the existence of zero vectors on the surface of a 2-sphere has been observed by Henri Poincaré. Topologists jokingly call this finding the "hairy ball theorem". More investigations in the fields of physics, meteorology, and other science have been conducted based on the hairy ball theorem. This project aims to present the definitions, proofs, and applications related to this beautiful mathematical theorem.

Introduction

The Hairy Ball Theorem, as a remarkable topological theorem, indicates that there is always a vanishing vector on an even-dimensional sphere. Pickover(2009, p.326) describes that “if a sphere is covered in hair and we try to smoothly brush the hair to make them all lie flat, we will always leave behind at least one hair standing up straight or a hole”. In other words, the sphere is a ball and the tangent vectors on the ball are hairs. You cannot comb the hairy ball without getting a cowlick, which explains that there is always a zero vector in the tangent vector field on a sphere. That is how the name of ”Hairy Ball” came from.

Non-vanishing vector fields

Since the hairy ball theorem is rejecting the continuous vector field on a sphere, it is necessary to introduce the definitions of continuous and discontinuous vectors. A vector field is defined as continuous or smooth, if all its components are continuously differentiable, otherwise it is discontinuous. A continuous vector field is observed that all of the vectors intend to go in the same direction with zooming in any spot on the vector field. Therefore, the vanishing point on a vector field leads to discontinuities in the vector field. (D’Agostino, 2020, pp.133-134)

Example 1: Cylider

Consider the map $f(t) : \mathbb{C} \to \mathbb{E}^3$ defined by:

$$(\eta, v) \longmapsto (x, y, z) = (\cos(\eta), \sin(\eta), v)$$

Suppose, we are keeping combing a hairy cylinder, then there will be a non-vanishing vector field which means $f(t) \neq 0$ for all t on the surface. (Crossley, 2005, p.112)

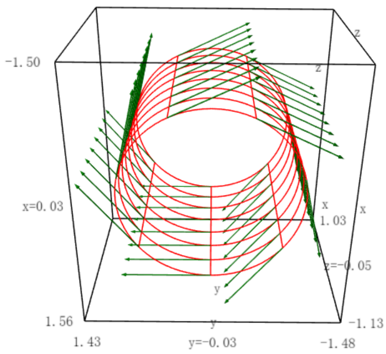

Figure 1: Cylinder with non-vanishing vector field

Example 2: Torus

Consider the map $\Psi : \mathbb{T}^2 \to \mathbb{E}^3$ defined by:

$$(u, v) \longmapsto (x, y, z) = (3(\cos(v) + 2)\cos(u), 3(\cos(v) + 2)\sin(u), 3(\sin(v + 2))$$

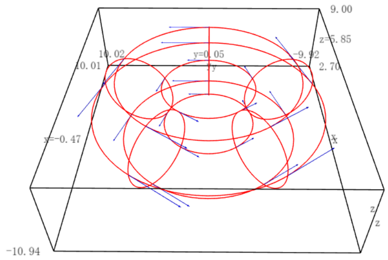

Figure 2: Torus with non-vanishing vector field

There is a continuous tangent vector field on this 2-Torus. However, a sphere is topologically different from a torus. In topology, any object which can be deformed into a ball without cutting and glutting is assumed the same as a ball. It is impossible to make a ball into a doughnut unless poking a hole on the ball and squashing its shape. This phenomenon might reveal the existence of vanishing points on a 2-sphere. (D'Agostino, 2020, pp.131-132)

Hairy Ball Theorem

If $v : \mathbb{S}^2 \to \mathbb{E}^3$ is a tangent vector field on the 2-sphere, then there exists a vanishing point $v(x, y, z) = 0$, such that v is not non-vanishing continuous.

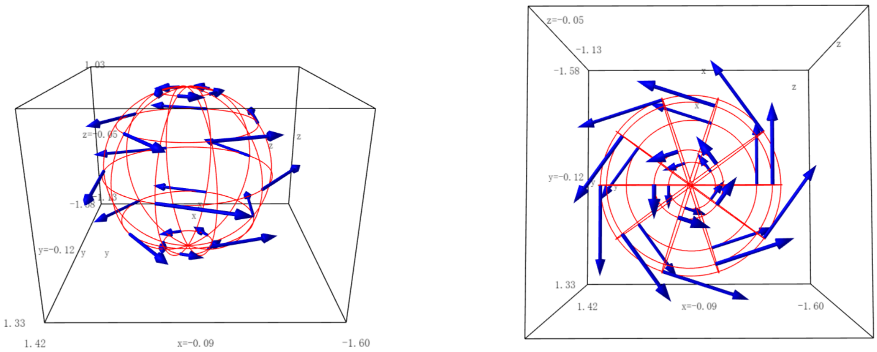

Figure 3: Sphere with tangent vector field

When looking at the top view of the sphere, the vector field seems like a whorl. By approaching to the center of the whorl, the vectors tend to take all possible directions, as they go around the saddle point p on the top. Additionally, the lengths of the vectors decrease until they reach 0 at p, such that $v_p = 0$.

Poincare-Hopf Theorem

The hairy ball theorem can be simply proved by the Poincare-Hopf theorem, so the following introduces its definition and concepts.

For any vector field v on an compact and diiferentiable manifold M with isolated vanishing points $\{x_i\}$, the sum of the indices of the zeros x_i of the vector field v is equal to the Euler characteristic of the manifold M, such that:

$$\sum_i \text{index}_{x_i} = \chi(M)$$

In particular, the existence of a non-vanishing vector field implies the Euler characteristic is equal to 0. This fact can be seen in the previous example of the torus.

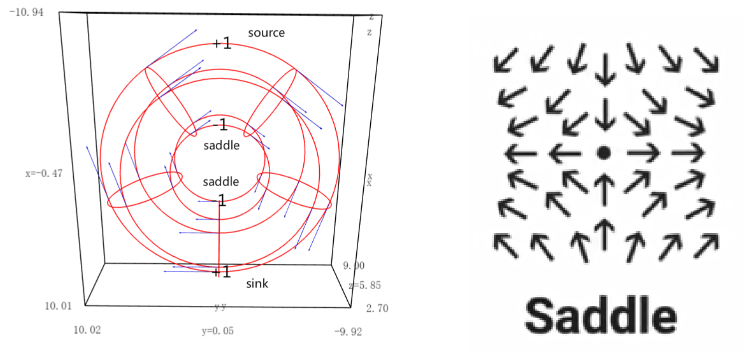

Figure 4: Torus with indices of zeros

The indices of the saddle points which are -1 cancel out these of the source and sink points which are +1. Therefore, the Euler characteristic of the torus is 0.

By the Poincare-Hopf theorem,the index sum of the sphere is equal to its Euler characteristic which is 2. If there is no zero vectors, the index sum should be 0. Hence, it proves the hairy ball theorem. (Libgober, pp.17-18)

Another proof

Curtin(2018) introduced a simple proof of the hairy ball theorem implementing winding numbers and equivalent notion of degrees of self-maps.

Consider the map $\boldsymbol{p} : \mathbb{S}^2 \to \mathbb{R}^{\nVdash}$ is defined on $(\theta, t) \in [0, 2\pi] \times [-1, 1]$, such that

$$\boldsymbol{p}(\theta, t) = (\sqrt{1-t^2}\cos\theta, \sqrt{1-t^2}\sin\theta, t).$$

The tangent plane to $\mathbb{S}^2$ at $\boldsymbol{p}$ has the basis $\{\boldsymbol{e}, \boldsymbol{n}\}$, where

$$\boldsymbol{e}(\theta, t) = (-\sin\theta, \cos\theta, 0),$$
$$\boldsymbol{n}(\theta, t) = (-t\cos\theta, -t\sin\theta, \sqrt{1-t^2}).$$

Since $\partial\boldsymbol{e}/\partial\theta = \boldsymbol{n}$ when $t = 1$, the basis $\{\boldsymbol{e}, \boldsymbol{n}\}$ rotates in the positive direction with increasing θ at the north pole, and oppositely it rotates in the negative direction with increasing θ at the south pole.

Suppose $\boldsymbol{v}$ is a non-vanishing tangent vector field on $\mathbb{S}^2$, then we have the unit vector $\boldsymbol{u}$ and its components H with respect to the orthonormal basis $\{\boldsymbol{e}, \boldsymbol{n}\}$, such that

$$\boldsymbol{u} = \boldsymbol{v} \circ \boldsymbol{p},$$

$$H = (\boldsymbol{u} \cdot \boldsymbol{e}, \boldsymbol{u} \cdot \boldsymbol{n}),$$

where H maps $\mathbb{S}^2$ to $\mathbb{S}^1 \subset \mathbb{R}^2 \backslash \{(0,0)\}$.

If v can be deformed continuously into one another, then the transformation is homotopic. A vector field without vanishing points on a spherical surface would provide such homotopy. Since the unit vector $\boldsymbol{u}(\theta, 1) = \boldsymbol{v}(0, 0, 1)$ is constant, it rotates in the negative direction with respect to the basis rotating in the positive direction. Thus, the homotopic transformation does not exist which proves there are vanishing points on the sphere.

Applications

In computer graphics, it is commonly required to generate a non-zero vector in $\mathbb{R}^3$ which is orthogonal to a given non-zero vector. According to the hairy ball theorem, there does not exist a continuous function that satisfies all vector inputs. The given vector can be considered as the radius of a sphere, so finding the non-vanishing tangential vector for every point on the sphere is impossible. (Kohulák, 2016)

The hairy ball theorem has also been applied in meteorology. Meteorologists place arrows to represent the wind on the surface of our earth and these are similar to the vector fields on a sphere. The wind directions are represented by the directions of arrows while the wind speeds are represented by the lengths of the arrows. According to the hairy ball theorem, there is always a place on the earth where the wind does not blow. In meteorology, such a point could be an eye of a cyclone or an anticyclone.

Bormashenko(2015) discussed that one application of the hairy ball theorem in the physics field is to analyze the surface instabilities for liquid or vapor interfaces. This kind of instabilities rises the pattern which can be seen as a set of n elementary cells. Since the cell shaped in a rectangular prism is homeomorphic to a sphere, the velocities can be defined as the tangent vectors on the surface of a cell. Based on the hairy ball theorem, there exist points where the velocities are zero o the surface of the liquid. The instabilities are influenced by the accumulation of the solid particles and pores in these zero-velocity points. The predictions of the zero-velocity points contribute to the analysis and visualization of the instabilities.

Conclusions

This paper has discussed the concepts and applications of the hairy ball theorem. The hairy ball theorem expects the existence of at least one vanishing vector on the surface of a 2-sphere. A variety of physical, meteorological, and other phenomena might be affected by this finding. However, the theorem has more been implemented to explain the facts instead of solving problems in science fields. It would be interesting to explore more investigations on its applications in the future.

References

Bormashenko, E. (2015). Surface instabilities and patterning at liquid/vapor interfaces: Exemplifications of the "hairy ball theorem". *Colloid and Interface Science Communications*, 5, 5–7. `https://doi.org/10.1016/j.colcom.2015.04.003`

Crossley, M. D. (2005). *Essential Topology* (1 ed.) (pp.112-115). Springer.

Curtin, E. (2018). Another Short Proof of the Hairy Ball Theorem. *The American Mathematical Monthly*, 125(5), 462–463. `https://doi.org/10.1080/00029890.2018.1436836`

D'Agostino, S. (2020). *Abandon perfectionism, because of the Hairy Ball Theorem* (pp.131-135). In *How to Free Your Inner Mathematician.* Oxford University Press. `https://doi.org/10.1093/oso/9780198843597.003.0023`

Kohulák, R. (2016). Hairy balls, cyclones and computer graphics. *Chalkdust Magazine.* `http://chalkdustmagazine.com/blog/hairy-balls-cyclones-computer-graphics`

Libgober, J. *The Euler characteristic, Poincare-Hopf theorem and applications.* (pp.17-18) `http://www.math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Libgober.pdf`

Pickover, C. A. (2009). *The math book: from Pythagoras to the 57th dimension, 250 milestones in the history of mathematics* (p.326). Sterling Pub.