Föreläsning 5

Signalbehandling i multimedia - ETI265

Kapitel 4

Fouriertransform av analog signal, FT
Fouriertransform av digital signal, DTFT

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Transformer i Digital Signalbehandling

Analoga signaler
Fouriertransform av analog signal, FT
(Laplacetransform)

Tidsdiskreta signaler (samplade, digitala signaler)
Fouriertransform av digital signal, DTFT, kap 4
Diskreta Fouriertransformen (DFT, FFT), kap 7
Z-transform av digital signal kap 3.

Fourierserieutveckling, (lite grann, senare)

Fourierserieutveckling av periodisk signal
Fourierserieutveckling av periodisk signal

Kapitel 4 Fouriertransform sid 236-238

Definition: **Fouriertransform av analog signal FT**

$$X(F) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j2\pi F t} dt$$

$$x(t) = \int_{F=-\infty}^{\infty} X(F)\, e^{j2\pi F t} dF =$$

$$= \frac{1}{2\pi} \int_{\Omega=-\infty}^{\infty} X(\Omega)\, e^{j\Omega t} d\Omega$$

$$\Omega = 2\pi F$$

Konvergens: **Om** $x(t)$ **stabil, dvs** $\int_t | x(t) | < \infty$

(Se även matte)

Lite svagare konvergens

$$\int_t | x(t) | \to \infty$$

men

$$\int_t | x(t) |^2 < \infty \qquad \textit{begränsad energi}$$

Definition: Fouriertransform av tidsdiskret signal DTFT, sid 251

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j2\pi f\ n} = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j\omega n} \qquad \omega = 2\pi\ f$$

$$x(n) = \int_{f=-1/2}^{1/2} X(f)\ e^{j2\pi f\ n}\, df = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega)\ e^{j\omega n}\, d\omega$$

Konvergens: **Om** $x(n)$ **stabil, dvs** $\sum_{n} |\,x(n)\,| < \infty$

Lite svagare konvergens

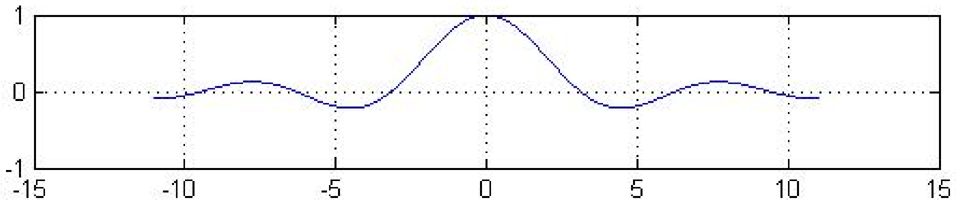
$$\sum_{n} |\,x(n)\,| \to \infty$$

men

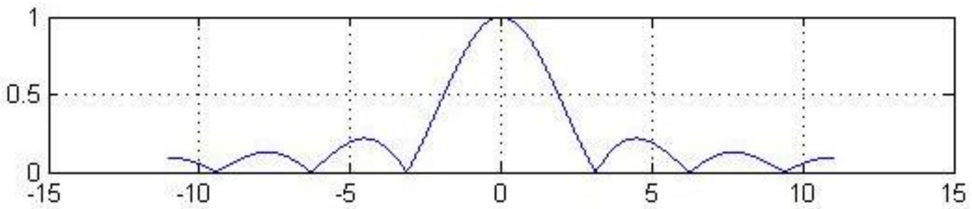
$$\sum_{n} |\,x(n)\,|^2 < \infty \qquad begränsad\ \ energi$$

En vanligt förekommande signalform i signalbehandling (och i många andra ämnen)

$$\frac{\sin(x)}{x}$$



$$\left|\frac{\sin(x)}{x}\right|$$



$$\frac{\sin(x)}{x}$$

kallas ofta 'sinc'

Känt gränsvärde

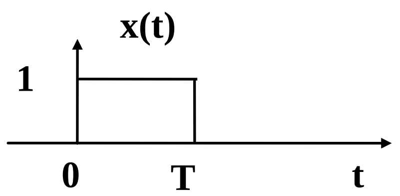
$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)=1$$

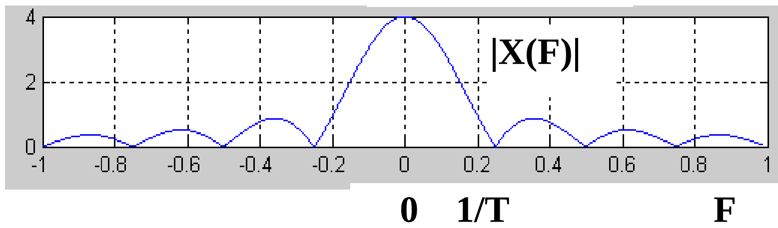
Fouriertransform av rektangelpuls sid 257-258

Tidsplanet

Frekvensplanet

Analog rektangelpuls (rektangulärt tidsfönster) **T=4 i figuren**





$$x(t) = \begin{cases} 1 & 0 \le \mathrm{t} \le \mathrm{T} \\ 0 & \text{för övrigt} \end{cases}$$

$$X(F) = T \frac{\sin(2\pi F \frac{T}{2})}{2\pi F \frac{T}{2}} e^{-j 2\pi F T/2}$$

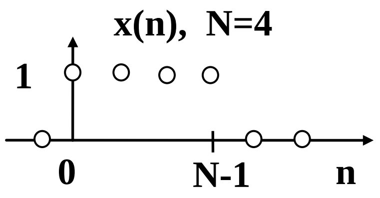
Beräkning av $X(F)$

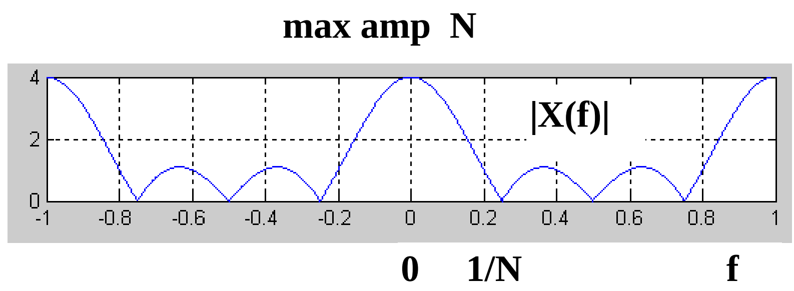
$$X(F) = \int_{t=-\infty}^{\infty} x(t) e^{-j 2\pi F t} \, dt = \int_{t=0}^{T} \underbrace{x(t)}_{=1} e^{-j 2\pi F t} \, dt =$$

$$= \frac{e^{-j 2\pi F T} - 1}{-j\, 2\pi F} = \frac{e^{-j 2\pi F T/2}\left(e^{j 2\pi F T/2} - e^{-j 2\pi F T/2}\right)}{j\, 2\pi F} =$$

$$= T \frac{\sin(2\pi F \frac{T}{2})}{2\pi F \frac{T}{2}} e^{-j 2\pi F T/2}$$

Tidsdiskret rektangelpuls (rektangulärt tidsfönster)





$$x[n]=\begin{cases}1 & 0\le \text{n} \le \text{N-1}\\ 0 & \text{för övrigt}\end{cases} \qquad X(f)=N\,\frac{\sin(2\pi f\frac{N}{2})}{N\,\sin(2\pi f\frac{1}{2})}\;e^{-j2\pi f\frac{N-1}{2}}$$

OBS. $X(f)$ **är periodiskt med perioden** $f=1,\ \omega=2\pi$

Beräkning av $X(f)$

$$X(f)=\sum_{n=-\infty}^{\infty}x(n)\,e^{-j2\pi f n}=\sum_{n=0}^{N-1}\underbrace{x(n)}_{=1}\,e^{-j2\pi f n}=$$

$$=\frac{1-e^{-j2\pi f N}}{1-e^{-j2\pi f}}=\frac{e^{-j2\pi f N/2}(e^{j2\pi f N/2}-e^{-j2\pi f N/2})}{e^{-j2\pi f 1/2}(e^{j2\pi f 1/2}-e^{-j2\pi f 1/2})}=$$

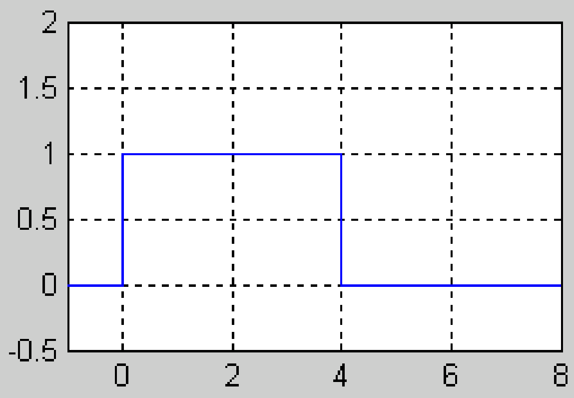
$$=N\,\frac{\sin(2\pi f\frac{N}{2})}{N\,\sin(2\pi f\frac{1}{2})}\;e^{-j2\pi f\frac{N-1}{2}}$$

Plot i MATLAB av Fouriertransform av rektangelpuls

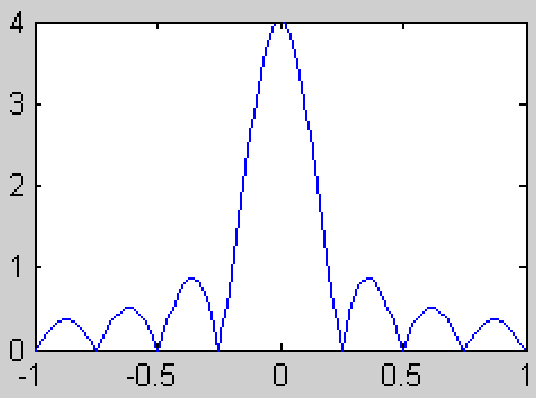
Analog rektangelpuls (rektangulärt tidsfönster) T=4

$$x(t) = \begin{cases} 1 & 0 \le \mathrm{t} \le \mathrm{T} \\ 0 & \text{för övrigt} \end{cases} \qquad X(F) = T \frac{\sin(2\pi F \frac{T}{2})}{2\pi F \frac{T}{2}} e^{-j 2\pi F \frac{T}{2}}$$

Tidssignal



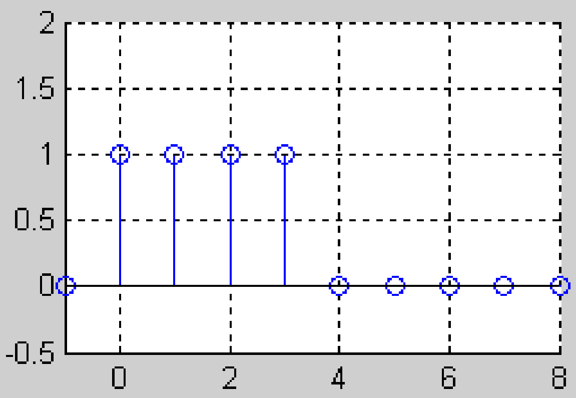
Spektrum



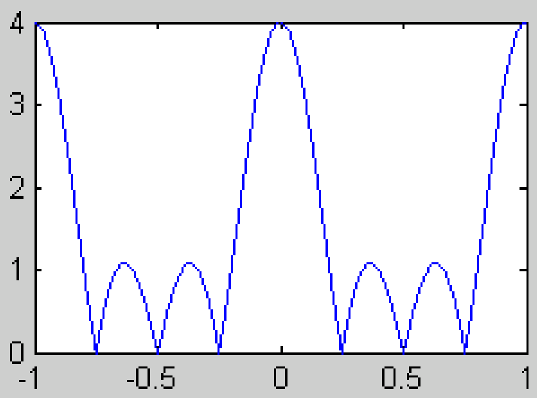
Tidsdiskret rektangelpuls (rektangulärt tidsfönster) N=4

$$x[n] = \begin{cases} 1 & 0 \le \mathrm{n} \le \mathrm{N\text{-}1} \\ 0 & \text{för övrigt} \end{cases} \qquad X(f) = N \frac{\sin(2\ \pi\ f\ \frac{N}{2})}{N\ \sin(2\pi\ f\ \frac{1}{2})}\ e^{-j\ 2\ \pi\ f\ \frac{N-1}{2}}$$

Tidssignal



Spektrum



Exempel på transformer DTFT.

Tidsfunktion Fouriertransform

Definition: **Fouriertransform av tidsdiskret signal DTFT**

$$x(n) = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi f n}\, df \,, \qquad X(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi f n}$$

$$x(n) = \{3 \;\; 2 \;\; 1\} \qquad X(f) = 3 + 2e^{-j2\pi f} + e^{-j2\pi 2f} = \\ = 3 + 2e^{-j\omega} + e^{-j2\omega}$$

deltafunktion

$$x(n) = \delta(n) = \{1\} \qquad X(f) = 1$$

skift

$$x(n) = \delta(n-1) = \{0 \;\; 1\} \qquad X(f) = e^{-j2\pi f} = e^{-j\omega}$$

$$x(n) = \delta(n-n_0) \qquad X(f) = e^{-j2\pi f n_0} = e^{-j\omega n_0}$$

$$y(n) = x(n-n_0) \qquad Y(\omega) = e^{-j\omega n_0} X(\omega)$$

faltning

$$y[n] = x[n] * h[n] \qquad Y(\omega) = H(\omega) X(\omega)$$

(se bevis nästa sida)

Se även formelsamlingen och läroboken

Faltning övergår i produkt vid transformering

Givet

$$y(n) = x(n) * h(n) = \sum_{l} x(l)\ h(n-l)$$

Sök

$$\begin{aligned} Y(\omega) &= \sum_{n} y(n)\, e^{-j\omega n} = \\ &= \sum_{n} \sum_{l} x(l)\ h(n-l) \quad \underbrace{e^{-j\omega n}}_{e^{-j\omega(n-l)} \cdot e^{-j\omega l}} = \\ &= \sum_{l} x(l)\, e^{-j\omega l} \sum_{n} h(n-l)\, e^{-j\omega (n-l)} = \\ &= X(\omega) H(\omega) \end{aligned}$$

Appendix: Visa Fouriertransformen

Definition: Fouriertransform av tidsdiskret signal DTFT,

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi f n} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \omega = 2\pi f$$

$$x(n) = \int_{f=-1/2}^{1/2} X(f)\, e^{j2\pi f n}\, df = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(f)\, e^{j\omega n}\, d\omega$$

Visa invers fouriertransform.

$$x_{invers}(n) = \int_{f=-1/2}^{1/2} X(f)\, e^{j2\pi f n}\, df = \int_{f=-1/2}^{1/2} \underbrace{\sum_{k=-\infty}^{\infty} x(k)\, e^{-j2\pi f k}}_{X(f)} \; e^{j2\pi f n}\, df =$$

$$= \sum_{k=-\infty}^{\infty} x(k) \int_{f=-1/2}^{1/2} e^{-j2\pi f k} \; e^{j2\pi f n}\, df = \sum_{k=-\infty}^{\infty} x(k) \underbrace{\int_{f=-1/2}^{1/2} e^{-j2\pi f (k-n)} \; df}_{\begin{cases} 0 & om\ k-n \neq 0 \\ 1 & om\ k-n = 0 \end{cases} = \delta(n-k)} =$$

$$= x(n)$$

Quiz

Visa att

$$\frac{1}{2\pi}\int_{\omega=-\pi}^{\pi}\frac{2e^{j\omega}}{1-0.5\,e^{-j\omega}}d\omega=\int_{f=-1/2}^{1/2}\frac{2e^{j2\pi f}}{1-0.5\,e^{-j2\pi f}}df\ =1$$

I Matlab (numerisk integration, dvs approximation);

$$\int_{f=-1/2}^{1/2}\frac{2e^{j2\pi f}}{1-0.5\,e^{-j2\pi f}}df$$

```
i=sqrt(-1);
z=2*exp(i*2*pi*(-0.5:0.01:0.5))./(1-0.5*exp(-i*2*pi*(-0.5:0.01:0.5)));
ct = cumtrapz(z*0.01);ct(end)

ans =

   1.0000 + 0.0000i
```