UNDER-GRADUATE PROJECT

PARTIAL DIFFERENTIAL EQUATIONS

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PARTIAL DIFFERENTIAL EQUATION

A partial differential equation is an equation involving an unknown function of two or more variables and certain of its derivatives.

K^{th} ORDER PDE

For a fixed integer k>=1, let U denote the open subset of R^n. An equation of the form

$$F(D^k u(x), D^{k-1} u(x), \ldots\ldots, Du(x), u(x), x) = 0$$

where x ϵ U is called kth order PDE where

$$F: R^{n^k} \text{x} R^{n^{k-1}} \text{x} R^{n^{k-2}} \text{x} \ldots\ldots \text{x} R^{n^1} \text{x } U \rightarrow R$$

is given and

$$u: U \rightarrow R$$

is unknown.

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Linear PDE

A PDE $F(D^k u(x), D^{k-1} u(x), \ldots\ldots, Du(x), u(x), x) = 0$ is called linear if it has the form

$$\sum_{r \leq k} a_r(x)\, D^r u = f(x)$$

for given functions $a_r(x)$ and f.

General Form: $a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)u = d(x, y)$

Examples:

1- $2u_{xx} - 3u_{tt} + 4x + u = 0$

2- $5u_{xx} + 31u_{tt} - 9y + \cos 2t = 0$

Quasi-Linear PDE-

A PDE F(D^ku(x), D^{k-1}u(x),……, Du(x), u(x),x) = 0 is called quasi-linear if it has the form

$\sum_{r=k}(a_r(D^{k-1}\mathrm{u(x)}, \dots \dots, \mathrm{Du(x)}, \mathrm{u(x)}, \mathrm{x}) D^r$u + $a_o(D^{k-1}\mathrm{u(x)}, \dots \dots, \mathrm{Du(x)}, \mathrm{u(x)}, \mathrm{x}))$= f(x)

for given functions $a_r(x)$ $and\ f$,i.e., all the terms with highest order derivatives occur linearly and their coefficients of such terms are only lower order derivatives of dependent variable.

General Form: a(x, y, u) u_x + b(x, y, u) u_y + c(x, y, u)=0

Examples 1- $2uu_x u_{xx}$-3$u * u_{tt}$+4x+u=0

2- 5$u_x u_{xx}$+31u_{tt}-9y+cos2t+=0

Semi-Linear PDE-

A PDE F(D^ku(x), D^{k-1}u(x),……, Du(x), u(x),x) = 0 is called semi-linear if it has the form

$\sum_{r=k}(a_r(x) D^r$u + $a_o(D^{k-1}\mathrm{u(x)}, \dots \dots, \mathrm{Du(x)}, \mathrm{u(x)}, \mathrm{x}))$= f(x)

for given functions $a_r(x)$ $and\ f$,i.e., highest order terms have coefficient as functions of independent variable alone.

General Form: a(x, y) u_x + b(x, y) u_y + c(x, y, u)=0

Examples 1- 2u_{xx}-3u_{tt}+4x + u^3=0

2- 5u_{xx}+31u_{tt}-9y+u^2 =0

Non-linear PDE-

A PDE F(D^ku(x), D^{k-1}u(x),……, Du(x), u(x),x) = 0 is called non-linear if the highest order derivative of dependent variable appear non-linearly in the equation.

Examples

1- $u_x * u_x$ + $u_y * u_y$ =1

2- 2*u*$u_x * u_x$ + $u_{yy} * u_{yy} + xyu$ =9

CLASSIFICATION OF SECOND ORDER PDEs

A second order PDE looks like

$$Au_{xx} + Bu_{xy} + Cu_{yy} + I(x, y, u, u_x, u_y) = 0.$$

The type of the above equation depends on the sign of the quantity given by

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y)$$

At the point (x0, y0) the second order linear PDE (1) is called

$$\begin{aligned}&\textit{hyperbolic}, \text{ if } \Delta(x_0, y_0) > 0\\&\textit{parabolic}, \text{ if } \Delta(x_0, y_0) = 0\\&\textit{elliptic}, \text{ if } \Delta(x_0, y_0) < 0\end{aligned}$$

Examples:

1-xyu_{xx}- $(x^2 - y^2)u_{xy}$-xyu_{yy}=2$(x^2 - y^2)$ is hyperbolic as B^2-4AC value is greater than 0.

2-2u_{xx}+4u_{xy}+3u_{yy}=4 is elliptic as B^2-4AC value is lesser than 0.

3-y^2u_{xx} -2xyu_{xy}+x^2u_{yy}=x^2p-y^2 q is parabolic as B^2-4AC value is equals 0.

TRANSPORT EQUATION

The *transport equation* models the concentration of a substance flowing in a fluid at a constant rate.

Homogeneous Solution:

$$\begin{cases} u_t + b \cdot Du = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ \qquad\quad u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

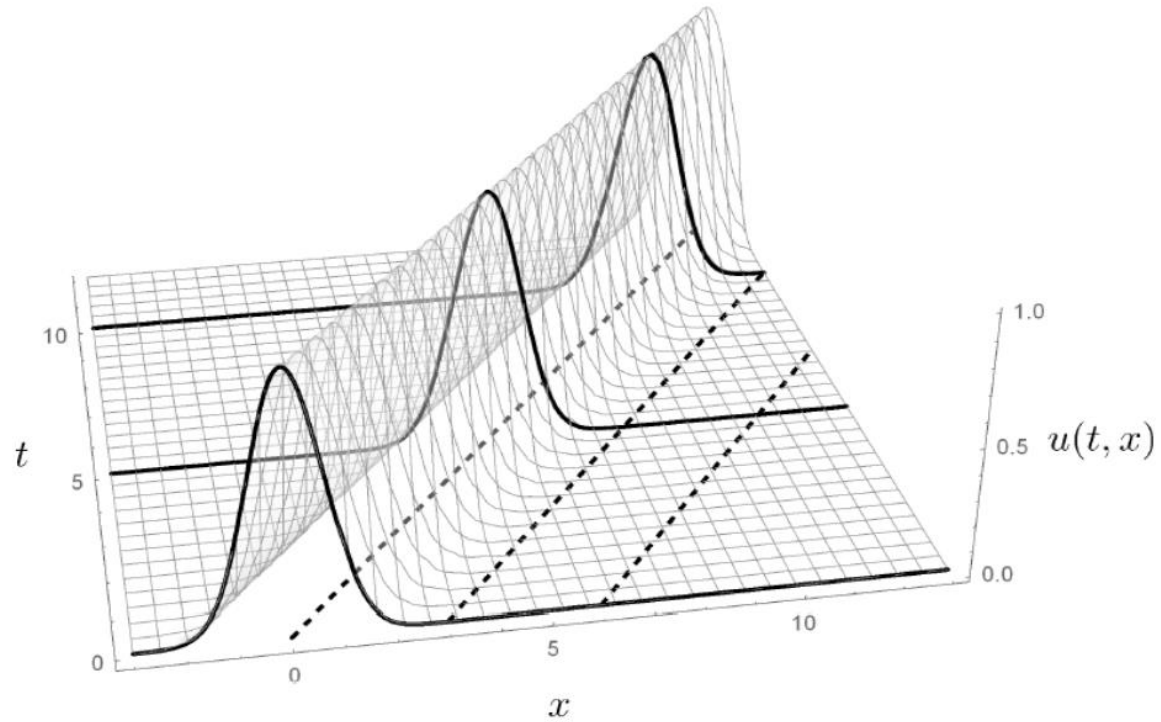
$$u(x, t) = g(x - tb) \quad (x \in \mathbb{R}^n, t \geq 0).$$

Example:

Non-Homogeneous Solution:

$$\begin{cases} u_t + b \cdot Du = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ \qquad\quad u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

$$u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s)\, ds \quad (x \in \mathbb{R}^n,\ t \geq 0)$$



The graph of u.

For each point (x,t), u(x,t) is constant on the line through (x,t) with the direction $(b, 1) \in \mathbb{R}^{n+1}$.

Examples for homogeneous transport equation:

1- u_t-3u_x=0, u(x,0) = 5^{-x^2} ,

therefore solution is u(x,t)= $5^{-(x+3t)^2}$

2- u_t+5u_x=0, u(x,0) = sinx,

therefore solution is u(x,t)=sin(x-5t)

LAPLACE EQUATION

Among the most important of all partial differential equations are undoubtedly *Laplace's equation*

$$(1) \qquad \Delta u = 0$$

Physical interpretation. Laplace's equation comes up in a wide variety of physical contexts. In a typical interpretation u denotes the density of some quantity (e.g. a chemical concentration) in equilibrium. Then if V is any smooth subregion within U, the net flux of u through ∂V is zero:

$$\int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu} \, dS = 0,$$

$\mathbf{F}$ denoting the flux density and $\boldsymbol{\nu}$ the unit outer normal field. In view of the Gauss–Green Theorem (§C.2), we have

$$\int_V \operatorname{div} \mathbf{F} \, dx = \int_{\partial V} \mathbf{F} \cdot \boldsymbol{\nu} \, dS = 0,$$

and so

$$(3) \qquad \operatorname{div} \mathbf{F} = 0 \quad \text{in } U,$$

since V was arbitrary. In many instances it is physically reasonable to assume the flux $\mathbf{F}$ is proportional to the gradient Du but points in the opposite direction (since the flow is from regions of higher to lower concentration). Thus

$$(4) \qquad \mathbf{F} = -aDu \quad (a > 0).$$

Substituting into (3), we obtain Laplace's equation

$$\operatorname{div}(Du) = \Delta u = 0.$$

If u denotes the $\begin{cases} \text{chemical concentration} \\ \text{temperature} \\ \text{electrostatic potential,} \end{cases}$

equation (4) is

$$\begin{cases} \text{Fick's law of diffusion} \\ \text{Fourier's law of heat conduction} \\ \text{Ohm's law of electrical conduction.} \end{cases}$$

Let us attempt to find the solution of Laplace equation in U=R^n of the from

$$u(x) = v(r),$$

where $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and v is to be selected (if possible) so that $\Delta u = 0$ holds. First note for $i = 1, \ldots, n$ that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)^{-1/2} 2x_i = \frac{x_i}{r} \quad (x \neq 0).$$

We thus have

$$u_{x_i} = v'(r)\frac{x_i}{r}, \; u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

for $i = 1, \ldots, n$, and so

$$\Delta u = v''(r) + \frac{n-1}{r}v'(r).$$

Hence $\Delta u = 0$ if and only if

$$v'' + \frac{n-1}{r}v' = 0. \tag{5}$$

If $v' \neq 0$, we deduce

$$\log(|v'|)' = \frac{v''}{v'} = \frac{1-n}{r},$$

and hence $v'(r) = \frac{a}{r^{n-1}}$ for some constant a. Consequently if $r > 0$, we have

$$v(r) = \begin{cases} b \log r + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3), \end{cases}$$

where b and c are constants.

Examples: $x^2 - y^2, \; x^3 - 3xy^2$

HARMONIC FUNCTION-

A C^2 function u satisfying (1) is called a harmonic function.

Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

Proof:

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y) + (-e^{-x})(x\sin y - y\cos y)$$

$$= e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y)$$

$$= -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y \quad \text{-----------------------------} \ (1)$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x}\cos y - ye^{-x}\sin y - e^{-x}\cos y)$$

$$= -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y \quad \text{-------------------------------} \ (2).$$

Adding (1) and (2) yield

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence u is harmonic.

MEAN VALUE FORMULAS-

It declares that u(x) equals both the average of u over the sphere $\partial B(x,r)$ and the average of u over the entire ball $B(x,r)$, provided $B(x,r)$ is a subset of U.

THEOREM: If U ϵ C^2 is harmonic, then

$$u(x) = \fint_{\partial B(x,r)} u\, dS = \fint_{B(x,r)} u\, dy$$

Proof. 1. Set

$$\phi(r) := \fint_{\partial B(x,r)} u(y)\, dS(y) = \fint_{\partial B(0,1)} u(x+rz)\, dS(z).$$

Then

$$\phi'(r) = \fint_{\partial B(0,1)} Du(x+rz)\cdot z\, dS(z),$$

and consequently, using Green's formulas, we compute

$$\begin{aligned}
\phi'(r) &= \fint_{\partial B(x,r)} Du(y)\cdot \frac{y-x}{r}\, dS(y) \\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\, dS(y) \\
&= \frac{r}{n}\fint_{B(x,r)} \Delta u(y)\, dy = 0.
\end{aligned}$$

Hence ϕ is constant, and so

$$\phi(r) = \lim_{t\to 0}\phi(t) = \lim_{t\to 0}\fint_{\partial B(x,t)} u(y)\, dS(y) = u(x).$$

And secondly,

$$\begin{aligned}
\int_{B(x,r)} u\, dy &= \int_0^r \left(\int_{\partial B(x,s)} u\, dS\right) ds \\
&= u(x)\int_0^r n\alpha(n)s^{n-1}ds = \alpha(n)r^n u(x).
\end{aligned}$$

THEOREM (CONVERSE TO MEAN VALUE PROPERTY): If U ϵ C^2 satisfies

$$u(x) = \fint_{\partial B(x,r)} u\, dS$$

for each ball $B(x,r)$ ϵ U, then u is harmonic.

Proof. If $\Delta u \not\equiv 0$, there exists some ball $B(x,r) \subset U$ such that, say, $\Delta u > 0$ within $B(x,r)$. But then for ϕ as above,

$$0 = \phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\, dy > 0,$$

a contradiction. Similarly for negative case of Laplacian of u also, we can check.

STRONG MAXIMUM PRINCIPLE-

It states that a harmonic function must attain its maximum on the boundary and can't attain maximum in the interior of a connected region unless it is constant.

Proof. Suppose there exists a point $x_0 \in U$ with $u(x_0) = M := \max_{\bar{U}} u$. Then for $0 < r < \operatorname{dist}(x_0, \partial U)$, the mean-value property asserts

$$M = u(x_0) = \fint_{B(x_0,r)} u\, dy \leq M.$$

As equality holds only if $u \equiv M$ within $B(x_0,r)$, we see $u(y) = M$ for all $y \in B(x_0,r)$. Hence the set $\{x \in U \mid u(x) = M\}$ is both open and relatively closed in U and thus equals U if U is connected. This proves assertion (ii), from which (i) follows. $\square$

Positivity. The strong maximum principle asserts in particular that if U is connected and $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U, \end{cases}$$

where $g \geq 0$, then u *is positive everywhere in* U *if* g *is positive somewhere on* ∂U.

An important application of the maximum principle is establishing the uniqueness of solutions to certain boundary-value problems for Poisson's equation.

THANK YOU