CHAPTER **2**

Functions of a Complex Variable

2.1 INTRODUCTION

We are familiar with the concepts of limit, differentiation and integration concerning the functions of real variables. Similar concepts can be defined with reference to complex variables also and their study constitutes "complex analysis". Here we will define functions of a complex variable and then introduce the concepts of limits, continuity and differentiability for those functions.

A basic understanding of complex variable theory will be useful in diverse branches of science and engineering.

2.2. FUNCTIONS OF COMPLEX VARIABLE

Def. Suppose D is a set of complex numbers. A rule f defined on D which assigns to every z in D, a complex number w, is called a function f or mapping f on D and we write $w = f(z)$.

Here z is a complex variable and can be written as $z = x + iy$ where x, y, are real and $i = \sqrt{-1}$. The set D is called **domain** of definition of f. The set of all $w = f(z)$ where $z \in D$ is called the **range** of f.

The image of z under the function f is $w = f(z)$ and as stated, this is also a complex number.

We write $w = f(z) = u + iv$ where u and v are real.

Since $z = x + iy$ and z depends on x and y, we notice that u and v also depend on x and y so that $u = u(x, y)$ and $v = v(x, y)$

Hence, in general, we write $w = f(z) = u(x, y) + iv(x, y)$. These u and v are called the real and imaginary parts of $w = f(z)$.

SOLVED EXAMPLES

Example 1 : *Let $w = f(z) = z^2, \forall z$. Find the values of w which correspond to*

(i) $z = 2 + i$ (ii) $z = 1 + 3i$

and also its real and imaginary parts.

Solution : Given $w = f(z) = z^2$

$\therefore \quad w = f(2+i) = (2+i)^2 = 4 + i^2 + 4i = 3 + 4i$

and $w = f(1+3i) = (1+3i)^2 = -8 + 6i$

Let $z = x + iy$. Then

$w = f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + 2ixy = u + iv$ (say)

Equating real and imaginary parts on either side, we have

Real part of $f(z) = u = x^2 - y^2$ and

Imaginary part of $f(z) = v = 2xy$

(i) Real part of $f(z) = 3$

Imaginary part of $f(z) = 4$

(ii) Real part of $f(z) = -8$

Imaginary part of $f(z) = 6$

Example 2 : *If $w = f(z) = z^2 + z$, find its real and imaginary parts. Also find $f(z)$ at $1 + i$.*

<u>Solution :</u> $w = f(z) = z^2 + z = (x+iy)^2 + (x+iy)$

$$= \left(x^2 - y^2 + 2ixy\right) + (x+iy) = \left(x^2 - y^2 + x\right) + i\,(2xy + y) = u + iv \text{ (say)}$$

Then $u = x^2 - y^2 + x$ and $v = 2xy + y$

and $f(1+i) = (1+i)^2 + (1+i) = 1 + 3i$

Example 3 : *Let $w = f(z) = iz + 3\bar{z}$. Find u and v and the value of $f(z)$ at $1 + 4i$.*

<u>Solution :</u> $w = f(z) = i(x+iy) + 3(x-iy) \quad (\because\ \bar{z} = x - iy)$

$= (3x - y) + i(x - 3y) = u + iv$ (say)

$\therefore$ We have $u = 3x - y$ and $v = x - 3y$

$\therefore \quad f(1+4i) = i(1+4i) + 3(1-4i) = i - 4 + 3 - 12i = -1 - 11i$

Note. The locus of z such that $|z - z_0| = R$ is a circle with centre z_0 and radius R in the z-plane

$\{z : |z - z_0| < R\} =$ the set of all points of z within the circle with centre z_0 and radius R in the z-plane.

2.3. $\in$ – DISC AROUND $w = w_0$ IN THE COMPLEX w-PLANE:

Fig. 2.1

Let us represent all the complex numbers $w = u + iv$ where u, v are real on a rectangular system of cartesian (u, v) coordinate plane. This is called the w-plane or (u, v) plane.

Let w_0 be represented on this plane. Then, $\{w \mid |w - w_0| < \in\}$ is called the $\in$ disc around w_0.

This is also called as an $\in$–neighbourhood of w_0. $\{w \mid 0 < |w - w_0| < \in\}$ is called the deleted $\in$ disc around w_0.

Similarly we represent the complex number $z = x + iy$ on the rectangular system of cartesian (x, y) plane, and this plane is referred to as complex z-plane or (x, y) plane. A δ-disc around z_0 in this plane can also be defined as above.

2.4. LIMIT AND CONTINUITY

Def. Limit of $f(z)$. A function $w = f(z)$ is said to tend to limit 'l' as z approaches a point z_0, if for every real $\in$, we can find a positive δ such that $|f(z) - l| < \in$ for $0 < |z - z_0| < \delta$ *i.e.*, for every $z \neq z_0$ in the δ-disc in z-plane, $f(z)$ has a value lying in the $\in$-disc of w-plane.

In this case symbolically we write $\underset{z \to z_0}{Lt} f(z) = l$

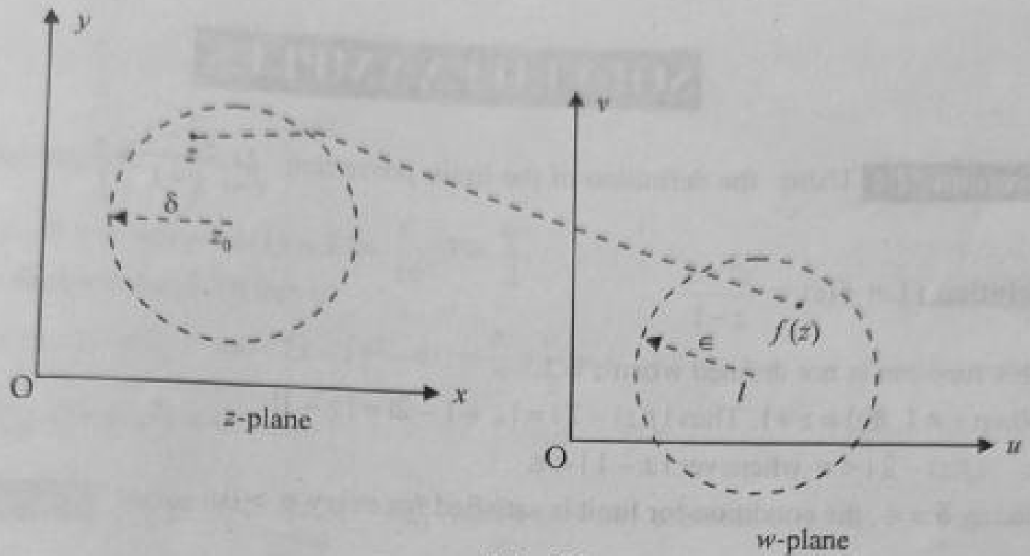

Fig. 2.2

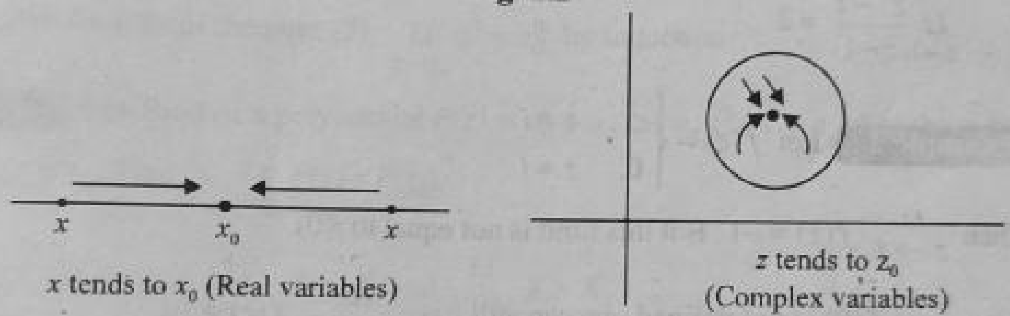

Fig. 2.3

Note. Though this definition is similar to the definition of limits, in the case of real number theory there is a big difference. In real case x can approach x_0 only along the real line either from its left or from its right, whereas in the above definition z may approach z_0 from any direction in the complex plane (fig. 2.3).

2.5. THEOREMS ON LIMITS

1. If the limit of a function exists as $z \to z_0$, then it is unique.

Note. For $\underset{z \to z_0}{\text{Lt}} f(z) = l$, it is imperative that $f(z)$ must be defined at all points z in a disc around z_0. But it is not required that $f(z_0)$ is defined.

2. Let $f(z) = u(x, y) + i\,v(x, y)$, $z = x + iy$, $z_0 = x_0 + iy_0$, then the condition that limit of $f(z)$ exists at z_0, $\underset{z \to z_0}{Lt} f(z) = u_0 + iv_0$ is satisfied if and only if $\underset{\substack{x \to x_0 \\ y \to y_0}}{Lt} u(x, y) = u_0$ and $\underset{\substack{x \to x_0 \\ y \to y_0}}{Lt} v(x,y) = v_0$.

3. Let f, F be functions whose limit exist at z_0 and $\underset{z \to z_0}{Lt} f(z) = W_0$ and $\underset{z \to 0}{Lt} F(z) = w_0$.

Then (*i*) $\underset{z \to z_0}{Lt} [f(z) + F(z)] = W_0 + w_0$ (*ii*) $\underset{z \to z_0}{Lt} \left[f(z) \cdot F(z)\right] = W_0 w_0$

(*iii*) $\underset{z \to z_0}{Lt} \left[\dfrac{f(z)}{F(z)}\right] = \dfrac{W_0}{w_0}$, $w_0 \neq 0$ (*iv*) $\underset{z \to z_0}{Lt} \text{C} f(z) = \text{C}.W_0$.

SOLVED EXAMPLES

Example 1 : Using the definition of the limit, prove that $\underset{z \to 1}{Lt} \frac{z^2-1}{z-1} = 2$

Solution : Let $f(z) = \frac{z^2-1}{z-1}$

This function is not defined when $z = 1$.

When $z \neq 1, f(z) = z+1$. Thus $|f(z) - 2| = |z+1-2| = |z-1|$

$\therefore \quad |f(z) - 2| < \epsilon$ whenever $|z-1| < \epsilon$

Taking $\delta = \epsilon$, the condition for limit is satisfied for every $\epsilon > 0$.

$$\therefore \quad \underset{z \to 1}{Lt} \frac{z^2-1}{z-1} = 2$$

Example 2 : Let $f(z) = \begin{cases} z^2, & z \neq i \\ 0, & z = i \end{cases}$

Then $\underset{z \to i}{Lt} f(z) = -1$. But this limit is not equal to $f(0)$.

Also, even $f(0)$ is not defined, we can still have $\underset{z \to i}{Lt} f(z) = -1$

Example 3 : Show that $\underset{z \to z_0}{Lt} z = z_0$

Solution : Let $f(z) = z$

$|f(z) - z_0| = |z - z_0|$

If we take $\delta = \epsilon$, we get that

$|f(z) - z_0| < \epsilon$ when $|z - z_0| < \delta$

Thus condition for limit is proved and hence the result.

Example 4 : Limit of a constant function is that constant.

Solution : Let $f(z) = K$, K is a constant. Then

$|f(z) - K| = |K - K| = 0 < \epsilon$.

This is true for all $\delta > 0$.

$\therefore \quad |z - K| < \delta \Rightarrow |f(z) - K| < \epsilon$

$$\therefore \quad \underset{z \to z_0}{Lt} f(z) = K.$$

Example 5 : Show that $\underset{z \to 2i}{Lt} (2x + iy^2) = 4i$

Solution : Consider $f(z) = 2x + iy^2$

$$\begin{aligned} |f(z) - 4i| &= |2x + iy^2 - 4i| \\ &\leq |2x| + |y^2 - 4| \\ &= 2|x| + |y-2||y+2| \end{aligned}$$

Let $\epsilon > 0$ and $|x| < \epsilon/4$, $|y - 2| < 1$

$\Rightarrow \quad 2|x| < \frac{\epsilon}{2}, |y+2| = |y-2+4| \le |y-2|+4 < 5$

Consider $\min\left(\frac{\epsilon}{10}, 1\right) = n$.

Then $|y-2| < n \Rightarrow |y-2||y+2| = \frac{\epsilon}{10} \cdot 5 = \frac{\epsilon}{2}$

We can find suitable δ for this ϵ.

$\therefore \quad 0 < |z-2i| < \delta \quad \Rightarrow |2x+iy^2-4i| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$

$\therefore \quad \underset{z \to 2i}{Lt} (2x+iy^2) = 4i$

Example 6 : Prove that $\underset{z \to z_0}{Lt} z^n = z_0^n$ where z_0 is any complex number.

Solution : We have from theorem (3), $\underset{z \to z_0}{Lt} z^n = z_0^n$ by induction.

Example 7 : The limit of a polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ is the value of the polynomial at z_0, $\forall z_0$. That is, $\underset{z \to z_0}{Lt} P(z) = P(z_0)$

Solution : We know that $\underset{z \to z_0}{Lt} z^n = z_0^n$ and $\underset{z \to z_0}{Lt} K = K$.

Also, $\underset{z \to z_0}{Lt} [f(z) + F(z)] = \underset{z \to z_0}{Lt} f(z) + \underset{z \to z_0}{Lt} F(z)$

Using this theorem, we get $\underset{z \to z_0}{Lt} P(z) = P(z_0)$

2.6. CONTINUITY

A function $f(z)$ is said to be ***continuous at*** $z = z_0$, if $f(z_0)$ is defined and $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$.

We observe that for the function $f(z)$ to be continuous at z_0 the function must be defined in some neighbourhood of z_0 including z_0 and $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$.

(or) A function f is continuous at a point z_0, if, corresponding to each positive number ϵ, a number δ exists, such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Def. $f(z)$ is said to be *continuous in a domain* if it is continuous at every point of that domain.

Theorem : If $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0 = x_0 + iy_0$, then u and v are also continuous at $z = z_0$, *i.e.*, at (x_0, y_0). Conversely, if $u(x, y)$ and $v(x, y)$ are both continuous at (x_0, y_0) then $f(z)$ will be continuous at $z = z_0$.

Properties of continuous functions :

In view of the above theorem, the properties of continuous functions of z can be deduced from the properties of continuous functions u and v of x and y.

1. If f is a continuous function of z at every point in a closed region R, then f is **bounded** in R for some constant M.
   $|f(z)| < M, \ \forall z$ in R.
2. If $f(z)$ and $F(z)$ are continuous at z_0, then

   (i) $f(z) + F(z)$ (ii) $f(z) - F(z)$ (iii) $f(z).F(z)$ (iv) $\dfrac{f(z)}{F(z)} (F(z) \neq 0)$ are also continuous at z_0.
3. Let D be the domain of the function $f(z)$, for all z in some neighbourhood N of a point z_0. Let the range of a function g be included in D. Then $f(g(z))$ is defined when z is in N. If g is continuous at z_0 and f is continuous at $g(z_0)$ then the **Composite function** of z, f(g) is continuous at z_0. **Thus a continuous function of a continuous function is continuous.**
4. Any polynomial function is continuous for all z.

   if $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots\cdots + a_n z^n$ then for every z_0, we have

   $$\underset{z \to z_0}{Lt} \; p(z) = p(z_0).$$

   Thus polynomial function is continuous.
5. Any irrational function (quotient of polynomials) is continuous whenever its denominator is non-zero.
6. All trigonometric and exponential functions are continuous
7. If $f(z)$ is continuous at z_0, then $|f(z)|$ is also continous at z_0.

SOLVED EXAMPLES

Example 1 : Show that the function $f(z) = \dfrac{\bar{z}}{z}$ is not continuous at $z = 0$.

Solution : Let $z = x + iy$

Suppose $z \to 0$ along x –axis. Then we have $y = 0$, $z = x$ and $\bar{z} = x$.

$$\therefore \underset{z\to 0}{Lt} \frac{\bar{z}}{z} = \underset{x\to 0}{Lt} \frac{x}{x} = 1$$

Again suppose $z \to 0$ along y–axis.

Then $x = 0$, $z = iy$ and $\bar{z} = -iy$

$$\therefore \underset{z\to 0}{Lt} \frac{\bar{z}}{z} = \underset{iy\to 0}{Lt} \left(\frac{-iy}{iy}\right) = -1$$

$$\therefore \underset{z\to 0}{Lt} \frac{\bar{z}}{z} \text{ does not exist.}$$

Example 2 : Show that the function $f(z) = \bar{z}$ is continuous over C.

Solution : We have $|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0|$. For any given $\epsilon > 0$ choose $\epsilon = \delta$, we get

$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$

$\therefore$ $f(z)$ is continuous at z_0.

Here z_0 is arbitrary. Thus $f(z)$ is continuous over C.

Example 3 : $f : C \to C$ is defined as $f(z) = |z|$. Then $f(z)$ is continuous on C.

Solution : We have $|f(z) - f(z_0)| = ||z| - |z_0|| \leq |z - z_0|$

For any $\epsilon > 0$ if we choose $\epsilon = \delta$, we get

$|z-z_0|<\delta \Rightarrow |f(z)-f(z_0)|<\epsilon.$

$\therefore$ $f(z)$ is continuous at z_0.

Here z_0 is arbitrary. Thus $f(z)$ is continuous over C.

Example 4 : Show that

(i) $f(z) = xy^2 + i(2x-y)$ (ii) $f(z) = e^x + i\,sin(xy)$ are continuous for all z.

Solution : (i) $f(z) = xy^2 + i(2x-y)$

Consider $f(z) = u(x, y) + i\,v(x, y)$

where $u(x, y) = xy^2$ and $v(x, y) = 2x - y$

which are continuous everywhere. By theorem $f(z)$ is also continuous everywhere.

(ii) $f(z) = e^x + i\sin(xy)$

Let $f(z) = u(x,y) + iv(x,y)$

$u(x, y) = e^x$, $v(x, y) = \sin xy$ which are continuous every where. By theorem $f(z)$ is also continuons everywhere.

Example 5 : *Is the function* $f(z) = \begin{cases} \dfrac{z^2+3iz-2}{z+i} & \text{for } z \neq i \\ 5, & \text{for } z = -i \end{cases}$ *continuous. If not can the function be refined to make it continuous at* $z = -i$?

Solution : $f(z) = \dfrac{g(z)}{h(z)}$ is continuous when $g(z)$ and $h(z)$ are continuous except at $h(z) = 0$.

So $f(z)$ is continuous every where except at $z = -i$, since $g(z)$, $h(z)$ are continuous.

Continuity at z = –i

$$\underset{z\to -i}{Lt} f(z) = \underset{\substack{x\to 0\\ y\to -1}}{Lt} \frac{(x+iy)^2+3i(x+iy)-2}{x+iy+i}$$

$$= \underset{y\to -1}{Lt} \frac{-y^2-3y-2}{i(y+1)} = \underset{y\to -1}{Lt} \frac{-y-2}{i} = -\frac{1}{i} = i$$

Also $$\underset{z\to -i}{Lt} f(z) = \underset{\substack{y\to -1\\ x\to 0}}{Lt} f(z) = \underset{x\to 0}{Lt} \frac{2(x-i)+3i}{1} = i$$

$$\therefore \quad \underset{z\to -i}{Lt} f(z) = i \neq 5 = f(-i)$$

$\therefore$ f is not continuous at $z = -i$.

Suppose we define $f(z)$ as $f(-i) = i$ instead of 5, then $f(z)$ is continuous at $z = -i$ and is therefore ntinuous everywhere. This discontinuity *at* $z = -i$ is known as removable discontinuity.

7. DERIVATIVE OF $f(z)$

Def. Let $w = f(z)$ be a given function defined for all z in a neighbourhood of z_0. If $\underset{\Delta z\to 0}{Lt} \dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ exists, the function $f(z)$ is said to be derivable at z_0 and the limit is noted by $f'(z_0)$. $f'(z_0)$ if exists is called the derivative of $f(z)$ at z_0.

Taking $z - z_0 = \Delta z$ we notice that $f'(z_0) = \underset{z\to z_0}{Lt} \dfrac{f(z)-f(z_0)}{z-z_0}$.

If f is differentiable at each point of a set, then we say that f differentiable on that set

SOLVED EXAMPLES

Example 1 : *Using the definition of derivative, find the derivative of z^2 for all z.*

Solution : $f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \dfrac{(z+\Delta z)^2 - z^2}{\Delta z} = \underset{\Delta z \to 0}{\text{Lt}} \dfrac{2z\,\Delta z + \Delta z^2}{\Delta z} = \underset{\Delta z \to 0}{\text{Lt}} (2z + \Delta z) = 2z$

Example 2 : *Find the derivative of $w = f(z) = z^3 - 2z$ at the point where*

(i) $z = z_0$ *(ii)* $z = 1$.

Solution : (*i*) We have

$$f'(z_0) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$f'(z_0) = \underset{\Delta z \to 0}{\text{Lt}} \frac{(z_0 + \Delta z)^3 - 2(z_0 + \Delta z) - (z_0^3 - 2z_0)}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \frac{z_0^3 + (\Delta z)^3 + 3z_0^2(\Delta z) + 3z_0(\Delta z)^2 - 2z_0 - 2\Delta z - z_0^3 + 2z_0}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} [3z_0^2 + 3z_0\,\Delta z + (\Delta z)^2 - z] = 3z_0^2 - 2$$

In general $f'(z) = 3z^2 - 2, \forall z$.

(*ii*) Substituting $z_0 = 1$, we get $f'(1) = 3 - 2 = 1$.

Example 3 : *Show that $f(z) = \bar{z}$ is not differentiable any where.*

Solution : Let $z = x + iy$. Then $\bar{z} = x - iy$ and $z = x + iy$, $\Delta z = \Delta x + i\Delta y$

$\therefore \quad f(z) = x - iy$

$\therefore \quad z + \Delta z = (x + \Delta x) + i(y + \Delta y)$

We have $f(z+\Delta z) = \overline{z + \Delta z} = (x + \Delta x) - i(y + \Delta y)$ and $f(z) = \bar{z} = x - iy$

$$\therefore \quad \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\underset{\Delta z \to 0}{\text{Lt}} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ exists, the limit must be same whatever be the direction along which $\Delta z \to 0$.

Let $\Delta z \to 0$ along x-axis; (*i.e.*) Take $\Delta z = \Delta x$ only.

$$\underset{\Delta z \to 0}{\text{Lt}} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \underset{\Delta z \to 0}{\text{Lt}} \frac{\Delta x}{\Delta x} = 1 \qquad (\because \Delta y = 0)$$

Let $\Delta z \to 0$ along y-axis; (*i.e.*) Take $\Delta z = i\Delta y$; $\Delta x = 0$.

$$\underset{\Delta z \to 0}{\text{Lt}} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \underset{\Delta y \to 0}{\text{Lt}} \frac{-i\Delta y}{i\Delta y} = -1$$

Thus, $\dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ does not tend to the same limit as $\Delta z \to 0$. Hence, $f'(z)$ does not exist at any z.

Example 4 : *Show that the function $f(z) = z^n$, where n is a positive integer is differentiable for all values of z, where n is a positive integer.*

Solution : Using the definition of the derivative, we have **[JNTU 2002, 2003]**

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \underset{\Delta z \to 0}{\text{Lt}} \frac{(z+\Delta z)^n - z^n}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \frac{z^n + nz^{n-1}\Delta z + \frac{n(n-1)}{2} z^{n-2}(\Delta z)^2 + \ldots + (\Delta z)^n - z^n}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \left[nz^{n-1} + \frac{n(n-1)}{2} z^{n-2}(\Delta z) + \ldots + (\Delta z)^{n-1} \right] = nz^{n-1}$$

Hence, $f'(z)$ exists for all values of z.

2.8. THEOREM

If a function is differentiable at a point, then it is continuous there.

Proof. Let $f(z)$ be differentiable at z_0.

Then $f'(z_0) = \underset{z \to z_0}{\text{Lt}} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

$\therefore$ $f(z_0)$ is well defined.

Let $\underset{z \to z_0}{\text{Lt}} [f(z) - f(z_0)] = \underset{z \to z_0}{\text{Lt}} \left[\frac{f(z) - f(z_0)}{z - z_0} . (z - z_0) \right] = \underset{z \to z_0}{\text{Lt}} \left[\frac{f(z) - f(z_0)}{z - z_0} \right] \underset{z \to z_0}{\text{Lt}} (z - z_0)$

$$= f'(z_0) . \underset{z \to z_0}{\text{Lt}} (z - z_0) = 0$$

$\therefore$ $\underset{z \to z_0}{\text{Lt}} f(z) = \underset{z \to z_0}{\text{Lt}} f(z_0) = f(z_0)$

$\therefore$ $f(z)$ is continuous at z_0.

Note. The converse of the above theorem is not true. (*i.e.*) a function can be continuous at a point, but not differentiable at that point.

Example 5 : $f(z) = |z|^2$ *is a function which is continuous at all z but not derivable at any* $z \neq 0$.

Solution : Let $f(z) = |z|^2 = z\bar{z}$. Then $f(z_0) = z_0\bar{z}_0$

$\underset{z \to 0}{Lt}$ $z\bar{z}$
$(x+iy)(x-iy)$
$x^2 - ixy + ixy + y^2$

$$\therefore f(z_0 + \Delta z) = (z_0 + \Delta z)(\bar{z}_0 + \overline{\Delta z}) = z_0\bar{z}_0 + z_0 \cdot \overline{\Delta z} + \Delta z \cdot \bar{z}_0 + \Delta z \cdot \overline{\Delta z}$$

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{z_0 \cdot \overline{\Delta z} + \Delta z \cdot \overline{z_0} + \Delta z \cdot \overline{\Delta z}}{\Delta z}$$

$\underset{\substack{x \to 0 \\ y \to 0}}{Lt} (x^2 + y^2) = 0$
$= 0$

Consider the limit as $\Delta z \to 0$.

Case I. Let $\Delta z \to 0$ along x-axis. Then, $\Delta x = \Delta x$, $\Delta y = 0$ and $\Delta z = \Delta x$

$$\therefore \quad \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z} = \underset{\Delta x \to 0}{\text{Lt}} \frac{z_0\Delta x + \Delta x\, \bar{z}_0 + \Delta x \cdot \Delta x}{\Delta x} = z_0 + \bar{z}_0 \quad \text{...(1)}$$

Case II. Let $\Delta z \to 0$ along y-axis.

Then $\Delta x = 0$; $\Delta y = \Delta y$; $\Delta z = i\Delta y$

$$\underset{\Delta z \to 0}{\text{Lt}} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z} = \underset{\Delta y \to 0}{\text{Lt}} \frac{z_0(-i\Delta y) + i\Delta y\, \bar{z}_0 + (i\Delta y)(-i\Delta y)}{i\Delta y} = -z_0 + \bar{z}_0 \quad ...(2)$$

Thus, from (1) and (2), for $f'(z_0)$ to exist $z_0 + \bar{z}_0 = -z_0 + \bar{z}_0$.

i.e., $z_0 = -z_0$ *i.e.*, $2z_0 = 0$ *i.e.*, $z_0 \neq 0$.

$\therefore f'(z)$ does not exist though $f(z) = |z|^2$ is continuous at all z.

2.9. RULES FOR DIFFERENTIATION

The rules of differentiation for functions of complex variables are similar to those of functions of real variables.

Theorem: If $f(z)$ and $g(z)$ are differentiable functions in a domain D, then their sum, product and quotient are also differentiable and we have,

(*i*) $\frac{d}{dz}[f(z) \pm g(z)] = \frac{d}{dz}[f(z)] \pm \frac{d}{dz}[g(z)]$

(*ii*) $\frac{d}{dz}[cf(z)] = c \cdot \frac{d}{dz}[f(z)]$, where '$c$' is a constant.

(*iii*) $\frac{d}{dz}[f(z) \cdot g(z)] = f(z) \cdot \frac{d}{dz}[g(z)] + g(z) \cdot \frac{d}{dz}[f(z)]$

(*iv*) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{\frac{d}{dz}[f(z)] \cdot g(z) - \frac{d}{dz}[g(z)] \cdot f(z)}{[g(z)]^2}$, $[g(z) \neq 0]$

(*v*) If $f(z) = F[g(z)]$, then $\frac{d}{dz}[f(z)] = F'[g(z)] \cdot g'(z)$ (chain rule)

2.10. ANALYTIC FUNCTIONS

[JNTU 2004, Nov. 2006, 2007S, Nov.2008S (Set No.2,3)]

Def. Let a function $f(z)$ be derivable at every point z in an $\in$ neighbourhood of z_0;

i.e., $f'(z)$ exists for all z such that $|z - z_0| < \in$ where $\in > 0$.

Then, $f(z)$ is said to be analytic at z_0.

The terms **Regular** or **Holomorphic** are synonyms of analytic.

Note. If $f(z)$ is analytic at z_0,

(*i*) $f'(z_0)$ exists and (*ii*) $f'(z)$ exists at every point z in a neighbourhood of z_0.

2.11. Def. Let D be a domain of complex numbers.

If $f(z)$ is analytic at every $z \in D$, $f(z)$ is said to be analytic in the domain D.

If $f(z)$ is analytic at every point z on the complex plane, $f(z)$ is said to be an ***entire function*** or ***integral function***.

That is, a point at which an analytic function ceases to have a derivative is called a **singular point**. An analytic function is also known as **regular** or **holomorphic**.

If $f'(z_0)$ does not exist then $z = z_0$ is called a singular point of $f(z)$.

If $f'(z)$ exists at every point in a neighbourhood of z_0 but $f'(z_0)$ does not exist, then z_0 is said to be an ***isolated singular point*** of $f(z)$.

e.g. $f(z)=\dfrac{1}{z}$ is analytic at every point $z \neq 0$.

$$\frac{\frac{1}{z+\Delta z}-\frac{1}{z}}{\Delta z} = \frac{z-z-\Delta z}{z(z+\Delta z)\cdot\Delta z} = \frac{-1}{z(z+\Delta z)} = \frac{-1}{z^2}$$

In fact $f'(z)=-\dfrac{1}{z^2}$ if $z \neq 0$.

At $z = 0$, $f'(z)$ does not exist.

$\therefore$ $z = 0$ is an isolated singular point of $f(z)$.

2.12. POLYNOMIALS AND RATIONAL FUNCTIONS

We know that $f(z) = z^n$ is differentiable for all z.

Applying the above formula for $z, z^2, z^3, \ldots$, and the rules of differentiation, we can see that a "polynomial"

$$f(z)=a_0+a_1z+a_2z^2+\ldots+a_nz^n \text{ is an analytic function of } z.$$

More generally, a "rational function" $f(z)=\dfrac{a_0+a_1z+a_2z^2+\ldots+a_nz^n}{b_0+b_1z+b_2z^2+\ldots+b_mz^m}$ is an analytic function of z throughout any finite domain in the complex plane where the denominator is not zero.

Also "partial fractions" $\dfrac{c}{(z-z_0)^m}\ (c \neq 0)$ where c and z_0 are complex numbers, m is a positive integer are special rational functions. They are analytic every where except at z_0.

2.13. CAUCHY—RIEMANN (C–R) EQUATIONS [JNTU 2003, 2008]

Having defined analytical function, we will prove a very important result to test the analyticity of a complex function.

Theorem : The necessary and sufficient condition for the derivative of the function $f(z)=w=u(x, y)+iv(x, y)$ to exist for all values of z in domain R are

(*i*) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x and y in R.

(*ii*) $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$ **[JNTU 2009S, JNTU (A) Nov. 2009 (Set No. 1)]**

The above relations are known as **Cauchy–Riemann equations.**

(Or) Derive the necessary and sufficient condition for $f(z)$ to be analytic in cartesian co-ordinates.

Proof : Necessity.

[JNTU 1998, 1996, 2002S, Nov. 2006, 2007S, Nov. 2008S, Nov. 2008 (Set No. 4)]

If $f(z) = u + iv$ is analytic in domain R, then u and v satisfy the equations

$\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$ provided the partial derivatives exist.

In order for $f(z)$ to be analytic, the limit

$$\underset{\Delta z\to 0}{\text{Lt}}\ \frac{f(z+\Delta z)-f(z)}{\Delta z} = f'(z)$$

$$= \underset{\substack{\Delta x\to 0\\ \Delta y\to 0}}{\text{Lt}}\ \frac{\{u(x+\Delta x, y+\Delta y)+iv(x+\Delta x, y+\Delta y)\}-\{u(x, y)+iv(x, y)\}}{\Delta x+i\Delta y}$$

must be existing ...(1)

Let $f'(z)$ exist, then $\underset{\Delta z \to 0}{\text{Lt}} \dfrac{f(z+\Delta z)-f(z)}{\Delta z}$ exists. Then limit is same for all the directions along which $\Delta z \to 0$.

Case I. Let $\Delta z \to 0$ along x-axis. Take $\Delta z = \Delta x$.

Taking limit as $\Delta z \to 0$ of (1), we get

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \left[\frac{u(x+\Delta x, y+\Delta y)-u(x,y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y)-v(x,y)}{\Delta x + i\Delta y} \right]$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \left[\frac{u(x+\Delta x, y)-u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y)-v(x,y)}{\Delta x} \right] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

[u_x, v_x exist as $f'(z)$ exists]

Case II. Let $\Delta z \to 0$ along y-axis. Take $\Delta z = i\Delta y$.

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \left[\frac{u(x+\Delta x, y+\Delta y)-u(x,y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y)-v(x,y)}{\Delta x + i\Delta y} \right]$$

$$= \underset{\Delta y \to 0}{\text{Lt}} \left[\frac{u(x, y+\Delta y)-u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y)-v(x,y)}{i\Delta y} \right]$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{...(2)}$$

Now $f(z)$ cannot possibly be analytic unless these two limits are identical. Thus, a necessary condition that $f(z)$ be analytic is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{or} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \ \text{at } (x, y)$$

Note : Cauchy Riemann conditions are necessary but not sufficient *i.e.* if the function $f(z) = u + iv$ is differentiable, then it must satisfy the C-R equations. But the converse is not true. A function may satisfy the C-R conditions at a point and yet it may not be differentiable at that point as illustrated in solved Example 14.

Sufficiency. Let $f(z) = u(x, y) + iv(x, y)$ be a function such that [*JNTU2005S(Set No.4)*]

(*i*) u_x, u_y, v_x, v_y all exist and are continuous in a neighbourhood of z.

and (*ii*) $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$ at z

Then $f'(z)$ exists.

Proof. Given $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are continuous.

We have $\Delta u = u(x+\Delta x, y+\Delta y) - u(x, y)$

$$= \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\} + \{u(x, y+\Delta y) - u(x, y)\}$$

$$= \left(\frac{\partial u}{\partial x} + \epsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \eta_1 \Delta y$$

where $\epsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

Similarly, we have $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ are continuous.

$$\Delta v=\left(\frac{\partial v}{\partial x}+\epsilon_2\right)\Delta x+\left(\frac{\partial v}{\partial y}+\eta_2\right)\Delta y=\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y+\epsilon_2\,\Delta x+\eta_2\,\Delta y$$

where $\epsilon_2\to 0$ and $\eta_2\to 0$ as $\Delta x\to 0$ and $\Delta y\to 0$. Then

$$\Delta w=\Delta u+i\Delta v=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y+\epsilon\,\Delta x+\eta\Delta y \qquad ...(2)$$

where $\epsilon=\epsilon_1+i\epsilon_2\to 0$ and $\eta=\eta_1+i\eta_2\to 0$ as $\Delta x\to 0$ and $\Delta y\to 0$.

By Cauchy–Riemann equations, we can write (2) as

$$\Delta w=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\Delta y+\epsilon\,\Delta x+\eta\Delta y$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\Delta x+i\Delta y)+\epsilon\,\Delta x+\eta\Delta y$$

Then on dividing by $\Delta z=\Delta x+i\Delta y$ and taking the limit as $\Delta z\to 0$, we see that

$$\frac{dw}{dz}=f'(z)=\underset{\Delta z\to 0}{\mathrm{Lt}}\ \frac{\Delta w}{\Delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

So that the derivative exists and is unique.

i.e., $f(z)$ is analytic in R.

Hence the theorem.

PROPERTIES OF ANALYTIC FUNCTIONS

1. If $f(z)$ and $g(z)$ are analytic functions, then $f\pm g$, $f.\,g$ and $\dfrac{f}{g}$ are also analytic functions, provided $g(z)\neq 0$.
2. Analytic function of an analytic function is analytic.
3. An entire function of an entire function is entire.
4. Derivative of an analytic function is itself analytic.

2.14. HARMONIC FUNCTIONS — LAPLACE EQUATION

Theorem : If $f(z)=u(x,y)+iv(x,y)$ is analytic in a domain D, then u and v satisfy Laplace equation $\nabla^2 u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ and $\nabla^2 v=\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=0$, respectively in D, and have continuous second order partial derivatives in D.

Proof. We have by Cauchy – Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad ...(1)$$

and

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad ...(2)$$

Differentiating (1) with respect to x and (2) with respect to y partially, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y} \qquad ...(3)$$

and $$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \text{... (4)}$$

Since the derivative of an analytic function is also analytic and second order derivatives

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

(3) + (4) gives $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *i.e.* $\nabla^2 u = 0$, where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

∇^2 is called the **Laplacian operator.**

Again differentiating (1) with respect to y, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$ ...(5)

Again differentiating (2) with respect to x, $\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$ ...(6)

(5) + (6) gives $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ *i.e.* $\nabla^2 v = 0$

Hence the theorem.

2.15. HARMONIC FUNCTIONS

Solutions of Laplace equations having continuous second order partial derivatives are called **Harmonic functions.** Their theory is called **Potential theory.** Hence, the real and imaginary parts of an analytic function are harmonic functions.

Thus the functions satisfying the Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

are known as **Harmonic functions.**

2.16. CONJUGATE HARMONIC FUNCTION

If two harmonic functions u and v satisfy the Cauchy–Riemann equations in a domain D and they are the real and imaginary parts of an analytic function f in D then v is said to be a conjugate Harmonic function of u in D. Two harmonic functions, u and v which are such that $u + iv$ is an analytic function are called conjugate harmonic functions. In other words, if $f(z) = u + iv$ is analytic and if u and v satisfy Laplace's equation, then u and v are called conjugate harmonic functions.

2.17. POLAR FORM OF CAUCHY–RIEMANN EQUATIONS

[JNTU 1998 Sept., 2001S, 2003, 2005, JNTU (H) Nov. 2009 (Set No. 1)]

If $f(z) = f\left(re^{i\theta}\right) = u(r, \theta) + iv(r, \theta)$ and $f(z)$ is derivable at $z_0 = r_0 e^{i\theta_0}$ then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof : Let $z = r\,e^{i\theta}$. Then $f(z) = u(r, \theta) + iv(r, \theta)$

Differentiating it with respect to r partially,

$$\frac{\partial}{\partial r} f(z) = f'(z) \frac{\partial z}{\partial r} = f'(z)\, e^{i\theta}$$

$$\therefore \quad f'(z) = \frac{1}{e^{i\theta}} \frac{\partial f}{\partial r} = \frac{1}{e^{i\theta}} (u_r + iv_r) \qquad \text{...(1)}$$

Similarly differentiating partially with respect to 'θ',

$$\frac{\partial f}{\partial \theta} = f'(z) \frac{\partial z}{\partial \theta} = f'(z) \,.\, r\, ie^{i\theta}$$

$$\therefore \quad f'(z) = \frac{1}{r\, ie^{i\theta}} (u_\theta + iv_\theta) \qquad \text{...(2)}$$

Using (1) and (2)

$$\therefore \quad \frac{1}{e^{i\theta}} (u_r + iv_r) = \frac{1}{r\, ie^{i\theta}} (u_\theta + iv_\theta)$$

$$\therefore \quad u_r + iv_r = \frac{1}{r} \frac{\partial v}{\partial \theta} - i \frac{1}{r} \frac{\partial u}{\partial \theta}$$

Equating real and imaginary parts, we get

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}}$$

$$\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}}$$

Hence, the result follows.

These are called **Cauchy–Riemann** equations in polar form.

Corollary. If $f''(z)$ exists

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

and $$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0$$

[JNTU Aug. 2006S (Set No. 1)]

Proof : We have $\dfrac{\partial u}{\partial r} = \dfrac{1}{r} \dfrac{\partial v}{\partial \theta}$ ...(1)

and $\dfrac{1}{r} \dfrac{\partial u}{\partial \theta} = -\dfrac{\partial v}{\partial r}$ ...(2)

Diff (1) partially w.r.t. 'r', we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2} \frac{\partial v}{\partial \theta} + \frac{1}{r} \frac{\partial^2 v}{\partial r \partial \theta}$$

From (1), we have $\dfrac{1}{r} \dfrac{\partial u}{\partial r} = \dfrac{1}{r^2} \dfrac{\partial v}{\partial \theta}$

Diff. (2) partially w.r.t. 'θ' and multiplying with $\dfrac{1}{r}$, we get $\dfrac{1}{r^2} \dfrac{\partial^2 u}{\partial \theta^2} = \dfrac{1}{r}\left(-\dfrac{\partial^2 v}{\partial r \partial \theta}\right)$

Adding the above three conditions, we get $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial u}{\partial r} + \dfrac{1}{r^2} \dfrac{\partial^2 u}{\partial \theta^2} = 0$

Similarly, we can prove that $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial v}{\partial r} + \dfrac{1}{r^2} \dfrac{\partial^2 v}{\partial \theta^2} = 0$

Note. If $f'(z)$ exists (and also $[f''(z)]$)

(*i*) u, v satisfy Laplace's equations.

(*ii*) u and v are harmonic functions.

2.18 ORTHOGONAL TRAJECTORIES

Two curves intersecting at a point P are said to be orthogonal, if their tangents are perpendicular at P. Two families of curves, or trajectories, are orthogonal if each curve of the first family is orthogonal to each curve of the second family where even an intersection occurs. Orthogonal families occur in many context. Parallels and meridians on a globe are orthogonal, as are equipotential and electric lines of force.

We can prove

Every analytic function of $(z) = u + iv$ defines two families of curves $u(x, y) = k_1$ and $v(x,y) = k_2$ forming an orthogonal system.

(OR)

if $W = f(z)$ is an analytic function, then the family of curves defined by $u(x, y) =$ constant cuts orthogonally the family of curves $v(x,y) =$ constant .

SOLVED EXAMPLES

Example 1 : *Show that $f(z) = xy + iy$ is everywhere continuous but is not analytic* [*JNTU 2000*]

Solution : $f(z_0) = x_0 y_0 + i y_0$ is well defined for any $z_0 = x_0 + i y_0$.

$$\operatorname*{Lt}_{z \to z_0} f(z) = \operatorname*{Lt}_{z \to z_0} (xy + iy)$$

$$= \operatorname*{Lt}_{\substack{x \to x_0 \\ y \to y_0}} (xy + iy) = x_0 y_0 + i y_0 = f(z_0)$$

$$= \operatorname*{Lt}_{\substack{y \to y_0 \\ x \to x_0}} (x_0 y_0 + i y_0) = f(z_0)$$

$\therefore$ f is continuous everywhere and $f(z) = u + iv = xy + iy$

so $u = xy,\ v = y$

Then $u_x = y,\ u_y = x,\ v_x = 0,\ v_y = 1$

Since Cauchy – Riemann conditions are not satisfied

$\therefore$ f is not analytic.

Example 2 : *Show that $f(z) = z^3$ is analytic for all z.*

Solution : Let $z = x + iy \Rightarrow z^3 = (x+iy)^3 = \left(x^3 - 3xy^2\right) + i\left(3x^2 y - y^3\right)$

If $z^3 = u + iv$, then we have $u = x^3 - 3xy^2,\ v = 3x^2 y - y^3$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2,\ \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial y} = -6xy,\ \frac{\partial v}{\partial x} = 6xy$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy–Riemann equations are satisfied and hence $f(z) = z^3$ is analytic.

Example 3 : *Show that z^2 is analytic for all z.*

Solution : Let $f(z) = u + iv = z^2 = (x+iy)^2 = (x^2 - y^2) + 2xy\,i$

so that $u = x^2 - y^2$ and $v = 2xy$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x \quad \text{and} \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y \quad \text{and} \quad \frac{\partial v}{\partial x} = 2y$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ The Cauchy–Riemann equations are satisfied.

$\therefore$ $f(z)$ is an analytic function.

Example 4 : *If $w = \log z$, find $\dfrac{dw}{dz}$ and determine where w is non-analytic.*

Solution : Let $w = u + iv$

We have $z = x + iy = r.e^{i\theta}$,

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$

$$\therefore \quad \log(x+iy) = \log(r.e^{i\theta}) = \log r + i\theta = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$$\therefore \quad w = \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

so that $u = \dfrac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1}(y/x)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2},$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} \text{ and } \frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$$

Hence the partial derivatives are continuous except at (0, 0).

Cauchy-Riemann equations are satisfied.

$\therefore$ w is analytic everywhere except at $z = 0$.

$$\therefore \quad \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z} \quad (z \neq 0)$$

Example 5 : *Show that $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.*

[*JNTU 2000, 2003* (*Set No. 2*)]

Solution : $\quad f(z) = u + iv = z + 2\bar{z} = (x+iy) + 2(x-iy) = 3x - iy$

$\Rightarrow \quad u = 3x, \; v = -y$

$\therefore \quad u_x = 3, \; v_x = 0, \; u_y = 0, \; v_y = 1$

$\therefore$ $f(z)$ is not analytic anywhere since Cauchy–Riemann conditions are not satisfied for any z.

Example 6 : *Prove that*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2 = 2|f'(z)|^2 \text{ where } \omega = f(z) \text{ is analytic}$$

[JNTU 2004, 2004S, 2005, 2008S (Set No. 2,3

Solution : Let $f(z) = u + iv$. Then Real $f(z) = u$

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + i v_x = u_x - i u_y$ (using C - R equations)

and $|f'(z)| = \sqrt{(u_x)^2 + (-u_y)^2}$

$\therefore \quad |f'(z)|^2 = u_x^2 + u_y^2$ ...(1)

We have $\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$

and $\frac{\partial^2}{\partial x^2}(u^2) = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]$ ...(2)

Similarly $\frac{\partial^2}{\partial y^2}(u^2) = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2}\right]$ ...(3)

(2) + (3) gives

$$\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right] \quad ...(4)$$

Since u is real part of $f(z)$, therefore it satisfies the Laplace's equation.

Hence $\nabla^2 u = 0$ *i.e.* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$ ...(5)

From (4) and (5), we have

$$\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] = 2(u_x^2+u_y^2) \text{ [by (1)] } = 2|f'(z)|^2$$

or $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2 = 2|f'(z)|^2$

or $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{R } f(z)|^2 = 2|f'(z)|^2$

Hence the result.

Example 7 : *Show that* $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$, *where f(z) is an analytic function.*

[JNTU 2002]

Solution : Taking $x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}=-\frac{i}{2}(z-\bar{z})$, we have

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial z}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

and $\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$

$$\therefore \quad \frac{\partial^2}{\partial z\,\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

Hence $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\log|f'(z)|)=4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left(\frac{1}{2}\log|f'(z)|^2\right)$

$$=2\frac{\partial^2}{\partial z\partial \bar{z}}[(\log f'(z)\,f'(\bar{z}))] \quad \left[\because |z|^2=z\bar{z}\right]$$

$$=2\frac{\partial^2}{\partial z\,\partial \bar{z}}[\log f'(z)+\log f'(\bar{z})]$$

$$=2\left[\frac{\partial}{\partial \bar{z}}\,\frac{f''(z)}{f'(z)}+\frac{\partial}{\partial z}\,\frac{f''(\bar{z})}{f'(\bar{z})}\right]=2(0+0)=0$$

Since $f(z)$ is analytic, $f'(z)$ is analytic, $f'(\bar{z})$ is also analytic and $\frac{\partial f'(z)}{\partial \bar{z}}=0$, $\frac{\partial f'(\bar{z})}{\partial z}=0$.

Example 8 : (*i*) Prove that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\,\partial \bar{z}}$

(*ii*) If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(x)|^2$

[JNTU (K) Nov. 2009 (Set No. 2)]

Solution : (*i*) Let $z=x+iy$. Then $\bar{z}=x-iy$

$$\therefore \quad x=\frac{z+\bar{z}}{2} \text{ and } y=\frac{z-\bar{z}}{2i}=\frac{-i}{2}(z-\bar{z})$$

Now $\frac{\partial x}{\partial z}=\frac{1}{2}, \frac{\partial x}{\partial \bar{z}}=\frac{1}{2}$ and $\frac{\partial y}{\partial z}=\frac{-i}{2}, \frac{\partial y}{\partial \bar{z}}=\frac{i}{2}$

Let $f = f(x, y)$. Then $f = f(z, \bar{z})$

We know that $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$

Similarly $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$

$$\therefore \quad \frac{\partial^2 f}{\partial z\,\partial \bar{z}} = \frac{\partial}{\partial z}\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f$$

$$\Rightarrow \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

(*ii*) We know that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^2$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}} f(z)\,.\,\overline{f(z)}$$

$$= 4\frac{\partial}{\partial \bar{z}} f'(z)\,.\,\overline{f(z)}$$

$$= 4f'(z)\overline{f'(z)} = 4|f'(z)|^2$$

Example 9 : *If $f(z) = u + iv$ is an analytic function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2$$

[JNTU 2003S (Set No. 4)]

Solution : We know that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$ (Refer Ex. 8)

Also $|f(z)|^2 = u^2 + v^2 = f(z)\,\overline{f(z)}$ ... (1)

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^p = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)\,\overline{f(z)}\right]^{p/2}, \text{ [by (1)]}$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\{f(z)\}^{p/2}\{\overline{f(z)}\}^{p/2}\right]$$

Differentiating first w.r.t. $\bar{z}$ taking $f(z)$ as constant and then w.r.t. z taking $f(\bar{z})$ as constant, we have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = 4\frac{\partial}{\partial z}\left[\{f(z)\}^{p/2}\cdot\frac{p}{2}\{f(\bar{z})\}^{\frac{p}{2}-1}\overline{f'(z)}\right]$$

$$=4\left[\frac{p}{2}\{f(z)\}^{\frac{p}{2}-1}f'(z)\cdot\frac{p}{2}\{f(\bar{z})\}^{\frac{p}{2}-1}\overline{f'(z)}\right]$$

$$=4\cdot\frac{p^2}{4}\left[\{f(z)\,f(\bar{z})\}^{\frac{p-2}{2}}f'(z)\,\overline{f'(z)}\right]$$

$$=p^2\left[\{|f(z)|^2\}^{\frac{p-2}{2}}|f'(z)|^2\right], \text{[from (1)]}$$

$$=p^2|f(z)|^{p-2}|f'(z)|^2$$

Example 10 : *If* $f(z)=u+iv$ *is an analytic function of* $z=x+iy$*, prove that*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|u|^p = p(p-1)|u|^{p-2}|f'(z)|^2$$

<u>**Solution :**</u> We have $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\,\partial\bar{z}}$

Also $f(z)+\overline{f(z)}=(u+iv)+(u-iv)=2u$

$$\therefore\quad |u|^2=\left|\frac{f(z)+\overline{f(z)}}{2}\right|^2$$

Hence $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|u|^p = 4\frac{\partial^2}{\partial z\,\partial\bar{z}}\left|\frac{f(z)+\overline{f(z)}}{2}\right|^p = \frac{4}{2^p}\frac{\partial^2}{\partial z\,\partial\bar{z}}\left|\{f(z)+\overline{f(z)}\}^2\right|^{p/2}$

$$=\frac{4}{2^p}\frac{\partial^2}{\partial z\,\partial\bar{z}}\left[\{f(z)+\overline{f(z)}\}\{f(z)+\overline{f(z)}\}\right]^{p/2}$$

$$=\frac{4}{2^p}\frac{\partial^2}{\partial z\,\partial\bar{z}}\left[\{f(z)+\overline{f(z)}\}^2\right]^{p/2}=\frac{4}{2^p}\frac{\partial^2}{\partial z\,\partial\bar{z}}\left[f(z)+\overline{f(z)}\right]^p$$

$$=\frac{4}{2^p}\frac{\partial}{\partial z}p\left[f(z)+\overline{f(z)}\right]^{p-1}\overline{f'(z)}$$

$$=\frac{4}{2^p}p(p-1)\left[f(z)+\overline{f(z)}\right]^{p-2}f'(z)\,\overline{f'(z)}$$

$$= \frac{1}{2^{p-2}} p(p-1)\left[\{f(z)+\overline{f(z)}\}^2\right]^{\frac{p-2}{2}} |f'(z)|^2$$

$$= p(p-1)\left[\left|\frac{f(z)+\overline{f(z)}}{2}\right|^2\right]^{\frac{p-2}{2}} |f'(z)^2|$$

$$= p(p-1)\left[|u|^2\right]^{\frac{p-2}{2}} |f'(z)|^2$$

$$= p(p-1)|u|^{p-2} |f'(z)|^2$$

Example 11 : *Find where the function*

(i) $w=\frac{1}{z}$ (ii) $w=\frac{z}{z-1}$ (iii) $w=z^3-4z+1$ (iv) $w=\frac{z+2}{z(z^2+1)}$ (v) $z=e^{-v}(\cos u+i\sin u)$

ceases (fails) to be analytic. **[JNTU 2004, JNTU (A) Nov. 2009 (Set No. 2)]**

Solution : A function $w=f(z)$ ceases to be analytic if $\frac{dw}{dz}=0$

(i) Given $w=\frac{1}{z}$

$\therefore \quad \frac{dw}{dz}=-\frac{1}{z^2}$ if $z\neq 0$

For $z=0$, $\frac{dw}{dz}$ does not exist. So, w is analytic everywhere except at the point $z=0$ which is a singular point of $f(z)$.

(ii) Given $w=\frac{z}{z-1}$

$\therefore \quad \frac{dw}{dz}=\frac{(z-1)\cdot 1-z\cdot 1}{(z-1)^2}=\frac{-1}{(z-1)^2}$ if $z\neq 1$

For $z=1$, $\frac{dw}{dz}$ does not exist. So, w is analytic everywhere except at the point $z=1$ which is a singular point of $f(z)$.

(iii) Given $w=z^3-4z+1$

$\therefore \quad \frac{dw}{dz}=3z^2-4$

$\frac{dw}{dz}$ exists at all points of z in the complex plane.

Hence w is analytic everywhere in the complex plane.

(iv) Given $w=\frac{z+2}{z(z^2+1)}$

$$\therefore \quad \frac{dw}{dz} = \frac{z(z^2+1)(1)-(z+2)(3z^2+1)}{z^2(z^2+1)^2} = \frac{-2(z^3+3z^2+1)}{z^2(z^2+1)^2} \text{ if } z \neq 0 \text{ and } z^2+1 \neq 0$$

For $z=0$ and $z=\pm i$, $\frac{dw}{dz}$ does not exist. So, w is analytic everywhere except at the points $z=0$, $z=i$ and $z=-i$ which are the singular points of $f(z)$.

(v) Given $z = e^{-v}(\cos u + i \sin u)$

$$= e^{-v} . e^{iu} = e^{iu-v} = e^{iu+i^2 v}$$

$$= e^{i(u+iv)} = e^{iw}, \text{ where } w = u+iv$$

$\therefore \log z = iw$. Hence $\frac{dw}{dz} = \frac{1}{iz}$.

Clearly w is not analytic when $iz=0$ i.e. when $z=0$

Example 12 : *Find all values of k, such that $f(z) = e^x(\cos ky + i \sin ky)$ is analytic*

[JNTU 2003 (Set No. 4), Reg-2008 (Set No. 2)]

Solution : Let $f(z) = u + iv = e^x(\cos ky + i \sin ky)$. Then

$u(x, y) = e^x \cos ky$ and $v(x, y) = e^x \sin ky$.

$\therefore \quad u_x = e^x \cos ky, \; u_y = -k e^x \sin ky$

and $\quad v_x = e^x \sin ky, \; v_y = k e^x \cos ky$

The Cauchy – Riemann equations are satisfied if $u_x = v_y$ *i.e.* $e^x \cos ky = k e^x \cos ky$ which is true for $k = 1$ and $u_y = -v_x$ *i.e.* $-k e^x \sin ky = -e^x \sin ky$ which is true for $k = 1$.

Thus f is analytic when $k = 1$.

Example 13 : *Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin, although Cauchy–Riemann equations are satisfied at that point.*

[JNTU Aug. 2007S, Nov. 2008, JNTU (K), (H) Nov. 2009 (Set No. 3,4)]

Solution : Let $f(z) = u(x,y) + iv(x,y)$ where $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$.

Then at the origin

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x,0)-u(0,0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0-0}{x} = 0$$

$$\text{Similarly } \frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0,y)-u(0,0)}{y} = \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{0-0}{x} = 0 \text{ and } \frac{\partial v}{\partial y} = 0$$

Hence, Cauchy–Riemann equations are satisfied at the origin.

$$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z)-f(0)}{z} = \underset{z \to 0}{\text{Lt}} \frac{\sqrt{|xy|}-0}{x+iy}$$

Now let $z \to 0$ along $y = mx$, we get $f'(0) = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|mx^2|}}{x(1+im)} = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|m|}}{1+im}$

This limit depends on m and hence is not unique.

$\therefore$ $f'(0)$ does not exist.

Note : $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist at $(x, y) = (0, 0)$.

Cauchy-Riemann conditions are satisfied.

However, they are not continuous at (0, 0).

Hence, $f'(z)$ does not exist at (0, 0).

Example 14 : Prove that the function $f(z)$ defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \ (z \neq 0) \qquad u = \frac{x^3-y^3}{x^2+y^2} + i\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

$$= 0, \ (z = 0)$$

is continuous and the Cauchy–Riemann equations are satisfied at the origin , yet $f'(0)$ does not exist.

[JNTU 99, Nov. 2006, 2007S, Nov. 2008S, Nov. 2008, JNTU (K) Nov. 2009 (Set No. 4)]

<u>**Solution :**</u> $\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \underset{y \to 0}{\text{Lt}} -\frac{y^3(1-i)}{y^2} = \underset{y \to 0}{\text{Lt}} \left[-y(1-i)\right] = 0$

and $\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to 0 \\ x \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i)}{x^2} = \underset{x \to 0}{\text{Lt}} x(1+i) = 0$

Also $f(0) = 0$ by given data.

Thus, we get $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$

When $x \to 0, \ y \to 0$ and $y \to 0, \ x \to 0$

Now take both x and y tend to 0 simultaneously along the path $y = mx$, then

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i) - m^3x^3(1-i)}{(1+m^2)x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x\left[1+i-m^3(1-i)\right]}{1+m^2} = 0$$

$\therefore$ Whatever is the manner in which $z \to 0$, we have $\underset{z \to 0}{\text{Lt}} f(z) = 0 = f(0)$

$\therefore$ The function is continuous at $z = 0$.

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \underset{x \to 0}{\text{Lt}} \frac{u(x,0) - u(0,0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{\text{Lt}} \frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{\text{Lt}} -\frac{y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \underset{x\to 0}{\text{Lt}} \frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{\text{Lt}} \frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{\text{Lt}} \frac{y}{y} = 1$$

∴ We have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

∴ Cauchy–Riemann equations are satisfied at the origin

$$f'(0) = \underset{z\to 0}{\text{Lt}} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{\text{Lt}} \frac{f(z)}{z} = \underset{z\to 0}{\text{Lt}} \frac{\left(x^3-y^3\right)+i\left(x^3+y^3\right)}{\left(x^2+y^2\right)(x+iy)}$$

Let $z \to 0$ along the path $y = mx$, then $f'(0) = \dfrac{1-m^3+i\left(1+m^3\right)}{\left(1+m^2\right)(1+im)}$

which depends on m and hence is not unique. Thus, $f'(z)$ does not exist at (0, 0).

Example 15 : Show that for the function $f(z) = \begin{cases} \dfrac{z^5}{|z|^4}, z \neq 0 \\ 0, \; z = 0 \end{cases}$

Cauchy-Rieman equations are satisfied at $z = 0$, but $f(z)$ is not differentiable at 0.

Solution : We have $f(z) = \begin{cases} \dfrac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

and $\quad z^5 = (x+iy)^5$

$$= (x^5 - 10x^3y^2 + 5xy^4) + i\,(5x^4y - 10x^2y^3 + y^5)$$

Thus $\quad f(z) = \dfrac{(x^5-10x^3y^2+5xy^4)+i(5x^4y-10x^2y^3+y^5)}{(x^2+y^2)^2}$

$$= u + iv$$

where $\quad u(x, y) = \dfrac{5xy^4-10x^3y^2+x^5}{(x^2+y^2)^2}, (x, y) \neq (0,0)$

and $\quad v(x, y) = \dfrac{y^5-10x^2y^3+5x^4y}{(x^2+y^2)^2}, (x, y) \neq (0,0)$

∴ $\quad u(0, 0) = v(0, 0) = 0.$

Computing the partial derivaties at origin, we get

$$\frac{\partial}{\partial x}[u(0,0)] = \underset{h\to 0}{Lt}\frac{u(h,0)-u(0,0)}{h} = \underset{h\to 0}{Lt}\frac{h^5}{h\cdot h^4} = 1$$

$$\frac{\partial}{\partial y}[u(0,0)] = \underset{h\to 0}{Lt}\frac{u(0,h)-u(0,0)}{h} = 0$$

$$\frac{\partial}{\partial x}[v(0,0)] = \underset{h\to 0}{Lt}\frac{v(h,0)-v(0,0)}{h} = 0$$

and $$\frac{\partial}{\partial y}[v(0,0)] = \underset{h\to 0}{Lt}\frac{v(0,h)-v(0,0)}{h} = \underset{h\to 0}{Lt}\frac{h^5}{h\cdot h^4} = 1$$

Thus Cauchy-Riemann equations are satisfied at origin.

Consider $\dfrac{f(0+h)-f(0)}{h\cdot 1} = \dfrac{h^5}{h|h|^4} = \dfrac{h^5}{h(h\bar{h})^2} = \dfrac{h^2}{(\bar{h})^2} = \left(\dfrac{h}{\bar{h}}\right)^2$

Converting $\left(\dfrac{h}{\bar{h}}\right)^2$ into polar form by taking $h = re^{i\theta}$ and $\bar{h} = re^{-i\theta}$, we get $\left(\dfrac{h}{\bar{h}}\right)^2 = e^{4i\theta}$

On the line making an angle θ with positive real axis, differential quotient $\dfrac{f(0+h)-f(0)}{h}$ has the constant value $e^{4i\theta}$. It approaches the value as $h \to 0$. Thus the differential quotient approaches different values along different paths. So it has no limit as $h \to 0$. Thus $f(z)$ is not differentiable at zero.

Example 16 : Show that the function $f(z) = z\,\bar{z}$ is differentiable but not analytic at $z = 0$.

Solution : We have $w = f(z) = z\,\bar{z} = (x+iy)(x-iy) = x^2+y^2$

or $\quad w = u + iv = x^2 + y^2$

Hence $\quad u = x^2+y^2$ and $v = 0$

$\therefore \quad \dfrac{\partial u}{\partial x} = 2x,\ \dfrac{\partial u}{\partial y} = 2y$

and $\quad \dfrac{\partial v}{\partial x} = 0,\ \dfrac{\partial v}{\partial y} = 0$

Now $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ only when $x = 0$

and $\quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ only when $y = 0$

Hence the Cauchy – Riemann equations are satisfied only at the origin. Hence $f(z)$ has a derivative at $z = 0$.

But the Cauchy – Riemann equations are not satisfied for $z \neq 0$.

Thus $f(z)$ is not analytic at $z = 0$.

$\therefore \quad f(z) = z\,\bar{z}$ is differentiable but not analytic at $z = 0$.

Example 17 : *Show that* $f(z)=\begin{cases}\dfrac{x\,y^2\,(x+i\,y)}{x^2+y^4}, & z\neq 0\\ 0, \text{ if } & z=0\end{cases}$

is not analytic at z = 0 although C – R equations are satisfied at the origin.

[JNTU 2003, 2007S, Nov. 2008 (Set No. 3)]

Solution : $\dfrac{f(z)-f(0)}{z-0}=\dfrac{f(z)-0}{z}=\dfrac{f(z)}{z}$

$$=\frac{x\,y^2\,(x+i\,y)}{(x^2+y^4)\cdot z}=\frac{x\,y^2(z)}{(x^2+y^4)\cdot z}=\frac{x\,y^2}{x^2+y^4}$$

Clearly $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}}\frac{x\,y^2}{x^2+y^4}=\lim_{\substack{y\to 0\\ x\to 0}}\frac{x\,y^2}{x^2+y^4}=0$.

Along the path $y = mx$,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{x\to 0}\frac{x\,(m^2\cdot x^2)}{x^2+m^4\cdot x^4}=\lim_{x\to 0}\frac{m^2\,x}{1+m^4}=0$$

Also along the path $x = m\,y^2$,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{y\to 0}\frac{(m\,y^2)\,y^2}{m^2\,y^4+y^4}=\lim_{y\to 0}\frac{m}{m^2+1}\neq 0$$

Limit value depends on m *i.e.* on the path of approach and is different for the different paths followed and therefore limit does not exist. Hence $f(z)$ is not differentiable at $z = 0$. Thus $f(z)$ is not analytic at $z = 0$.

To prove that $C - R$ conditions are satisfied at the origin.

Let $f(z)=u+i\,v=\dfrac{x\,y^2(x+i\,y)}{x^2+y^4}$

Then $u\,(x,y)=\dfrac{x^2\,y^2}{x^2+y^4}$ and $v\,(x,y)=\dfrac{x\,y^3}{x^2+y^4}$; for $z\neq 0$

Also $u\,(0,0)=0$ and $v\,(0,0)=0$ $\quad[\because f(z)=0 \text{ at } z=0]$

Now $\dfrac{\partial u}{\partial x}=\displaystyle\lim_{x\to 0}\frac{u\,(x,0)-u\,(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u\,(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v\,(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

and $\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$

Thus Cauchy – Riemann equations are satisfied at the origin.

Hence $f(z)$ is not analytic at $z = 0$ eventhough C-R equations are satisfied at the origin.

Example 18 : *Show that the function*

$$f(z) = e^{-z^{-4}},\ z \neq 0 \quad \text{and} \quad f(0) = 0$$

is not analytic at $z = 0$, although Cauchy – Riemann equations are satisfied at this point.

Solution : We have

$$f(z) = e^{-z^{-4}} = e^{-(x+iy)^{-4}} \quad (z \neq 0)$$

or $\quad f(z) = u(x,y) + i\,v(x,y) = e^{-(x+iy)^{-4}}$

$\Rightarrow \quad u(x,0) + i\,v(x,0) = e^{-x^{-4}}$ and $u(0,y) + i\,v(0,y) = e^{-y^{-4}}$.

$\Rightarrow \quad u(x,0) = e^{-x^{-4}},\ v(x,0) = 0$ and $u(0,y) = e^{-y^{-4}},\ v(0,y) = 0$

Also we have $f(0) = 0$

i.e. $\quad f(0) = u(0,0) + i\,v(0,0) = 0$

$\Rightarrow \qquad u(0,0) = 0$ and $v(0,0) = 0$

Now at $\quad z = 0, \frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_{x=0,\,y=0} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$

$$= \lim_{x\to 0} \frac{e^{-x^{-4}} - 0}{x} = \lim_{x\to 0} \frac{1}{x\,e^{x^{-4}}} = \lim_{x\to 0} \frac{1}{x}\cdot\frac{1}{e^{1/x^4}}$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{1}{1+\frac{1}{x^4}+\frac{1}{x^8\,2!}+\ldots}\right]$$

$$= \lim_{x\to 0} \frac{1}{x+\frac{1}{x^3}+\frac{1}{x^7\,2!}+\ldots} = 0$$

Similarly $\quad \left(\frac{\partial u}{\partial y}\right)_{x=0,\,y=0} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y}$

$$= \lim_{y\to 0} \frac{1}{y}\cdot\frac{1}{e^{1/y^4}} = 0$$

At $z = 0$, $\frac{\partial v}{\partial x} = \left(\frac{\partial v}{\partial x}\right)_{x=0,\,y=0} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$

Similarly at $z = 0$, $\dfrac{\partial v}{\partial y} = 0$

Hence at (0, 0) *i.e.* at $z = 0$, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

So the Cauchy – Riemann equations are satisfied at the origin.

Now we have to show that $f(z)$ is not analytic at $z = 0$

$$\underset{z\to 0}{Lt}\, f(z) = \underset{z\to 0}{Lt}\, e^{-z^{-4}}$$

Let $z \to 0$ along the path $z = r\,e^{i\pi/4}$ so that $r \to 0$ as $z \to 0$

$$\therefore \quad \underset{z\to 0}{\text{Lt}}\, f(z) = \underset{r\to 0}{\text{Lt}}\, e^{-r^{-4}e^{-i\pi}} = \infty$$

Which shows that $\underset{z\to 0}{\text{Lt}}\, f(z)$ does not exist

So $f(z)$ is not continuous at $z = 0$. Hence $f(z)$ is not necessarily differentiable at $z = 0$.

$\therefore \quad f(z)$ is not analytic at $z = 0$

Example 19 : *If* $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ *prove that* $\dfrac{f(z)-f(0)}{z} \to 0$ *as* $z \to 0$ *along any radius vector but not as* $z \to 0$ *along the curve* $y = ax^3$.

(or)

Test for analyticity at the origin for $f(z) = \dfrac{x^3y(y-ix)}{x^6+y^2}$ *for* $z \neq 0 = 0$ for $z = 0$

[*JNTU April 2006* (*Set No. 4*)]

Solution : Given $f(z) = \dfrac{x^3y(y-ix)}{x^6+y^2} = \dfrac{-ix^3y(x+iy)}{x^6+y^2} = \dfrac{-ix^3yz}{x^6+y^2}$

$$\therefore \quad \frac{f(z)}{z} = \frac{-ix^3y}{x^6+y^2} \qquad \text{...(1)}$$

Now $\quad \dfrac{f(z)-f(0)}{z} = \dfrac{f(z)-0}{z} = \dfrac{f(z)}{z} = \dfrac{-ix^3y}{x^6+y^2}$ [by (1)]

Along the path $y = mx$ (radius vector),

$$\underset{z\to 0}{Lt}\, \frac{f(z)-f(0)}{z} = \underset{x\to 0}{Lt}\, \frac{-ix^3(mx)}{x^6+m^2x^2} = \underset{x\to 0}{Lt}\, \frac{-imx^2}{m^2+x^4} = 0$$

Along the path $y = ax^3$,

$$\underset{z\to 0}{Lt}\, \frac{f(z)-f(0)}{z} = \underset{x\to 0}{Lt}\, \frac{-ix^3\cdot ax^3}{x^6+a^2x^6} = \underset{x\to 0}{Lt}\, \frac{-ia}{1+a^2} \neq 0$$

Hence $\underset{z\to 0}{Lt}\, \dfrac{f(z)-f(0)}{z} \to 0$ along any radius vector but not along the path $y = ax^3$.

Since the limit is not unique, therefore the given function is not analytic at the origin.
Note : Though the given function is not analytic at the origin, it satisfies C–R equations.

Given $\quad f(z) = \dfrac{x^3y(y-ix)}{x^6+y^2} = \dfrac{x^3y^2}{x^6+y^2} - i\dfrac{x^4y}{x^6+y^2}$

$= u(x, y) + iv(x, y)$

So $\quad u(x, y) = \dfrac{x^3y^2}{x^6+y^2}$ and $v(x, y) = -\dfrac{x^4y}{x^6+y^2}$

Since $f(0) = 0$, $u(0, 0) = 0$ and $v(0, 0) = 0$

Now
$$\left(\frac{\partial u}{\partial x}\right)_{x=0,y=0} = \underset{x\to 0}{Lt}\,\frac{u(x,0)-u(0,0)}{x} = \underset{x\to 0}{Lt}\,\frac{0}{x^6\cdot x} = 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{x=0,y=0} = \underset{y\to 0}{Lt}\,\frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{Lt}\,\frac{0}{y^2\cdot y} = 0$$

$$\left(\frac{\partial v}{\partial x}\right)_{x=0,y=0} = \underset{x\to 0}{Lt}\,\frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{Lt}\,\frac{0}{x^6\cdot x} = 0$$

$$\left(\frac{\partial v}{\partial y}\right)_{x=0,y=0} = \underset{y\to 0}{Lt}\,\frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{Lt}\,\frac{0}{y^2\cdot y} = 0$$

Hence at (0, 0), $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

So the Cauchy-Riemann equations are satisfied at the origin.

Example 20 : Show that an analytic function of constant absolute value is constant.

[JNTU (H) Nov. 2009 (Set No. 4)]

Solution : Suppose $f(z)$ is analytic in a domain D and $|f(z)| = k$ = constant in D. Then we want to prove that $f(z)$ = constant in D.

$$|f(z)| = k \Rightarrow u^2 + v^2 = k^2$$

Differentiating partially w.r.t. x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \qquad \text{...(1)}$$

and
$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \qquad \text{...(2)}$$

Using Cauchy–Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

we get
$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \qquad \text{...(3)}$$

$$u\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial x}=0 \qquad ...(4)$$

Eliminating $\frac{\partial u}{\partial y}$ from (3) and (4), we get $\left(u^2+v^2\right)\frac{\partial u}{\partial x}=0$

Again eliminating $\frac{\partial u}{\partial x}$, we get $\left(u^2+v^2\right)\frac{\partial u}{\partial y}=0.$

If $k^2=u^2+v^2=0$ then $u = 0$ and $v = 0$ and hence $f(z) = 0$.

If $k \neq 0$ then $u_x=0=u_y$. From Cauchy–Riemann equations $v_x=v_y=0$. All these yield u = constant and v = constant so that $f(z)$ is constant.

Example 21 : *(i) An analytic function with constant real part is constant.*

(ii) An analytic function with constant imaginary part is constant.

__Solution :__ (*i*) Let $w = u + iv$ be an analytic function.

$\therefore$ By Cauchy–Riemann equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ ...(1)

By given data real part is constant.

Let $u=k_1$ then $\frac{\partial u}{\partial x}=0$ and $\frac{\partial u}{\partial y}=0.$

From (1), we get $\frac{\partial v}{\partial y}=0$ and $\frac{\partial v}{\partial x}=0.$

Thus, v is independent of x and y.

$\therefore$ We can take $v = k_2$, a constant.

$\therefore$ $w = u + iv = k_1 + ik_2$

$\Rightarrow$ $w = k$.

$\therefore$ w is constant.

(*ii*) Similarly, we can show that an analytic function with constant imaginary part is constant.

Example 22 : *Show that both the real and imaginary parts of an analytic function are harmonic.*

[JNTU 1998]

__Solution :__ Let $f(z) = u + iv$ be an analytic function.

$\therefore$ $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ ...(1) and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ by Cauchy–Riemann equations. ...(2)

Differentiating (1) partially w.r.t. x, $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}$ ...(3)

Differentiating (2) partially w.r.t. y, $\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x}$ ... (4)

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \left(\text{since } \frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 v}{\partial y\partial x}\right)$

$\therefore$ u satisfies Laplace's equation, hence, u is harmonic.

Again differentiating (1) w.r.t. y, $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 v}{\partial y^2}$ ...(5)

Again differentiating (2) w.r.t. x, $\frac{\partial^2 u}{\partial x\partial y}=-\frac{\partial^2 v}{\partial x^2}$ ...(6)

(5) – (6) gives, $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=0$

$\therefore$ v satisfies Laplace's equation. Hence, v is harmonic.

Thus, both u and v are harmonic functions.

Example 23 : *Every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = k_1$ and $v(x, y) = k_2$ forming an orthogonal system.* **[JNTU 1999, 1995 Sep.]**

(Or) If $w = f(z)$ is an analytic function, then prove that the family of curves defined by $u(x, y)$ = constant cuts orthogonally the family of curves $v(x, y)$ = constant **[JNTU 2004, Aug. 2007S (Set No. 1)]**

Solution : Consider the two families of curves

$$u(x, y) = k_1 \quad \text{...(1)}$$

$$v(x, y) = k_2 \quad \text{...(2)}$$

Let $f(z) = u + iv$ be the analytic function

By Cauchy–Riemann equations, $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ ...(3)

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \text{...(4)}$$

Differentiating (1) w.r.t. x,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=m_1 \quad \text{(say)}$$

Differentiating (2) w.r.t. x,

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}=\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}=m_2 \quad \text{(say)} \quad \left[\because\ \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y},\ \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}\right]$$

$$\text{Now } m_1\times m_2=\frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\times\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}=-1$$

$\therefore m_1 m_2 = -1$

Hence, the curves (1) and (2) cut each other orthogonally

i.e., They form an orthogonal system, which proves the result.

The two families are said to be the orthogonal trajectories of one another.

Example 24 : *For $\omega=\exp(z^2)$, find u and v and prove that the curves $u(x, y)=c_1$ and $v(x, y)=c_2$ where c_1 and c_2 are constants cut orthogonally.*

[JNTU 2003 (Set No. 2)]

Solution : Let $\omega = u(x, y) + i\, v(x, y) = e^{z^2}$

i.e., $u + i\, v = e^{(x+iy)^2} = e^{x^2-y^2+i\,2xy}$

i.e., $u + i\, v = e^{x^2-y^2} \cdot e^{i\,2xy} = e^{x^2-y^2}\,(\cos 2xy + i \sin 2xy)$

$\therefore\ u(x, y) = e^{x^2-y^2} \cdot \cos 2xy$ and $v(x, y) = e^{x^2-y^2} \cdot \sin 2xy$

We know that analytic function of an analytic function is analytic.

Since z^2 is analytic and e^z is analytic, therefore e^{z^2} is also analytic function.

We know that if $\omega = u + i\, v$ is an analytic function, the curves of the family $u(x, y) = c_1$ cut orthogonally the curves of the family $v(x, y) = c_2$.

Thus $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system.

Example 25 : *Show that the curves* $r^n = \alpha \sec n\theta$ *and* $r^n = \beta\, \text{cosec}\, n\theta$ *cut orthogonally*

[*JNTU 2003 (Set No. 3)*]

Solution : Given curves can be written as

$$r^n \cos n\theta = \alpha \quad \text{and} \quad r^n \sin n\theta = \beta$$

i.e.,
$$u(r, \theta) = \alpha \quad \text{and} \quad v(r, \theta) = \beta$$

Now
$$u(r, \theta) + i\, v(r, \theta) = \alpha + i\beta = r^n \cos n\theta + i r^n \sin n\theta$$
$$= r^n (\cos n\theta + i \sin n\theta) = r^n e^{i n\theta} = (r e^{i\theta})^n = z^n$$

which is an analytic function.

We know that if $\omega = u + i\, v$ is an analytic function, the curves of the family $u(x, y) = c_1$ cut orthogonally the curves of the family $v(x, y) = c_2$. Hence the result.

Example 26 : If $\omega = u + i\, v = z^3$ *prove that* $u = c_1$ *and* $v = c_2$ *where* c_1 *and* c_2 *are constants, cut each other orthogonally.* **[JNTU 2003 (Set No. 3)]**

Solution : $\omega = u + i\, v = z^3 = (x + i\, y)^3 = (x^3 - 3xy^2) + i\,(3x^2y - y^3)$.

$\therefore\ u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

We have two families of curves

$$x^3 - 3xy^2 = c_1 \quad \text{... (1)}$$

and
$$3x^2y - y^3 = c_2 \quad \text{... (2)}$$

Let a particular curve of the family $x^3 - 3xy^2 = c_1$ intersect a particular curve of the family $3x^2y - y^3 = c_2$ at (x_0, y_0).

Differentiating (1) with respect to x, we get

$$3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0 \ \Rightarrow\ \frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

Slope of the tangent at (x_0, y_0) for the first curve,

$$m_1 = \left(\frac{x^2 - y^2}{2xy}\right)_{(x_0, y_0)} = \frac{x_0^2 - y_0^2}{2x_0\, y_0}$$

Differentiating (2) with respect to x, we get

$$3\left(2xy + x^2\frac{dy}{dx}\right) - 3x^2\frac{dy}{dx} = 0 \quad i.e.\ \frac{dy}{dx} = -\frac{2xy}{x^2 - y^2}$$

Slope of the tangent at (x_0, y_0) for the second curve,

$$m_2 = -\frac{2x_0 y_0}{x_0^2 - y_0^2}$$

Product of the two slopes $= m_1 m_2 = -1$

So these curves intersect orthogonally. Thus the two families of curves intersect orthogonally.

Example 27 : Show that (*i*) $f(z) = e^z$ (*ii*) $f(z) = e^{\bar{z}}$ **[JNTU (H) Nov. 2009 (Set No. 1)]** is analytic everywhere in the complex plane and find $f'(z)$.

Solution : (*i*) Let $w = f(z) = e^z = e^{x+iy} = e^x e^{iy}$

or $\quad w = u + iv = e^x(\cos y + i\sin y)$

$\Rightarrow \quad u = e^x \cos y$ and $v = e^x \sin y$

$$\therefore \quad \frac{\partial u}{\partial x} = e^x \cos y, \ \frac{\partial u}{\partial y} = -e^x \sin y$$

and
$$\frac{\partial v}{\partial x} = e^x \sin y, \ \frac{\partial v}{\partial y} = e^x \cos y$$

Hence
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus the Cauchy – Riemann equations are satisfied. Also the partial derivatives are continuous.

Hence $\frac{dw}{dz} = f'(z)$ exists at all points in the complex plane

i.e. $f(z)$ is analytic everywhere.

$$\therefore \quad f(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i\, e^x \sin y$$
$$= e^x(\cos y + i\sin y) = e^x e^{iy} = e^{x+iy} = e^z$$

(*ii*) This is left as an exercise to the student.

Note. If a complex function is analytic, then it can be differentiated just in the ordinary way.

Example 28 : *Prove that z^n (n is a positive integer) is analytic and hence find its derivative.* **[JNTU 2003S, 2004, Aug. 2007S, JNTU (H) Nov. 2009 (Set No. 2)]**

Solution : Let $f(z) = z^n$.

Using polar co-ordinates, let $f(z) = u(r, \theta) + i\, v(r, \theta)$

Then as, $z = r e^{i\theta}$,

$$f(z) = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

$$\therefore \quad u = r^n \cos n\theta \text{ and } v = r^n \sin n\theta$$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta, \ \frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$$

and $\dfrac{\partial u}{\partial \theta} = -nr^n \sin n\theta,\ \dfrac{\partial v}{\partial \theta} = nr^n \cos n\theta$

Now $\dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = \dfrac{1}{r}\cdot n\,r^n \cos n\theta = n\,r^{n-1} \cos n\theta = \dfrac{\partial u}{\partial r}$

and $\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = \dfrac{1}{r}(-n\,r^n \sin n\theta) = -n\,r^{n-1} \sin n\theta = -\dfrac{\partial v}{\partial r}$

Since the Cauchy – Riemann equations are satisfied and the partial derivatives are continuous, therefore $f(z) = z^n$ is analytic.

Differentiating $f(z) = u\,(r, \theta) + i\,v\,(r, \theta)$ partially with respect to 'r', we get

$$\frac{\partial f}{\partial r} = f'(z)\,\frac{\partial z}{\partial r} = f'(z)\,e^{i\theta}$$

$$\Rightarrow \quad f'(z) = \frac{1}{e^{i\theta}}\frac{\partial f}{\partial r} = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right)$$

$$= e^{-i\theta}(n\,r^{n-1} \cos n\theta + i\,n\,r^{n-1} \sin n\theta) = e^{-i\theta} n\,r^{n-1}(\cos n\theta + i \sin n\theta)$$

$$= n\,r^{n-1} e^{-i\theta} e^{in\theta} = n\,r^{n-1} e^{i(n-1)\theta} = n\,(r\,e^{i\theta})^{n-1} = n\,z^{n-1}$$

Example 29 : *Prove that the function $f(z) = \bar{z}$ is not analytic at any point.*

[JNTU 2003, 2005, JNTU (A) Dec. 2009 (Set No. 3)]

Solution : Let $f(z) = u + i\,v = \bar{z} = x - i\,y$

Thus $u = x$ and $v = -y$

$$\therefore \quad \frac{\partial u}{\partial x} = 1,\ \frac{\partial u}{\partial y} = 0$$

and $$\frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = -1$$

Clearly $$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy – Riemann conditions are not satisfied. Therefore $f\,(z)$ is not analytic at any point.

Example 30 : *Determine whether the function $2xy + i\,(x^2 - y^2)$ is analytic*

[JNTU 2003 (Set No. 4)]

Solution : Let $f(z) = u + i\,v = 2xy + i\,(x^2 - y^2)$

So $u\,(x, y) = 2xy$ and $v\,(x, y) = x^2 - y^2$

$$\therefore \quad \frac{\partial u}{\partial x} = 2y,\ \frac{\partial u}{\partial y} = 2x \quad \text{and} \quad \frac{\partial v}{\partial x} = 2x,\ \frac{\partial v}{\partial y} = -2y$$

Hence Cauchy – Riemann conditions are not satisfied.
Therefore $f(z)$ is not analytic.

Example 31 : Determine p such that the function $f(z)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{px}{y}\right)$ be an analytic function. **[JNTU 2003, Aug. 2007S (Set No. 2)]**

Solution : Let $f(z)=u+iv=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{px}{y}\right)$.

Then $u(x,y)=\frac{1}{2}\log(x^2+y^2)$ and $v(x,y)=\tan^{-1}\left(\frac{px}{y}\right)$

$$\therefore \quad u_x=\frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x=\frac{x}{x^2+y^2},\quad u_y=\frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2y=\frac{y}{x^2+y^2}$$

$$\text{and } v_x=\frac{1}{1+\left(\frac{px}{y}\right)^2}\cdot\frac{p}{y}=\frac{py}{y^2+p^2x^2},\quad v_y=\frac{1}{1+\left(\frac{px}{y}\right)^2}\left(\frac{-px}{y^2}\right)=\frac{-px}{y^2+p^2x^2}$$

Clearly $u_x=v_y$ if $p=-1$

and $u_y=-v_x$ if $p=-1$

Hence $f(z)$ is analytic when $p=-1$.

Example 32 : Show that $f(z)=\sin z$ is analytic everywhere in the complex plane and find $f'(z)$.

Solution : We have

$$w=f(z)=\sin z=\sin(x+iy)$$

or $w=u+iv=\sin x\cdot\cos iy+\cos x\cdot\sin iy=\sin x\cdot\cosh y+i\cos x\cdot\sinh y$

Hence $u=\sin x\cosh y,\ v=\cos x\sinh y$

$$\therefore \quad \frac{\partial u}{\partial x}=\cos x\cosh y,\ \frac{\partial u}{\partial y}=\sin x\sinh y$$

$$\text{and } \frac{\partial v}{\partial x}=-\sin x\sinh y,\ \frac{\partial v}{\partial y}=\cos x\cosh y$$

Clearly $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

The Cauchy – Riemann equations are identically satisfied. Also the partial derivatives are continuous.

Hence $\frac{dw}{dz}=f'(z)$ exists at all points of the z-plane *i.e.* $f(z)$ is analytic everywhere.

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\cos x\cosh y-i\sin x\sinh y$$

$$=\cos x\cos iy-\sin x\sin iy=\cos(x+iy)=\cos z$$

Example 33 : If $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ is an analytic function. **[JNTU 2003S (Set No. 1)]**

(or) If u and v are functions of x and y satisfying Laplace's equation, show that $(s+it)$ is analytic where $s=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}$ and $t=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$ **[JNTU June 2009 S (Set No 3)]**

__Solution :__ Since u and v are harmonic functions, we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad\Rightarrow\quad \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 u}{\partial x^2} \qquad \text{... (1)}$$

and $$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \quad\Rightarrow\quad \frac{\partial^2 v}{\partial y^2}=-\frac{\partial^2 v}{\partial x^2} \qquad \text{... (2)}$$

Let $$U=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x} \qquad \text{... (3)}$$

and $$V=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \qquad \text{... (4)}$$

Differentiating (3) and (4) partially with respect to x, we get

$$\frac{\partial U}{\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}-\frac{\partial^2 v}{\partial x^2} \qquad \text{... (5)}$$

and $$\frac{\partial V}{\partial x}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y} \qquad \text{... (6)}$$

Differentiating (3) and (4) partially with respect to y, we get

$$\frac{\partial U}{\partial y}=\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial y\partial x} \qquad \text{... (7)}$$

and $$\frac{\partial V}{\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial^2 v}{\partial y^2} \qquad \text{... (8)}$$

From (1) and (7), we have

$$\frac{\partial U}{\partial y}=-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}\right)=-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right)=-\frac{\partial V}{\partial x} \quad \text{[From (6)]}$$

From (2) and (5), we have

$$\frac{\partial U}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial V}{\partial y} \quad \text{[From (8)]}$$

Hence $\dfrac{\partial U}{\partial x}=\dfrac{\partial V}{\partial y}$ and $\dfrac{\partial U}{\partial y}=-\dfrac{\partial V}{\partial x}$

Since Cauchy – Riemann equations are satisfied and partial derivatives $\dfrac{\partial U}{\partial x}, \dfrac{\partial U}{\partial y}, \dfrac{\partial V}{\partial x}$ and $\dfrac{\partial V}{\partial y}$ are continuous, therefore $U + i\,V = \left(\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial x}\right)+i\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)$ is an analytic function.

Example 34 : Show that xy^2 cannot be real part of an analytic function.

Solution : Let $u(x, y) = xy^2$.

If $u(x, y)$ be the real part of an analytic function $f(z)=u+iv$ then $\nabla^2 u = 0$.

We have $\dfrac{\partial u}{\partial x}=y^2, \dfrac{\partial u}{\partial y}=2xy$

and $\dfrac{\partial^2 u}{\partial x^2}=0, \dfrac{\partial^2 u}{\partial y^2}=2x$

Clearly $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\neq 0$

i.e. $\nabla^2 u \neq 0$

Hence $u(x, y) = xy^2$ cannot be the real part of an analytic function.

Example 35 : Prove that, if $u=x^2-y^2$, $v=\dfrac{-y}{x^2+y^2}$ both u and v satisfy Laplace's equation, but $u + iv$ is not a regular (analytic) function of z. **[JNTU Aug. 2007S (Set No. 4)]**

Solution : Given $u=x^2-y^2$ and $v=\dfrac{-y}{x^2+y^2}$

$$\therefore \quad \frac{\partial u}{\partial x}=2x,\ \frac{\partial u}{\partial y}=-2y,\ \frac{\partial^2 u}{\partial x^2}=2,\ \frac{\partial^2 u}{\partial y^2}=-2$$

and
$$\frac{\partial v}{\partial x}=-y\left[\frac{-2x}{(x^2+y^2)^2}\right]=\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2}=2y\left[\frac{(x^2+y^2)^2-2x(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}\right]=\frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y}=\frac{-(x^2+y^2)-2y(-y)}{(x^2+y^2)^2}=\frac{-(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2}=\frac{(x^2+y^2)^2(2y)-(y^2-x^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4}=\frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

Clearly $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ and $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=0$

Hence both u and v satisfies the Laplace's equation.

We observe that $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

Since u and v do not satisfy the Cauchy – Riemann equations, therefore $u + i\,v$ is not an analytic (regular) function of z.

Example 36 : Show that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.

Solution : We have $u = e^{-x}(x \sin y - y \cos y)$

Differentiating partially, w.r.t x and y, we get

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y) - e^{-x}(x \sin y - y \cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = -2e^{-x} \sin y + x\,e^{-x} \sin y - y\,e^{-x} \cos y \quad \text{...(1)}$$

and $$\frac{\partial u}{\partial y} = e^{-x}\left(x \cos y + y \sin y - \cos y\right)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}\left(-x \sin y + 2 \sin y + y \cos y\right) \quad \text{...(2)}$$

Adding (1) & (2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ u is a harmonic function.

Example 37 : Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a conjugate harmonic function v of u.

Solution : Given $u = x^2 - y^2 - y$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x \text{ and } \frac{\partial^2 u}{\partial x^2} = 2 \text{ and } \frac{\partial u}{\partial y} = -2y - 1 \text{ and } \frac{\partial^2 u}{\partial y^2} = -2$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u is a harmonic function.

To find the conjugate harmonic function v, we must have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \quad \text{...(1)} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y + 1 \quad \text{...(2)}$$

We will integrate the first equation w.r.t. y and then differentiate w.r.t. x, we get

$$v = 2xy + h(x) \text{ and } \frac{\partial v}{\partial x} = 2y + \frac{dh}{dx}$$

Comparing with (2), we get

$$\frac{dh}{dx} = 1 \Rightarrow h(x) = x + c$$

where c is any real constant. Hence, $v = 2xy + x + c$ is the most general conjugate harmonic function of given u.

Example 38: Show that the function $u(x, y) = e^x \cos y$ is harmonic. Determine its harmonic conjugate $v(x, y)$ and the analytic function $f(z) = u + iv$. [JNTU 1996]

Solution : Given $u(x, y) = e^x \cos y$

Differentiating with respect to x and y, we get

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Hence, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus, u is a harmonic function. Let v be the harmonic conjugate of u. Then, by Cauchy-Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y$$

Integrating, $v = e^x \sin y + f(y)$ ...(1)

$$\therefore \quad \frac{\partial v}{\partial y} = e^x \cos y + f'(y) \quad \text{...(2)}$$

Again $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y \quad \text{...(3)}$$

From (2) and (3), we get

$$e^x \cos y = e^x \cos y + f'(y)$$

or $$f'(y) = 0 \Rightarrow f(y) = c$$

Hence, from (1), we get

$$v = e^x \sin y + c$$

$$\therefore \quad f(z) = u + iv = e^x \cos y + ie^x \sin y + ic$$

$$= e^x (\cos y + i \sin y) + ic = e^x e^{iy} + ic = e^z + ic$$

Example 39: If $f(z)$ is a regular function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2$$

[JNTU 2004 (Set No. 1), 2005, Aug. 2007, Nov. 2008S (Set No. 1)]

or $$\nabla^2 \left[|f(z)|^2\right] = 4|f'(z)|^2$$

Solution : Let $f(z) = u(x, y) + iv(x, y)$ is a regular function. Then

$$|f(z)|^2 = u^2 + v^2 = \phi(x, y) \quad \text{(say)}$$

$\therefore \quad \dfrac{\partial \phi}{\partial x} = 2u\dfrac{\partial u}{\partial x} + 2v\dfrac{\partial v}{\partial x}$

and $\quad \dfrac{\partial^2 \phi}{\partial x^2} = 2\left[u\dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\partial u}{\partial x}\right)^2 + v\dfrac{\partial^2 v}{\partial x^2} + \left(\dfrac{\partial v}{\partial x}\right)^2\right]$

Similarly, $\dfrac{\partial^2 \phi}{\partial y^2} = 2\left[u\dfrac{\partial^2 u}{\partial y^2} + \left(\dfrac{\partial u}{\partial y}\right)^2 + v\dfrac{\partial^2 v}{\partial y^2} + \left(\dfrac{\partial v}{\partial y}\right)^2\right]$

Adding, we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right]$$
$$+2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \qquad \ldots(1)$$

Since u and v have to satisfy Cauchy-Riemann equations and the Laplace equation, we have

$$\left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2, \left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2$$

and $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0 \qquad \ldots(2)$

Thus; $\quad \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 4\left[\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right]$ [using (2) in (1)]

Hence, $\quad \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Example 40 : Show that $\left\{\dfrac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\dfrac{\partial}{\partial y}|f(z)|\right\}^2 = \{f'(Z)\}^2$ [JNTU 2000]

Solution : Take $f(z) = u + iv \Rightarrow |f(z)| = \sqrt{u^2 + v^2}$

Differentiating partially $|f(z)|$ w.r.t. 'x', we get

$$\frac{\partial}{\partial x}(|f(z)|) = \frac{1}{2}\cdot\frac{2uu_x + 2vv_x}{\sqrt{u^2+v^2}} = \frac{uu_x + vv_x}{\sqrt{u^2+v^2}} \qquad \ldots(1)$$

Differentiating partially $|f(z)|$ w.r.t. 'y', $\dfrac{\partial}{\partial y}(|f(z)|) = \dfrac{uu_y + vv_y}{\sqrt{u^2+v^2}} \qquad \ldots(2)$

$(1)^2 + (2)^2$ gives,

$$\left[\frac{\partial}{\partial x}(|f(z)|)\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{(u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x) + (u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y)}{|f(z)|^2}$$

Assuming f is analytic, Causchy–Riemann conditions $u_x = v_y$ and $u_y = -v_x$ are satisfied.

So $\quad 2uv\, u_xv_x = -2uvu_yv_y$

$$\therefore \quad \left[\frac{\partial}{\partial x}|f(z)|\right]^2+\left[\frac{\partial}{\partial y}|f(z)|\right]^2=\frac{(u^2+v^2)(u_x^2+v_x^2)}{|f(z)|^2}=u_x^2+v_x^2=|f'(z)|^2$$

$[\because f'(z)=u_x+iv_x$ and $|f'(z)|=\sqrt{u_x^2+v_x^2}$]

Example 41 : Show that $u(x, y)=x^3-3xy^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z.

Solution : Given $u=x^3-3xy^2$

$$\therefore \quad \frac{\partial u}{\partial x}=3x^2-3y^2, \frac{\partial u}{\partial y}=-6xy$$

and $$\frac{\partial^2 u}{\partial x^2}=6x, \frac{\partial^2 u}{\partial y^2}=-6x$$

Clearly $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Hence u is harmonic.

Let v be the conjugate of u. Then

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy \quad \text{(Using } C-R \text{ equations)}$$

or $$dv=6xy\,dx+(3x^2-3y^2)\,dy$$

Taking $M = 6xy$ and $N = 3x^2 - 3y^2$, we get

$$dv=M\,dx+N\,dy \qquad \dots (1)$$

Now $$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=6x$$

Hence (1) is exact differential equation

Integrating, $$v=\int M\,dx + \int N\,dy+c$$

y constant only those terms which do not contain x.

$$=\int 6xy\,dx+\int(-3y^2)\,dy+c$$

y constant

or $$v=6y\left(\frac{x^2}{2}\right)-3\left(\frac{y^3}{3}\right)+c=3x^2y-y^3+c$$

$$\therefore \quad f(z)=u+iv=x^3-3xy^2+i(3x^2y-y^3)+ic$$

$$=x^3+3x^2(iy)+3x(iy)^2+(iy)^3+ic$$

or $f(z)=(x+iy)^3+k$ where $k = ic$.

Note : The above result may be obtained by using Milne – Thomson's method.

Example 42 : Find k such that $f(x,y)=x^3+3kxy^2$ may be harmonic and find its conjugate.

[JNTU 2004 (Set No. 4), 2004S]

Solution : We have $f(x,y)=x^3+3kxy^2$

$$\therefore \quad \frac{\partial f}{\partial x}=3x^2+3ky^2, \frac{\partial f}{\partial y}=6kxy$$

and $$\frac{\partial^2 f}{\partial x^2}=6x, \frac{\partial^2 f}{\partial y^2}=6kx$$

Since $f(x,y)$ is harmonic, therefore $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$

i.e. $6x+6kx=0$ *i.e.* $x(1+k)=0$

i.e. $1+k=0$ $(\because x\neq 0)$ or $k=-1$

Hence $f(x,y)=x^3-3xy^2$

Let $g(x,y)$ be the conjugate of $f(x,y)$. Then

$$dg=\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy \;=-\frac{\partial f}{\partial y}dx+\frac{\partial f}{\partial x}dy \quad \text{(Using } C-R \text{ equations)}$$

$$=-6kxy\,dx+(3x^2+3ky^2)\,dy$$

or $dg=6xy\,dx+(3x^2-3y^2)\,dy \qquad (\because k=-1)$

This is exact differential equation.

Integrating, $g=\displaystyle\int 6xy\,dx+\int -3y^2\,dy+c$

y constant only those terms which do not contain x.

$$=6y\left(\frac{x^2}{2}\right)-3\left(\frac{y^3}{3}\right)+c=3x^2y-y^3+c$$

Example 43 : If u is a harmonic function, show that $w=u^2$ is not a harmonic function unless u is a constant.

[JNTU 2005S, Aug. 2006S (Set No. 1)]

Solution : Given u is a harmonic function.

$\therefore$ By definition, we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{...(1)}$$

Also given $w=u^2$

$$\therefore \quad \frac{\partial w}{\partial x}=2u\frac{\partial u}{\partial x} \text{ and } \frac{\partial w}{\partial y}=2u\frac{\partial u}{\partial y}$$

Now $$\frac{\partial^2 w}{\partial x^2}=2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right] \text{ and } \frac{\partial^2 w}{\partial y^2}=2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$\therefore \quad \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[u(0)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] \text{ using (1)}$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] \quad \ldots(2)$$

$$\neq 0$$

Hence w is not a harmonic function.

If u = constant = K(say) then $\frac{\partial u}{\partial x}=0,\ \frac{\partial u}{\partial y}=0$

From (2) it is clear that $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0$

$\therefore$ w is a harmonic function if u is a constant.

Thus $w = u^2$ is not a harmonic function unless u is a constant.

Example 44 : Show that the function $u = 2\log(x^2+y^2)$ is harmonic and find its harmonic conjugate ((or) find $f(z)=u+iv$). **[JNTU 2006 (Set No. 1)]**

Solution : Given $u = 2\log(x^2+y^2)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{2}{x^2+y^2}\cdot 2x = \frac{4x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 4-4x\cdot 2x}{(x^2+y^2)^2} = \frac{4(y^2-x^2)}{(x^2+y^2)^2}$$

and $$\frac{\partial u}{\partial y} = \frac{4y}{x^2+y^2},\ \frac{\partial^2 u}{\partial y^2} = \frac{4(x^2-y^2)}{(x^2+y^2)^2}$$

Now $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{4(y^2-x^2)+4(x^2-y^2)}{(x^2+y^2)^2}=0$$

Hence u is a harmonic function.

Let $v(x, y)$ be the harmonic conjugate of $u(x, y)$. Then

$$dv = \frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy \text{ (using } C\text{-}R \text{ equations)}$$

i.e. $$dv = \frac{-4y}{x^2+y^2}dx+\frac{4x}{x^2+y^2}dy = \frac{4(x\,dy-y\,dx)}{x^2+y^2}$$

Integrating both sides, we get $v = 4\tan^{-1}\frac{y}{x}+c$

Example 45 : Find the orthogonal trajectories of the family of curves $x^3 y - xy^3 = C$ = constant.

Solution : Consider $u(x, y) = x^3 y - xy^3$.

if $f(z) = u + iv$ is analytic, then v = constant will be the required orthogonal trajectories.

Then $\quad u_x = 3x^2 y - y^3, u_y = x^3 - 3xy^2$

By C-R equations, $\quad v_y = u_x = 3x^2 y - y^3$,

Integrating, $$v = \frac{3x^2y^2}{2} - \frac{y^4}{4} + C(x)$$

Differentiating, $$v_x = 3xy^2 + \frac{dc}{dx} = -u_y = -x^3 + 3xy^2$$

$$\Rightarrow \frac{dC}{dx} = -x^3 \Rightarrow C(x) = -\frac{x^4}{4} + K, \text{ where K is any constant.}$$

Thus required orthogonal trajectories is

$$v(x, y) = \frac{3x^2y^2}{2} - \frac{y^4}{4} - \frac{x^4}{4} + K = \text{constant} \quad \text{or} \quad x^4 + y^4 - 6x^2 y^2 = \text{constant}$$

Example 46 : Find the orthogonal trajectories of the family of cunves $r^2 \cos 2\theta = C$.

Solution : Consider $u(r, \theta) = r^2 \cos 2\theta$. Then $\dfrac{\partial u}{\partial r} = 2r \cos 2\theta$

$\dfrac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta$. Using Cauchy-Riemann equations.

we have $\dfrac{\partial v}{\partial \theta} = r\dfrac{\partial u}{\partial r} \Rightarrow \dfrac{\partial v}{\partial \theta} = 2r^2 \cos 2\theta$

Integrating, $v(r, \theta) = r^2 \sin 2\theta + C_1(r)$

Differentating with respect to r, we get

$$\frac{\partial v}{\partial r} = 2r \sin 2\theta + \frac{dC_1}{dr} = \frac{-1}{r}\frac{\partial u}{\partial \theta} = \frac{-1}{r}(-2r^2 \sin 2\theta) = 2r \sin 2\theta$$

which gives $\dfrac{dC_1}{dr} = 0 \Rightarrow C_1$ = constant

Thus the orthogonal trajectories are given by $v = r^2 \sin 2\theta$

2.18 CONSTRUCTION OF ANALYTIC FUNCTION WHOSE REAL OR IMAGINARY PART IS KNOWN

Suppose $f(z) = u + iv$ is an analytic function, whose real part u is known. We can find v, the imaginary part and also the function $f(z)$. The procedure is as follows :

Method 1 : We know that $dv = \dfrac{\partial v}{\partial x} dx + \dfrac{\partial v}{\partial y} dy$

But $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$ (Cauchy – Riemann equations)

$$\therefore \quad dv = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$$

Taking $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$, we get

$$dv = M\,dx + N\,dy \qquad \text{...(1)}$$

Now $\quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = -\nabla^2 u$

As u satisfies Laplace's equation, $\nabla^2 u = 0$

Hence $\quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$ or $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus, equation(1) is an exact differential equation.

Hence equation(1) can be integrated and v can be determined.

Now u and v are known and hence the function $f(z) = u + i\,v$ is determined.

Method 2 : Milne – Thomson's method

By this method $f(z)$ is directly constructed without finding v (or u) and the method is given below :

Let $\quad f(z) = u(x, y) + i\,v(x, y) \qquad$ ...(1)

Since $\quad z = x + iy$, $\bar{z} = x - iy$, we have

$$x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2i}$$

$$\therefore \quad f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + i\,v\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \qquad \text{...(2)}$$

Now considering this relation as a formal identity in two independent variables z and $\bar{z}$.

Putting $\quad \bar{z} = z$ in (2), we get

$$f(z) = u(z, 0) + i\,v(z, 0) \qquad \text{...(3)}$$

$\therefore \quad$ (3) is same as (1), if we replace x by z and y by 0.

Thus to express any function in terms of z, replace x by z and y by 0.

Now $\quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$, by Cauchy-Riemann equations

Let $\quad \frac{\partial u}{\partial x} = \phi_1(x, y)$ and $\frac{\partial u}{\partial y} = \phi_2(x, y)$

Then $\quad f'(z) = \phi_1(x, y) - i\,\phi_2(x, y) \qquad$ ...(4)

Now, to express $f'(z)$ completely in terms of z, we replace x by z and y by 0 in (4)

$\therefore \quad f'(z) = \phi_1(z, 0) - i\,\phi_2(z, 0)$

Hence $\quad f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\,dz + c_1$

where c_1 is a complex constant.

Similarly if $v(x, y)$ is given, we can find u such that $u + i\,v$ is analytic. Let us use Milne-Thomson's method.

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$, by $C - R$ equations

$= \psi_1(x, y) + i\,\psi_2(x, y)$ where $\frac{\partial v}{\partial y} = \psi_1(x, y)$ and $\frac{\partial v}{\partial x} = \psi_2(x, y)$

$= \psi_1(z, 0) + i\,\psi_2(z, 0)$

$\therefore\ f(z) = \int [\psi_1(z, 0) + i\,\psi_2(z, 0)]\,dz + c_2$

where c_2 is a complex constant.

SOLVED EXAMPLES

Example 1 : Find most general analytic function whose real part is $u = x^2 - y^2 - x$.

Solution : We have $u_x = \frac{\partial u}{\partial x} = 2x - 1 = v_y = \frac{\partial v}{\partial y}$ by the Cauchy–Riemann equations.

Then integrating, we get $v = 2xy - y + k(x)$

From this, $\frac{\partial v}{\partial x} = 2y + \frac{dk}{dx}$

By Cauchy–Riemann equations $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore\ \frac{\partial u}{\partial y} = -2y - \frac{dk}{dx}$

Given $u = x^2 - y^2 - x$ and $\frac{\partial u}{\partial y} = -2y$

$\therefore\ \frac{dk}{dx} = 0 \Rightarrow k =$ real constant . Hence $v = 2xy - y + k$

Thus, $f(z) = u + iv = (x^2 - y^2 - x) + i(2xy - y + k) = z^2 - z + ik$

Aliter.

We will now use Milne – Thomson Method

Let $f(z) = u + i\,v$ where $u = x^2 - y^2 - x$

Then $\frac{\partial u}{\partial x} = 2x - 1$ and $\frac{\partial u}{\partial y} = -2y$

$\therefore\ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$, using Cauchy – Riemann equations

$= (2x - 1) - i(-2y) = 2x - 1 + i\,2y$

By Milne – Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $f'(z) = 2z - 1$

Integrating, $f(z) = 2\int z\,dz - \int dz = z^2 - z + C$

where C is a complex constant.

Example 2 : Find an analytic function whose real part is $e^{-x}(x \sin y - y \cos y)$.

Solution : Let $f(z) = u + iv$ where $u = e^{-x}(x \sin y - y \cos y)$

Differentiating partially with respect to x and y, we get

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y) + (x \sin y - y \cos y)(-e^{-x})$$

and $$\frac{\partial u}{\partial y} = e^{-x}(x \cos y + y \sin y - \cos y)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(Using Cauchy–Riemann equations)}$$

$$= e^{-x}(\sin y - x \sin y + y \cos y) - i\,e^{-x}(x \cos y + y \sin y - \cos y)$$

By Milne-Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0

$$\therefore \quad f'(z) = 0 - i\,e^{-z}(z-1) = -i\left(ze^{-z} - e^{-z}\right)$$

Integrating with respect to z, we have $f(z) = i\,z\,e^{-z} + \text{constant.}$

Example 3 : If $u = e^{x}\left[(x^2 - y^2)\cos y - 2xy \sin y\right]$ is real part of an analytic function, find the analytic function. **[JNTU June 2002]**

Solution : Let $f(z) = u + i\,v$ where $u = e^{x}\left[(x^2 - y^2)\cos y - 2xy \sin y\right]$

Then $$\frac{\partial u}{\partial x} = e^{x}\left[(x^2 - y^2)\cos y - 2xy \sin y\right] + e^{x}\left[2x \cos y - 2y \sin y\right]$$

and $$\frac{\partial u}{\partial y} = e^{x}\left[-2y \cos y + (x^2 - y^2)(-\sin y) - 2x \sin y - 2xy \cos y\right]$$

Now $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(using } C - R \text{ equations)}$$

$$= e^{x}\left[(x^2 - y^2)\cos y - 2xy \sin y + 2x \cos y - 2y \sin y\right]$$
$$- i\,e^{x}\left[-2y \cos y + (y^2 - x^2)\sin y - 2x \sin y - 2xy \cdot \cos y\right]$$

By Milne – Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $f'(z) = e^{z}[z^2 + 2z] - i\,e^{z}[0] = e^{z}(z^2 + 2z)$

Integrating by parts with respect to 'z', we get

$$f(z) = e^{z} z^2 + C \quad \text{where } C \text{ is a complex constant.}$$

Example 4 : Find an analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ **[JNTU 2003]**

Solution : Let $f(z) = u + iv$ where $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(Using Cauchy–Riemann equations)}$$

$$= \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} - i\,\frac{\sin 2x\,(-2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2\sin 2x \cdot \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

Using Milne–Thomson method, we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore \quad f'(z) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} + i\,(0) = \frac{-2}{1 - \cos 2z} = -\operatorname{cosec}^2 z$$

Integrating, we get $f(z) = \cot z + ic$.

Here the integration constant is taken as imaginary since u does not contain any constant.

Example 5 : Find the analytic function whose imaginary part is e^x ($x \sin y + y \cos y$).

[JNTU 1998S]

Sol. Given $v = e^x\,(x \sin y + y \cos y)$ ...(1)

Differentiating (1) partially w.r.t. x, $\dfrac{\partial v}{\partial x} = e^x\left[(x+1)\sin y + y\cos y\right]$

Differentiating (1) partially w.r.t. y, $\dfrac{\partial v}{\partial y} = e^x\left[(x+1)\cos y - y\sin y\right]$

But $f'(z) = \dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} + i\,\dfrac{\partial v}{\partial x} = e^x\left[(x+1)\cos y - y\sin y\right] + i\,e^x\left[(x+1)\sin y + y\cos y\right]$

Using Milne–Thomson method $f'(z) = e^z\,(z+1)$ [Putting $x = z$ and $y = 0$]

Integrating, $f(z) = ze^z + c$

i.e. $u + iv = (x+iy)\,e^{x+iy} + c = (x+iy)\cdot e^x \cdot e^{iy} + c = e^x\,(x+iy)(\cos y + i\sin y) + c$

$\therefore \quad u + iv = e^x\,(x\cos y - y\sin y) + i\,e^x\,(x\sin y + y\cos y) + c$

Equating real parts, $u = e^x\,(x\cos y - y\sin y) + c$

Example 6 : If $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$ is imaginary part of an analytic function, find analytic function and its real part.

[JNTU 1999]

Solution : Given $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$ ...(1)

Differentiating (1) partially w.r.t. x, $\dfrac{\partial v}{\partial x} = 2x + \dfrac{(x^2+y^2)\,1 - x\cdot 2x}{(x^2+y^2)^2} = \dfrac{2x + y^2 - x^2}{(x^2+y^2)^2}$

Differentiating (1) partially w.r.t. y, $\dfrac{\partial v}{\partial y} = -2y + \dfrac{(x^2+y^2)\,(0) - x\cdot 2y}{(x^2+y^2)^2} = -2y - \dfrac{2xy}{(x^2+y^2)^2}$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\,\frac{\partial v}{\partial x}$$

$$= -2y - \frac{2xy}{(x^2+y^2)^2} + i\left[2x + \frac{y^2 - x^2}{(x^2+y^2)^2}\right]$$

Using Milne–Thomson method $f'(z)=i(2z)-\frac{iz^2}{z^4}=i(2z)-\frac{i}{z^2}$

Integrating, $f(z)=iz^2+\frac{i}{z}+c$

i.e. $u+iv = i(x+iy)^2+\frac{i}{x+iy}+c=i\left(x^2-y^2+2ixy\right)+\frac{i(x-iy)}{x^2+y^2}+c$

or $u+iv = -2xy+\frac{y}{x^2+y^2}+c+i\left[x^2-y^2+\frac{x}{x^2+y^2}\right]$

Equating real parts, we get $u=-2xy+\frac{y}{x^2+y^2}+c$

Example 7 : Prove that the function $v(x,y)=\sin x\cdot\cosh y+2\cos x\cdot\sinh y+x^2-y^2+4xy$ satisfies Laplace equation. Determine the corresponding analytic function $u+iv$. **[JNTU 2003 (Set No. 3)]**

Solution : Given $v=\sin x\cdot\cosh y+2\cos x\cdot\sinh y+x^2-y^2+4xy$

$$\therefore \quad \frac{\partial v}{\partial x}=\cos x\cdot\cosh y-2\sin x\cdot\sinh y+2x+4y$$

and $$\frac{\partial^2 v}{\partial x^2}=-\sin x\cdot\cosh y-2\cos x\cdot\sinh y+2$$

Similarly $$\frac{\partial v}{\partial y}=\sin x\cdot\sinh y+2\cos x\cdot\cosh y-2y+4x$$

and $$\frac{\partial^2 v}{\partial y^2}=\sin x\cdot\cosh y+2\cos x\cdot\sinh y-2$$

Clearly $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

Hence v satisfies Laplace's equation (*i.e.* v is harmonic).

Let $f(z)=u+iv$ where $v=\sin x\cdot\cosh y+2\cos x\cdot\sinh y+x^2-y^2+4xy$

$$\therefore \quad f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}, \text{ using } C-R \text{ equations}$$

$$=(\sin x\cdot\sinh y+2\cos x\cdot\cosh y-2y+4x)+i(\cos x\cdot\cosh y-2\sin x\cdot\sinh y+2x+4y)$$

By Milne – Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $f'(z)=(2\cos z+4z)+i(\cos z+2z)$

Integrating, $f(z)=2\sin z+4\cdot\frac{z^2}{2}+i\left(\sin z+2\cdot\frac{z^2}{2}\right)+C=2\sin z+2z^2+i(\sin z+z^2)+C$

where C is a complex constant.

Example 8 : Find the analytic function whose real part is (*i*) $\frac{x}{x^2+y^2}$ (*ii*) $\frac{y}{x^2+y^2}$

[JNTU 2003, Nov. 2008 (Set No. 2)]

Solution : (i) Let $f(z)=u+i\,v$ where $u=\frac{x}{x^2+y^2}$

$$\therefore \quad f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \text{ (using Cauchy – Riemann equations)}$$

$$=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{2i\,x\,y}{(x^2+y^2)^2}=\frac{(y+i\,x)^2}{(x^2+y^2)^2}=\frac{(y+i\,x)^2}{(y+i\,x)^2(y-i\,x)^2}=\frac{1}{(y-i\,x)^2}$$

$$=\frac{1}{(i\,x-y)^2}=\frac{i^2}{(x+i\,y)^2}=-\frac{1}{z^2}$$

Integrating, $f(z)=-\int\frac{dz}{z^2}+c=\frac{1}{z}+c$ where c is a complex constant.

(ii) Let f(z) = $u+iv$ where $u=\frac{y}{x^2+y^2}$

$$\therefore \; f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \text{ (using C-R equations)}$$

$$= \frac{-2xy}{(x^2+y^2)^2} - \frac{i(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{1}{(x^2+y^2)^2}\left[i^2 2xy+i(y^2-x^2)\right]$$

$$= \frac{i}{(x^2+y^2)^2}[(y+ix)^2]$$

$$= \frac{i(y+ix)^2}{(y+ix)^2(y-ix)^2}$$

$$= \frac{i}{(y-ix)^2}=\frac{i}{(ix-y)^2}$$

$$= \frac{i}{(ix+i^2y)^2}=\frac{i}{i^2(x+iy)^2}$$

$$= -\frac{1}{z^2}i$$

Hence $f(z) = -i\int\frac{dz}{z^2}=\frac{i}{z}+c$, where c is a complex constant.

Example 9 : Find the regular function whose imaginary part is $\log(x^2+y^2)+x-2y$

Solution : Given $v = \log(x^2+y^2)+x-2y$ ... (1)

$$\therefore \quad \frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2}+1$$

$$\text{and} \quad \frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2}-2 \quad \text{... (2)}$$

Let $f(z)=u+iv$. Then $f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$ (Using $C-R$ equations)

$$= \frac{2y}{x^2+y^2}-2+i\left(\frac{2x}{x^2+y^2}+1\right) \text{ [Using (1) and (2)]}$$

By Milne – Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $\quad f'(z) = -2+i\left(\frac{2z}{z^2}+1\right) = -2+i\left(\frac{2}{z}+1\right)$

Integrating, $f(z) = \int\left[-2+i\left(\frac{2}{z}+1\right)\right]dz + c$

$$= -2z+i(2\log z+z)+c = 2i\log z-(2-i)z+c$$

Example 10 : Find a function w such that $w=u+iv$ is analytic, given

$$u = \sin x\cosh y+2\cos x\sinh y+x^2-y^2+4xy$$

Solution : Given $u = \sin x\cosh y+2\cos x\sinh y+x^2-y^2+4xy$

$$\therefore \quad \frac{\partial u}{\partial x} = \cos x\cosh y-2\sin x\sinh y+2x+4y$$

$$\text{and} \quad \frac{\partial u}{\partial y} = \sin x\cdot\sinh y+2\cos x\cdot\cosh y-2y+4x$$

Let $\quad w=u+iv$. Then

$$\frac{dw}{dz} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \text{ (Using } C-R \text{ equations)}$$

$$= (\cos x\cosh y-2\sin x\sinh y+2x+4y)-i(\sin x\sinh y+2\cos x\cosh y-2y+4x)$$

Now to express $\frac{dw}{dz}$ completely in terms of z, we replace x by z and y by 0

Hence $\frac{dw}{dz} = (\cos z+2z)-i(2\cos z+4z)$

Integrating, $w = \sin z+2\cdot\frac{z^2}{2}-i\left(2\sin z+4\cdot\frac{z^2}{2}\right)+c = \sin z+z^2-2i(\sin z+z^2)+c$

Example 11 : If $f(z)=u+iv$ is analytic, and $v=\dfrac{2\sin x\sin y}{\cos 2x+\cosh 2y}$, find u.

(Or) Find the analytic function whose imaginary part is $\dfrac{2\sin x\sin y}{\cos 2x+\cosh 2y}$

Solution : Let $f(z)=u+iv$ where $v=\dfrac{2\sin x\sin y}{\cos 2x+\cosh 2y}$

$$\therefore \quad f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

$$=\frac{(\cos 2x+\cosh 2y)(2\sin x\cos y)-2\sin x\sin y\,(2\sinh 2y)}{(\cos 2x+\cosh 2y)^2}$$

$$+i\,2\sin y\left[\frac{\cos x\,(\cos 2x+\cosh 2y)-\sin x\,(-2\sin 2x)}{(\cos 2x+\cosh 2y)^2}\right]$$

By Milne – Thomson's method, we express $f'(z)$ in terms of z by putting $x=z$ and $y=0$.

$$\therefore \quad f'(z)=\frac{(\cos 2z+1)\,2\sin z}{(\cos 2z+1)^2}=\frac{2\sin z}{\cos 2z+1}$$

$$=\frac{2\sin z}{(2\cos^2 z-1)+1}=\frac{2\sin z}{2\cos^2 z}=\tan z\cdot\sec z$$

Integrating w.r.t. 'z', we get

$f(z)=\int \sec z\tan z\,dz=\sec z+ic$, taking the constant of integration as imaginary, since u does not contain any constant.

$$\therefore \quad f(z)=u+iv=\sec(x+iy)+ic=\frac{1}{\cos(x+iy)}+ic,$$

$$=\frac{1}{\cos x\cos(iy)-\sin x\sin(iy)}+ic=\frac{1}{\cos x\cosh y-i\sin x\sinh y}+ic$$

$$=\frac{\cos x\cosh y+i\sin x\sinh y}{\cos^2 x\cosh^2 y+\sin^2 x\sinh^2 y}+ic$$

Equating real parts,

$$u=\frac{\cos x\cosh y}{\cos^2 x\cosh^2 y+\sin^2 x\sinh^2 y}$$

$$=\frac{\cos x\cosh y}{\cos^2 x(1+\sinh^2 y)+(1-\cos^2 x)\sinh^2 y}=\frac{\cos x\cosh y}{\cos^2 x+\sinh^2 y}$$

$$=\frac{2\cos x\cosh y}{(2\cos^2 x-1)+(2\sinh^2 y+1)}=\frac{2\cos x\cosh y}{\cos 2x+\cosh^2 y}$$

Example 12 : *Find the analytic function $f(z)=u+iv$ if $u=a(1+\cos\theta)$* **[JNTU 2000]**

Solution : Given $u=a(1+\cos\theta)$

Diff. w.r.t. θ and r, we get

$$\frac{\partial u}{\partial \theta} = u_\theta = -a\sin\theta, \ \frac{\partial u}{\partial r} = u_r = 0$$

The Cauchy–Riemann equations in polar coordinates are $u_r = \frac{1}{r}v_\theta, \ v_r = -\frac{1}{r}u_\theta$

$\Rightarrow \qquad rv_r = -u_\theta = a\sin\theta$

$\therefore \qquad rv_r = a\sin\theta$

Integrating w.r.t. 'r', $v(r, \theta) = a\sin\theta . \log r + c(\theta)$ ... (1)

Differentiating w.r.t. 'θ',

$$v_\theta = a\cos\theta . \log r + \frac{dc}{d\theta} = ru_r = r.0 = 0$$

$$\Rightarrow \qquad \frac{dc}{d\theta} = -a\cos\theta . \log v$$

Again integrating, we get $c(\theta) = a\sin\theta \log r + c_1$, where c_1 is a constant.

Substituting $c(\theta)$ in (1), we get

$$v(r, \theta) = a\sin\theta\log r + a\sin\theta\log r + c_1 = 2a\sin\theta\log r + c_1$$

$$\therefore \qquad f(z) = u + iv = a\,(1 + \cos\theta + 2\sin\theta\log r) + c_1$$

Example 13 : Find the analytic function $f(z) = u\,(r, \theta) + iv\,(r, \theta)$ when $v(r, \theta) = r^2\cos 2\theta - r\,\cos\theta + 2$.

Solution : We have by Cauchy–Riemann equations in polar coordinates

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2\sin 2\theta + r\sin\theta \qquad ...(1)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta \qquad ...(2)$$

$\therefore$ From (1), we get $\frac{\partial u}{\partial r} = -2r\sin 2\theta + \sin\theta$

Integrating with respect to r, we get

$$u = -r^2\sin 2\theta + r\sin\theta + \phi(\theta) \text{ where } \phi(\theta) \text{ is a constant.}$$

Differentiating u w.r.t. θ, we get $\frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta + r\cos\theta + \phi'(\theta)$ ...(3)

From (2) and (3), we get $-2r^2\cos 2\theta + r\cos\theta = \frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta + r\cos\theta + \phi'(\theta)$

$\therefore \quad \phi'(\theta) = 0 \Rightarrow \phi(\theta) = c$

$\therefore \quad u = -r^2\sin 2\theta + r\sin\theta + c$

Hence, $f(z) = u + iv = r^2(-\sin 2\theta + i\cos 2\theta) + r(\sin\theta - i\cos\theta) + c + 2i$

$$= i\left(r^2 e^{2i\theta} - r e^{i\theta}\right) + c + 2i$$

Example 14 : Find the conjugate harmonic function of the harmonic function $u = x^2 -$ [JNTU 1998]

Solution : Given $\qquad u = x^2 - y^2$ ...(1)

Differentiating (1) partially w.r.t. x, $\frac{\partial u}{\partial x} = 2x$

Again differentiating $\frac{\partial^2 u}{\partial x^2} = 2$ ...(2)

Differentiating (1) partially w.r.t. y, $\frac{\partial u}{\partial y} = -2y$

Again differentiating, $\frac{\partial^2 u}{\partial y^2} = -2$ ...(3)

(2) + (3) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ u is harmonic.

Let v be its harmonic conjugate. $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2x + i2y$

Using Milne–Thomson method, $f'(z) = 2z$

Integrating, we get $f(z) = z^2 + c$

$\therefore$ $u + iv = (x+iy) + ik$ where $c = ik$

$= x^2 - y^2 + i\,2xy + ik = x^2 - y^2 + i(2xy + k)$

Equating imaginary parts, $v = 2xy + k$ is the required form.

Example 15 : Show that $u(x, y) = e^{2x}(x\cos 2y - y\sin 2y)$ is harmonic and find its harmonic conjugate. **[JNTU 1998]**

(or) Find the analytic function whose real part is $u = e^{2x}(x\cos 2y - y\sin 2y)$.

[JNTU (H) Nov. 2009 (Set No. 3)]

Solution : Given $u(x, y) = e^{2x}(x\cos 2y - y\sin 2y)$ ...(1)

Differentiating (1) partially w.r.t. x, $\frac{\partial u}{\partial x} = e^{2x}\left[(2x+1)\cos 2y - 2y\sin 2y\right]$

Again differentiating, $\frac{\partial^2 u}{\partial x^2} = e^{2x}\left[(4x+4)\cos 2y - 4y\sin 2y\right]$ ...(2)

Differentiating (1) partially w.r.t. y, $\frac{\partial u}{\partial y} = e^{2x}\left[-2x\sin 2y - 2y\cos 2y - \sin 2y\right]$

Again differentiating,

$$\frac{\partial^2 u}{\partial y^2} = e^{2x}\left[-(4x+4)\cos 2y + 4y\sin 2y\right] = -\frac{\partial^2 u}{\partial x^2} \text{ [using (2)]}$$

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

$\therefore$ u is harmonic.

Let v be its harmonic conjugate.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \text{ using Cauchy-Riemann equations}$$

$$= e^{2x}\left[(2x+1)\cos 2y - 2y\sin 2y\right] - i\,e^{2x}\left[-2x\sin 2y - 2y\cos 2y - \sin 2y\right]$$

Using Milne–Thomson method

$$f'(z)=e^{2z}\left[(2z+1)\cos 0-0\right]-i\,e^{2z}(0)=e^{2z}(2z+1)=e^{2z}\cdot 2z+e^{2z}$$

Integrating, we get $f(z)=z\cdot e^{2z}+c$

or $$u+iv=(x+iy)\cdot e^{2x+2iy}+c$$

$$=(x+iy)\,e^{2x}(\cos 2y+i\sin 2y)+c$$

$$=e^{2x}(x\cos 2y-y\sin 2y)+i\left[e^{2x}(x\sin 2y+y\cos 2y)+k\right] \text{ when } c=ik$$

Equating imaginary parts, we get $v=e^{2x}(2\sin 2y+y\cos 2y)+k.$

Example 16 : Show that the function $u = 4xy - 3x + 2$ is harmonic. Construct the corresponding analytic function $f(z)=u+iv$ in terms of z. **[JNTU 1995S]**

Solution : Given $u = 4xy - 3x + 2$ ...(1)

Differentiating (1) partially w.r.t. x, $\frac{\partial u}{\partial x}=4y-3$

Again differentiating $\frac{\partial^2 u}{\partial x^2}=0$

Differentiating (1) partially w.r.t. y, $\frac{\partial u}{\partial y}=4x$

Again differentiating, $\frac{\partial^2 u}{\partial y^2}=0$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$\therefore$ u is harmonic.

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\Rightarrow f'(z)=4y-3-i\cdot 4x$$

Using Milne–Thomson method,

$$f'(z)=-3-i\,4z \text{ (putting } x=z \text{ and } y=0)$$

Integrating, $f(z)=-3z-i\,2z^2+c.$

Example 17 : Find the conjugate harmonic of $u=e^{x^2-y^2}\cos 2xy$. Hence find $f(z)$ in terms of z. **[JNTU 2003S (Set No. 1)]**

Solution : We have

$$u=e^{x^2-y^2}\cos 2xy$$

$\therefore$ $$\frac{\partial u}{\partial x}=e^{x^2-y^2}(-\sin 2xy)(2y)+\cos 2xy\,e^{x^2-y^2}(2x)$$

$$=2e^{x^2-y^2}(x\cos 2xy-y\sin 2xy)$$

and $$\frac{\partial u}{\partial y}=e^{x^2-y^2}(-\sin 2xy)(2x)+\cos 2xy\,e^{x^2-y^2}(-2y)$$

$$=-2\,e^{x^2-y^2}(x\sin 2xy+y\cos 2xy)$$

Let v be the conjugate of u. Then

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \qquad \text{(Using Cauchy – Riemann equations)}$$

$$=2e^{x^2-y^2}(x\cos 2xy-y\sin 2xy)+i2e^{x^2-y^2}(x\sin 2xy+y\cos 2xy)$$

By Milne – Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $f'(z)=2e^{z^2}(z-0)+i\,2e^{z^2}(0+0)=2ze^{z^2}$

Integrating, $f(z)=\int 2ze^{z^2}dz+c=\int e^t dt+c$ (putting $z^2=t$)

$$=e^t+c=e^{z^2}+c$$

or $$u+iv=e^{(x+iy)^2}+c=e^{(x^2-y^2)+i2xy}+c$$

$$=e^{x^2-y^2}(\cos 2xy+i\sin 2xy)+c$$

$$=e^{x^2-y^2}\cos 2xy+i(e^{x^2-y^2}\sin 2xy+k), \text{ where } c=ik$$

Equating imaginary parts, $v=e^{x^2-y^2}\sin 2xy+k$

Example 18 : If $f(z)=u+iv$ is an analytic function of z and if $u-v=e^x(\cos y-\sin y)$, find $f(z)$ in terms of z. **[JNTU 1995S, June 2009S, JNTU (K) Nov. 2009 (Set No. 4)]**

(or) Find the analytic function $f(z)=u+iv$ if $u-v=e^x(\cos y-\sin y)$

[JNTU 2005S (Set No.2)]

Solution : Given $u-v=e^x(\cos y-\sin y)$ ...(1)

Differentiating partially w.r.t. x and y,

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=e^x(\cos y-\sin y) \qquad ...(2)$$

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=e^x(-\sin y-\cos y)$$

But $$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x} \qquad \left[\because f(z) \text{ is analytic}\right]$$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=e^x(\sin y+\cos y) \qquad ...(3)$$

(2) + (3) gives,

$$2\frac{\partial u}{\partial x}=2e^x\cos y \text{ or } \frac{\partial u}{\partial x}=e^x\cos y \qquad ...(4)$$

(3) – (2) gives,

$$\frac{\partial v}{\partial x}=e^x\sin y. \qquad ...(5)$$

Integrating (4) and (5) keeping y as constant gives

$$u=e^x\cos y+\psi_1(y) \text{ and } v=e^x\sin y+\psi_2(y)$$

$$\therefore \quad u-v=e^x(\cos y-\sin y)+\psi_1(y)-\psi_2(y) \qquad ...(6)$$

From (1) and (6)

$$\psi_1(y)-\psi_2(y)=0$$

$$\Rightarrow \quad \psi_1(y)=\psi_2(y)$$

We have $u=e^x \cos y+\psi_1(y)$

Differentiating partially w.r.t. y, we get

$$\frac{\partial u}{\partial y}=-e^x \sin y+\psi_1'(y)$$

$$-\frac{\partial v}{\partial x}=-e^x \sin y+\psi_1'(y) \qquad \text{[from Cauchy – Riemann equations]}$$

$$\Rightarrow \quad \frac{\partial v}{\partial x}=e^x \sin y-\psi_1'(y) \qquad ...(7)$$

From (7) and (5), $\psi_1'(y)=0$

Integrating $\psi_1(y)=c$

Hence $u=e^x \cos y+c$ and $v=e^x \sin y+c$

$$\therefore \quad f(z)=u+iv$$

$$=e^x \cos y+c+i\left(e^x \sin y+c\right)$$

$$=e^x (\cos y+i \sin y)+(1+i)c$$

$$=e^x \cdot e^{iy}+k=e^{x+iy}+k$$

or $f(z)=e^z+k.$

Example 19 : *If $f(z)=u+iv$ be an analytic function of z and if $u-v=(x-y)\left(x^2+4xy+y^2\right)$, find f(z) in terms of z.* **[JNTU 1994S]**

Solution : Given $f(z)=u+iv$ ...(1)

$\therefore \quad i f(z) = iu - v$ ...(2)

(1) + (2) gives,

$$f(z)(i+1)=(u-v)+i(u+v)$$

$\Rightarrow \quad (1+i) f(z)=U+iV$ where $U=u-v,\ V=u+v$

Differentiating,

$$(1+i) f'(z)=\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}=\frac{\partial U}{\partial x}-i\frac{\partial U}{\partial y}$$

$$=\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}-i\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}\right)$$

$$=(x^2+4xy+y^2)+(x-y)(2x+4y)-i\left[-\left(x^2+4xy+y^2\right)+(x-y)(4x+2y)\right]$$

$$\therefore \quad (1+i) f'(z)=3x^2+6xy-3y^2-i\left\{3x^2-6xy-3y^2\right\}$$

Using Milne-Thomson method

$$(1+i)f'(z) = 3z^2 - i\cdot 3z^2 \quad \text{(putting } x = z \text{ and } y = 0\text{)}$$

$$= (1-i)\cdot 3z^2$$

Integrating, $(1+i)f(z) = (1-i)z^3 + c$

$$\therefore \quad f(z) = \frac{1-i}{1+i}z^3 + \frac{c}{1+i} = \frac{(1-i)^2}{1+1}z^3 + c = -iz^3 + c$$

where c is a complex constant.

Example 20 : *Find the analytic function* $f(z) = u(x, y) + iv(x, y)$

if $\quad u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}}$ and $f\left(\dfrac{\pi}{2}\right) = 0$.

(or) If $f(z) = u + iv$ is an analytic function and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}}$, find $f(z)$ subject to the condition $f(\pi/2) = 0$. **[JNTU Nov. 2006, Nov. 2008S (Set No. 4)]**

Solution : We have $\quad f(z) = u + iv$ ...(1)

$$\therefore \quad i f(z) = iu - v \quad \text{...(2)}$$

(1) + (2) gives $(1+i)f(z) = (u-v) + i(u+v)$ ...(3)

Putting $(1+i)f(z) = F(z)$, $u - v = U, u + v = V$,

(3) becomes $F(z) = U + iV$

It is given that $U = u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}} = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

$$\therefore \quad \frac{\partial U}{\partial x} = \frac{(\cos x - \cosh y)(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-\sin x)}{2(\cos x - \cosh y)^2}$$

and $$\frac{\partial U}{\partial y} = \frac{(\cos x - \cosh y)e^{-y} - (\cos x + \sin x - e^{-y})(-\sinh y)}{2(\cos x - \cosh y)^2}$$

Now $$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$$= \frac{1}{2(\cos x - \cosh y)^2}[(\cos x - \cosh y)(-\sin x + \cos x) + \sin x(\cos x + \sin x - e^{-y})$$

$$-i[(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y]$$

By Milne – Thomson's method, we express $F'(z)$ is terms of z by putting $x = z$ and $y = 0$.

$$\therefore \quad F'(z) = \frac{(\cos z - 1)(-\sin z - \cos z) + \sin z(\cos z + \sin z - 1) - i[(\cos z - 1) + 0]}{2(\cos z - 1)^2}$$

$$= \frac{\cos z(\cos z - 1) + \sin^2 z - i(\cos z - 1)}{2(\cos z - 1)^2}$$

$$= \frac{(1-\cos z) - i(\cos z - 1)}{2(\cos z - 1)^2} = \frac{-1-i}{2(\cos z - 1)}$$

i.e. $(1+i)\, f'(z) = \dfrac{-(1+i)}{2(\cos z - 1)}$

or $\quad f'(z) = -\dfrac{1}{2(\cos z - 1)} = -\dfrac{1}{2\left(1 - 2\sin^2 \frac{z}{2} - 1\right)} = \dfrac{1}{4}\operatorname{cosec}^2\left(\dfrac{z}{2}\right)$

Integrating with respect to z, we get

$$f(z) = \frac{1}{4}\int \operatorname{cosec}^2\left(\frac{z}{2}\right) dz + c = -\frac{1}{2}\cot\left(\frac{z}{2}\right) + c$$

Given $f\left(\dfrac{\pi}{2}\right) = 0 \quad \Rightarrow \quad 0 = -\dfrac{1}{2}\cot\dfrac{\pi}{4} + c \quad \Rightarrow \quad c = \dfrac{1}{2}$

Hence $\quad f(z) = \dfrac{1}{2} - \dfrac{1}{2}\cot\left(\dfrac{z}{2}\right)$

or $\quad f(z) = \dfrac{1}{2}\left[1 - \cot\left(\dfrac{z}{2}\right)\right]$

Example 21 : Find $f(z) = u + iv$ given that $u + v = \dfrac{\sin 2x}{\cos h 2y - \cos 2x}$

[JNTU 2003S, 2008S (Set No. 1)]

Solution : Given $\quad f(z) = u + iv \qquad \text{... (1)}$

$\therefore \qquad i f(z) = iu - v \qquad \text{... (2)}$

(1) + (2) gives

$$(1+i)\, f(z) = (u - v) + i(u + v)$$

Letting $(1+i)\, f(z) = F(z), u - v = U$ and $u + v = V$, we obtain

$$F(z) = U + iV$$

It is given that $V = u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\therefore \quad \frac{\partial V}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

and $\quad \dfrac{\partial V}{\partial y} = \sin 2x . \dfrac{\partial}{\partial y}\left(\dfrac{1}{\cosh 2y - \cos 2x}\right) = \dfrac{-2\sin 2x \sinh y}{(\cosh 2y - \cos 2x)^2}$

Now $\quad F'(z) = \dfrac{\partial U}{\partial x} + i\dfrac{\partial V}{\partial x} = \dfrac{\partial V}{\partial y} + i\dfrac{\partial V}{\partial x}$

$$= \frac{(-2\sin 2x \sinh y) + i\left[2\cos 2x(\cosh 2y - \cos 2x) - 2\sin^2 2x\right]}{(\cosh 2y - \cos 2x)^2}$$

By Milne – Thomson's method, we express $F'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore \quad F'(z) = \frac{i[2\cos 2z(1-\cos 2z) - 2\sin^2 2z]}{(1-\cos 2z)^2} = \frac{i\,2(\cos 2z - 1)}{(1-\cos 2z)^2}$$

$$= \frac{2i}{\cos 2z - 1} = \frac{2i}{-2\sin^2 z} = -i\,\text{cosec}^2 z$$

Integrating, $F(z) = i\cot z + c$

i.e. $(1+i)\,f(z) = i\cot z + c$ or $f(z) = \frac{i}{1+i}\cot z + \frac{c}{1+i} = \frac{i(1-i)}{2}\cot z + c_1$

$$\therefore \quad f(z) = \left(\frac{1+i}{2}\right)\cot z + c_1$$

Example 22 : Find a and b if $f(z) = (x^2 - 2xy + ay^2) + i(bx^2 - y^2 + 2xy)$ is analytic. Hence find $f(z)$ in terms of z. **[JNTU 2003S (Set No. 3)]**

Solution : Let $f(z) = u + iv$ be an analytic function so that

$$u = x^2 - 2xy + ay^2 \text{ and } v = bx^2 - y^2 + 2xy$$

Hence $\quad \frac{\partial u}{\partial x} = 2x - 2y, \ \frac{\partial u}{\partial y} = -2x + 2ay$

and $\quad \frac{\partial v}{\partial x} = 2bx + 2y, \ \frac{\partial v}{\partial y} = -2y + 2x$

Since $f(z)$ is analytic, Cauchy – Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{... (1)}$$

and $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad$ ... (2)

are satisfied.

(2) $\Rightarrow \quad -2x + 2ay = -2bx - 2y$

This is possible only when $a = -1$, $b = 1$

Hence $a = -1$ and $b = 1$.

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x - 2y + i(2bx + 2y) = 2[(x-y) + i(x+y)] \quad (\because \ b = 1)$

By Milne – Thomson's method,

$f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $f'(z) = 2(z + iz) = 2z(1+i)$

Integrating, $f(z) = 2(1+i)\frac{z^2}{2} + C = (1+i)z^2 + C$

where C is a complex constant.

Example 23 : Find an analytic function $f(z)$ such that $\text{Re}[f'(z)] = 3x^2 - 4y - 3y^2$ and $f(1+i) = 0$ [JNTU 2003, JNTU (A) Dec. 2009 (Set No. 3)]

Solution : Since $f(z)$ is analytic, therefore $f'(z)$ is also analytic.

Let $f'(z) = U + iV$. Then

$$U = 3x^2 - 4y - 3y^2.$$

$\therefore \quad U_x = 6x$ and $U_y = -4 - 6y$

Since U and V satisfy Cauchy – Riemann equations,

$\therefore \quad U_x = 6x = V_y$

Integrating with respect to 'y', we get

$$V = 6xy + c_1(x) \qquad \text{...(1)}$$

Now $\dfrac{\partial V}{\partial x} = V_x = 6y + \dfrac{dc_1}{dx}$

Since $V_x = -U_y$, we have

$$6y + \frac{dc_1}{dx} = 4 + 6y$$

$\Rightarrow \quad c_1(x) = 4x + c_2 \qquad \text{...(2)}$

where c_2 is an arbitrary constant.

From (1) and (2), we have $V = 6xy + 4x + c_2$

$\therefore \quad f'(z) = U + iV = (3x^2 - 4y - 3y^2) + i(6xy + 4x + c_2)$

By Milne – Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $f'(z) = 3z^2 + i\,4z + c_2$

Integrating, $f(z) = 3\dfrac{z^3}{3} + i\,4\cdot\dfrac{z^2}{2} + c_2 z + c_3 = z^3 + 2iz^2 + c_2 z + c_3 \qquad \text{... (3)}$

Given $\quad f(1+i) = 0 \quad \Rightarrow \quad 0 = (1+i)^3 + 2i(1+i)^2 + c_2(1+i) + c_3$

Thus $\quad f(z) = z^3 + 2iz^2 + c_2 z - c_2(1+i) - 6 + 2i$ [by (3)]

Example 24 : *Show that the function* $f(x, y) = x^3y - xy^3 + xy + x + y$ *can be the imaginary part of an analytic function of* $z = x + iy$. *[JNTU 2004, 2004S, 2005, 2008S (Set No. 2, 3)]* Determine the real part and also the complex function.

Solution : Let $\phi(z)$ be an analytic function of z.

Given $\quad f(x, y) = x^3y - xy^3 + xy + x + y$

$f(x, y)$ will be the imaginary part of $\phi(z)$ if $\nabla^2 f = 0$

Now $\dfrac{\partial f}{\partial x}=3x^2y-y^3+y+1,\ \dfrac{\partial f}{\partial y}=x^3-3xy^2+x+1$

and $\dfrac{\partial^2 f}{\partial x^2}=6xy,\ \dfrac{\partial^2 f}{\partial y^2}=-6xy$

Clearly $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$ *i. e.*, $\nabla^2 f=0$

So $f(x, y)$ is the imaginary part of an analytic function of z.

Let $g(x, y)$ be the real part of $\phi(z)$. Then

$$dg = \frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy=-\frac{\partial f}{\partial y}dx+\frac{\partial f}{\partial x}dx \quad \text{(Using Cauchy – Riemann equations)}$$

or $$dg = -(x^3-3xy^2+x+1)\,dx+(3x^2y-y^3+y+1)\,dy$$

The above equation is exact differential equation.

Integrating, $g=-\int(x^3-3xy^2+x+1)\,dx+\int(-y^3+y+1)\,dy+c$

(y constant) (only those terms which do not contain x)

$$=-\left(\frac{x^4}{4}-\frac{3x^2y^2}{2}+\frac{x^2}{2}+x\right)+\left(\frac{-y^4}{4}+\frac{y^2}{2}+y\right)+c$$

$$=-\frac{1}{4}(x^4+y^4+6x^2y^2)+\frac{1}{2}(y^2-x^2)+y-x+c$$

Now $\phi(z)=g(x, y)+i\,f(x, y)$ where $f(x, y)=x^3y-xy^3+xy+x+y$

$$\phi'(z)=\frac{\partial g}{\partial x}+i\frac{\partial f}{\partial x}=-\frac{\partial f}{\partial y}+i\frac{\partial f}{\partial x} \quad \text{(Using } C-R \text{ equations)}$$

$$=-(x^3-3xy^2+x+1)+i(3x^2y-y^3+y+1)$$

By Milne – Thomson's method, $\phi'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $\phi'(z)=-(z^3+z+1)+i$

Integrating, $\phi(z)=-\left(\dfrac{z^4}{4}+\dfrac{z^2}{2}+z\right)+iz+A$

where A is a complex constant

Example 25 : If $f(z)=u+iv$ is an analytic function of z, find $f(z)$ if

$$2u+v=e^{2x}\left[(2x+y)\cos 2y+(x-2y)\sin 2y\right]$$ [JNTU (A) Dec. 2009 (Set No. 2)]

Solution : Let $f(z)=u+iv$

$\therefore \quad (1+2i)\,f(z)=(u+iv)(1+2i)=(u-2v)+i\,(2u+v)$

Letting $(1+2i)\,f(z)=F(z)$, $u-2v=U$ and $2u+v=V$, we obtain

$$F(z)=U+iV$$

$$\therefore \quad V=2u+v=e^{2x}\left[(2x+y)\cos 2y+(x-2y)\sin 2y\right]$$

$$\frac{\partial V}{\partial x}=e^{2x}(2\cos 2y+\sin 2y)+2e^{2x}\left[(2x+y)\cos 2y+(x-2y)\sin 2y\right]$$

and $$\frac{\partial V}{\partial y}=e^{2x}\left[-2\,(2x+y)\sin 2y+\cos 2y+2\,(x-2y)\cos 2y-2\sin 2y\right]$$

Now $F'(z)=\dfrac{\partial U}{\partial x}+i\dfrac{\partial V}{\partial x}=\dfrac{\partial V}{\partial y}+i\dfrac{\partial V}{\partial x}$ (Using $C-R$ equations)

$$=e^{2x}\left[-2\,(2x+y)\sin 2y+\cos 2y+2\,(x-2y)\cos 2y-2\sin 2y\right]$$

$$+i\,e^{2x}\left[2\cos 2y+\sin 2y+2\,(2x+y)\cos 2y+2\,(x-2y)\sin 2y\right]$$

By Milne – Thomson's method, we express $F'(z)$ in terms of z by putting $x=z$ and $y=0$

$$\therefore \quad F'(z)=e^{2z}(1+2z)+i\,e^{2z}(2+4z)=e^{2z}(1+2z)(1+2i)$$

i.e. $\quad (1+2i)\,f'(z)=e^{2z}(1+2z)(1+2i)$

or $\quad f'(z)=e^{2z}(1+2z)$

Integrating, $f(z)=\displaystyle\int e^{2z}dz+2\int z\,e^{2z}dz+c=\frac{e^{2z}}{2}+2\left[\frac{z\,e^{2z}}{2}-\frac{1}{4}e^{2z}\right]+c=z\,e^{2z}+c$

Example 26 : Prove that $u=e^{-x}[(x^2-y^2)\cos y+2xy\sin y]$ is harmonic and find the analytic function whose real part is u.

Solution : Given $u=e^{-x}\,[(x^2-y^2)\cos y+2xy\sin y]$

$$\therefore \quad \frac{\partial u}{\partial x}=-e^{-x}[(x^2-y^2)\cos y+2xy\sin y]+e^{-x}\,[2x\cos y+2y\sin y]$$

and $$\frac{\partial u}{\partial y}=e^{-x}[(x^2-y^2)(-\sin y)-2y\cos y+2x\sin y+2xy\cos y]$$

Now

$$\frac{\partial^2 u}{\partial x^2}=e^{-x}[(x^2-y^2)\cos y+2xy\sin y]-e^{-x}[2x\cos y+2y\sin y]$$

$$-e^{-x}[2x\cos y+2y\sin y]+e^{-x}[2\cos y]$$

$$= e^{-x}[(x^2-y^2)\cos y+2xy\sin y+2\cos y-4x\cos y-4y\sin y]$$

and $$\frac{\partial^2 u}{\partial y^2}=e^{-x}[(x^2-y^2)(-\cos y)+(-\sin y)(-2y)-2(\cos y-y\sin y)+2x\cos y+2x(\cos y-y\sin y)]$$

$$= e^{-x}[-(x^2-y^2)\cos y+4y\sin y+4x\cos y-2xy\sin y-2\cos y]$$

Clearly $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

Hence u is a harmonic function.

Now we have to find the analytic function $f(z)$.

Let $f(z) = u+iv$ where $u = e^{-x}[(x^2-y^2)\cos y+2xy\sin y]$

$$\therefore\ f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \quad \text{[using } C\text{-}R \text{ equations]}$$

$$= e^{-x}[-(x^2-y^2)\cos y-2xy\sin y+2x\cos y+2y\sin y]$$

$$-i\,e^{-x}[-(x^2-y^2)\sin y+2xy\cos y-2y\cos y+2x\sin y]$$

By Milne-Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $f'(z) = e^{-z}[-z^2+2z]-i\,e^{-z}(0)$

$$= e^{-z}(-z^2+2z)$$

Integrating w.r.t z, we get

$$f(z) = \int e^{-z}(-z^2-2z)dz = z^2e^{-z}+C$$

where C is a complex constant.

Example 27 : *Show that e^{z^2} is entire. Find its derivative.*

Solution : We know that $f(z) = u+iv$

$$\therefore\ u+iv = e^{z^2} = e^{(x^2-y^2)+i2xy} = e^{x^2-y^2}.e^{i2xy} = e^{x^2-y^2}(\cos 2xy+i\sin 2xy)$$

Then $u = e^{x^2-y^2}\cos 2xy$, $v = e^{x^2-y^2}\sin 2xy$

Now $u_x = 2x\cdot e^{x^2-y^2}\cdot\cos 2xy-2ye^{x^2-y^2}\cdot\sin 2xy$

$\Rightarrow\ u_x = 2xu-2yv$, in the same way $v_x = 2xv+2yu$, $v_y = -2yv+2xu$.

Thus C - R conditions are satisfied for all x, y.

$\therefore e^{z^2}$ is an entire function

$$f' = u_x + iv_x = (2xu - 2yv) + i(2xv + 2yu)$$

By Milne -Thomson method, replace x by z and y by 0.

$$\therefore f'(z) = (2ze^{z^2} - 0) + i(0+0) = 2ze^{z^2}.$$

Example 28 : Determine the analytic function $w = u + iv$

where $u = \dfrac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$ given that $f(0) = 1$. [JNTU 2006 (Set No. 3

Solution : Let $w = f(z) = u + iv$ where $u = \dfrac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{[by } C-R \text{ equations]}$$

$$= \frac{(\cos 2x + \cosh 2y)(-2\sin x \cosh y) - 2\cos x \cosh y\,(-2\sin 2x)}{(\cos 2x + \cosh 2y)^2}$$

$$-i\,\frac{(\cos 2x + \cosh 2y)(2\cos x \sinh y) - 2\cos x \cosh y\,(2\sinh 2y)}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{-2\sin x \cosh y(\cos 2x + \cosh 2y) + 4\cos x \sin 2x \cosh y}{(\cos 2x + \cosh 2y)^2}$$

$$-i\,\frac{[2\cos x \sinh y(\cos 2x + \cosh 2y) - 4\cos x \sinh 2y \cosh y]}{(\cos 2x + \cosh 2y)^2}$$

By Milne-Thomson's method, we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$

$$\therefore f'(z) = \frac{-2\sin z(\cos 2z + 1) + 4\cos z \sin 2z}{(\cos 2z + 1)^2} - i(0)$$

$$= \frac{-2\sin z \cos 2z - 2\sin z + 4\cos z(2\sin z \cos z)}{(2\cos^2 z - 1 + 1)^2}$$

$$= \frac{[-2\sin z(2\cos^2 z - 1) - 2\sin z] + 8\cos^2 z \sin z}{4\cos^4 z}$$

$$= \frac{4\cos^2 z \sin z}{4\cos^4 z} = \frac{\sin z}{\cos^2 z} = \tan z.\sec z$$

Integrating w.r.t. z, we get

$f(z) = \sec z + ic$, taking the constant of integration as imaginary, since u does not contain any constant.

Given that $f(0) = 1 \Rightarrow 1 = \sec 0 + ic$ or $ic = 0 \therefore c = 0$

Hence $f(z) = \sec z$

Example 29 : Determine the analytic function $f(z) = u + iv$ given that $3u + 2v = y^2 - x^2 + 16x$. **[JNTU Feb. 2008S, June 2009 S (Set No.2)]**

Solution : We have

$$3u + 2v = y^2 - x^2 + 16x \quad \text{... (A)}$$

Differentiating (A) w.r.t. 'x', we get

$$3\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x} = -2x + 16 \quad \text{...(1)}$$

Differentiating (A) w.r.t. 'y', we get

$$3\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} = 2y$$

Using C – R equations,

$$-3\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial x} = 2y \quad \text{...(2)}$$

(1) × 2 – (2) × 3 gives

$$\frac{\partial v}{\partial x} = \frac{1}{13}(-4x - 6y + 32)$$

(1) × 3 + (2) × 2 gives

$$\frac{\partial u}{\partial x} = \frac{1}{5}(-6x - 4y + 48)$$

Thus $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{1}{5}(-6x - 4y + 48) + \dfrac{i}{13}(-4x - 6y + 32)$

By Milne-Thomson's method, we express $f'(z)$ in terms of z, on replacing x by z and y by 0.

$$\therefore \quad f'(z) = \frac{1}{5}(-6z + 48) + \frac{i}{13}(-4z + 32)$$

$$= \frac{6}{5}(8 - z) + \frac{i4}{13}(8 - z) = \left(\frac{6}{5} + i\frac{4}{13}\right)(8 - z)$$

Integrating w.r.t. z, we get

$$f(z) = \left(\frac{6}{5} + i\frac{4}{13}\right)\frac{(8 - z)^2}{-2} + c, \text{ where c is a complex constant}$$

$$= \left(\frac{-3}{5} - \frac{i2}{13}\right)(z - 8)^2 + c.$$

Example 30 : Find the analytic function whose real part is $y + e^x \cos y$.

[JNTU Nov. 2008 (Set No.1)]

Solution : Let $f(z) = u + iv$ where $u = y + e^x \cos y$.

$$\text{Then } \frac{\partial u}{\partial x} = e^x \cos y, \ \frac{\partial u}{\partial y} = 1 + e^x(-\sin y) = 1 - e^x \sin y$$

$$\text{Now } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(using C-R equations)}$$

$$= e^x \cos y - i\,(1 - e^x \sin y)$$

By Milnne – Thyomson's method,

$f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

$$\therefore \quad f'(z) = e^z - i$$

Integrating with respect to z, we have

$$f(z) = e^z - iz + c \quad \text{where c is complex constant}$$

EXERCISE 2(A)

1. Find whether
   (i) $f(z) = \sin x \sin y - i \cos x \cos y$ [JNTU (A), (H) Nov. 2009 (Set No. 3,4)]
   (ii) $f(z) = \dfrac{x - iy}{x^2 + y^2}$ is analytic or not [JNTU 2002, JNTU (H) Nov. 2009 (Set No. 1)]
2. Show that $f(z) = |z|^2$ is not analytic. [JNTU 2002]
3. Find where the function $w = \dfrac{z-2}{(z+1)(z^2+1)}$ fails to be analytic.
4. Show that $f(z) = \cos z$ is analytic everywhere in the complex plane and find $f'(z)$.
5. Show that the function $e^x(\cos y + i \sin y)$ is holomorphic (analytic) and find its derivative
6. Show that the function $u = \dfrac{1}{2}\log(x^2 + y^2)$ is harmonic and find its conjugate.
7. Show that (i) $u = y^3 - 3x^2 y$ (ii) $u = (x-1)^3 - 3xy^2 + 3y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z.
8. Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic, find the conjugate function 'v' and express $u + iv$ as an analytic function of z. [JNTU 2001S]
9. Show that the function (i) $u(x, y) = \log(x^2 + y^2)$ (ii) $u(x, y) = \cos x \cosh y$ can be the real part of an analytic function of $z = x + iy$. Find the imaginary part and also the complex function.
10. Find a function w such that $w = u + iv$ is analytic, given
    (i) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ (ii) $u = x^4 - 6x^2y^2 + y^4$
    (iii) $u = \dfrac{y}{x^2 + y^2}$ [JNTU Reg. 2008 (Set No. 2)]
    (iv) $u = \cos x \cosh y$ [JNTU (K) Nov. 2009 (Set No. 2)]
    (v) $u = e^{-x}\left[(x^2 - y^2)\cos y + 2xy \sin y\right]$

(vi) $v=-\sin x \sinh y$ (vii) $u=\dfrac{2\cos x\cosh y}{\cos 2x+\cosh^2 y}$ and $f(0)=1$

11. Determine the analytic function whose real part is [JNTU Nov. 2006 (Set No. 3)]

(i) x^3-3xy^2+y+1 (ii) $e^x \cos y$ (iii) $e^{2x}(x\cos 2y - y\sin 2y)$

12. Find the regular function whose imaginary part is

(i) $\dfrac{x-y}{x^2+y^2}$ **[JNTU (A) Dec. 2009 (Set No. 4)]**

(ii) $e^{-x}(x\cos y + y\sin y)$ (iii) $e^x \sin y$

13. Find the analytic function $f(z)=u(r,\theta)+i\,v(r,\theta)$ such that

(i) $u(r,\theta)=-r^3\sin 3\theta$ (ii) $v(r,\theta)=\left(r-\dfrac{1}{r}\right)\sin\theta,\ r\neq 0$

(iii) $u(r,\theta)=r^2\cos 2\theta - r\cos\theta + 2$ [JNTU Aug. 2007S (Set No. 2)]

14. If $f(z)=u+iv=\dfrac{1}{z}$, show that the curves $u(x,y)=c_1$ and $v(x,y)=c_2$ intersect orthogonally.

ANSWERS

1. (i) not analytic (ii) analytic
3. $z=-1, \pm i$
4. $-\sin z$
5. e^z
6. $\tan^{-1}\left(\dfrac{y}{x}\right)+c$
7. (i) $v=-3xy^2+x^3+c$, $f(z)=iz^3+ic$ (ii) $v=3x^2y-6xy+3y-y^3+c$
8. $v=-e^{-2xy}\cos(x^2-y^2)$, $f(z)=-ie^{iz^2}+c$
9. (i) $2\tan^{-1}\left(\dfrac{y}{x}\right)$ (ii) $-\sin x\sinh y$
10. (i) z^3+3z^2+1+ic (ii) z^4+c (iii) $\dfrac{i}{z}+c$ (iv) $\cos z+c$ (v) $z^2e^{-z}+c$ (vi) $\cos z+c$ (vii) $\sec z$
11. (i) z^3-iz+c (ii) e^z (iii) $ze^{2z}+ic$
12. (i) $\dfrac{3-i}{z}+c$ (ii) $1+ize^{-z}+c$ (iii) e^z+c
13. (i) iz^3+ic (ii) $\left(r+\dfrac{1}{r}\right)\cos\theta+\left(r-\dfrac{1}{r}\right)\sin\theta+c$

2.19. COMPLEX POTENTIAL FUNCTION

Let $w=\phi(x,y)+i\psi(x,y)$. If this function is analytic then it is called **complex potential function**. Its real part $\phi(x,y)$ is called velocity potential function and imaginary part $\psi(x,y)$ is called stream function in the context of fluid mechanism. Both ϕ and ψ will satisfy Laplace's equation. If one function is given, we can find other function.

SOLVED EXAMPLES

Example 1 : If $w=\phi+i\psi$ represents the complex potential for an electric field and $\psi=x^2-y^2+\dfrac{x}{x^2+y^2}$ determine the function ϕ.

Solution : It is readily verified that ψ satisfies the Laplace's equation.

$\therefore$ ϕ and ψ must satisfy the Cauchy-Riemann equations :

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y} \quad ...(1) \qquad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x} \quad ...(2)$$

$\therefore$ From (1), $\dfrac{\partial\phi}{\partial x}=\dfrac{\partial}{\partial y}\left[x^2-y^2+\dfrac{x}{x^2+y^2}\right]=-2y-\dfrac{2xy}{\left(x^2+y^2\right)^2}$

Integrating w.r.t. x, $\phi=-2xy+\dfrac{y}{x^2+y^2}+\eta(y)$

where $\eta(y)$ is an arbitrary function of y.

$\therefore$ (2) gives, $-2x+\dfrac{x^2-y^2}{\left(x^2+y^2\right)^2}+\eta'(y)=-2x+\dfrac{x^2-y^2}{\left(x^2+y^2\right)^2}$

$\therefore$ $\eta'(y)=0$, *i.e.*, $\eta(y)=0$, an arbitrary constant.

Thus, $\phi=-2xy+\dfrac{y}{x^2+y^2}+c$

An alternative method :

We have $\dfrac{\partial\psi}{\partial x}=2x+\dfrac{y^2-x^2}{(x^2+y^2)^2}$

and $\dfrac{\partial\psi}{\partial y}=-2y-\dfrac{2xy}{(x^2+y^2)^2}$

Also we know that $d\phi=\dfrac{\partial\phi}{\partial x}dx+\dfrac{\partial\phi}{\partial y}dy$

$$=\frac{\partial\psi}{\partial y}dx-\frac{\partial\psi}{\partial x}dy \text{ (using C-R equations)}$$

$$=\left[-2y-\frac{2xy}{(x^2+y^2)^2}\right]dx-\left[2x+\frac{y^2-x^2}{(x^2+y^2)^2}\right]dy$$

This is an exact differential.

$$\therefore \phi=\int\left[-2y-\frac{2xy}{(x^2+y^2)^2}\right]dx+c=-2xy+\frac{y}{x^2+y^2}+c$$

Example 2 : If the potential function is $\log\sqrt{x^2+y^2}$, find the flux function and the complex potential function.

Solution : Let ϕ and ψ be the potential function and flux function respectively, then the complex potential function is given by $\omega=\phi+i\psi$ ... (1)

Given $\phi = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2)$

$$\therefore \frac{\partial\phi}{\partial x} = \frac{1}{2}\cdot\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2} \quad \text{and} \quad \frac{\partial\phi}{\partial y} = \frac{1}{2}\cdot\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}$$

From (1), we have

$$\frac{d\omega}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y} \quad \text{(Using } C-R \text{ equations)}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

By Milne – Thomson's method, $\frac{d\omega}{dz}$ is expressed in terms of z by replacing x by z and y by 0.

Hence $\frac{d\omega}{dz} = \frac{z}{z^2} = \frac{1}{z}$

Integrating, $\omega = \int\frac{dz}{z} + c = \log z + c$, where c is a complex constant which is the required complex potential function.

Now $\omega = \log z + c = \log(x+iy) + (P+iQ)$ (Taking $c = P + iQ$)

or $\phi + i\psi = \log(x+iy) + P + iQ$

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\phi + i\psi = \log(r\cos\theta + ir\sin\theta) + P + iQ = \log r(\cos\theta + i\sin\theta) + P + iQ$$

$$= \log re^{i\theta} + P + iQ = \log r + \log e^{i\theta} + P + iQ = \log r + i\theta + P + iQ$$

$$= \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x) + P + iQ$$

Comparing the imaginary part, $\psi = \tan^{-1}(y/x) + Q$, which is the required flux function.

Example 3 : In a two – dimensional flow of a fluid, the velocity potential $\phi = x^2 - y^2$. Find the stream function ψ.

Solution : Since ϕ is velocity potential, it must satisfy the Laplace's equation,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Now $\frac{\partial\phi}{\partial x} = 2x$, $\frac{\partial^2\phi}{\partial x^2} = 2$ and $\frac{\partial\phi}{\partial y} = -2y$, $\frac{\partial^2\phi}{\partial y^2} = -2$

Clearly $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

Hence ϕ satisfies the Laplace's equation.

Now ψ is the stream function.

Let $\omega = \phi + i\psi$

$$\therefore \quad \frac{d\omega}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y} \qquad \text{(Using } C - R \text{ equations)}$$

$$= 2x + i\,2y = 2\,(x + i\,y) = 2z$$

Integrating, $\omega = 2\int z\,dz + c = z^2 + c$ where c is a complex constant.

or $\phi + i\,\psi = (x + i\,y)^2 + c = (x^2 - y^2) + i\,2xy + A + i\,B$ (where $c = A + i\,B$)

Equating imaginary parts on both sides, we get $\psi = 2xy + B$,which is the required stream function.

Note. Some times the constant in velocity potential or stream function is omitted.

Example 4 : If $w = \varphi + i\psi$ represents the complex potential for an electric field and $\psi = 3x^2 y - y^3$ find φ.

Solution : Let $w = \phi + i\,\psi$ represent complex potential of an electric field where $\psi = 3x^2 y - y^3$.

This implies that w is an analytic function.

Hence $\phi_x = \psi_y = 3x^2 - 3y^2$ and $\phi_y = -\psi_x = -(6xy)$

Since $d\phi = \phi_x dx + \phi_y d_y$, we have

$d\phi = (3x^2 - 3y^2)\,dx - 6xydy$

$$\Rightarrow \quad \phi = \left(3\frac{x^3}{3} - 3y^2 x\right) + c$$

$$\Rightarrow \quad \phi = x^3 - 3xy^2 + c$$

where c is a constant.

EXERCISE 2(B)

1. In a two dimensional flow of a fluid, the stream function is $\psi = \dfrac{-y}{x^2 + y^2}$. Find the velocity potential ϕ.
2. If the potential function is $\log(x^2 + y^2)$ find the complex potential function. [JNTU 2003 (Set No. 2)]

ANSWERS

1. $\dfrac{x}{x^2 + y^2}$ 2 $\log(x^2 + y^2) + 2\tan^{-1}\left(\dfrac{y}{x}\right) + c$