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CONTRACTION COEFFICIENTS FOR COMPRESSIBLE FLOW THROUGH AXISYMMETRIC ORIFICES

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Summary—Accurate evaluation of the contraction coefficient for discharge of a compressible fluid through an axisymmetric sharp-edged orifice is difficult. It requires the lengthy numerical solution of a set of equations based on three-dimensional theory. An approximate, but relatively simple, analytical expression has been obtained in this paper using one-dimensional theory. The solution depends on a knowledge of two results from incompressible inviscid flow theory for the same orifice–reservoir configuration; that is,

(i) the incompressible contraction coefficient, and
(ii) the velocity distribution on an equipotential surface which commences at the orifice edge.

Other one-dimensional solutions to this problem are available in the literature, but they have difficulty in allowing for the effects of compressibility upstream of the orifice. This difficulty is avoided in the present solution.

Comparison of theoretical results with experiment shows good agreement, in some respects superior to the earlier solutions.

NOTATION

A area
C_c contraction coefficient; C_D coefficient of discharge
I $\int_0^1 (u_{1i}/u_{2i})^2\, y\, dy$
I_{I} $\int_0^1 (u_{1i}/u_{2i})^4\, y\, dy$
I_{II} $\int_0^1 (u_{1i}/u_{2i})^6\, y\, dy$
m mass flow rate
P pressure
R radius
r pressure ratio P_2/P_s; r_c critical pressure ratio $\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}$
u velocity
X $(1-r_c)$
x $(1-r)$
y radius ratio R/R_0
γ ratio of specific heats
θ angle of the velocity vector to the axis of symmetry
ρ density

Subscripts

a ambient downstream conditions
c conditions at the critical pressure
i value for the incompressible case
s stagnation conditions
o orifice
1 upstream boundary conditions
2 conditions at the vena contracta

INTRODUCTION

THE MASS flow rate through an orifice is usually less than the value for isentropic flow through an area equal to the orifice area. Experimental measurements of mass flow rate may be expressed in non-dimensional form, thus

$$C_D = \frac{m_{\text{experimental}}}{A_0(\rho_2 u_2)_{\text{isentropic}}}, \tag{1}$$

where C_D is the coefficient of discharge and is usually less than unity. The theoretical model shows that the flow passes through an effective area (the vena contracta) having a value less than the orifice area. A contraction coefficient is defined therefore, such that

$$C_c = \frac{\text{cross-sectional area of vena contracta}}{\text{cross-sectional area of orifice}}$$

$$= \frac{m_{\text{theoretical}}}{A_0(\rho_2 u_2)_{\text{isentropic}}}. \tag{2}$$

It has been found that where the contraction coefficient can be calculated exactly its value is in good agreement with the coefficient of discharge when viscous effects in the flow may be neglected. In this paper, an approximate method of evaluating the contraction coefficient is developed. Its validity is examined by comparing values with the coefficient of discharge, on the assumption that the exact value of the contraction coefficient is closely equal to the coefficient of discharge.

One-dimensional theory has been used successfully by a number of authors (Buckingham,[1] Cunningham,[2] Jobson,[3] Bragg[4]) to predict the contraction coefficient for axisymmetric compressible flow through sharp-edged orifices where the Reynolds number is high enough for viscous effects to be neglected. In all cases the result has been given in terms of the *incompressible* contraction coefficient which may be determined from incompressible flow theory.

In one-dimensional theory, a control volume is defined across which the equations of motion are applied. The previous authors have chosen this volume such that its upstream boundary surface is normal to the streamlines in a plane sufficiently far upstream for the flow to be unaffected by the orifice (Fig. 1, A–A). In consequence the force on the orifice wall must be included in the equation of motion.

Buckingham, for subcritical flow, and Cunningham, for supercritical flow, assumed that for given upstream conditions and for a given mass flow rate the force on the orifice wall was unaffected by compressibility. By combining the mass continuity and momentum equations they obtained an expression for the contraction coefficient without the need to estimate the wall force.

Jobson assumed that the force on the orifice wall was the same in compressible and incompressible flow for given upstream conditions and a given pressure ratio across the orifice. The integrated value of the difference between the upstream stagnation pressure and the pressure acting normal to the orifice wall was termed the "force defect" and, in dimensionless form, the "force defect coefficient". Jobson expressed the latter in terms of the incompressible contraction coefficient in order to facilitate a solution of the momentum equation for the comparable compressible flow case.

Bragg recognized that an allowance for upstream compressibility was necessary. He assumed that, at any cross-section in the reservoir normal to the jet axis, the mass flux at the wall was proportional to the mean flux through the cross-sectional area. He modified the force defect coefficient accordingly.

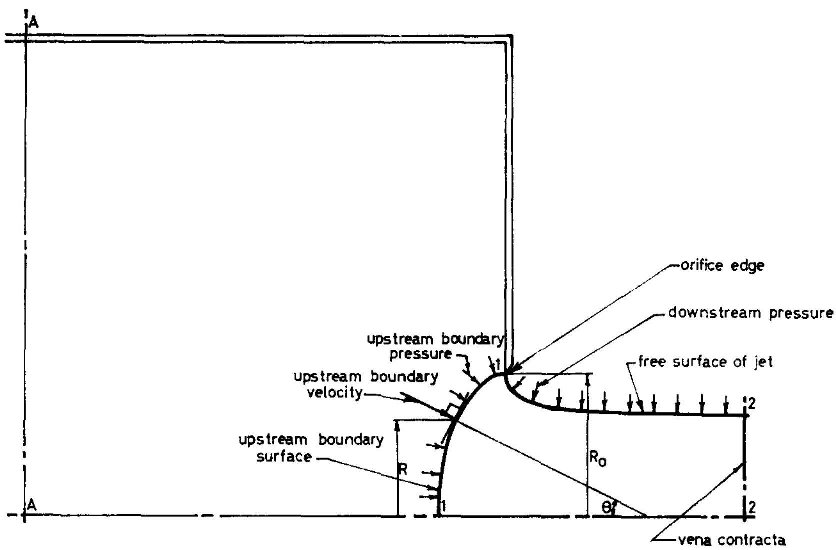

FIG. 1. Model of the flow through a sharp-edged orifice.

The method proposed in this paper is similar to those described above, in that the compressible coefficient is expressed in terms of the incompressible value. It differs, however, in that the upstream boundary surface of the control volume is chosen to be attached to the orifice edge (Fig. 1, 1–1) and to be everywhere normal to the flow through it. The shape of the boundary may change with pressure ratio. Nevertheless, provided the shape is known for the incompressible flow, it is not required explicitly for compressible flow. However, it is necessary to relate the two flows by assuming that the radial pressure distribution is independent of the compressibility effects. The validity of this assumption is discussed below.

In the theories described above, including the present one, the flow conditions across the vena contracta (Fig. 1) are assumed to be uniform, the pressure there being equal either to the downstream pressure or to the critical pressure in the supercritical case. In the latter case, the fluid is assumed to expand to the critical pressure at the orifice edge, the pressure on the free surface of the jet, however, being taken equal to the downstream value. These assumptions are reasonable when compared with experimental measurements by Stanton,[5] who showed that the static pressure across the major part of the minimum cross-section of a supercritical jet was substantially equal to the critical pressure.

Benson and Pool[6] have shown the assumptions made both by Jobson and by Bragg to be reasonable for a two-dimensional slit. The results of Bragg, being typical of the two, have therefore been used in this paper, together with published experimental results from a number of workers, for comparison with the present theory.

THEORY

The coefficient of contraction of area is defined by (equation 2)

$$C_c = \frac{\text{cross-sectional area of vena contracta}}{\text{cross-sectional area of orifice}}.$$

The vena contracta is the plane in the subcritical jet at which the streamlines of the flow downstream of the orifice are considered to become parallel. The pressure here is uniform across the jet and equal to the downstream pressure. For supercritical flow, the vena contracta is defined as a jet cross-section across which the pressure is uniform and equal to the critical value.

(i) *Incompressible flow*

Consider the steady irrotational flow of an incompressible fluid from a large reservoir at a uniform pressure P_s into a space maintained at a uniform pressure P_2. It is assumed that conditions are uniform in the plane of the vena contracta (Fig. 1, 2–2) and correspond to complete expansion to the pressure P_2. The control volume is chosen to be bounded by the area of the jet at the vena contracta, by the free surface of the jet between the vena contracta and the orifice edge, and by an upstream boundary surface, Fig. 1, 1–1, from the orifice edge and everywhere normal to the fluid velocity, u_{1i}.

The momentum equation for the axial component of momentum transfer through the control volume is

$$\int_{A_{1i}} P_1 \cos\theta \,\mathrm{d}A - P_2 A_0 = \rho u_{2i}^2 C_{ci} A_0 - \int_{A_{1i}} \rho u_{1i}^2 \cos\theta \,\mathrm{d}A. \tag{3}$$

The local velocity at any point on the upstream boundary, 1–1, is given by

$$P_1 = P_s - \frac{\rho u_{1i}^2}{2} \tag{4a}$$

and in the plane of the vena contracta by

$$P_2 = P_s - \frac{\rho u_{2i}^2}{2}. \tag{4b}$$

Noting that

$$\cos\theta \,\mathrm{d}A/A_0 = 2y \,\mathrm{d}y$$

and substituting for P_1 and P_2 from equations (4a) and (4b) then the momentum equation reduces to

$$\int_0^1 (u_{1i}/u_{2i})^2 y \,\mathrm{d}y = C_{ci} - 0{\cdot}5 \tag{5a}$$

$$= I. \tag{5b}$$

(ii) *Subcritical compressible flow*

By choosing an upstream boundary surface, 1–1, of the control volume from the orifice edge such that it is everywhere normal to the fluid velocity crossing it, the momentum equation for the axial component of momentum transfer through the control volume may be again written. As the velocity distribution in compressible flow may be expected to differ from that in incompressible flow, then the shape of the upstream boundary may be expected to differ also. The momentum equation is

$$\int_{A_1} P_1 \cos\theta \,\mathrm{d}A - P_2 A_0 = \rho_2 u_2^2 C_c A_0 - \int_{A_1} \rho_1 u_1^2 \cos\theta \,\mathrm{d}A. \tag{6}$$

For isentropic expansion from stagnation conditions, (i) to the upstream boundary

$$\rho_1 u_1^2 = \frac{2\gamma}{\gamma-1} P_s\left[\left(\frac{P_1}{P_s}\right)^{1/\gamma} - \frac{P_1}{P_s}\right] \tag{7a}$$

and (ii) to the vena contracta

$$\rho_2 u_2^2 = \frac{2\gamma}{\gamma-1} P_s\,(r^{1/\gamma}-r). \tag{7b}$$

Substituting from equations (7a) and (7b) in equation (6), and noting that

$$\cos\theta\,\mathrm{d}A/A_0 = 2y\,\mathrm{d}y,$$

then

$$\gamma(r^{1/\gamma}-r)\,C_c + \frac{\gamma-1}{2}r = \int_0^1\left[2\gamma\left(\frac{P_1}{P_s}\right)^{1/\gamma} - \frac{P_1}{P_s}\right] y\,\mathrm{d}y. \tag{7}$$

The pressure ratio (P_1/P_s) in equation (7), may be expressed in terms of the incompressible velocity distribution (u_{1i}/u_{2i}), if the radial pressure distribution is assumed to be the same in compressible as in incompressible flow. Thus from equations (4a) and (4b)

$$\frac{P_1}{P_s} = 1 - x\left(\frac{u_{1i}}{u_{2i}}\right)^2, \tag{8a}$$

whence, upon substituting in equation (7),

$$C_c = \frac{\gamma-1}{2\gamma(r^{1/\gamma}-r)}\left[\!\left[x - 2xI - \frac{2\gamma}{\gamma-1}\left\{1 - 2xI + 2\int_0^1\left[1 - x\left(\frac{u_{1i}}{u_{2i}}\right)^2\right]^{1/\gamma} y\,\mathrm{d}y\right\}\right]\!\right], \tag{8b}$$

where I is given by equation (5).

The quantity x under the integral sign has a value which is always less than 0·5 because, for subcritical flow, the pressure ratio, r must be greater than 0·5. In addition, over the greater part of the upstream boundary $(u_{1i}/u_{2i})^2$ has a value considerably less than unity. Thus, a binomial expansion of the expression under the integral sign may be expected to give a rapidly converging series expansion. Equation (8b) may therefore be written in the approximate form

$$C_c \simeq \frac{(\gamma-1)\,x}{(r^{1/\gamma}-r)}\left(C_{ci} - \frac{I_{\mathrm{I}}\,x}{\gamma} - \frac{2\gamma-1}{3\gamma^2} I_{\mathrm{II}}\,x^2\right), \tag{9}$$

where

$$I_{\mathrm{I}} = \int_0^1 (u_{1i}/u_{2i})^4\, y\,\mathrm{d}y \tag{10}$$

and

$$I_{\mathrm{II}} = \int_0^1 (u_{1i}/u_{2i})^6\, y\,\mathrm{d}y. \tag{11}$$

(iii) *Supercritical flow*

For supercritical flow, the pressure on the wall at the orifice edge has been assumed to equal the critical pressure, P_c, which is also assumed to act across the vena contracta. The pressure on the free jet between these two points is assumed to equal the ambient pressure, P_a. As in the preceding analysis, the upstream boundary surface is chosen to be normal to the flow velocity, and the radial pressure distribution is assumed to be the same as in incompressible flow.

The momentum equation for the axial component of momentum transfer through the control volume is

$$\int_{A_1} P_1\cos\theta\,\mathrm{d}A - P_a A_0(1-C_c) - P_c A_0 C_c = \rho_2 u_2^2 A_0 C_c - \int_{A_1}\rho_1 u_1^2\cos\theta\,\mathrm{d}A. \tag{12}$$

From equations (2a) and (2b), assuming no change in radial pressure distribution,

$$P_1 = P_s[1 - X(u_{1i}/u_{2i})^2] \tag{13}$$

and for sonic velocity at the vena contracta

$$\rho_2 u_2^2 = \gamma P_s r_c. \tag{14}$$

For isentropic expansion from reservoir conditions to the upstream boundary

$$\rho_1 u_1^2 = \frac{2\gamma}{\gamma-1} P_s \left\{ \left[1 - X\left(\frac{u_{1i}}{u_{2i}}\right)^2\right]^{1/\gamma} - 1 + X\left(\frac{u_{1i}}{u_{2i}}\right)^2 \right\}. \tag{15}$$

Combining equations (12) to (15) and using the binomial expansion for the expression $[1 - X(u_{1i}/u_{2i})^2]^{1/\gamma}$, then the compressible contraction coefficient may be expressed as

$$C_c \simeq \frac{1}{(\gamma+1)\, r_c - r} \left\{ x + 2X(C_{ci} - 0{\cdot}5) - \frac{2}{\gamma} I_{\mathrm{I}} X^2 - \frac{2}{3\gamma} \frac{2\gamma-1}{\gamma} I_{\mathrm{II}} X^3 \right\}. \tag{16}$$

Equations (9) and (16) give expressions for the compressible contraction coefficient for subcritical and supercritical discharge respectively. They are identical when the pressure ratio has the critical value, and the solutions are therefore continuous.

The compressible contraction coefficient is given in terms of the pressure ratio and the ratio of specific heats, together with the incompressible contraction coefficient and the incompressible velocity distribution on the upstream boundary. The latter two values may be determined for a given orifice–reservoir configuration and are independent of the pressure ratio.

RESULTS

(i) *Theoretical*

The shape of the upstream boundary for incompressible flow, defined as being everywhere normal to the streamlines, was obtained using the relaxation method developed by Southwell and Vaisey.[7] With this method the shape of the free surface has to be established prior to commencement of the relaxation procedure. Hunt[8] obtained measurements of the free surface which were consistent with his theoretical values and those of Abul-Fetouh.[9] They were used, therefore, to define the shape of the free jet surface. The corresponding contraction coefficient was 0·595. The relaxation solution, using the assumed shape, must yield a constant velocity along the free surface and this may be evaluated from the derivative of the stream function normal to the free stream surface. Previous workers have used this boundary condition to establish by iteration the true shape of the free surface. Due to the difficulty of evaluating the normal gradient from the discrete relaxation net, however, the accuracy of the shape of the free surface obtained in this way is open to question. In this work, therefore, the values from Hunt were taken to be the more accurate and the boundary conditions were used simply as a check.

For an orifice to pipe diameter ratio of 1 : 6, which closely approximates the flow from an infinite reservoir, the distribution of the velocity crossing the upstream boundary is shown in Fig. 2. The accuracy of the solution was checked by evaluating the integral, I, from the upstream boundary shape and from the incompressible contraction coefficient (equations 5a and 5b). Thus

$$I = \int_0^1 (u_{1i}/u_{2i})^2\, y\, \mathrm{d}y = 0{\cdot}0943$$

and

$$I = C_{ci} - 0{\cdot}5 = 0{\cdot}095.$$

A further check on the accuracy of the upstream boundary shape was obtained by integrating the velocity distribution with respect to the upstream boundary surface area. The resulting mass flow through the upstream boundary surface area agreed with the mass flow through the vena contracta to within 2 per cent. Values of the integrals I_{I} and I_{II} were then evaluated as follows:

$$I_{\mathrm{I}} = \int_0^1 (u_{1i}/u_{2i})^4\, y\, \mathrm{d}y = 0{\cdot}0293,$$

$$I_{\mathrm{II}} = \int_0^1 (u_{1i}/u_{2i})^6\, y\, \mathrm{d}y = 0{\cdot}0127.$$

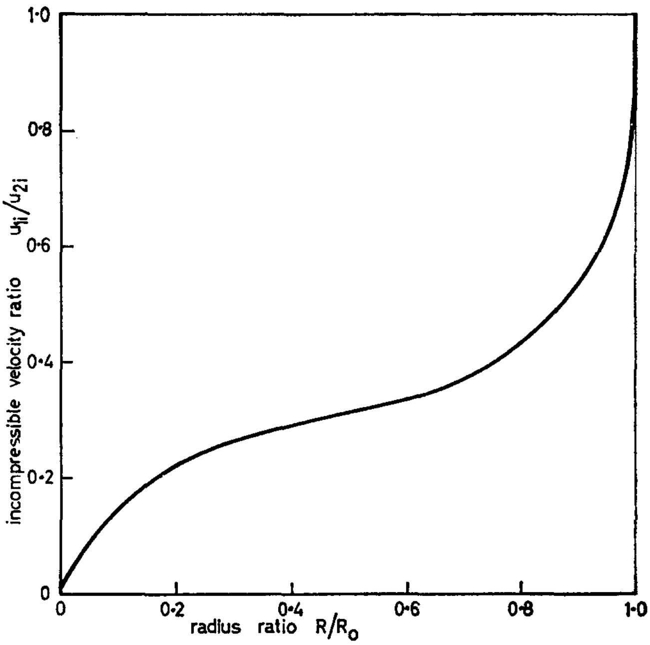

FIG. 2. Velocity distribution on the upstream boundary surface in incompressible flow.

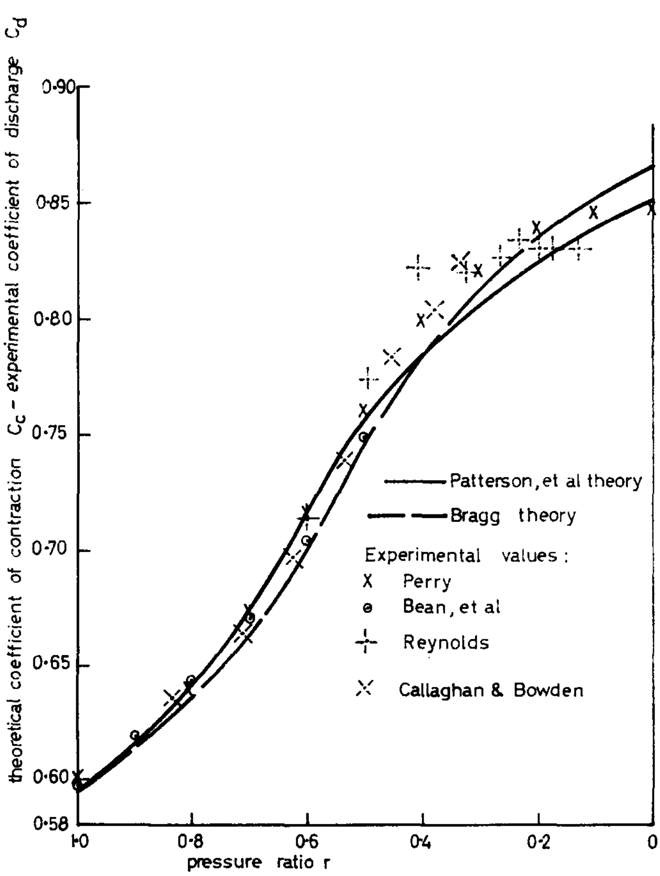

FIG. 3. Comparison of experimental results for coefficient of discharge with theoretical results for coefficient of contraction.

Substitution of these values into equations (8) and (15), together with the ratio of specific heats for air ($\gamma = 1{\cdot}4$), led to the graph of compressible contraction coefficient against pressure ratio shown in Fig. 3.

(ii) *Experimental*

Experimental results of the coefficient of discharge from Reynolds,[10] Perry,[11] Bean *et al.*[12] and Callaghan and Bowden[13] have been superimposed on Fig. 3, together with the theoretical values for the contraction coefficient obtained by Bragg. The theoretical curve appears to agree well with experimental values except in the just supercritical region where the theoretical values are low. Theoretical values are slightly higher than those of Bragg in the subcritical region and slightly lower in the supercritical region.

DISCUSSION

The radial pressure distribution obtained from the pressure on the upstream boundary for incompressible flow is plotted in Fig. 4 for a number of pressure ratios from the critical 0·528 to unity. The pressure is not greatly different to the stagnation pressure except in

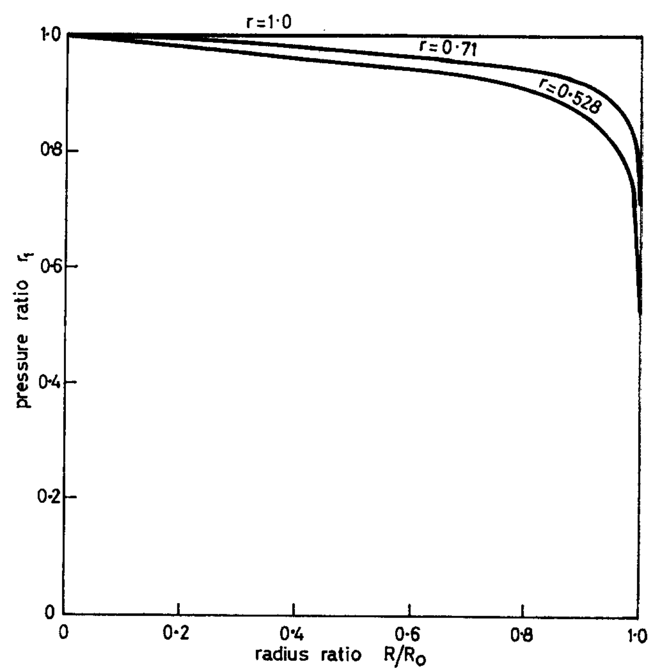

FIG. 4. Radial pressure distribution on the upstream boundary surface in incompressible flow.

the region near the orifice edge, where it falls off sharply. Experimental measurements have not been made in this region, but Page[14] has taken measurements of the pressure distribution in the plane of the orifice which also show the same trend (Fig. 5). Thus, it is only in the region near the orifice edge that compressibility may have a significant effect. In the momentum equation radial pressure distribution is integrated over the orifice area, and, therefore, small local differences in pressure distribution due to compressibility near the orifice edge will have little effect.

It is very difficult to establish the magnitude of these local pressure differences. If, however, the velocity is assumed to remain the same in compressible and in incompressible flow at any orifice radius, then the resulting difference in pressure at this radius due to compressibility may be calculated. The result is shown in Fig. 6 where the percentage change in pressure is shown to have a maximum value of 2·6 per cent at a pressure ratio of 0·71. By considering this error as a small perturbation in the theoretical analysis it may

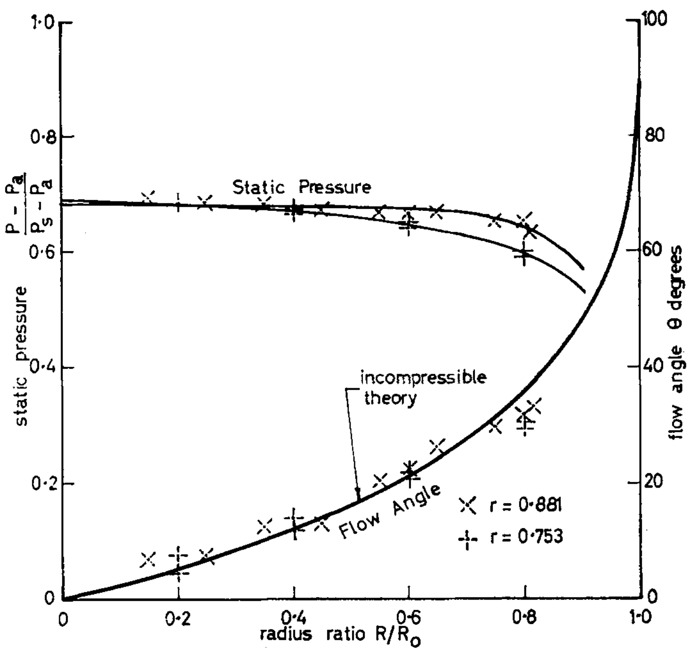

FIG. 5. Experimental values of static pressure and flow angle in the plane of the orifice in compressible flow.

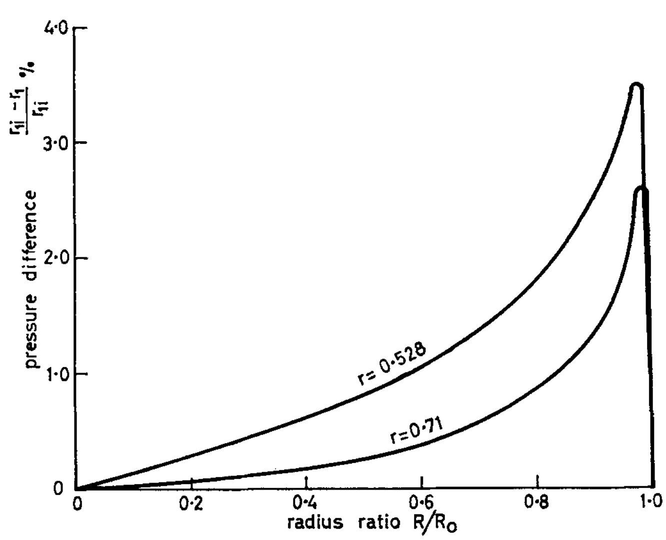

FIG. 6. Percentage difference in pressure due to compressibility.

be shown that the resultant error in the coefficient of contraction is 0·9 per cent. At the critical pressure ratio, although the percentage change in pressure is 3·5 per cent, the resultant error in the coefficient of contraction is zero from the perturbation analysis. Similarly, at a pressure ratio of unity, the error in the coefficient of contraction is zero because the compressible and incompressible distributions are identical.

It is to be emphasized that in the determination of the contraction coefficient using the momentum equation, the radial distribution of pressure is the only physical quantity of the flow required. If the contraction coefficient is determined from mass continuity, then the upstream boundary shape for compressible flow is also required. The change in the latter from the incompressible flow shape is dependent on the change in flow angle in the two cases. Experimental measurements of flow angle immediately upstream of the orifice are not available, but Page has measured values in the plane of the orifice in subcritical flow (Fig. 5). The results show little variation from the theoretical incompressible value, and it may be concluded that the shape of the upstream boundary in subcritical compressible flow is not greatly different from the incompressible shape. Calculations of the contraction coefficient for subcritical pressure ratios using the mass continuity equation give good agreement with those from the momentum equation analysis, thus supporting this view.

In the supersonic region, a comparison of theoretical values obtained by Benson and Pool[15] with those of Bragg for a two-dimensional slit (Fig. 7) shows that the latter are

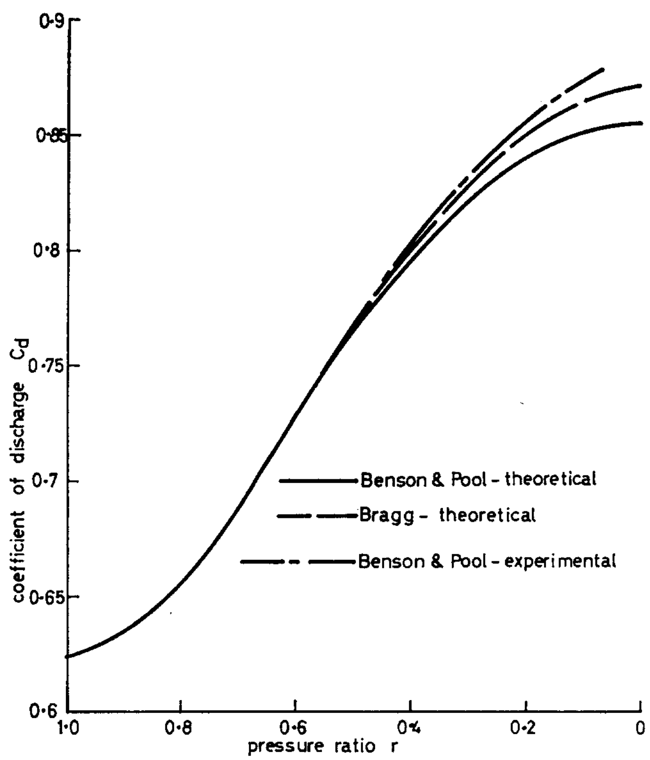

FIG. 7. Experimental and theoretical values for compressible flow through a slit.

higher at low-pressure ratios. Experimental results obtained by Benson and Pool seem to agree more closely with those of Bragg than with those of their theory, but these results, they believe, may not be reliable because of suspected errors in flow-rate measurement. In the present paper, for the axisymmetric case, the results are also lower than those of Bragg and, in addition, are in good agreement with experiment. The evidence suggests, therefore, that the results of Bragg are too high in this region and the present theory is more correct. The assumption that the critical pressure always acts at the vena contracta leads to a fixed radial velocity distribution on the upstream boundary irrespective of the pressure ratio. Increase in the contraction coefficient with increasing pressure ratio occurs as a result of the reduction in pressure, relative to stagnation, acting on the free surface of the jet. With a fixed upstream radial velocity distribution the change in contraction coefficient will cause a change in flow angle with a consequent change in the shape of the upstream boundary.

CONCLUSIONS

An adequate one-dimensional theory may be developed to predict the compressible contraction coefficient for discharge from a reservoir. The theory is based on an upstream control surface attached to the corners of the orifice and everywhere normal to the flow through it. For a particular physical configuration a knowledge of the incompressible contraction coefficient and the shape of the upstream control surface is necessary in order to obtain the compressible contraction coefficient. These may be determined from incompressible flow theory.

The form of the theoretical expression is simple. However, because an "exact" solution for axisymmetric flow through an orifice is not available, the accuracy of the theory cannot be completely established. Nevertheless, the results appear to agree closely with experiment and with a previous theory developed by Bragg. At very low pressure ratios in supercritical flow, and in the subcritical region, the theory appears to be superior to previous one-dimensional theories.

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