Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT

ELSEVIER

Simulation Modelling Practice and Theory 10 (2002) 141–152

SIMULATION MODELLING PRACTICE AND THEORY

www.elsevier.com/locate/simpat

An orifice flow model for laminar and turbulent conditions

W. Borutzky [a,*], B. Barnard [b], J. Thoma [c]

[a] *Department of Applied Computer Science, Bonn-Rhein-Sieg University of Applied Sciences, D-53754 Sankt Augustin, Germany*

[b] *Department of Mechanical Engineering, Monash University, Caulfield Campus, Vic. 3145, Australia*

[c] *Department of Systems Design, University of Waterloo, Ont., Canada*

Received 19 July 2001; received in revised form 1 February 2002

Abstract

The square root characteristic commonly used to model the flow through hydraulic orifices may cause numerical problems because the derivative of the flow with respect to the pressure drop tends to infinity when the pressure drop approaches zero. Moreover, for small values of the pressure drop it is more reasonable to assume that the flow depends linearly on the pressure drop.

The paper starts from an approximation of the measured characteristic of the discharge coefficient versus the square root of the Reynolds number given by Merritt and proposes a single *empirical* flow formula that provides a *linear* relation for small pressure differences and the conventional square root law for turbulent conditions. The transition from the laminar to the turbulent region is smooth. Since the slope of the characteristic is finite at zero pressure difference, numerical difficulties are avoided. The formula comprises physical meaningful terms and employs parameters which have a physical meaning. The proposed orifice model has been used in a bond graph model of a hydraulic sample circuit. Simulation results have proved to be accurate. The orifice model is easily implemented as a library model in a modern modeling language. Ultimately, the model can be adapted to approximate pipe flow losses as well.

Keywords: Hydraulic orifices; Laminar and turbulent flow; Treatment of discontinuities and singularities in ordinary differential equations; Empirical formula; Bond graphs; Library model

* Corresponding author. Tel.: +49-2241-865-288; fax: +49-2241-865-8288.
E-mail address: borutzky@uni-koeln.de (W. Borutzky).

PII: S1569-190X(02)00092-8

1. Introduction

Orifices, sometimes of variable cross-section, are an essential component in hydraulic circuits. Frequently the fluid flow Q through an orifice is assumed to be proportional to the square root of the pressure drop across the orifice $\sqrt{\Delta p}$.

$$Q = c_{\mathrm{d}} A \sqrt{\frac{2}{\rho} |\Delta p| \mathrm{sign}(\Delta p)} \tag{1}$$

(In Eq. (1) c_{d} denotes the discharge coefficient, A is the cross-section area of the restriction.) This formula strictly holds for an incompressible steady-state flow. It describes the flow with good accuracy for turbulent flow conditions. However, since the actual form of the fluid flow strongly depends on the geometry of the restriction, in particular whether it is a sharp edged one, and since small disturbances may lead to a change from laminar to turbulent flow conditions, the formula is utilized even when the flow might be laminar. While for the above reasons it might be thought not worthwhile to distinguish between laminar and turbulent flow, a drawback of using this formula for both regimes of flow is that during the simulation of a hydraulic circuit numerical difficulties may be encountered due to the fact that the derivative of the flow with regard to the pressure drop tends to infinity while the pressure drop approaches zero. The automatic time step adjustment of a non-stiffly stable integration algorithm will drastically reduce the time step. This is due to the fact that for $\Delta p = 0$ higher derivatives of Q with respect to Δp do not exist. As all numerical integration algorithms are based on the prerequisite of the existence of higher derivatives up to a certain order they are in principle not applicable. This problem, however, does not become transparent when using numerical integration techniques as they circumvent the usage of higher derivatives by using only function evaluations of the right hand sides of the ordinary differential equations at appropriate locations within each step. Nevertheless, in the vicinity of $\Delta p = 0$ simulation will be slowed down considerably or may even fail [1]. (A method that can be applied without any difficulty in the vicinity of singularities or poles is the "Bulirsch–Stoer algorithm" which is based on extrapolation by means of rational functions.) If a stiffly stable integration algorithm with a fixed step size is chosen, numerical stability is ensured, but the accuracy depends on the chosen step size. The square root characteristic not only introduce the potential danger of numerical problems; for laminar flow it is more reasonable to assume that the flow depends linearly on pressure drop for small values of the pressure drop. In the following we propose an empirical formula that effectively provides a linear relation for small pressure differences and the conventional square root law for turbulent conditions. The transition from the laminar to the turbulent region is smooth, that is, the derivative exhibits no discontinuity. Other formulae have been given by Ellman and Piché [1], and Ellman and Vilenius [2]. They use a switching method and two equations with a smoothed transition point. (A discontinuity in the derivative can be located and handled when using algorithms with root-finding capabilities as the code LSODAR by Hindmarsh [3] or almost all of the integrators implemented in MATLAB [4].)

2. A flow formula for laminar and turbulent flow conditions

The above square root law, Eq. (1) is derived from Bernoulli's energy equation for an incompressible steady-state flow. The coefficient c_d accounts for energy losses. It depends on the geometry of the restriction and of the Reynolds number R which characterizes the mode of the flow. Often a constant value holding for *turbulent* conditions is adopted. However, it is known that the discharge coefficient c_d is a non-linear function of $\sqrt{R}$ [5]. If the so-called hydraulic diameter D_h of the orifice is known, the Reynolds number can be expressed by the volume flow rate Q

$$R = \frac{D_h}{A\nu} Q \tag{2}$$

In Eq. (2) the kinematic viscosity ν depends on the temperature and the pressure. Often an average value is used. Observing that c_d is a non-linear function of the Reynolds number R and by combining Eqs. (1) and (2) we see that the volume flow rate through an orifice is determined by an *implicit* non-linear equation of the form

$$Q = f(Q) A \sqrt{\frac{2}{\rho} |\Delta p| \text{sign}(\Delta p)} \tag{3}$$

A calculation of the volume flow rate based on Eq. (3) is costly in regard to computational time since numerical iteration is required. By looking at the plot of the discharge coefficient versus the square root of the Reynolds number given by Merritt [5] (cf. Fig. 1) we see that the relation may be approximated by

$$c_d = k\sqrt{R} \tag{4}$$

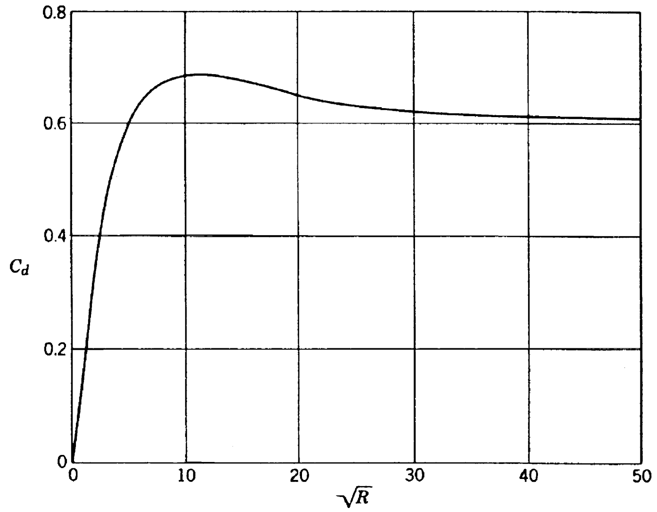

Fig. 1. Discharge coefficient versus the square root of the Reynolds number (Merritt, 1967).

$k \in \mathbb{R}$, $k > 0$, for small Reynolds numbers, and by a constant c_{turb} for large values of R.

$$c_{\text{d}} = c_{\text{turb}} := 0.61 \tag{5}$$

By determining the Reynolds number R_{t} at which the linear characteristic intersects with the constant value of c_{d} the curve in Fig. 1 might be approximated by a piecewise linear function (cf. [5], p. 45). However, at the transition point R_{t} where the relation changes from one equation to the other, we have a discontinuity in the derivative. As an alternative we propose an *empirical* approximation of the non-linear relation $c_{\text{d}} = f_1(\sqrt{R})$ in Fig. 1 with the following features.

1. The approximation is given by a single relation for all Reynolds numbers.
2. For small pressure differences it provides a linear relation between the flow through the orifice and the pressure drop. For turbulent flow conditions it matches the conventional square root characteristic.
3. The transition from the laminar flow region to turbulent flow conditions is smooth.
4. The parameters employed have a physical meaning.

A simple formula that meets the above requirements is

$$c_{\text{d}} = \frac{c_{\text{turb}}\sqrt{R}}{\sqrt{R} + \sqrt{R_{\text{t}}}} \tag{6}$$

For small values of R this approximation reduces to

$$c_{\text{d}} \approx \frac{c_{\text{turb}}}{\sqrt{R_{\text{t}}}}\sqrt{R} \tag{7}$$

By substituting (7) into (1) and eliminating R by means of (2) in fact, we obtain a linear relation between the volume flow rate Q and the pressure drop Δp across the orifice, for laminar flow.

$$Q = \left(\frac{c_{\text{turb}}}{\sqrt{R_{\text{t}}}}\right)^2 \frac{2AD_{\text{h}}}{\rho v}\Delta p \tag{8}$$

For large Reynolds numbers $R \gg R_{\text{t}}$ we have $c_{\text{d}} \approx c_{\text{turb}}$. In that case the discharge coefficient in Eq. (1) is a constant. Hence, in the turbulent region the flow is determined by the conventional square root characteristic. Finally, substituting Eq. (2) into (6) and the result into Eq. (1) yields a quadratic equation for $\sqrt{|Q|}$ instead of an implicit non-linear relation for Q of the form given by Eq. (3).

$$(\sqrt{|Q|})^2 + \sqrt{\frac{R_{\text{t}}Av}{D_{\text{h}}}}\sqrt{|Q|} = c_{\text{turb}}A\sqrt{\frac{2}{\rho}|\Delta p|} \tag{9}$$

As can be clearly seen from this equation, it is the second term that makes the difference to the conventional square root law for turbulent flow conditions. The quadratic equation (9) for $\sqrt{|Q|}$ has one unique solution. If the volume flow rate, Q,

($Q > 0$) is differentiated with respect to Δp, after a lengthy calculation including l' Hospital's rule we obtain for the gradient at zero pressure drop

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}(\Delta p)}\right|_{\Delta p=0} = \frac{2Ac_{\mathrm{turb}}^2 D_{\mathrm{h}}}{\rho \nu R_{\mathrm{t}}} =: \frac{1}{a} \tag{10}$$

In Fig. 2 the volume flow rate, Q, obtained from Eq. (9) is plotted (lower line) versus $\sqrt{\Delta p}$ for positive pressure differences. For comparison the flow through an orifice according to Eq. (1) with $c_{\mathrm{d}} = c_{\mathrm{turb}}$ (purely turbulent case) is given by the upper line. As can be seen the values of the flow according to Eq. (9) are below those according to Eq. (1) with $c_{\mathrm{d}} = c_{\mathrm{turb}}$. In regard to the conventional purely turbulent flow model our proposed formula (6) now results in an over-estimate of energy losses in the orifice, rather than the traditional under-estimate. This larger pressure drop corresponding to a given flow now includes the viscosity effects which dominate laminar flow and which are not properly taken into account by the purely turbulent flow model. Still, the proposed approximation of c_{d} approaches the asymptotic value c_{turb} too slowly. The relative deviation from c_{turb} is

$$\epsilon := \frac{c_{\mathrm{turb}} - c_{\mathrm{d}}}{c_{\mathrm{d}}} = \frac{\sqrt{R_{\mathrm{t}}}}{\sqrt{R} + \sqrt{R_{\mathrm{t}}}} \tag{11}$$

For laminar flow through round sharp-edged orifices Wuest ([6], p. 366) has theoretically determined the relation $\Delta p = 50.4Q\eta/(\pi D^3)$ allowing to determine the coefficient k in Eq. (4).

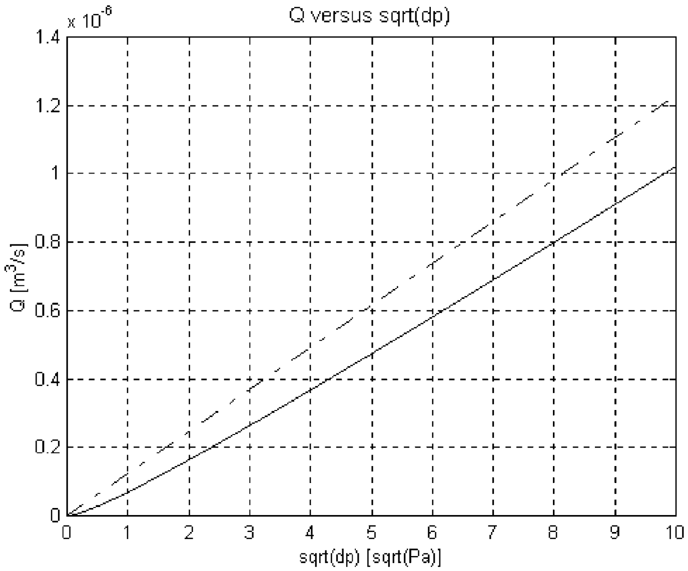

Fig. 2. Flow Q through an orifice versus $\sqrt{\Delta p}$ for the turbulent case (upper line) and the laminar–turbulent model (lower line).

$$c_{\mathrm{d}} = 0.2\sqrt{R} \tag{12}$$

By comparing this relation with Eq. (7) we obtain $R_{\mathrm{t}} = 9.33$. Hence, at $R = R_{\mathrm{t}}$ the relative deviation is 50%. For $R = 49R_{\mathrm{t}}$ it is 12.5%, and for $R = 2000$ it is still 6.4%.

Better results may be obtained by a slight modification of Eq. (6).

$$c_{\mathrm{d}} = \frac{c_{\mathrm{turb}}\sqrt{R}}{\sqrt{R + R_{\mathrm{t}}}} \tag{13}$$

This formula also meets the above requirements.

Replacing the Reynolds number R by the volume flow rate Q by means of Eq. (2) and substituting the resulting expression for c_{d} into Eq. (1) gives a quadratic equation for the flow Q

$$\Delta p = \frac{\rho \nu R_{\mathrm{t}}}{2Ac_{\mathrm{turb}}^2 D_{\mathrm{h}}} Q + \frac{\rho}{2A^2 c_{\mathrm{turb}}^2} Q^2 \mathrm{sign}(Q) \tag{14}$$

which has the unique solution

$$Q = \left(c_{\mathrm{turb}} A \sqrt{\frac{2}{\rho}|\Delta p| + \left(\frac{\nu R_{\mathrm{t}}}{2c_{\mathrm{turb}} D_{\mathrm{h}}} \right)^2} - A\nu \frac{R_{\mathrm{t}}}{2D_{\mathrm{h}}} \right) \mathrm{sign}(\Delta p) \tag{15}$$

By looking at Eq. (14) we see that we have determined the parameters a, b in the formula

$$p = a\dot{V} + b\dot{V}^2 \mathrm{sign}\dot{V} \tag{16}$$

given in [7]. (Note, in [7] the pressure drop across the orifice is denoted by p.) Eq. (14) clearly shows the term, *linear* in Q, accounting for laminar flow that is missing from Eq. (1). It accounts for the viscosity, ν, of the fluid while the second term, quadratic in Q and accounting for turbulent flow, is independent of ν. For small flow values the linear term is dominant over the quadratic term. For small $Q > 0$ Eq. (16) reads

$$\Delta p = aQ\left(1 + \frac{b}{a}Q\right) = aQ\left(1 + \frac{R}{R_{\mathrm{t}}}\right) \approx aQ \tag{17}$$

By differentiating Eq. (14) ($Q > 0$), which uses the improved Eq. (13) for c_{d} we obtain for the same gradient of the Q versus Δp characteristic at zero pressure drop an identical result to that of Eq. (10).

With

$$Q_{\mathrm{t}} := \frac{A\nu}{D_{\mathrm{h}}} R_{\mathrm{t}} \tag{18}$$

$$p_{\mathrm{t}} := \frac{Q_{\mathrm{t}}^2}{c_{\mathrm{turb}}^2 A^2 \frac{2}{\rho}} \tag{19}$$

Eq. (14) can be written in the form

$$\frac{\Delta p}{p_{\mathrm{t}}} = \left(\frac{Q}{Q_{\mathrm{t}}}\right) + \left(\frac{Q}{Q_{\mathrm{t}}}\right)^2 \tag{20}$$

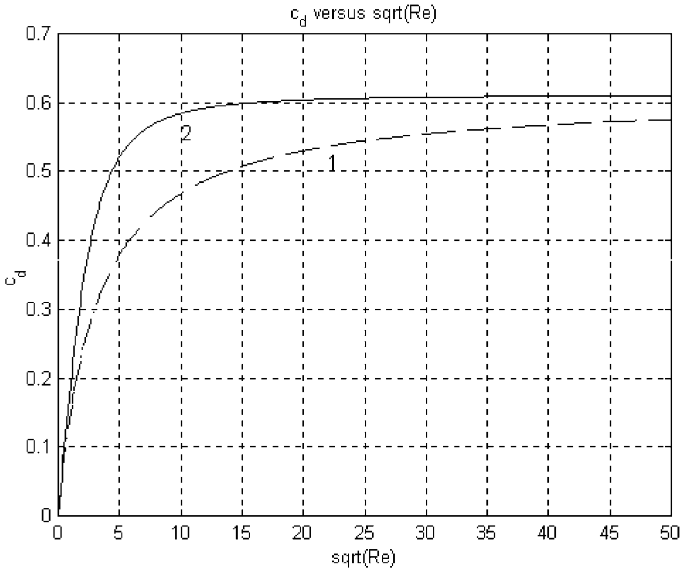

Fig. 3. A comparison of both approximations of c_d versus $\sqrt{R}$.

From Eq. (20) we see that when $Q = Q_t$ the normalized pressure drop $\Delta p/p_t$ is twice the normalized pressure drop we obtain from the purely turbulent model (Eq. (1) with $c_d = c_{turb}$).

For Eq. (13) the relative deviation ϵ from c_{turb} is smaller than for Eq. (6).

$$R = R_t: \quad \epsilon = 29.3\%$$

$$R = 49R_t: \quad \epsilon = 1\%$$

Fig. 3 shows a comparison of both approximations. In regard to the curve given by Merritt (cf. Fig. 1) both approximations stay below the asymptotic value c_{turb}. That is, energy losses are somewhat over-estimated especially near the transition region.

3. An example

Fig. 4 shows the circuit schematic and a simple bond graph model of a hydraulic sample system, e. g. a large wood chipping machine with a large rotor. This example system has been inspired by a bleed-off flow control circuit from Vickers *Industrial Hydraulic Manual 935/00-A*. The system operates as a speed regulator. At no load there is only a small flow that passes through the orifice (R_0) and the pressure is low. If the load is almost totally inertia, the pressure is high during the start and then falls. As there are two regimes of high pressure and low pressure the system is suitable to test the orifice model we propose. The manually operated two-way valve

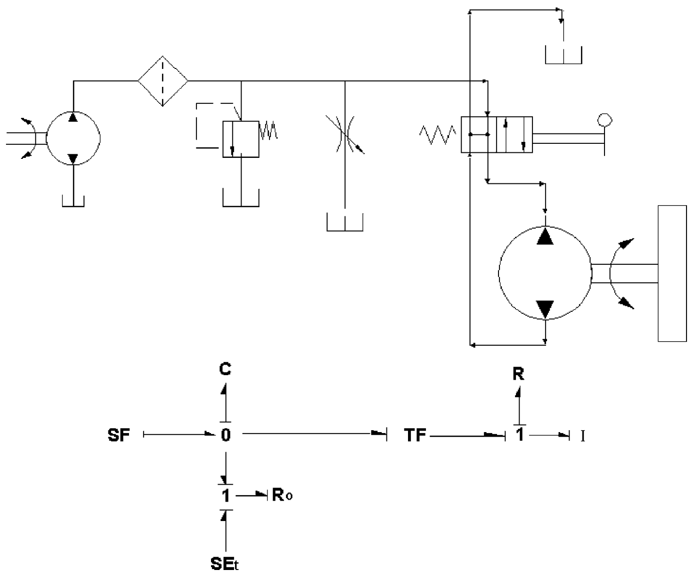

Fig. 4. Hydraulic circuit and simple bond graph of a speed regulator.

allows for changing the orientation of the rotation. The bond graph model is kept intentionally as simple as possible in order to study the effect of the new orifice model. The C element represents the compressibility of the fluid in the system volume of the filter and the hydraulic line. Losses due to the pressure relief valve and the two-way valve are neglected. The hydraulic motor is modeled just as an ideal transformer. The R element attached to the 1-junction of the angular velocity of the load accounts for efficiency losses of motor.

The bond graph model including our orifice model for laminar and turbulent conditions has been entered into the 20-sim modeling and simulation package [8] which is particularly suited for bond graph models. Fig. 5 shows the dynamic behavior of the system due to a jump of the volume flow rate, Q_{pump}, from 0 to 30 l/min delivered by the pump at $t = 0.5$ s. As can be clearly seen, the pressure indeed rises swiftly to high values during the start and falls to low values when the angular velocity of the load, ω, approaches a steady-state value of about 5 rad/s. At the maximum of the pressure drop, $\mathrm{d}p_{R0}$, across the orifice the volume flow rate, Q_{R0}, through the orifice is about 0.29 l/s. That is, 58% of the volume flow rate, Q_{pump}, delivered by the pump goes through the orifice at $t = 0.78$ s. The corresponding Reynolds number is about 63.5×10^3. Hence the flow through the orifice is turbulent at that time point. At steady state ($t = 1.5$ s) $Q_{R0} \approx 0.0023$ l/s which is only 0.45% of Q_{pump}. The Reynolds number is about 497. That is, at steady state the flow through the orifice is laminar!

If we replace our orifice model for laminar and turbulent conditions by the standard square root characteristic valid for turbulent flow only, we obtain results close to that depicted in Fig. 5 as can be seen by comparison with Fig. 6. That is, simulation runs using our new orifice model provide accurate results to be expected. The

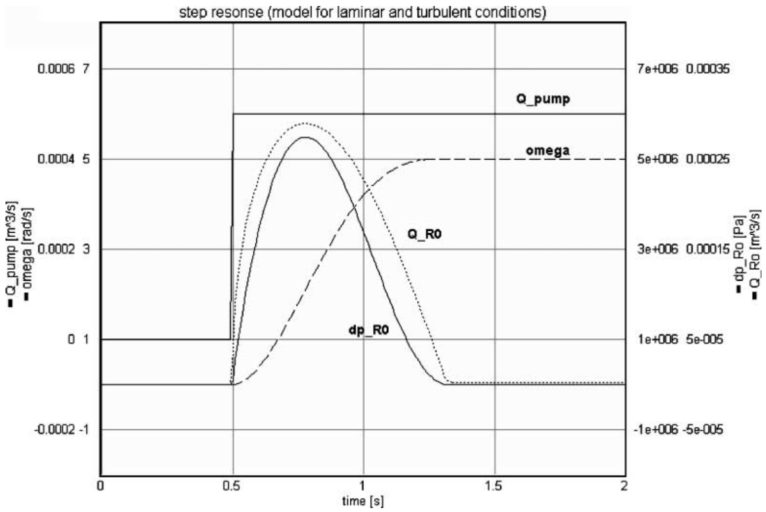

Fig. 5. Step response of the hydraulic system (model for laminar and turbulent conditions).

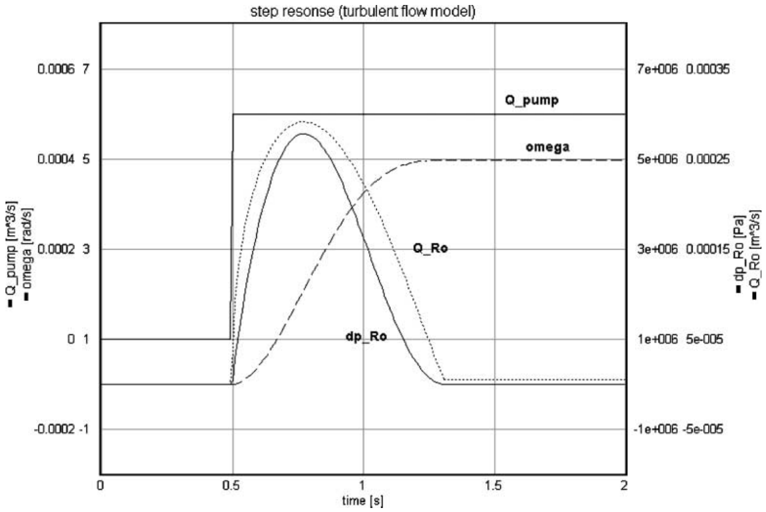

Fig. 6. Step response of the hydraulic system (turbulent flow model).

volume flow rate versus pressure drop characteristic however has a finite gradient at zero pressure drop. Hence, numerical problems in the vicinity of small pressure drops are avoided. Given the parameters of the example system the orifice characteristic is depicted in Fig. 7. The gradient at zero pressure drop is $1/a = 0.359 \times 10^{-6}$ ($\mathrm{m^3/s\,Pa}$). The different orifice flow rates in each model can just be detected near the steady-state flow in Figs. 5 and 6. The proposed new model shows a slightly higher

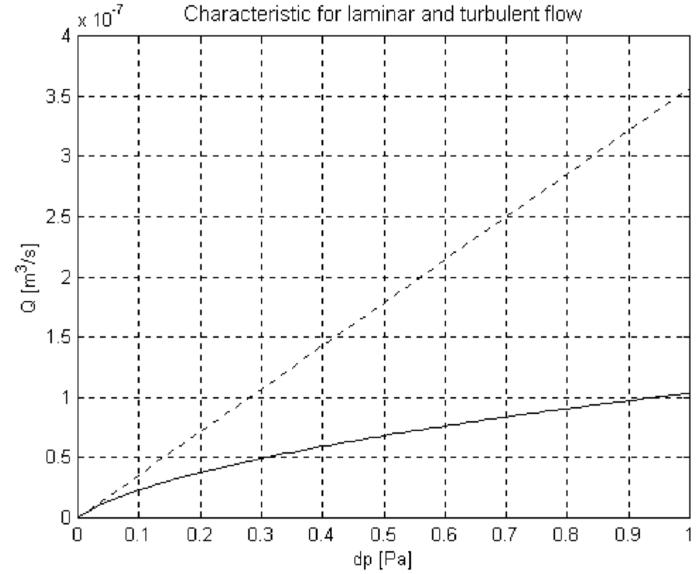

Fig. 7. Characteristic of the orifice with finite gradient at zero pressure drop.

pressure drop through the orifice owing to the laminar flow portion of the model. The new model's benefits are also important at start-up when the pressure is low.

4. Conclusions

The function of all resistor models in the dynamic behavior of a system model is to dissipate energy. This energy loss reduces overshoots and stabilizes oscillations. The proposed models use an averaged approximation to the true c_d curves, which depend both on orifice geometry and Reynolds number. This approximation is a substantial improvement on totally neglecting laminar flow. Using such an approximation can be justified because it is the total energy dissipated in one oscillation which matters.

While the model has been developed for an orifice, notice that it is easy to find an equivalent term for pipe losses. The well known Darcy pipe flow/pressure drop formula, which gives head loss as a function of the flow velocity, can be rearranged to match Eq. (1) above. It can be shown that the discharge coefficient, c_d, can be replaced by $\sqrt{D_h/(8fL)}$ where L is the pipe length. Although friction factor, f, is a function of Reynolds number, its laminar portion is accounted for in Eq. (6) by suitably choosing R_t. The limiting value of f can be used to find the equivalent c_{turb}. Thus this model can be used to approximate pipe flow losses as well as for orifices; the transition R needed for a a pipe is near 2000. Again, it is the total energy that matters.

The errors introduced by the proposed models are greatest at flow rates near R_t. The value of the discharge coefficient c_d is lower than the true value for low Reynolds

Numbers. Unless the proposed models are used, small flow rates calculated with the turbulent flow equation will seriously under-estimate the energy losses. The proposed models over-estimate the losses. However, since many small losses are often neglected, it may be preferable to over-estimate losses than the reverse, when using numerical integration. Fig. 2 clearly shows how neglecting laminar flow equations predict a much lower pressure drop than really occurs due to the viscosity effects which dominate laminar flow. In order to achieve stability in the numerical integration of a model, it is important not to neglect stabilizing energy losses which occur during the early stages of the integration. This orifice model is a contribution to that aim.

Ultimately, though the new model does not account for the overshoot of Merritt's curve in the transition region (cf. Fig. 1), modeling and simulation of the industrial example system indicate results that are very close to those obtained with the conventional pure turbulent model. Small differences can be detected only for small pressure drops. At the same time the new model ensures efficient and reliable numerical integration. Thus, there is no need for an improved curve fitting of Merritt's c_d versus $\sqrt{R}$ relation although this could be achieved by a modification of equation (13). The result would be a $\Delta p - Q$ relation more complicated than the one given by Eq. (14). Moreover, the actual form of c_d versus $\sqrt{R}$ curves depend on the geometry of the orifice and on steady-state conditions. Owing to small disturbances the laminar–turbulent transition region can vanish. Weule [9] has given measured c_d versus $\sqrt{R}$ characteristics which do not exhibit the overshoot in Merritt's curve. Also, Ellman and Piché [1] in their approach assume a $c_d - \sqrt{R}$ characteristic with no overshoot. Thus, in our view a more complex approach that also accounts for the overshoot appears to be not worthwhile. Our approach sufficiently approximating Merritt's c_d versus $\sqrt{R}$ characteristic by a single formula nicely results in an equations in which a linear term for laminar flow is just added to the quadratic term for turbulent flow (Eq. (14)). Both terms have a physical meaning and for small pressure drops the linear term dominates the quadratic one as to be expected.

The model is easily described in a modeling language, e.g. SIDOPS [10] or Modelica [11], to be included in a library of bond graph models of hydraulic devices.

Acknowledgements

The authors are grateful to the three anonymous referees for their helpful comments and suggestions which have improved the paper.

Appendix A. Parameters of the hydraulic system

S_f: $Q_{pump} = 0.5e - 3\ \mathrm{m^3/s} \triangleq 30\ \mathrm{l/min}$
C: $(V/\beta) = 9.6e - 12\ \mathrm{m^3/Pa}$
Se_t: $p_t = 0$
R_0: $D = 2.25e - 3$ m (orifice diameter)

TF: $V_m = 0.1e - 3$ m^3/rad (motor displacement)
R: $R_{effic} = 5e - 3$ N m s/rad ($\triangleq$ motor efficiency of 88%)
I: $J = 50$ kg m^2 (motor load inertia)
oil: density: $\rho = 780$ kg/m^3, dynamic viscosity: $\eta = 2$ mPa s

References

[1] A. Ellman, R. Piché, A two regime orifice flow formula, Trans. ASME J. Dyn. Syst., Measure., Control 121 (1999) 721–724.
[2] A. Ellman, M.J. Vilenius, Methods for simulating the steady-state and dynamic behavior of two-way cartridge valve circuits, SAE J. Commer. Vehicles 99 (1990) 384–393.
[3] A.C. Hindmarsh, Odepack a systemized collection of ode solvers, preprint UCRL-88007, Lawrence Livermore National Laboratory, August 1982.
[4] A. Biran, M. Breiner, MATLAB 5 for Engineers, Addison Wesley, 1999. See also: http://www.Mathworks.com.
[5] H.E. Merritt, Hydraulic Control Systems, Wiley & Sons, 1967.
[6] W. Wuest, Strömung durch Schlitz- und Lochblenden bei kleinen Reynolds-Zahlen, Ingenieur Archiv (22) (1954) 357–367.
[7] J.U. Thoma, H.J. Halin, Bondgraphs and practical simulation, Simulat. Practice Theory 7 (5–6) (1999) 401–417.
[8] J.F. Broenink, 20-sim software for hierarchical bond-graph/block-diagram models, Simulat. Practice Theory 7 (5–6) (1999) 481–492. See also: http//www.20sim.com.
[9] H. Weule, Theoretische und experimentelle Untersuchung digitaler hydraulischer Positionierantriebe. PhD thesis, Technische Universität Braunschweig, 1972.
[10] A.P.J. Breunese, J.F. Broenink. Modeling mechatronic systems using the SIDOPS + language, in: J.J. Granda, G. Dauphin-Tanguy (Eds.), 1997 International Conference on Bond Graph Modeling, and Simulation (ICBGM'97), Simulation Series, vol. 29, no. 1, ISBN: 1-56555-103-6, 1997, pp. 301–306.
[11] H. Elmqvist et al., Modelica™–A Unified Object-Oriented Language for Physical Systems Modeling. http://www.Modelica.org.