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On the One-Dimensional Theory of Steady Compressible Fluid Flow in Ducts with Friction and Heat Addition*

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Steady, diabatic (non-adiabatic), frictional, variable-area flow of a compressible fluid is treated in differential form on the basis of the one-dimensional approximation. The basic equations are first stated in terms of pressure, temperature, density, and velocity of the fluid. Considerable simplification and unification of the equations is then achieved by choosing the square of the local Mach number as one of the variables to describe the flow.

The transformed system of equations thus obtained is first examined with regard to the existence of a solution. It is shown that, in general, a solution exists whose calculation requires knowledge only of the variation with position of any three of the dependent variables of the system. The direction of change of the flow variables can be obtained directly from the transformed equations without integration. As examples of this application of the equations, the direction of change of the flow variables is determined for two special flows.

In the particular case when the local Mach number $M=1$, a special condition must be satisfied by the flow if a solution is to exist. This condition restricts the joint rate of variation of heating, friction, and area at $M=1$. Further analysis indicates that when a solution exists at this point it is not necessarily unique.

Finally, it is shown that the physical phenomenon of choking, which is known to occur in certain simple flow situations, is related to restrictions imposed on the variables by the form of the transformed equations. The phenomenon of choking is thus given a more general significance in that the transformed equations apply to a more general type of flow than has hitherto been treated.

INTRODUCTION

THE rational experimental development of jet- and rocket-propulsion power plants requires adequate knowledge of the theoretical mechanics of diabatic (non-adiabatic), frictional, variable-area compressible fluid flow. The differential equations describing this type of flow are well known.[1a, 1b, 2–4] Their solution in the three-dimensional case, however, is so difficult that some simplification is necessary to permit development of the theory in a form immediately useful for technical applications.

In the present paper, such simplification is effected by generalizing the familiar "one-dimensional" or hydraulic treatment of steady fluid flow to include the simultaneous effects of heat addition, friction, and area change upon the flow of a compressible fluid rather than by attempting to show that the one-dimensional approximation follows from a simplification of the hydrodynamic and heat-flow equations in their general three-dimensional form. The generalization leads to one-dimensional equations in differential form, which are identical with equations previously used by other investigators in less general cases.

Generalized conservation equations have been derived in Appendix A in order that a complete and logical basis for the theory may be accessible to the reader. The resulting theory is intended to serve as a foundation in differential form for calculation of all types of mathematically continuous (that is, shockless) flow of perfect gases to which the one-dimensional approximation is applicable. Thus the theory applies directly to

* This paper is a revised report of theoretical work performed by the authors at the Cleveland Laboratory of the National Advisory Committee for Aeronautics in 1944–45, and was issued in slightly different form as NACA TN No. 1336, 1947.

[1] W. F. Durand, ed. *Aerodynamic Theory* (Julius Springer Verlag, Berlin, 1935): (a) Vol. I, p. 237; (b) Vol. III, pp. 47–49.

[2] C. Eckart, Phys. Rev. **58**, 267 (1940).

[3] Wien-Harms, *Handbuch der Experimental Physik* (Akaddemische Verlagsgesellschaft, Leipzig, 1931), Bd. 4, pp. 343–369.

[4] A. Vazsonyi, Quart. Appl. Math. **3**, 29 (1945).

compressible flow in combustion chambers and also, with but slight modification, to flow in turbines, compressors,[5] nozzles, and diffusers, whenever the one-dimensional approximation is valid.

In order to obtain convenient and unified equations, the generalized relations are transformed by introducing a new basic variable, the square of the local Mach number $M^2=N$. Pressure and temperature are chosen as the additional basic variables; other relevant flow variables (for example, density, velocity, mass flow) may be expressed in terms of Mach number squared, pressure, and temperature. Values of M from zero to infinity are considered; the treatment is therefore applicable to both subsonic and supersonic flow.

The variable M has been used throughout differential treatments by Gukhman,[6] Bailey,[7] Nielsen,[8] and Chambre and Lin,[9] who investigated various examples of frictional, diabatic compressible flow.** A related variable

$$Z=(\gamma-1)M^2/[2+(\gamma-1)M^2],$$

which can be used alternatively with M, will be discussed briefly in Appendix B. Pertinent papers in which M is not used extensively are those by Binder[10] and Keenan and Neumann,[11] which report studies of isothermal and adiabatic frictional flow, respectively. A treatment of frictionless, diabatic compressible flow carried out by Szczeniowski[12] is partly in differential form. The same subject, in M language without differential formulation, is discussed by Hicks[13] and Wood.[14] The variable M has also been employed to advantage by Sorg[5] for the analysis of compressible flow through turbines and compressors, a related field that is not specifically discussed in the present paper. Mach and Crocco vectors, the square of whose magnitudes are N and Z, have been found useful for the description of three-dimensional adiabatic and diabatic flows.[15a, 15b]

In the general case, the differential equations obtained in the present treatment do not permit of formal integration; but being of first order, they are particularly amenable to numerical methods. A solution of the system is shown to exist, except possibly at sonic velocity, and the behavior of the solution in this neighborhood is investigated. From the differential equations useful information may easily be obtained about direction of changes in the flow variables. Choking is shown to be a consequence of a certain property of the equations.

THE ONE-DIMENSIONAL APPROXIMATION

Basic Equations

The "one-dimensional" steady-flow theory utilizes a model consisting of a perfect gas contained within a duct, across any section of which the flow variables are constant. Only the component of velocity normal to the section is considered; body forces are neglected, and heat, whether supplied by combustion, conversion of frictional work, or conduction from the walls, is assumed to be transferred instantaneously and completely but only transversely throughout the cross section, which may be of variable area. Each flow variable can thus be considered as a function of a single parameter, say the distance along the axis of the tube, whence the term "one-dimensional."

The conventional variables—pressure, temperature, density, and velocity in one-dimensional flow—are connected by four relations derivable from the first law of thermodynamics, the conservation of mass, the second law of motion, and the thermal equation of state for a perfect gas. The four relations are:

[5] K. W. Sorg, Forschung **10**, 270 (1939).
[6] A. A. Gukhman, J. Tech. Phys. U.S.S.R. **9**, 411 (1939).
[7] N. P. Bailey, J. Ae. Sci. **2**, 227 (1944).
[8] J. N. Nielsen, NACA ARR No. L4C16 (1944).
[9] P. Chambre and C. C. Lin, J. Ae. Sci. **13**, 537 (1946).
** The authors regret that explicit reference could not be made to several valuable (classified) papers by E. R. Hawthorne. It is our understanding that Dr. Hawthorne and Professor Shapiro have been invited to present a comprehensive paper on flow of compressible fluids with heat transfer at a meeting of the American Society of Mechanical Engineers later this year.
[10] R. C. Binder, A.S.M.E. Trans. **66**, 221 (1944).
[11] J. H. Keenan and E. P. Neumann, NACA Tech. Note No. 963 (1945).
[12] B. Szczeniowski, Can. J. Research **23**, 1 (1945).
[13] B. L. Hicks, NACA ACR No. E5A29 (1945) (Wartime Report No. E-88).
[14] G. P. Wood, J. Ae. Sci. **14**, 24 and 63 (1947).
[15] (a) B. L. Hicks, P. E. Guenther, and R. H. Wasserman, Quart. Appl. Math. (1947); (b) B. L. Hicks, Quart. Appl. Math., accepted for publication in 1948, see also Phys. Rev. **69**, 135 (A), 250 (A) (1946); **71**, 476 (A) (1947); Ballistic Research Laboratories Report No. 633 (1947).

Conservation of energy $c_p dT + VdV = dQ$, (1)

Conservation of mass $d(\rho VA) = 0$, (2)

Equation of motion $-dp = \rho VdV + \rho dF$, (3)

Equation of state $d(p/R\rho T) = 0$. (4)

The specific heat at constant pressure c_p and the gas constant R do not vary in the flow. The symbols ρ, V, p, and T, respectively, stand for density, velocity, absolute static pressure, and absolute static temperature. The pipe area, which may be variable, is represented by A. Heat added per unit mass is indicated by Q, and work per unit mass done against friction by F. Consistent units are used throughout. In Eqs. (1)–(4) each variable is to be considered as a function of a single parameter, such as the distance x along the tube considered positive in the direction of flow; and, of course, the meaning of each differential du is then given by

$$du = u'(x)dx.$$

Equations (1)–(3) are customarily used without explicit recognition of their true meaning with regard to the one-dimensional approximation. The interpretation of the quantity dF in particular is often obscure. In order to provide a logical, unified basis for the theory, Eqs. (1)–(3) are derived in Appendix A; special care is taken to keep the derivations within the framework of the one-dimensional approximation.

Applicability

The validity of the one-dimensional approximation depends upon the assumption of the uniformity of flow conditions across a plane normal to the direction of flow. Experience has shown that this assumption constitutes an adequate approximation in many special cases; in particular, with subsonic turbulent flow in long pipes without separation, the reasonably flat velocity profile permits the use of equations derived on this basis. Van Driest[16] has shown how the results of one-dimensional theory for incompressible fluids can be corrected for the effects of turbulence and non-uniformity of velocity distribution in cases where these factors have been evaluated experimentally. The corrections to the energy equation are quite complicated, and their complexity would be increased for compressible flow. In general there is insufficient experimental data available at present to permit formulation of such corrections where compressibility, incomplete growth of the boundary layer, or separation of the flow must be considered. Although compressibility and boundary layer effects are somewhat amenable to calculation, the occurrence of sparation is difficult or impossible to predict, and the question of applicability, in these cases, of the one-dimensional approximation must be determined by experiment or estimated by experience.

The one-dimensional approximation would not be valid if oblique shocks occur in the flow. Nor can normal shocks, if treated as flow discontinuities, be handled in the differential form of the present approximation. If, however, in Eqs. (1) and (3), dQ and dF are considered to depend upon the derivatives of T and V and if heat and momentum transfer in the direction of flow is allowed, then the equations for continuous normal shock[17] can be put in the form of Eqs. (1)–(4).

In the development and use of Eqs. (1)–(4) various approximations are made, such as neglecting the squares of velocity components normal to the direction of flow, replacing the square of the cosine of the half-angle by unity, and assuming the constancy of R and c_p. In this paper no attempt is made to state under what circumstances such approximations are suitable.

TRANSFORMATION OF EQUATIONS

Change of Variables

A canonical form for Eqs. (1)–(4) is obtained by taking logarithmic derivatives and choosing as a variable the square of the local Mach number,

$$M^2 \equiv N = V^2/\gamma RT, \qquad (5)$$

where γ is the ratio of the specific heats. This choice to obtain simplification of the equations is not unique; similar advantages result with other dimensionless combinations of velocity squared and a temperature. For instance, some workers have used the ratio of dynamic tem-

[16] E. R. Van Driest, A.S.M.E. Trans. **68A**, 231 (1946).

[17] Reference 1, p. 219.

perature to total temperature; in Appendix B to the present report, the canonical differential equations in terms of this variable are given.

If Eqs. (1)–(4) are divided by c_pT, ρVA, p, and $p/R\rho T$, respectively, there result

$$(\gamma-1)\frac{V^2}{\gamma RT}\frac{dV}{V}+\frac{dT}{T}=\frac{dQ}{c_pT}, \tag{6}$$

$$\frac{dV}{V}+\frac{d\rho}{\rho}=-\frac{dA}{A}, \tag{7}$$

$$\gamma\frac{V^2}{\gamma RT}\frac{dV}{V}+\frac{dp}{p}=-\frac{\rho dF}{p}, \tag{8}$$

$$-\frac{dp}{p}+\frac{dT}{T}+\frac{d\rho}{\rho}=0. \tag{9}$$

With use of Eq. (5) and the expression for dV/V obtained by logarithmic differentiation of Eq. (5),

$$\frac{dV}{V}=\frac{1}{2}\left(\frac{dN}{N}+\frac{dT}{T}\right) \tag{10}$$

and, upon elimination of $d\rho/\rho$, there are found

$$\frac{(\gamma-1)N}{2}\frac{dN}{N}+\left[1+\frac{(\gamma-1)N}{2}\right]\frac{dT}{T}=\frac{dQ}{c_pT}\equiv d\theta, \tag{11}$$

$$\frac{1}{2}\frac{dN}{N}+\frac{dp}{p}-\frac{1}{2}\frac{dT}{T}=-\frac{dA}{A}\equiv d\alpha, \tag{12}$$

$$-\frac{\gamma N}{2}\frac{dN}{N}-\frac{dp}{p}-\frac{\gamma N}{2}\frac{dT}{T}=\frac{dF}{RT}\equiv d\mu, \tag{13}$$

where the dimensionless quantities $d\theta$, $d\alpha$, and $d\mu$ have been introduced to simplify the following analysis.

Solution for Logarithmic Differentials

If the determinant formed by the coefficients of dN/N, dp/p, and dT/T in Eqs. (11), (12), and (13) is not identically zero, the equations may be solved uniquely for these three differentials. As the determinant in question is proportional to $(1-N)$, which vanishes only for $N=1$, the solution is obtained as follows:

$$dN/N=(1-N)^{-1}\{(1+\gamma N)d\theta+[2+(\gamma-1)N]d\mu+[2+(\gamma-1)N]d\alpha\}, \tag{14}$$

$$dp/p=(1-N)^{-1}\{-\gamma Nd\theta-[1+(\gamma-1)N]d\mu-\gamma Nd\alpha\}, \tag{15}$$

$$dT/T=(1-N)^{-1}[(1-\gamma N)d\theta-(\gamma-1)Nd\mu-(\gamma-1)Nd\alpha]. \tag{16}$$

It is also convenient to record the differential expressions for the density ρ and velocity V:

$$d\rho/\rho=dp/p-dT/T=(1-N)^{-1}\times(-d\theta-d\mu-Nd\alpha), \tag{17}$$

$$dV/V=(dN/N+dT/T)/2=(1-N)^{-1}(d\theta+d\mu+d\alpha). \tag{18}$$

Application of Second Law of Thermodynamics

The first law of thermodynamics was used in the formulation of the basic equations; the second law of thermodynamics may be employed to furnish additional information. The entropy differential dS for a perfect gas is given[18] by

$$dS/c_p=dT/T-[(\gamma-1)/\gamma]dp/p=d\theta+[(\gamma-1)/\gamma]d\mu. \tag{19}$$

The second law of thermodynamics then states

$$0\leqslant dS/c_p-dQ/c_pT=[(\gamma-1)/\gamma]d\mu. \tag{20}$$

The relation, according to Eq. (19), that

$$dS/c_p=(d\theta+d\mu+d\alpha)-d\mu/\gamma-d\alpha$$

when used with Eq. (20), results in the inequalities

$$d\theta\leqslant dS/c_p=d\beta-d\mu/\gamma-d\alpha\leqslant d\beta-d\alpha, \tag{21}$$

where $d\beta\equiv d\theta+d\mu+d\alpha$.

DISCUSSION OF EQUATIONS

Remarks on Integration of Equations

Equations (14)–(16) can be rewritten as

$$N'=\frac{N}{1-N}\frac{1+\gamma N}{c_pT}Q'+\frac{N}{1-N}\frac{2+(\gamma-1)N}{RT}F'-\frac{N}{1-N}\frac{2+(\gamma-1)N}{A}A', \tag{22}$$

[18] P. S. Epstein, *Textbook of Thermodynamics* (John Wiley and Sons, Inc., New York, 1937), p. 63.

$$p' = -\frac{p}{1-N}\frac{\gamma N}{c_pT}Q' - \frac{p}{1-N}\frac{1+(\gamma-1)N}{RT}F' + \frac{p}{1-N}\frac{\gamma N}{A}A', \quad (23)$$

$$T' = \frac{T}{1-N}\frac{1-\gamma N}{c_pT}Q' - \frac{T}{1-N}\frac{(\gamma-1)N}{RT}F' + \frac{T}{1-N}\frac{(\gamma-1)N}{A}A', \quad (24)$$

where the primes indicate differentiation with respect to x. This system clearly satisfies, except at $N=1$, the conditions of the fundamental existence theorem[19] when Q, F, and A are differentiable. Hence a solution exists except at sonic velocity and may be obtained formally when possible, and by standard numerical methods otherwise, as soon as the functions Q, F, and A (or their derivatives) are specified. More generally, the system may be solved in similar fashion for any three of the variables N, p, T, Q, F, and A as functions of x, when the variation with x of the other three is prescribed. Also it may be noted that as all the foregoing variables are functions of one parameter, any two may be considered as functions of each other under suitable circumstances.

Direction of Change of Flow Variables

In practical as well as in theoretical work it is frequently useful to be able to determine the direction of change of flow quantities with respect to heat addition, friction, or area variation without troubling to get quantitative information from integrated forms. Equations (14)–(18) or (22)–(24) permit the specification of signs of derivatives at any particular point and also throughout certain regions of flow. Thus Eq. (14) shows that in subsonic flow the effect of positive θ', μ', or α', is to increase N, whereas for supersonic flow the effect is to decrease N. When the derivatives have different signs, the net effect will depend upon the algebraic sum of the separate contributions.

As an example of the use of this technique, suppose heat is added to a fluid in a constant-area pipe, with negligible friction; that is, $\theta' \neq 0$, $\mu' = \alpha' \equiv 0$. It is easily seen from Eqs. (14) through (18) that for the entire range of N from zero to infinity

$$(1-N)dN/dQ \geqq 0, \quad (25)$$

$$dp/dN \leqq 0, \quad (26)$$

$$(1-\gamma N)^{-1}dT/dN \geqq 0, \quad (27)$$

$$d\rho/dN \leqq 0, \quad (28)$$

$$dV/dN \geqq 0. \quad (29)$$

These results are given by Hicks.[13] By use of the chain rule for the derivative of a function of a function, the sign of the derivative of any of the flow variables with respect to any of the others may be obtained; thus, from Eqs. (25) and (29) it is clear that

$$(1-N)\frac{dV}{dQ} = (1-N)\frac{dV}{dN}\frac{dN}{dQ} \geqq 0. \quad (30)$$

As another example, consider the flow in circular cylindrical pipes with heat addition and with friction; that is, $\theta' \neq 0$, $\mu' \neq 0$, $\alpha' \equiv 0$. (See also related discussion by Nielsen.)[8] Equation (14) will be used to determine the direction of change of N with respect to x. If the heat addition is only through the wall, which is at temperature T_w, the heat added per unit mass of fluid in passing a distance dx along the tube is given by

$$\rho VAdQ = h(T_w - T)(\pi Ddx), \quad (31)$$

where D is the tube diameter, and h the local surface-to-fluid coefficient of heat transfer in heat units transferred per unit temperature difference, per unit area.[20a] In conjunction with Eq. (11), Eq. (31) leads to

$$d\theta \equiv \frac{dQ}{c_pT} = \frac{h[(T_w/T)-1]\pi Ddx}{c_p\rho V\pi D^2/4} = \frac{4h[(T_w/T)-1]dx}{c_p\rho VD}. \quad (32)$$

The expression for frictional work done is

[19] L. Bieberbach, *Theorie der Differentialgleichungen* (Julius Springer Verlag, Berlin, 1930), pp. 34–35.

[20] W. H. McAdams, *Heat Transmission* (McGraw-Hill Book Company, Inc., New York, 1942). (a) Eq. 2, p. 135; (b) Eq. 8, p. 119; (c) Eq. 1, p. 162.

assumed to be given[20b] (see also Appendix A) by

$$dF=\frac{fV^2dx}{2(D/4)}, \tag{33}$$

where f is the Fanning friction factor. From Eqs. (13) and (33) it follows that

$$d\mu\equiv\frac{dF}{RT}=\frac{2f\gamma Ndx}{D}. \tag{34}$$

If Reynolds' analogy is valid, h may be replaced[20c] by $c_p\rho Vf/2$, whence Eq. (14) becomes

$$dN/dx=[N/(1-N)]\{(1+\gamma N)[(T_w/T)-1] +[2+(\gamma-1)N]\gamma N\}2f/D. \tag{35}$$

This equation may be used to determine the direction of change of N with x, and hence of other flow quantities, for various ranges of N and of T_w/T. For values of $(T_w/T)\ll 1$, (maximum rate of cooling), dN/dx is positive for values of

$$1>N>[-\gamma+(\gamma(5\gamma-4))^{\frac{1}{2}}]/2\gamma(\gamma-1)=0.58$$

for $\gamma=1.4$; that is, the effects of friction in increasing the Mach number overbalance the effects of the cold walls in lowering it if $1>M=N^{\frac{1}{2}}>0.76$ for $\gamma=1.4$ If $N>1$ then $(dN/dx)<0$, and acceleration of frictional, supersonic flow by convective cooling appears to be impossible. This behavior of compressible gases affects the design of supersonic wind tunnels.[21] Acceleration of frictionless supersonic flow by cooling should, however, be possible[9,13].

Behavior of Solution at Sonic Velocity

The differential equations (14)–(18) must be examined as to behavior at the singular point $N=1$. In order that the logarithmic differentials may be defined at this point, it is necessary that

$$d\beta\equiv d\theta+d\mu+d\alpha=0 \text{ at } N=1, \tag{36}$$

because each logarithmic differential is proportional to $d\beta/(1-N)$ there. If $d\beta\neq 0$ upstream of the end of the duct, N can equal 1 only at the end of the duct. This situation is illustrated by the "choked" converging nozzle and by the frictional diabatic flow, which is treated in the previous section. Equation (36) is formally satisfied at the end of a duct where $d\theta$, $d\mu$, and $d\alpha$ may be considered to vanish for all values of N.

Between the ends of a duct, however, $d\beta$ must always vanish where $N=1$. This condition shows that at $N=1$ arbitrary variations of $d\theta$, $d\mu$, and $d\alpha$ are not possible. A specific illustration is the ideal nozzle in which $d\theta=d\mu=0$; according to Eq. (36), $d\alpha$ is then restricted to the value 0, which means that the area has a stationary value at $N=1$. This is the well-known result that sonic velocity can be attained only in the throat of an ideal nozzle in shockless flow. A quite similar treatment applies for the cases where $d\theta$ and $d\mu$ are the quantities to be investigated (see pertinent material in references 5–14 and 21). Condition (36), which was necessitated by the presence of the determinant $(1-N)/2$ in Eqs. (14)–(18), is thus seen to provide a unification of the treatment of the flow behavior in the neighborhood of sonic velocity.

Combination of the second law of thermodynamics with Eq. (36) also yields limitations on the behavior of the flow at $N=1$. According to Eqs. (21) and (36), at sonic velocity

$$d\theta\leqq dS/c_p\leqq -d\alpha=dA/A. \tag{37}$$

These results may be stated in words to the effect that in converging or constant-area channels at $N=1$, neither the heat term $d\theta=dQ/c_pT$ nor the entropy term dS/c_p can be positive. In diverging channels these two terms may be either positive or negative. If either $d\theta$ or $d\mu$ is everywhere 0, relation (21) yields more detailed results. For example, if $d\theta\equiv 0$, then at $N=1$, by Eqs. (36) and (20)

$$\begin{aligned} d\mu&=-d\alpha\\ &=[\gamma/(\gamma-1)]dS/c_p\geqq 0, \quad (d\theta=0,\ N=1). \end{aligned}$$

Continuous flow with friction and without heat addition at sonic velocity cannot, therefore, occur in a converging channel.[7]

A more complete treatment of the behavior of the flow when $N=1$ between the ends of the duct is obtained by considering second-order terms. As N approaches unity, Eq. (14), which may be written

$$N'/N=\{(1+\gamma N)\theta'+[2+(\gamma-1)N]\mu' +[2+(\gamma-1)N]\alpha'\}/(1-N),$$

[21] F. Clauser, Phys. Rev. 71, 465A (1947).

takes the form 0/0. For the evaluation of this limit, L'Hopital's rule gives after some calculation

$$N_0' = [\theta_0' N_0' + (1+\gamma)\beta_0'']/(-N_0'),$$

where subscript 0 denotes value of function at $N=1$. The solution for N_0' is

$$N_0' = -\theta_0'/2 \pm [(\theta'_0/2)^2 - (1+\gamma)\beta_0'']^{\frac{1}{2}}. \quad (38)$$

The double-valuedness of the derivative at $N=1$ will have important consequences in that a unique solution of the equations may not be obtained when $N=1$ along the flow path. In general, it will be possible to continue the solution from $N=1$ along either of two paths, depending on the choice of sign. In certain cases, depending on the signs of θ_0' and β_0'', one sign will correspond to continuation into subsonic flow, the other into supersonic; otherwise the two choices will correspond to different continuations into flow of the same character. This result means that specification of initial conditions and of variation of $d\theta$, $d\mu$, and $d\alpha$ alone is not sufficient to insure a unique solution if N becomes unity along the flow. In the event that the radicand is zero, it is possible that only one solution is obtained; or it may happen that some higher derivative is double-valued with resulting ambiguity of solution. The analysis for this case is somewhat involved and will not be continued here.

It is interesting to note that a less general problem of the same nature has been presented by Lorenz[22] and Prandtl and Proell.[23] Some of this work is possibly more accessible in Stodola's book.[24]

The Phenomenon of Choking

The general equations (11)–(13) impose restrictions on the relations between the flow variables and the heat, friction, and area variation. When these restrictions take the form of upper or lower limits on mass flow, the associated phenomena are termed "choking" processes. As an example, it is well known that the ideal nozzle has for given subsonic entry conditions a maximum mass flow beyond which the discharge cannot be increased no matter how much the exit pressure is lowered. Another case is "thermal choking," wherein the entrance Mach number and hence mass flow in diabatic, frictionless, constant-area flow is limited for given heat addition despite indefinite reductions in outlet pressure.[13]

The nature of choking may be studied with the help of Eq. (22), which was derived simply from the basic equations. It will be shown that unless heat, friction, and area variation are such that $(1-N)$ times the right-hand side of Eq. (22) changes from positive to negative as x increases, the Mach number in the tube cannot become greater than 1 if the entrance velocity is subsonic and cannot become less than 1 if the entrance velocity is supersonic, provided that the flow variables remain continuous.

For convenience, designate by Y the factor $(1-N)$ times the right-hand side of Eq. (22). The quantity Y is seen to consist of a sum of terms in Q', F', and $-A'$ multiplied by functions of N that are always positive. (In the event that only one of the terms Q', F', and $-A'$ is not 0, Y becomes merely the derivative of the heat added, the frictional work, or the area, multiplied by a simple function of the flow variables; then positive Y corresponds to the case of heat addition, friction, or a converging duct.)

Suppose first that Y is always negative. Then if the flow at the entrance section x_1 is subsonic, $dN/dx = (1-N)^{-1}Y < 0$, and the Mach number decreases; if the entering flow is supersonic, $dN/dx = (1-N)^{-1}Y > 0$, and the Mach number increases.

Suppose now that Y is always positive. Then, if the entering flow at x_1 is subsonic, $dN/dx = (1-N)^{-1}Y > 0$, and the Mach number increases. But N cannot increase past unity as x increases. For suppose $N=1$ at $x=x_0$ and is greater than 1 in the right-hand neighborhood of x_0 (exclusive of x_0); then dN/dx is negative in this neighborhood, because $(1-N)$ is less than 0 and Y is greater than 0. Now N is equal to unity at $x=x_0$, is continuous, and has a negative derivative in the neighborhood mentioned. Hence N is less than 1 in this neighborhood, which contradicts the assumption. Therefore N cannot be

[22] H. Lorenz, Physik. Zeits. **4**, 333 (1903).
[23] L. Prandtl and R. Proell, V.D.I. Zeits. **48**, 348 (1904).
[24] A. Stodola, *Steam and Gas Turbines* (McGraw-Hill Book Company, Inc., New York, 1927), pp. 98–101.

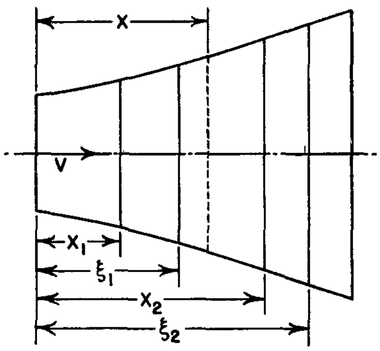

FIG. 1. Fluid elements considered in deriving the equations of energy and mass conservation.

greater than 1 if Y is always positive, N is continuous, and $N(x_1)$ is less than 1. In general, no continuous solution exists for values of $x>x_0$ if Y is always positive. This statement, and the foregoing proof, are valid even if $N'(x_0)$ does not exist. An analogous development may be made for $N(x_1)$ greater than 1 with the conclusion that, with Y positive and N continuous, N cannot be less than 1.

If Y changes from positive to negative at $x=x_0$, however, the value of N may cross unity at that point, but if Y is initially negative, N goes away from unity as previously shown and can only turn toward unity if Y changes from negative to positive; this change must be made at some value of N other than 1. After Y has changed to positive, the situation reduces to the case that Y is always positive, with the N where Y changes sign now taken as the entrance N.

It has been shown that up to some fixed point in the tube, which can be either the exit or the point at which Y changes from positive to negative, N and, therefore, the Mach number do not become greater than 1 if the entering velocity is subsonic nor less than 1 if the entering velocity is supersonic. Furthermore, if Y is positive up to the point at which N is limited, the derivative of N before this point is always positive if the entering flow is subsonic and is always negative if the entering flow is supersonic. Thus, for positive Y and subsonic entrance velocity the entrance N cannot exceed some limiting value less than 1 determined by the particular Q, F, $-A$ variation, for N is always increasing from its initial value and cannot exceed 1; and, by analogous considerations for positive Y and supersonic entrance velocity N cannot be less than some limiting value greater than 1. This limitation is essentially the choking phenomenon.

The specific form this limitation takes is not easily stated in the general case, because the choice of which factors are to be held constant and of which variables are to be considered limited determines the particular form of the restrictions. Numerical results for some particular cases are given in references 7, 13, and 14, among others, to illustrate the nature of possible results. It is felt that additional special cases should be investigated before a thorough study of the general case is attempted.

APPENDIX A

Derivations of Three Basic Equations

Conservation of Energy

The first law of thermodynamics applies to energy changes between two states of a system enclosed within a surface. Let the system (Fig. 1) be the gas of mass Δm that is contained in the initial state within the tube walls and the sections at x_1 and x_2 and in the final state within the tube walls and the sections at ξ_1 and ξ_2; x_1, x_2, and ξ_1 are arbitrary, and ξ_2 is determined by the condition that the mass between x_1 and ξ_1 equal the mass between x_2 and ξ_2. When $m(x)$ is defined as the total mass of gas contained within the duct between the sections $x=0$ and $x=x$, the definition of Δm becomes

$$\Delta m \equiv m(x_2)-m(x_1)=m(\xi_2)-m(\xi_1). \quad (39)$$

Let $U(x)$ denote the internal (thermal) energy per unit mass of fluid, $p(x)$ the static pressure, $V(x)$ the gas velocity, $A(x)$ the cross-sectional area of the duct, each taken at the section x; and $Q(x)$ the heat (in mechanical-energy units) added from walls or by combustion*** to a unit mass of the fluid during its passage from $x=0$ to $x=x$. Then the first law of thermodynamics says for

*** Actually, the heat liberated by combustion might be considered as part of the internal energy; or the external surroundings might be considered to include the fuel; or the first law might be generalized to include heat sources. The treatment given here is convenient but must be understood to require some justification on one of the bases mentioned.

steady flow that

$$\int_{m(\xi_1)}^{m(\xi_2)}(U+\tfrac{1}{2}V^2)dm-\int_{m(x_1)}^{m(x_2)}(U+\tfrac{1}{2}V^2)dm$$
$$=\int_{m(\xi_1)}^{m(\xi_2)}Qdm-\int_{m(x_1)}^{m(x_2)}Qdm$$
$$-\left[\int_{x_2}^{\xi_2}pAdx-\int_{x_1}^{\xi_1}pAdx\right]. \quad (40)$$

As $Adx=(1/\rho)dm$, where $\rho(x)$ is the density of the gas, Eq. (40) becomes

$$\int_{m(x_2)}^{m(\xi_2)}(U+\tfrac{1}{2}V^2-Q+p/\rho)dm$$
$$=\int_{m(x_1)}^{m(\xi_1)}(U+\tfrac{1}{2}V^2-Q+p/\rho)dm \quad (41)$$

when the limits of integration are changed.

Provided that the integrand is continuous (which requirement excludes shock), the expression on the right-hand side of Eq. (41) may be written, by the theorem of the mean for integrals:

$$[m(\xi_1)-m(x_1)]f(m_1^*),$$

where $f(m_1^*)$ indicates the value of the integrand at some $m=m_1^*$, $m(x_1)<m_1^*<m(\xi_1)$. The integral on the left-hand side yields a similar result, with subscript 1 replaced by subscript 2, whence, by virtue of Eq. (39)

$$f(m_1^*)=f(m_2^*);\quad m(x_1)<m_1^*<m(\xi_1),$$
$$m(x_2)<m_2^*<m(\xi_2). \quad (42)$$

As $\xi_1\rightarrow x_1$, $\xi_2\rightarrow x_2$, this equation becomes

$$f[m(x_1)]=f[m(x_2)]. \quad (43)$$

But x_1 and x_2 are arbitrary. Hence f is a constant, and

$$\frac{d}{dx}(U+\tfrac{1}{2}V^2-Q+p/\rho)=0$$

or, since $d(U+p/\rho)=dH$, where H is the enthalpy per unit mass,

$$dQ=dH+VdV. \quad (44)$$

For a perfect gas, $dH=c_pdT$, whence the energy equation is finally

$$dQ=c_pdT+VdV. \quad (45)$$

Conservation of Mass

The conservation of mass, in the form useful here, states that in a steady flow the mass entering a closed surface during any time interval Δt is equal to the mass leaving during the interval Δt. Let the closed surface consist of the sections at arbitrary ξ_1 and x_2 (Fig. 1) and the portions of the duct between these sections, and let Δt be the time required for the mass $m(\xi_1)-m(x_1)$ to enter the surface while mass $m(\xi_2)-m(x_2)$ flows out. The value of x_1 is arbitrary, whereas ξ_2 is fixed by the conditions on the time interval. Upon definition of $t(x)$ as the time required for a fluid particle to travel from origin $x=0$ to $x=x$, Δt may be defined by

$$\Delta t\equiv t(\xi_1)-t(x_1)=t(\xi_2)-t(x_2). \quad (46)$$

The law of conservation of mass says that

$$\int_{x_1}^{\xi_1}\rho Adx=\int_{x_2}^{\xi_2}\rho Adx. \quad (47)$$

Upon change of variable, this equation becomes

$$\int_{t(x_1)}^{t(\xi_1)}\rho A\frac{dx}{dt}dt=\int_{t(x_2)}^{t(\xi_2)}\rho A\frac{dx}{dt}dt. \quad (48)$$

As before, by the theorem of the mean for integrals and condition (46), the integrand must be constant, whence

$$d(\rho VA)=0. \quad (49)$$

Equation of Motion

The vector form of the second law of motion for continuous media states that the integral of the density of surface forces over a closed surface

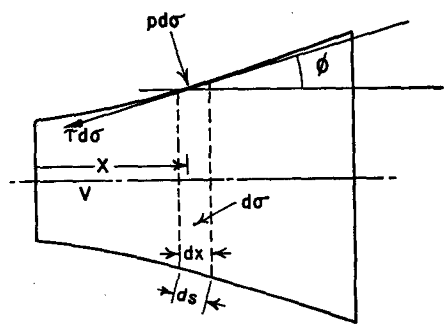

FIG. 2. Forces acting upon a fluid element described in the one-dimensional approximation.

is equal to the integral of the density times the particle derivative of the velocity over the volume enclosed by the surface.[25] For the mass of gas contained within the sections at x_1 and x_2 (Fig. 1) and the walls of the duct, the horizontal component of the equation of motion becomes

$$p(x_1)A(x_1)-p(x_2)A(x_2)+\int_{x_1}^{x_2} Rdx = \int_{x_1}^{x_2} \rho\frac{dV}{dt}Adx, \quad (50)$$

where $R(x)dx$ is the horizontal component of the force exerted on the gas by the portion of the duct between x and $x+dx$. For steady flow, the integral on the right-hand side may be transformed as follows:

$$\int_{x_1}^{x_2} \rho\frac{dV}{dt}Adx=\int_{x_1}^{x_2} \rho\frac{dV}{dx}\frac{dx}{dt},$$

$$Adx=\rho VA\int_{x_1}^{x_2}\frac{dV}{dx}dx=\rho VA[V(x_2)-V(x_1)]. \quad (51)$$

It follows from Eqs. (50) and (51) that the equation of motion may be written in differential form as

$$pdA+Adp+\rho VAdV=Rdx. \quad (52)$$

Now Rdx may be resolved into two constituent parts, the horizontal components of the tangential frictional drag and of the normal

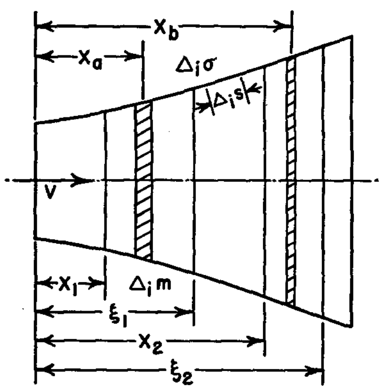

FIG. 3. Fluid elements considered in deriving the equation of momentum conservation.

[25] H. V. Craig, *Vector and Tensor Analysis* (McGraw-Hill Book Company, Inc., New York, 1943), pp. 355–356.

pressure reaction. If the half-angle of the duct is denoted by $\phi(x)$, the wall surface by $\sigma(x)$, and the tangential frictional drag per unit area by $\tau(x)$, from Fig. 2 it is clear that

$$Rdx=-(\tau d\sigma)\cos\phi+(pd\sigma)\sin\phi,$$

and that

$$d\sigma=dA/\sin\phi,$$

hence

$$Rdx=-(\tau d\sigma)\cos\phi+pdA. \quad (53)$$

It is possible to use Eq. (53) directly without further analysis if the friction factor is related to τ by the equation $\tau=f\rho V^2/2$. In many engineering treatments, however, f is defined in terms of the "energy loss due to friction." In order to make this concept of energy loss more precise and to make possible derivation of a rigorous connection between τ and energy loss, define $F(x)$ as the work done by unit mass of the fluid against friction in moving from the origin to position x. The work done in moving the entire mass of fluid between x_1 and ξ_1 to the region between x_2 and ξ_2 (see Fig. 3) will be computed in terms of the original variables τ and σ and in terms of the new variables F and m. If the two quantities are equated and suitably transformed, a relation will be obtained between dF and $\tau d\sigma$.

Let x_1, ξ_1, and x_2 be chosen arbitrarily, and let ξ_2 be determined by the condition that the total mass Δm between x_1 and ξ_1 equal the total mass between x_2 and ξ_2. Let x_a be an arbitrary point between x_1 and ξ_1, and let x_b be determined by the condition that $m(x_a)-m(x_1)=m(x_b)-m(x_2)$. In particular, if $x_a=x_1$, then $x_b=x_2$; if $x_a=\xi_1$, then $x_b=\xi_2$. Thus it is seen that, as x_a runs from x_1 to ξ_1, x_b runs from x_2 to ξ_2.

In order to determine the work done in terms of the variables τ and σ, the procedure is to move thin sections from their original positions, given in each case by x_a, to their final positions x_b, to find the work done by each of them, and then to add up the work done for all the sections.

First, divide the mass Δm between x_1 and ξ_1 into smaller elements of mass $\Delta_1 m$, $\Delta_2 m$, $\cdots\Delta_n m$. (See Fig. 3.) Let $\Delta_i\sigma$ be the wall surface corresponding to the element of mass $\Delta_i m$, and let the duct wall between the initial and the final position of $\Delta_i m$ be divided into elements of length $\Delta_j s$. For a given $\Delta_i m$ it follows from Fig. 3

that

$$\Delta_i m=\rho_i A_i \Delta_i x,$$
$$\Delta_i \sigma=2\pi r_i \Delta_i s,$$
$$\Delta_i x=\cos\phi_i \Delta_i s,$$

so that

$$\Delta_i \sigma=g_i \Delta_i m,$$

where $g_i \equiv 2\pi r_i/\rho_i A_i \cos\phi_i$. Each of the quantities ρ_i, r_i, ϕ_i, and g_i is to be evaluated at the proper point within $\Delta_i x$.

Finally, from Fig. 3 there exists on $\Delta_i m$ a force $\Delta F_i(s_j)$, to be overcome by the work against friction, equal to

$$\tau(s_j)\Delta_i\sigma(s_j)=\tau(s_j)g_i(s_j)\Delta_i m,$$

where it is convenient to consider τ and σ as functions of s, inasmuch as frictional drag and the extension of an element of fixed mass depends on the position of the element in the tubes.

For any element of mass $\Delta_i m$ then, an approximation (the accuracy of which depends on the size of $\Delta_i m$) to the work done by $\Delta_i m$ in going from section x_a to section x_b is

$$\lim_{\Delta_j s\to 0}\sum_{s(x_a)}^{s(x_b)}\Delta F_i(s_j)\Delta_j s=\lim_{\substack{k\to\infty\\ \Delta_j s\to 0}}\sum_{j=1}^{k}\tau(s_j)g_i(s_j)\Delta_i m\Delta_j s$$
$$=\Delta_i m\int_{s[m(x_{a_i})]}^{s[m(x_{b_i})]}\tau(s)g_i(s)ds.$$

It follows that the corresponding approximation to the work done by the entire mass Δm is

$$\sum_{i=1}^{n}\left\{\int_{s[m(x_{a_i})]}^{s[m(x_{b_i})]}\tau(s)g_i(s)ds\right\}\Delta_i m$$

and, as the approximation becomes more and more accurate as the largest $\Delta_i m$ becomes smaller and smaller, the expression for the work done against friction when the entire mass between x_1 and ξ_1 is transported to the position between x_2 and ξ_2 becomes

$$\lim_{\substack{n\to\infty\\ \Delta_i m\to 0}}\sum_{i=1}^{n}\left\{\int_{s[m(x_{a_i})]}^{s[m(x_{b_i})]}\tau(s)g_i(s)ds\right\}\Delta_i m$$
$$=\int_{m(x_1)}^{m(\xi_1)}\int_{s[m(x_a)]}^{s[m(x_b)]}\tau(s)g(s)dsdm. \quad (54)$$

It is clear that the work done against friction when the entire mass is moved as previously described is equal also to

$$\lim_{\substack{\Delta_j F\to 0\\ \Delta_i m\to 0}}\sum_i\sum_j\Delta_j F\Delta_i m=\int_{m(x_1)}^{m(\xi_1)}\int_{F[m(x_a)]}^{F[m(x_b)]}dFdm$$
$$=\int_{m(x_1)}^{m(\xi_1)}\int_{F[m(x_a)]}^{F[m(x_b)]}(dF/ds)dsdm \quad (55)$$

whence it follows, since the intervals for both integrations are arbitrary, that

$$dF/ds=\tau g \quad (56)$$

and

$$dF=\frac{2\pi r}{\rho A\cos\phi}\frac{\tau d\sigma}{2\pi r}$$

whence

$$(\tau d\sigma)\cos\phi=\cos^2\phi\rho AdF. \quad (57)$$

Because, for the small angles usually under consideration, $\cos^2\phi$ is very nearly unity (for a half-angle of 6°, $\cos^2\phi=0.989$), the retention of this factor except for particularly precise work would not seem justifiable. Hence Eq. (53) may be written

$$Rdx=-\rho AdF+pdA. \quad (58)$$

The differential equation of motion is finally

$$-dp=\rho VdV+\rho dF. \quad (59)$$

The connection between dF and the differential loss in stagnation pressure $(-dp_t)$ can, with the help of Eqs. (14) and (15), be expressed in the form

$$\frac{-dp_t}{p_t}=\frac{dF}{RT}+\frac{\gamma Nd\theta}{[2+(\gamma-1)N]}. \quad (60)$$

Thus except for the limiting case of incompressible flow, $(-dp_t)$ and dF cannot be used interchangeably in defining the friction factor even for the adiabatic case where $d\theta=0$.

APPENDIX B

The *Z* Language

In the place of $N=V^2/\gamma RT$ the equations may be formulated in terms of Z, defined through

$$Z=(V^2/2c_p)/(T+V^2/2c_p). \quad (61)$$

The numerator is the so-called dynamic temperature, the denominator is the total temperature. Z and N are related by

$$Z=[(\gamma-1)N]/[2+(\gamma-1)N], \quad (62)$$

$$N=2Z/(\gamma-1)(1-Z). \quad (63)$$

This replacement represents a one-to-one transformation of N into Z in the range 0 to infinity for N, 0 to 1 for Z.

In order to illustrate the form that some of the earlier equations take under this transformation, Eqs. (14)–(16) are written in terms of Z,

$$\frac{dZ}{Z}=\frac{2}{1-[(\gamma+1)/(\gamma-1)]Z}\left\{\frac{1-Z}{2}\left[1+\left(\frac{\gamma+1}{\gamma-1}\right)Z\right]d\theta+(1-Z)d\mu+(1-Z)d\alpha\right\}, \quad (64)$$

$$\frac{dp}{p}=\frac{2}{1-[(\gamma+1)/(\gamma-1)]Z}\left(-\frac{\gamma}{\gamma-1}Zd\theta-\frac{1+Z}{2}d\mu-\frac{\gamma}{\gamma-1}Zd\alpha\right), \quad (65)$$

$$\frac{dT}{T}=\frac{2}{1-[(\gamma+1)/(\gamma-1)]Z}\left[\left(\frac{1-Z}{2}-\frac{\gamma Z}{\gamma-1}\right)d\theta-Zd\mu-Zd\alpha\right]. \quad (66)$$

Crystalline Aggregation of Cobalt Powder[a]

J. T. McCartney[b] and R. B. Anderson[c]

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In electron microscopic studies of Fischer-Tropsch catalysts, an interesting phenomenon was observed in cobalt metal powder reduced from cobaltous oxide. The oxide particles sintered into larger smooth droplets of cobalt that were aggregated into thin hexagonal-shaped platelets. X-ray diffraction analysis showed the presence of the hexagonal crystal phase of cobalt. A possible explanation is that the aggregates are formed by forces similar to those operating in normal crystal formation, but of reduced magnitude insufficient to destroy the identity of the component particles.

In the course of examination, with the electron microscope, of various components of cobalt catalysts used in the Fischer-Tropsch synthesis of liquid and solid hydrocarbons from mixtures of carbon monoxide and hydrogen, an interesting phenomenon was observed in cobalt metal powder reduced from the oxide. The cobalt was prepared by precipitation of cobaltous oxide from cobaltous nitrate with aqueous ammonia, reduction of the oxide in hydrogen at 250°C, and stabilization of the metallic cobalt against rapid oxidation by exposure to carbon dioxide at liquid nitrogen temperature. Figures 1 and 2 are electron micrographs showing the relative sizes and shapes of the particles of the oxide and the reduced metal. Surface area measurements by a nitrogen adsorption method[1] gave values of 67 square meters per gram for the oxide and 2 square meters per gram for the reduced cobalt. Thus, during reduction, there was an evident sintering of the irregular particles of the oxide into larger smooth droplets of cobalt metal.

Such droplets have been observed in several other metals prepared and reduced in a similar way. In cobalt, however, these particles are more or less closely aggregated into hexagonal platelets such as shown in Figs. 3 and 4. Varying degrees of dispersion in different specimens are illustrated in Figs. 2–4. The hexagons vary in diameter from 7 to 25 microns, and their thickness is apparently of the order of the diameter of the component particles. The hexagonal shape can be discerned also in the optical microscope. X-ray diffraction analysis shows that the crystal phase present is hexagonal cobalt. The hexagons are quite stable,

[a] Contribution from the Central Experiment Station, Pittsburgh, Pennsylvania. Published by permission of the Director, Bureau of Mines, U. S. Department of the Interior.

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